Parametric Biobjective Linear Programming

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Abstract

Linear parametric optimization has been used for decades to combine multiple objective functions into a single problem. The solution to this problem is a set of optimal solutions, containing a solution that is optimal for each parameter. *Multi-objective optimization* is commonly used when multiple objectives are optimized simultaneously, and these objectives are often conflicting. Because these objective functions are conflicting, there is usually no unique optimal solution. Instead, the goal is to find all *nondominated images* that represent the trade-offs among the objectives. The *weighted sum scalarization* method is a well-known approach for finding nondominated images by transforming a multi-objective optimization problem into a single-objective optimization problem.

In this thesis, we consider linear parametric programming problems with multiple objective functions depending linearly on some parameters. Both parametric (single-objective) linear programming and (non-parametric) multi-objective linear programming are well-researched topics. However, literature on the combination of both, parametric linear programming with multiple objectives, is scarce. This research gap encourages our work in this field. More precisely, we examine linear parametric programs with multiple objective functions that depend linearly on some parameters. We investigate various cases of parametric biobjective linear programs and multi-parametric biobjective linear programs. We establish a connection of these problems to non-parametric multi-objective problems. Using the so-called *weight set decomposition*, we are able to explain the behavior of parametric biobjective linear programs when the parameter value is variated. We prove that there is a one-to-one correspondence between the solution of some parametric biobjective programs and the solution of the corresponding multi-objective linear program using the weighted sum scalarization.

We provide structural insights to the solution of parametric biobjective linear programs with respect to extreme weights of the weight set of the multi-objective linear program and develop solution strategies for the parametric program. Similarly, we extend our analysis to biparametric biobjective linear programs and a generalization of our findings to parametric multi-objective linear programs. We characterize the structure of the parameter set of both single and biparametric problems using the weight set of the multi-objective linear programs. Finally, we develop algorithms to solve parametric biobjective linear programs based on the weight-set decomposition.

Zusammenfassung

Lineare parametrische Optimierung wird seit Jahrzehnten verwendet, um mehrere Zielfunktionen zu einem einzigen Problem zu kombinieren. Die Lösung dieses Problems ist eine Menge optimaler Lösungen, die für jeden Parameter eine optimale Lösung enthält. *Multiobjektive Optimierung* wird häufig verwendet, wenn mehrere Ziele gleichzeitig optimiert werden und diese Ziele oft miteinander in Konflikt stehen. Da diese Zielfunktionen miteinander in Konflikt stehen, gibt es in der Regel keine eindeutige optimale Lösung. Stattdessen besteht das Ziel darin, alle *nicht dominierten Bilder* zu finden, die die Kompromisse zwischen den Zielen darstellen. Die Methode der *gewichteten Summenskalierung* ist ein bekannter Ansatz zur Suche nach nicht dominierten Bildern, indem ein multikriterielles Optimierungsproblem in ein einkriterielles Optimierungsproblem umgewandelt wird.

In dieser Arbeit betrachten wir lineare parametrische Programmierungsprobleme mit mehreren Zielfunktionen, die linear von bestimmten Parametern abhängen. Sowohl die parametrische (einzielige) lineare Programmierung als auch die (nicht-parametrische) mehrzielige lineare Programmierung sind gut erforschte Themen. Die Literatur zur Kombination beider Ansätze, also zur parametrischen linearen Programmierung mit mehreren Zielen, ist jedoch rar. Diese Forschungslücke motiviert unsere Arbeit auf diesem Gebiet. Genauer gesagt untersuchen wir lineare parametrische Programme mit mehreren Zielfunktionen, die linear von bestimmten Parametern abhängen. Wir untersuchen verschiedene Fälle von parametrischen biobjektiven linearen Programmen und multiparametrischen biobjektiven linearen Programmen. Wir stellen einen Zusammenhang zwischen diesem Problem und nichtparametrischen Mehrzielproblemen her. Mit Hilfe der sogenannten *Gewichtssatzzerlegung* können wir das Verhalten parametrischer biobjektiver linearer Programme bei Variation des Parameterwerts erklären. Wir beweisen, dass es eine Eins-zu-Eins-Entsprechung zwischen der Lösung des (einzigen) parametrischen biobjektiven Programms und der Lösung des triobjektiven linearen Programms unter Verwendung der gewichteten Summenskalierung gibt.

Wir liefern strukturelle Einblicke in die Lösung des parametrischen, biobjektiven, linearen Programms in Bezug auf extreme Gewichte des Gewichtssatzes des multikriteriellen, linearen Programms. Darüber hinaus entwickeln wir Lösungsstrategien für das parametrische Programm. In ähnlicher Weise erweitern wir unsere Analyse auf biparametrische biobjektive lineare Programme und verallgemeinern unsere Ergebnisse auf parametrische multikriterielle lineare Programme. Unter Verwendung des Gewichtssatzes multikriterieller linearer Programme charakterisieren wir die Struktur des Parametersatzes für ein- und zweiparametrische Probleme. Abschliessend entwickeln wir Algorithmen für parametrische biobjektive lineare Programme auf Basis der Gewichtssatzzerlegung.

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Contents

Chapter 1

Introduction

The work presented in this thesis lies at the intersection of two well-studied branches of optimization and mathematical programming: *parametric optimization* and *multi-objective optimization*. Before describing the subject of our research, we first outline the development and current status of these two disciplines.

Parametric optimization is a classical research topic in optimization. Its purpose is to study the behavior of solutions when the objective functions or constraints of an optimization problem depend on a parameter or multiple parameters. We call a problem *linear parametric* if the objectives are affine linearly dependent on the parameters and the constraints are independent of the parameters. For a detailed overview of the literature on linear parametric optimization, we refer to the recent survey of Nemesch et al. [Nem+25]. In the 1950s, Gass and Saaty are among the first to explore the structure of solutions when coefficients in the objective function are parametrized, introducing the concept of a parametric objective function [SG54]. They present a technique for finding, for each parameter value, the optimal solution to the corresponding linear optimization problem [GS55]. Murty [Mur76] adapts the parametric simplex method to solve linear parametric problems. Eisner and Severance [ES76] develop an efficient algorithm to solve parametric optimization problems by computing parameter values where the optimal solution changes without exhaustively searching the entire parameter set. Later, Gusfield [Gus83] develops a general method to find the sequence of optimal solutions for the entire parameter set. His algorithm builds upon the parametric search method described by Megiddo [Meg78]. In addition to exact algorithms, several approximation algorithms have been developed for parametric optimization problems. For a comprehensive overview of approximation algorithms for parametric problems, we refer again to Nemesch et al. [Nem+25]. Several prominent combinatorial optimization problems have been studied in a parametric setting, including the parametric shortest path problem [YTO91], the parametric assignment problem [GK10], and the parametric minimum cost flow problem [Car83].

Although parametric optimization predominantly addresses problems with one scalar parameter, it is the multi-parametric optimization that deals with parameter vectors. There are two formulations of multi-parametric linear programs in literature; one with a parametric objective function and the second with the parametric right hand sides in the constraints. Gal and Nedoma [GN72] are among the first to consider the computational aspects of multi-parametric linear programming where they present a method to find all non-intersecting regions for each optimal solution in the parameter set for both formulations. Later, Katoh and Ibaraki [KI87] develop an approximation algorithm to partition the parameter set into subregions. More recently, Helfrich et. al [Hel+22] provide an approximation algorithm for a general class of multi-parametric optimization problems with a parametric objective function. Multi-parametric optimization is also used to address multi-level optimization problems, wherein one optimization problem is constrained by another. This process continues depending on the problem level [AP20].

A second, well-studied field of optimization comprises multi-objective optimization problems (MOP) which deals with optimization of multiple, often conflicting objectives. Typically, no single feasible solution is optimal for all objectives simultaneously due to their conflicting nature. Instead, trade-off must be made to balance the objectives. Thus, the notion of optimality is usually understood in the sense of Pareto optimality. The Pareto optimal solutions represent the set of all optimal compromises and informs a decision maker about reasonable alternatives. The goal of multi-objective optimization is to compute the Pareto front or the set of nondominated images in the image set.

In 1975, Yu and Zeleny [YZ75] show that all nondominated images of a multi-objective linear problem (MOLP) with linear objective functions form a subset of the convex hull of the extreme nondominated images. Benson [Ben98] presents an algorithm, known as the "Benson's Outer Approximation Algorithm" to generate the set of all extreme nondominated images in the image set for a MOLP. Later, Heyde and Löhne [HL08] present a geometric approach to duality for multi-objective linear programming, closely related to weighted sum scalarization. Another method for finding extreme nondominated images of multi-objective mixed integer programs is developed by Özpeynirci and Köksalan [ÖK10]. Ehrgott et al. [ELS12] present a dual variant of Benson's outer approximation algorithm using the geometric duality theory of multi-objective linear program. One of the well known methods to find the extreme nondominated images of MOLP with two objectives is the dichotomic approach [Coh04]. Most of the time, solving a multi-objective optimization problem is a challenging task. For example, in multi-objective linear programming, the number of nondominated images usually is infinite and even the number of *extreme nondominated images* can be superpolynomial in the input size Ruhe [Ruh88]. For a discussion on the hardness of general multi-objective combinatorial optimization problems, see [Bök+17].

A common approach to solve multi-objective optimization problems is by scalarization which transforms the problem into a scalar-valued optimization problem. One such scalarization is the weighted sum scalarization method introduced by Zadeh [Zad63]. A single objective is obtained by taking a non-negative linear combination of the objectives. This weighted sum objective is optimized over the same feasible set. Geoffrion [Geo68] proves that an optimal solution of the weighted sum scalarization of the MOP for some weight with strictly positive components is an efficient solution of a MOP.

In terms of the weight set, Benson and Sun [BS02] develop a weight set decomposition algorithm to generate the extreme nondominated images of a MOLP. Przybylski et al. [PGE10] present structural results of the weight set decomposition and propose an iterative algorithm to enumerate all extreme nondominated images by shrinking supersets of the actual weight set components. More recently, Halffmann et al. [Hal+20] propose a weight set decomposition algorithm of triobjective mixed-integer problem.

The combination of these two fields of optimization, i. e. parametric and multi-objective optimization seem apparent, yet it remains largely unexplored in literature. The only closely related literature is that of sensitivity analysis, which typically examines how changes in objective coefficients affect the entire efficient set or a single solution. As data such as costs, risk, and demand might be uncertain in many real-life problems, sensitivity analysis has been proposed to address this data uncertainty. Wendell [Wen84] presents a tolerance approach in linear programming that caters to independent as well as simultaneous variation in several parameters in the coefficients of the objective function. This tolerance approach is further extended to multi-objective linear programs by Hansen et al. [HLW89]. They use the weighted sum method to find the maximum tolerance percentage for the weights to deviate so that a basic solution remains optimal. Sitarz [Sit08] analyzes changes associated to one objective function coefficient and proves that the parameter set in which a given solution remains efficient in a multi-objective linear program is convex. However, these studies examine the sensitivity region with respect to only one extreme efficient solution. Another closely related work is that of Andersen et al. [And+25] where they study the sensitivity of the cost coefficients in multi-objective integer problems. They show that the sensitivity region with respect to a single objective function coefficient is a convex set. They obtain a sensitivity region in terms of permissible changes to coefficients so that a set of efficient solutions remains efficient.

This limited research motivates the study of theoretical and algorithmic approaches for linear parametric multi-objective programming (PMOP). The research in this thesis is an effort to lay the foundation in this area based on parametric and multi-objective optimization techniques. The goal of PMOP is to find, for each parameter value, a set of Pareto optimal solutions. We make use of the weight set decomposition and relate structures in the weight set to explain the optimal solution sets of the parametric biobjective programs.

We consider two variants of the parametric biobjective linear program where the objective functions depend on a real-valued parameter. More precisely, we aim for a sub-division of the parameter set into subintervals such that for every subinterval the same solutions are Pareto optimal. The parameter values where the Pareto optimal sets change are of particular interest. We develop two approaches to obtain these so-called breakpoints, and subintervals determined by two consecutive breakpoints that correspond to a unique set of Pareto optimal solutions. To do so, we relate each of the two parametric biobjective programs to a triobjective program. In fact, each of the biobjective problems can be seen as a partial weighted sum scalarization [Hal+20] of the corresponding triobjective linear problem.

We also examine two distinct cases of the biparametric biobjective linear program by relating each to their respective multi-objective linear program. This time the goal is to find, for each combination of parameters, a set of Pareto optimal solutions. As a result we divide the parameter set into subregions that correspond to each Pareto optimal solution. The structural investigations from our study result in two algorithms. The first algorithm uses any of the established multi-objective programming algorithms and solves linear programs to determine breakpoints of the parametric problem. In the second algorithm, we adapt an existing weighted sum decomposition algorithm.

An overview of the parametric biobjective linear programs and the corresponding multi-objective linear programs, which are considered in this thesis, is listed in Table 1.1.

Problem	**Notation**	**Weighted sum**	**Weight set**	**Parameters**
$\min_{x\in X}\begin{pmatrix}c_1x\\c_2x\\d_1x\end{pmatrix}$	TOLP	$\mathrm{WS}(\mathrm{TOLP}, w^*)$	$\mathcal{W}(\mathrm{TOLP})$	None
$\min_{x\in X}\begin{pmatrix}c_1x+\lambda d_1x\\c_2x\end{pmatrix}$	PBLP^1	$\mathrm{WS}\left(\mathrm{PBLP}^1(\lambda), w\right)$	$\mathcal{W}\left(\mathrm{PBLP}^1(\lambda)\right)$	λ
$\min_{x\in X}\begin{pmatrix}c_1x+\lambda d_1x\\c_2x+\lambda d_1x\end{pmatrix}$	PBLP^2	$\mathrm{WS}\left(\mathrm{PBLP}^2(\lambda), w\right)$	$\mathcal{W}\left(\mathrm{PBLP}^2(\lambda)\right)$	λ, λ
$\min_{x\in X}\begin{pmatrix}c_1x+\lambda d_1x\\c_2x+\lambda d_2x\end{pmatrix}$	PBLP^3	$\mathrm{WS}\left(\mathrm{PBLP}^3(\lambda), w\right)$	$\mathcal{W}\left(\mathrm{PBLP}^3(\lambda)\right)$	λ
$\min_{x\in X}\begin{pmatrix}c_1x\\c_2x\\d_1x\\d_2x\end{pmatrix}$	4-OLP	$\mathrm{WS}\left(\text{4-OLP}, w^*\right)$	$\mathcal{W}(\text{4-OLP})$	None
$\min_{x\in X}\begin{pmatrix}c_1x+\lambda d_1x\\c_2x+\mu d_1x\end{pmatrix}$	BBLP^1	$\mathrm{WS}\left(\mathrm{BBLP}^1(\lambda,\mu), w\right)$	$\mathcal{W}\left(\mathrm{BBLP}^1(\lambda,\mu)\right)$	λ, μ
$\min_{x\in X}\begin{pmatrix}c_1x+\lambda d_1x\\c_2x+\mu d_2x\end{pmatrix}$	BBLP^2	$\mathrm{WS}\left(\mathrm{BBLP}^2(\lambda,\mu), w\right)$	$\mathcal{W}\left(\mathrm{BBLP}^2(\lambda,\mu)\right)$	λ, μ

Table 1.1: Parametric and multi-objective problems with their notations

1.1 Outline

The remainder of this thesis is structured as follows. In Chapter 2, we present relevant definitions and terminology related to multi-objective and parametric optimization. In particular, we state some general properties of the solution set to a parametric linear optimization problem. Then, we briefly introduce the weight set decomposition and explain the connection between its structure and the set of nondominated images of a MOLP.

In Chapter 3, we consider linear parametric biojective programs with two objective functions and a single parameter. Our focus is mainly on two distinct cases of the parametric problem with a single parameter. We establish a connection between parametric biobjective problems and the corresponding triobjective linear program. We develop a theoretical framework relating the solution set of the parametric program to a multi-objective linear program using the structure of weight sets of both these problems. This approach partitions the parameter set into intervals, each associated with a distinct optimal solution. It is followed by an extension to parametric multi-objective linear programs. Next, we consider a special parametric biobjective linear program with same parameter in different parametric objective functions in Chapter 4.

In Chapter 5, we extend our analysis of parametric linear programs to two parameters. Here, we still consider multi-objective linear programs with two objective functions and two associated parameters. We use the weighed sum scalarization method to relate the biparametric biobjective linear program and the corresponding multi-objective linear programs. We characterize the structure of the parameter set of the parametric problem with respect to the weight set of the multi-objective linear programs. We obtain a subdivision of the parameter set into regions such that each region is associated to a unique optimal solution. Furthermore, we extend our results to multi-parametric biobjective linear programs.

Subsequently, we propose algorithms to solve parametric biobjective linear programs in Chapter 6. The *Breakpoint Enumeration Algorithm* relies on a given set of extreme nondominated images of the corresponding triobjective linear program, and the *Adapted Weight Set Decomposition* modifies an existing weight set decomposition algorithm. We also extend these algorithms to parametric multi-objective linear programs of Chapter 3 and biobjective biparametric linear programs in Chapter 4. We illustrate the Breakpoint Enumeration Algorithm with an example.

We conclude this thesis in Chapter 7 and discuss future research in this topic.

1.2 Contributions and credits

Some parts of this thesis are based on following publications:

[YNR25] K. Yuden, L. Nemesch, and S. Ruzika. *Parametric Biobjective Linear Programming*. (submitted). 2025.

[YR26] K. Yuden and S. Ruzika. *Biparametric Biobjective Linear Programming*. (Submitted). 2026.

In particular, Chapter 3 and 6 are joint work with Levin Nemesch and are an extended version of [YNR25]. Most of the content in Chapter 4 is based on [YR26].

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Chapter 2

Preliminaries and definitions

In this chapter, we introduce basic definitions and relevant concepts of linear parametric programming as well as multi-objective programming. We assume the reader to have basic knowledge about linear programming and polyhedral theory. For an overview on linear programming we refer to the book by Hamacher and Klamroth [HK06]. It is followed by some important results related to the weighted sum scalarization. In particular, we recapitulate results concerning the so-called weight set, i. e. the structure of the set of parameters needed for the weighted sum scalarization.

We use the following variants of the componentwise ordering to compare objective function vectors and to establish a notion of optimality. For $y^1, y^2 \in \mathbb{R}^k$ with $k > 1$, the *weak componentwise order*, the *componentwise order*, and the *strict componentwise order* are defined by

$$\begin{aligned} y^1 \leqq y^2 &\iff y^1_i \leq y^2_i \text{ for all } i = 1, \dots, k, \\ y^1 \leq y^2 &\iff y^1 \leqq y^2 \text{ but } y^1 \neq y^2, \text{ and} \\ y^1 < y^2 &\iff y^1_i < y^2_i \text{ for all } i = 1, \dots, k, \end{aligned}$$

respectively. The non-negative orthant is denoted by $\mathbb{R}^k_{\geqq} := \left\{x \in \mathbb{R}^k : x \geqq 0\right\}$ and, likewise, the sets $\mathbb{R}^k_{\geq}$ and $\mathbb{R}^k_{>}$ are defined analogously using the componentwise and strict componentwise ordering.

2.1 Parametric programming

We consider linear programming problems having an objective function which is subject to a non-negative parameter λ.

2.1 Definition (Parametric Linear Program). Let $\lambda \in \mathbb{R}_{\geq}$ be a non-negative parameter. A *parametric linear program* is defined as

$$\begin{aligned} \min \quad & c^\top x + \lambda \cdot d^\top x \\ \text{s. t.} \quad & Ax = b \\ & x \geqq 0, \end{aligned} \tag{PLP}$$

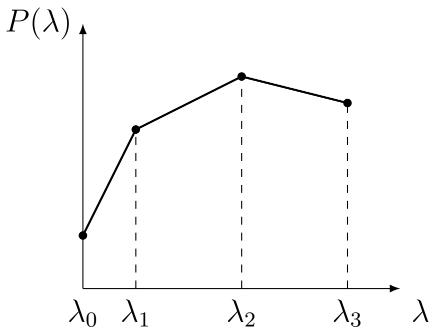


Figure 2.1: An illustration of an optimal value function $P(\lambda)$ with three breakpoints.

where $c, d \in \mathbb{Q}^n$ are coefficient vectors, $A \in \mathbb{Q}^{m\times n}$, $m, n \in \mathbb{N} \setminus \{0\}$, $b \in \mathbb{Q}^m$, and the feasible set is denoted by $X := \{x \in \mathbb{R}^n : Ax \geqq b, x \geqq 0\}$. ◁

For a fixed parameter value $\lambda \in \mathbb{R}_{\geq}$, the problem PLP is a non-parametric single objective optimization problem. We denote it by PLP(λ).

2.2 Definition. The *optimal value function* $P : \mathbb{R}_{\geq} \to \mathbb{R}$ maps each specific parameter value $\lambda^* \in \mathbb{R}_{\geq}$ to the optimal solution value $P(\lambda^*) := c^\top x + \lambda^* \cdot d^\top x$ where x is the *optimal solution* for the (non-parametric) linear program PLP(λ^*). ◁

The function $P(\lambda)$ can be obtained as the lower envelope of all the linear functions associated with basic feasible solutions of the feasible set X (see Figure 2.1). The function $P(\lambda)$ is piecewise linear and concave in λ [Gus83]. The parameter values where the slope of $P(\lambda)$ changes are called *breakpoints*.

The ordered breakpoints of $P(\lambda)$ are denoted by $\lambda_1 < \lambda_2 < \ldots < \lambda_k$ for some $k \in \mathbb{N}$. Thus, the parameter set is decomposed into *parameter intervals* such that a solution $x \in X$ is optimal for the problem PLP(λ) for every parameter value in that interval. We denote a parameter interval by $\mathcal{I}(x)$.

We are interested in finding a set of optimal solutions for a PLP for all values of λ. More precisely, we define a *minimal solution set* in order to have minimal cardinality amongst all such sets of optimal solutions.

2.3 Definition. A *solution set* $S \subseteq X$ of PLP is a set such that for every $\lambda \geq 0$, S contains a solution for the (non-parametric) linear program PLP(λ). It is called *minimal* if, additionally, there is no other solution set $S' \subseteq X$ for PLP with $|S'| < |S|$. ◁

Therefore, the solution to a PLP involves finding a minimal solution set S and the corresponding parameter intervals for each $x \in S$.

Murty [Mur80] establishes that the number of intervals in the parameter set, and, as a result, the number of breakpoints of the parametric linear program can be exponential in the input size.

Building on the parametric linear programming, we now introduce the mutli-parametric linear program in which the objective function depends on a parameter vector, $\lambda \in \mathbb{R}^s_{\geqq}$. Multi-parametric programming deals with minimizing a parametrized objective function while satisfying a set of constraints and obtain an exact mapping of optimal solutions to the set of parameters. We refer to Gal and Nedoma [GN72] for some definitions and notations.

2.4 Definition (Multi-parametric Linear Program). Let $\lambda \in \mathbb{R}^s_{\geqq}$ be a parameter vector. The *multi-parametric linear program* is defined as

$$\begin{aligned} \min \quad & cx + \sum_{i=1}^{s} \lambda_i d_i x \\ \text{s. t.} \quad & Ax = b \\ & x \geqq 0, \end{aligned} \tag{MPLP}$$

where $c, d_i \in \mathbb{R}^n_{\geqq}$ are coefficient row vectors, $A \in \mathbb{Q}^{m \times n}$, $s, m, n \in \mathbb{N} \setminus \{0\}$ and $b \in \mathbb{Q}^m$ and $X \coloneqq \{x \in \mathbb{R}^n : Ax = b, x \geqq 0\}$ is the feasible set. ◁

If we fix the parameter vector $\lambda \in \mathbb{R}^s_{\geqq}$, then the problem is a non-parametric single objective program MPLP(λ). As a result we are interested in optimal solution for MPLP(λ) for each $\lambda \in \mathbb{R}^s_{\geqq}$.

2.5 Definition. A *solution set* $S(\text{MPLP}) \subseteq X$ of MPLP is a set such that for every vector $\lambda \geq 0$, $S(\text{MPLP})$ contains a solution for the single objective program MPLP(λ). It is called *minimal* if, additionally, there is no other solution set $S' \subseteq X$ for MPLP with $|S'| < |S(\text{MPLP})|$. ◁

2.6 Definition. Let $x \in S(\text{MPLP})$ be a solution in a minimal solution set of MPLP. A *critical region* $\mathcal{R}(x) \subseteq \mathbb{R}^s_{\geqq}$ is a set of all feasible parameter vectors for which x is an efficient solution of MPLP(λ). ◁

In case of the single parametric program, the critical region for each solution in a minimal solution set is a parameter interval in $\mathbb{R}$.

The goal in multi-parametric linear programming consists of the following:

i.) a minimal solution set $S(\text{MPLP}) \subseteq X$, and
ii.) critical regions in the parameter set $\mathbb{R}^s_{\geqq}$ corresponding to each solution in a minimal solution set.

2.2 Multi-objective optimization

We introduce some relevant definitions and concepts of multi-objective optimization and polyhedral theory. For a detailed introduction to multi-objective optimization, we refer the reader to the book of Ehrgott [Ehr05].

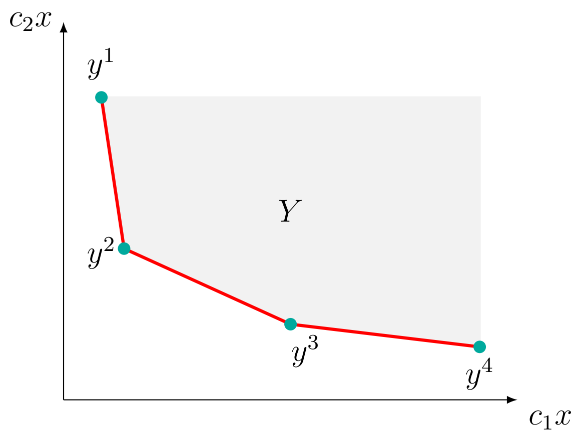


Figure 2.2: An illustration of a nondominated image set of a BOLP. The front in red represents nondominated image set, Y_N and the blue points are extreme nondominated images y^1, y^2, y^3 and y^4.

2.7 Definition (Multi-objective Linear Program). A *multi-objective linear program* is defined as

$$\begin{aligned} \min \quad & Cx \\ \text{s. t.} \quad & Ax \geqq b \\ & x \geqq 0, \end{aligned} \tag{MOLP}$$

where $C \in \mathbb{Q}^{k\times n}$ is the objective matrix consisting of the rows $c_i, i = 1,\ldots,k$, $A \in \mathbb{Q}^{m\times n}$, $m,n,k \in \mathbb{N} \setminus \{0\}$, $b \in \mathbb{Q}^m$. ◁

We call the set $X := \{x \in \mathbb{R}^n : Ax = b, x \geqq 0\}$ the feasible set and the set $Y := \{Cx : x \in X\} \subseteq \mathbb{R}^k$ the image set of the MOLP. For $k = 2$ and $k = 3$, a MOLP is called a biobjective linear program (MOLP) and a triobjective linear program (TOLP), respectively.

As the objective functions of a MOLP are typically conflicting, there is usually not a single optimal solution x minimizing all the individual objectives c_ix for all $i = 1,\ldots,k$ simultaneously.

2.8 Definition (Efficiency and nondominance). Let $x^* \in X$ be a feasible solution with corresponding image $y^* = Cx^*$. Then, $y^* \in Y$ is *nondominated* (weakly nondominated), if there is no other image $y \in Y$ such that $y \leq y^*(y < y^*)$. Correspondingly, $x^* \in X$ is *efficient* (weakly efficient), if y^* is nondominated (weakly nondominated). ◁

The set of nondominated (weakly nondominated) images is denoted by Y_N (Y_{wN}) and the set of efficient (weakly efficient) solutions by X_E (X_{wE}). Figure 2.2 illustrates nondomianted image set and extreme nondomianted images of a biobjective linear program.

For linear problems, the feasible set X and also its image $Y = CX$ are both polyhedra. We recall some basic concepts of polyhedral theory [Zie12]. The dimension of a polyhedron $P \subseteq \mathbb{R}^k$ is the maximum number of affinely independent points of P minus one. A bounded polyhedron is called a polytope. For $w \in \mathbb{R}^k$ and $t \in \mathbb{R}$, the inequality $w^\top y \leq t$ is called valid for P if $P \subseteq \left\{y \in \mathbb{R}^k : w^\top y \leq t\right\}$. A set $F \subset P$ is a face of P if there is some valid inequality $w^\top y \leq t$ such that $F = \left\{y \in P : w^\top y = t\right\}$. An extreme point is a face of dimension 0, an edge is a face of dimension 1 and a facet is a face of dimension $k-1$ where k is a dimension of P.

In MOLP, the set of non-dominated images is a convex hull defined by the *extreme nondominated images*. An image $y \in Y$ is an extreme nondominated image if $y \in Y_N$ and is an extreme point of Y. We denote the set of extreme nondominated images by Y_{EN}. Consequently, the search for all nondominated images can be restricted to finding a set of efficient solutions that correspond to such extreme nondominated images.

2.9 Definition. A *solution set* $S \subseteq X$ of a MOLP is a set such that for every $y \in Y_{\text{EN}}$ there is an $x \in S$ such that $Cx = y$. It is called *minimal* if, additionally, there is no other solution set $S' \subseteq X$ with $|S'| < |S|$. ◁

2.3 Weight set decomposition

Scalarization methods are useful techniques to compute nondominated images of a MOLP. A scalarization method transforms a multi-objective linear program into a single objective problem in a structured way by using additional parameters, constraints, or reference points. One commonly used scalarization method is the weighted sum scalarization.

2.10 Definition. Consider a MOLP and let a non-negative vector of weights $w \in \mathbb{R}^k_{\geq}$ be given. The *weighted sum scalarization problem* of the MOLP with respect to the weight w is the single objective linear program

$$\min_{x \in X} \quad w^\top Cx. \tag{WS(MOLP, w)}$$

◁

A feasible solution that is optimal for some weighted sum scalarization problem is called *supported* and a feasible solution whose image is uniquely optimal for some weighted sum scalarization problem is called *extreme supported* (cf. [Ehr05]). For linear programs, all extreme nondominated images are supported. Consequently, the set of extreme nondominated images of a MOLP, Y_{EN} coincides with the set of extreme supported nondominated images.

2.11 Theorem (Isermann [Ise74]). A feasible solution $x^* \in X$ is an efficient solution of MOLP if and only if there exists some $w \in \mathbb{R}^k_{>}$ such that $w^\top Cx^* \leq w^\top Cx$ for all $x \in X$. ◁

By Isermann's theorem, any efficient solution to MOLP can be found as an optimal solution of WS(MOLP, w) for some properly chosen weight w. Therefore, we are interested in all weights and, also, its structure.

2.12 Definition. The *(normalized) weight set* is denoted by $\mathcal{W}$ and defined as

$$\mathcal{W} := \left\{ w \in \mathbb{R}^k_{\geqq} : \sum_{i=1}^{k} w_i = 1 \right\}. \qquad \triangleleft$$

We consider the normalized weight set because the normalization of the weight vectors does not affect the optimality of a solution obtained via the weighted sum scalarization, i. e., x is optimal for WS(MOLP,w) with $w \in \mathbb{R}^k_{\geq}$, if and only if x is optimal for $w' := \frac{1}{\sum_{i=1}^k w_i} w \in \mathcal{W}$. We denote the weight sets of BOLP and TOLP by $\mathcal{W}$(BOLP) and $\mathcal{W}$(TOLP), respectively.

There exist a bijection between $\mathcal{W}$ and the set $\left\{ w \in \mathbb{R}^{k-1}_{\geqq} : \sum_{i=1}^{k-1} w_i \leq 1 \right\}$ and the set $\mathcal{W}$ is a polytope of dimension $k-1$ (see Figure 2.2). So, in case of a weighted sum scalarization of a triobjective linear program, it is enough to consider the projection of (w_1, w_2, w_3) onto (w_1, w_2) because w_3 is uniquely determined by the values of w_1 and w_2, i. e. $w_3 = 1 - w_1 - w_2$.

A *weight set decomposition* is a subdivision of the weight set into so-called weight set components. All weights in a single weight set component map to the same nondominated image. For every $y \in Y$, the *weight set component* of y is denoted by $\mathcal{W}(y)$, and defined as

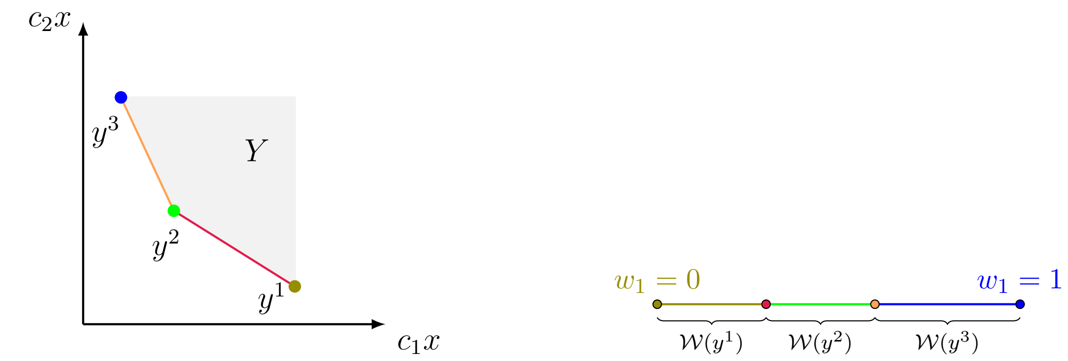
$$\mathcal{W}(y) := \left\{ w \in \mathcal{W} : w^\top y = \min \left\{ w^\top y' : y' \in Y \right\} \right\}.$$

A weight set component $\mathcal{W}(y)$ consists of the subset of weights $w \in \mathcal{W}$ for which the corresponding nondominated image y is an optimal value of WS(MOLP, w). A weight w is called an *extreme weight*, if it is an extreme point of $\mathcal{W}(y)$. Important properties of these weight set components are presented by Przybylski et al. [PGE10].

2.13 Proposition (Przybylski et al. [PGE10]). Let $y \in Y_N$. Then, the following statements hold:

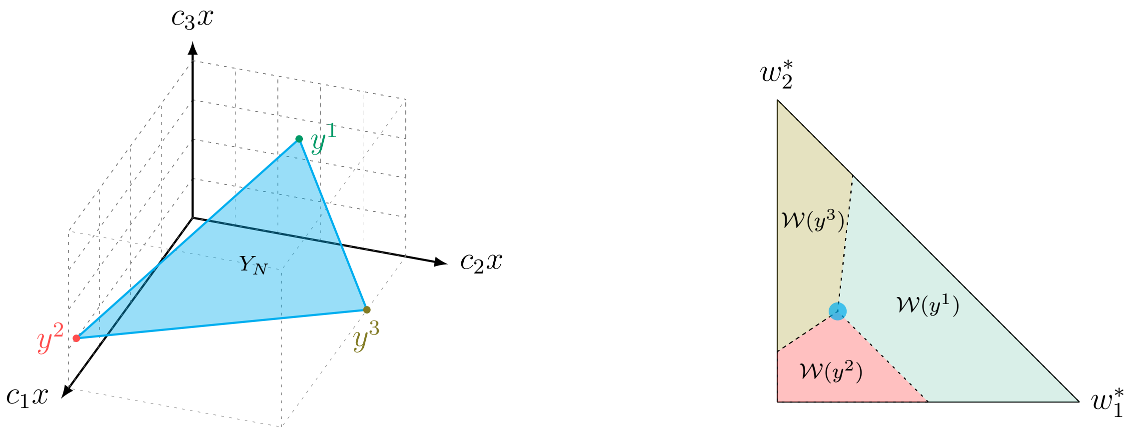
(i) $\mathcal{W}(y) = \left\{ w \in \mathcal{W} : w^\top y \leq w^\top y' \text{ for all } y' \in Y_{\text{EN}} \right\}$.
(ii) $\mathcal{W}(y)$ is a convex polytope.
(iii) A nondominated image y is an extreme nondominated image of Y if and only if $\mathcal{W}(y)$ has dimension $k-1$. $\triangleleft$

Proposition 2.13.(i) implies that it is sufficient to consider only extreme nondominated images to define weight set components. Proposition 2.13.(iii) implies that the weight set components corresponding to the extreme nondominated images are full-dimensional polytopes in $\mathbb{R}^{k-1}_{\geqq}$.



(a) Nondominated image set of a BOLP **(b)** Weight set of a BOLP

Figure 2.3: An illustration of the bijection between the nondominated image set of a BOLP and its weight set. The images y^1, y^2 and y^3 in Y_{EN} correspond to weight set components $\mathcal{W}(y^1)$ $\mathcal{W}(y^2)$ and $\mathcal{W}(y^3)$ in $\mathcal{W}(\mathrm{BOLP})$.



(a) Nondominated image set of a TOLP **(b)** Weight set of a TOLP

Figure 2.4: An illustration of the bijection between the nondominated image set of a TOLP and its corresponding weight set. The extreme nondominated images y^1, y^2, y^3 in Y_{EN} correspond to weight set components $\mathcal{W}(y^1), \mathcal{W}(y^2)$, and $\mathcal{W}(y^3)$ in $\mathcal{W}(\mathrm{TOLP})$.

We observe from these properties that the structure of the weight set is closely linked to the structure of the nondominated image set of a MOLP by a one-to-one mapping between the weight set components and the extreme nondominated images.

A visualization of the mapping between the weight set and the nondominated images of a biobjective linear program in Figure 2.3. Similarly, Figure 2.4 illustrates the mapping between the weight set and the extreme nondominated images of TOLP.

As stated in the introduction, dichotomic search is a commonly used method to solve biobjective linear programs. Consequently, it is used to solve the partial weighted sum scalarisation of the triobjective linear program. We now describe the dichotomic search method to find a set of extreme nondominated images for BOLP. We start with two optimal solutions obtained by minimizing the objective functions c_1x and c_2x, respectively.

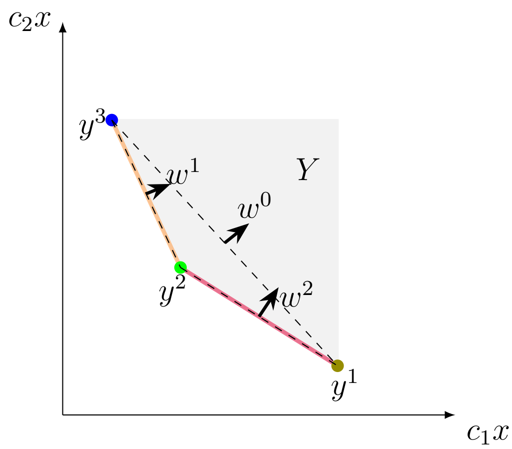


Figure 2.5: An illustration of the dichotomic search for a BOLP. We begin with the initial optimal images y^1 and y^3 and compute the weight w^0, which is the normal vector of the line passing though y^1 and y^3. Next, we solve the weighted sum problem using w^0 to get y^2. This results in two new weights w^1 and w^2. We solve the weighted sum problems again and yield already known images and thus, the search terminates.

Equivalently, these two solutions are optimal solutions to the weighted sum problem of BOLP with weights $(1, 0)$ and $(0, 1)$. The normal vector of the line segment joining these images is then used to solve the weighted sum scalarization problem. It results in two cases; either the two images are optimal or a new nondominated image is computed with a better optimal value. If it is the first case, then the procedure stops but if it is the second case, the procedure recursively continues on each of the two subintervals determined by the new image. We illustrate the working of dichotomic search in Figure 2.5.

Chapter 3

Parametric biobjective linear programs

In this chapter, we introduce parametric biobjective linear programs and use multi-objective programming techniques to analyse the parametric behaviour with respect to the solution set. We formulate a general parametric biobjective linear program and investigate different cases to solve these problems. We will follow up with an extension of our analysis to parametric triobjective linear programs and parametric multi-objective linear programs.

We consider a parametric multi-objective linear program with two objective functions dependent on a single parameter $\lambda \in \mathbb{R}_{\geq}$.

3.1 Definition (Parametric Biobjective Linear Program). Let $\lambda \in \mathbb{R}_{\geq}$ be a parameter. A *parametric biobjective linear program* is defined as

$$\begin{aligned} \min \quad & Cx + \lambda Dx \\ \text{s. t.} \quad & Ax \geqq b, \\ & x \geqq 0, \end{aligned} \tag{PBLP}$$

where $C, D \in \mathbb{Q}^{2\times n}$ consist of rows c_i, d_i $i = 1, 2$, $A \in \mathbb{Q}^{m\times n}$, $m, n \in \mathbb{N} \setminus \{0\}$, and $b \in \mathbb{Q}^m$. Again, the feasible set is denoted by $X := \{x \in \mathbb{R}^n : Ax \geqq b, x \geqq 0\}$. ◁

As the literature frequently adopts a case-based approach, see [Sit08], we focus on two special cases of the general PBLP with a parametric dependency in either one or both objectives. In the first case, only one of the two objectives depends linearly on the parameter. In the second case, we consider the same linear parametric dependency in both objectives. More precisely, we consider the following two special cases.
Case I : One parametric objective

$$\begin{aligned} \min \quad & \begin{pmatrix} c_1x + \lambda d_1x \\ c_2x \end{pmatrix} \\ \text{s. t.} \quad & x \in X. \end{aligned} \tag{PBLP^1}$$

Case II : Same parameter in both objectives

$$\begin{aligned} \min \quad & \begin{pmatrix} c_1 x + \lambda d_1 x \\ c_2 x + \lambda d_1 x \end{pmatrix} \\ \text{s.t.} \quad & x \in X. \end{aligned} \tag{PBLP^2}$$

For notational consistency and to avoid repetition, we adopt the convention of denoting both cases of PBLP together as PBLP^j for $j = 1, 2$ simultaneously (e.g., when the results hold for both problems).

The problem PBLP^j, for some fixed λ, is a non-parametric biobjective linear program, and we denote it by PBLP$^j(\lambda)$. Furthermore, we denote the set of extreme non-dominated images of PBLP$^j(\lambda)$ by $Y_{\text{EN}}(\text{PBLP}^j(\lambda))$ and a corresponding minimal solution set by $S(\text{PBLP}^j(\lambda))$. With some abuse of notation, we write

$$S(\text{PBLP}^j(\lambda_1)) = S(\text{PBLP}^j(\lambda_2))$$

if PBLP$^j(\lambda_1)$ and PBLP$^j(\lambda_2)$ share a minimal solution set, and

$$S(\text{PBLP}^j(\lambda_1)) \neq S(\text{PBLP}^j(\lambda_2))$$

if they do not.

We analyse a minimal solution set that corresponds to a set of extreme nondominated images of PBLP$^j(\lambda)$. Therefore, a minimal solution set of PBLP^j equates to a set that contains efficient solutions of PBLP$^j(\lambda)$ for each fixed value of $\lambda \geq 0$. We denote this set by S.

3.2 Definition. A *solution set* $S \subseteq X$ of PBLP^j is a set such that for every $\lambda \geq 0$, S contains, as a subset, a solution set for the biobjective linear program PBLP$^j(\lambda)$. It is called *minimal* if, additionally, there is no other solution set $S' \subseteq X$ for PBLP^j with $|S'| < |S|$. ◁

We relate PBLP^j to a corresponding triobjective linear program using the weighted sum scalarization. To this end, we consider the triobjective linear problem with the objective functions $c_1 x, c_2 x$, and $d_1 x$, i. e.

$$\begin{aligned} \min \quad & (c_1 x, c_2 x, d_1 x)^\top \\ \text{s. t.} \quad & x \in X, \end{aligned} \tag{TOLP}$$

and denote its set of extreme nondominated images by $Y_{\text{EN}}(\text{TOLP})$. We denote a minimal solution set that corresponds to the set $Y_{\text{EN}}(\text{TOLP})$ by $S(\text{TOLP})$. We define the weight set $\mathcal{W}(\text{TOLP})$ of TOLP as

$$\mathcal{W}(\text{TOLP}) := \left\{ w^* \in \mathbb{R}^3_{\geqq} : \sum_{i=1}^{3} w_i^* = 1 \right\}.$$

The weighted sum scalarization of TOLP with a normalized weight $w^* \in \mathcal{W}(\text{TOLP})$ is

$$\min_{\mathrm{x} \in \mathrm{X}} w_1^* c_1 x + w_2^* c_2 x + w_3^* d_1 x. \tag{WS(TOLP, w^*)}$$

3.1 Case I : One parametric objective

We approach the problem by applying the weighted sum scalarization to the parametric biobjective linear program PBLP^1 and formally characterize its relationship to the corresponding triobjective linear program TOLP.

For a fixed value of λ, the weighted sum scalarization of $\text{PBLP}^1(\lambda)$ is

$$\min_{\mathrm{x} \in \mathrm{X}} w_1(c_1 x + \lambda d_1 x) + w_2 c_2 x \tag{WS(PBLP$^1(\lambda)$, w)}$$

where $w \in \mathbb{R}^2_{\geqq}$ and $w_1 + w_2 = 1$.
We reformulate this problem using $w_2 = 1 - w_1$ and obtain

$$\min_{\mathrm{x} \in \mathrm{X}} w_1 c_1 x + (1 - w_1) c_2 x + w_1 \lambda d_1 x.$$

This problem can be interpreted as a specific weighted sum scalarization problem of the TOLP with weight vector $(w_1, 1 - w_1, w_1 \lambda)$. It holds that

$$w_1 + (1 - w_1) + w_1 \lambda = 1 + w_1 \lambda \geq 1$$

since $w_1 \geq 0$ and $\lambda \geq 0$. We normalize the weights and get

$$\min_{\mathrm{x} \in \mathrm{X}} \frac{w_1}{1 + w_1 \lambda}(c_1 x) + \frac{w_2}{1 + w_1 \lambda}(c_2 x) + \frac{w_1 \lambda}{1 + w_1 \lambda}(d_1 x),$$

which is a particular case of WS(TOLP, w^*) with the corresponding weight vector

$$w^* = \left(\frac{w_1}{1 + w_1 \lambda}, \frac{w_2}{1 + w_1 \lambda}, \frac{w_1 \lambda}{1 + w_1 \lambda} \right)^\top \in \mathcal{W}(\text{TOLP}). \tag{3.1}$$

We now present our main result, which establishes the equivalence between the efficient solutions of the problem PBLP^1 and the efficient solutions of the corresponding triobjective linear program.

3.3 Theorem. A feasible solution x^* is optimal for $\mathrm{WS}(\mathrm{TOLP}, w^*)$ with non-negative weights w_1^*, w_2^*, w_3^*, where $w_1^* > 0$ if and only if there exist a parameter $\lambda \geq 0$ and non-negative weights w_1, w_2 where $w_1 > 0$ such that x^* is optimal for $\mathrm{WS}\big(\mathrm{PBLP}^1(\lambda), w\big)$. $\triangleleft$

Proof. We first show that x^* is optimal for $\mathrm{WS}(\mathrm{TOLP}, w^*)$, implying that there exist $\lambda \geq 0$ and non-negative weights w_1, w_2 such that x^* is optimal for $\mathrm{WS}\big(\mathrm{PBLP}^1(\lambda), w\big)$.
Let x^* be optimal for $\mathrm{WS}(\mathrm{TOLP}, w^*)$.
Note that $w_1^* > 0$ and we define

$$\begin{aligned} \lambda &:= \frac{w_3^*}{w_1^*}, \\ w_1 &:= \frac{w_1^*}{w_1^* + w_2^*}, \\ w_2 &:= \frac{w_2^*}{w_1^* + w_2^*}. \end{aligned} \tag{3.2}$$

Suppose x^* is not optimal for $\mathrm{WS}\big(\mathrm{PBLP}^1(\lambda), w\big)$. Then there exists some x' that is feasible for $\mathrm{WS}\big(\mathrm{PBLP}^1(\lambda), w\big)$ where $x' \neq x^*$ such that

$$w_1 c_1 x' + w_2 c_2 x' + w_1 \lambda d_1 x' < w_1 c_1 x^* + w_2 c_2 x^* + w_1 \lambda d_1 x^*.$$

We plug in Equation (3.2) to get

$$\begin{aligned} &\frac{w_1^*}{w_1^* + w_2^*} c_1 x' + \frac{w_2^*}{w_1^* + w_2^*} c_2 x' + \frac{w_1^*}{w_1^* + w_2^*} \left(\frac{w_3^*}{w_1^*} d_1 x'\right) \\ &\qquad < \frac{w_1^*}{w_1^* + w_2^*} c_1 x^* + \frac{w_2^*}{w_1^* + w_2^*} c_2 x^* + \frac{w_1^*}{w_1^* + w_2^*} \left(\frac{w_3^*}{w_1^*} d_1 x^*\right). \end{aligned}$$

Then, we multiply by $w_1^* + w_2^*$ to get

$$w_1^* c_1 x' + w_2^* c_2 x' + w_3^* d_1 x' < w_1^* c_1 x^* + w_2^* c_2 x^* + w_3^* d_1 x^*.$$

This leads to a contradiction that x^* is optimal for $\mathrm{WS}(\mathrm{TOLP}, w^*)$.
Conversely, let x^* be optimal for $\mathrm{WS}\big(\mathrm{PBLP}^1(\lambda), w\big)$ with non-negative weights w_1, w_2 and for some non-negative λ.
We define w_1^*, w_2^*, w_3^* using Equation (3.1),

$$\begin{aligned} w_1^* &:= w_1, \\ w_2^* &:= w_2, \\ w_3^* &:= w_1 \lambda. \end{aligned} \tag{3.3}$$

We do not normalize for simplicity. Suppose x^* is not optimal for $\mathrm{WS}(\mathrm{TOLP}, w^*)$. Then there exists x' that is feasible for $\text{WS}(\text{TOLP}, w^*)$ where $x' \neq x^*$ such that

$$w_1^* c_1 x' + w_2^* c_2 x' + w_3^* d_1 x' < w_1^* c_1 x^* + w_2^* c_2 x^* + w_3^* d_1 x^*.$$

We plug in Equation (3.3) to get

$$w_1 c_1 x' + w_2 c_2 x' + w_1 \lambda d_1 x' < w_1 c_1 x^* + w_2 c_2 x^* + w_1 \lambda d_1 x^*.$$

This leads to a contradiction that x^* is optimal for $\text{WS}\big(\text{PBLP}^1(\lambda), w\big)$. □

Observe that we assume $w_1^* > 0$ and $w_1 > 0$ in Theorem 3.3. Based on the construction in the proof, the parameter λ is undefined if $w_1^* = 0$, and w_1^* approaching 0 is equivalent to the parameter λ approaching ∞. And indeed, the one-to-one correspondence from Theorem 3.3 does not hold if $w_1^* = 0$: there exist $w^* \in \mathbb{R}^3_{\geqq}$ with $w_1^* = 0$ and an optimal solution x^* for $\text{WS}(\text{TOLP}, w^*)$ such that there is no $\lambda \geq 0$ and $w \in \mathbb{R}^2_{\geqq}$ where x^* is optimal for $\text{WS}(\text{PBLP}^1(\lambda), w)$. This can be illustrated by the following example:

3.4 Example. Let $X \subseteq \mathbb{R}^3$ and $X = \text{conv}(\{(1,1,1),(0,1,1),(0,0,2)\})$ as shown in the Figure 3.1.
Let $c_1 = (1,0,0), c_2 = (0,1,0), d_1 = (0,0,1)$.
Then for $x = (x_1, x_2, x_3) \in X$, it holds that $c_1 x = x_1$, $c_2 x = x_2$ and $d_1 x = x_3$ and are the objective functions of TOLP.
Let $w^* = (0,0,1)$. Then the solutions $(1,1,1)$ and $(0,1,1)$ are both optimal solutions of $\text{WS}(\text{TOLP}, w^*)$ with an optimal solution value of 1.
However, consider now the problem $\text{WS}(\text{PBLP}^1(\lambda), w)$. For every value of $\lambda \geq 0$ and $w \in \mathbb{R}^2_{\geqq}$ with $w_1 > 0$, it holds that $(0,1,1)$ is a better solution than $(1,1,1)$ because the value of $\text{WS}(\text{PBLP}^1(\lambda), w)$ for $(0,1,1)$ is

$$w_1((0) + (\lambda)(1)) + w_2(1) = w_2 + w_1\lambda$$

which is less than the value for $(1,1,1)$, i. e.

$$w_1((1) + \lambda(1)) + w_2(1) = w_1 + (w_2 + w_1\lambda).$$

Additionally, for $w_1 = 0$, it holds $w_2 = 1$ and the problem $\text{WS}(\text{PBLP}^1(\lambda), w)$ is reduced to $\min_{x \in X} c_2 x$. Since then the solution $(0,0,2)$ is optimal for $\text{WS}(\text{PBLP}^1(\lambda), w)$ with an optimal solution value of 0. Therefore, the solution $(1,1,1)$ is never optimal for $\text{WS}(\text{PBLP}^1(\lambda), w)$. ◁

In the converse case, for any given parameter value $\lambda \geq 0$ and a weight $w \in \mathbb{R}^2_{\geqq}$ with $w_1 = 0$ for $\text{WS}\big(\text{PBLP}^1(\lambda), w\big)$, we can see that it is possible to construct a weight $w^* \in \mathbb{R}^3_{\geqq}$ for $\text{WS}(\text{TOLP}, w^*)$ using Equation (3.3). In fact, an optimal solution x for $\text{WS}\big(\text{PBLP}^1(\lambda), w\big)$ with a weight $w \in \mathbb{R}^2_{\geqq}$ such that $w_1 = 0$ remains optimal for

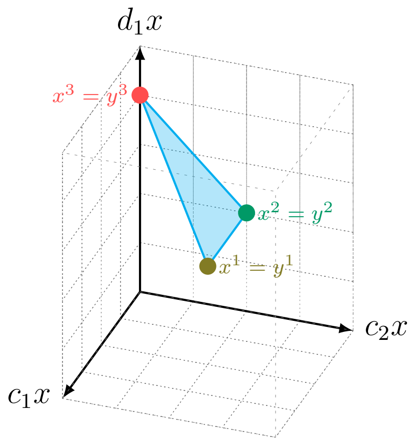


Figure 3.1: Illustration of the nondominated image set of the TOLP in Example 3.4 where $Y_N(\text{TOLP}) = \text{conv}(\{(1,1,1),(0,1,1),(0,0,2)\})$

WS(TOLP, w^*) with the weight $w^* = (0, w_2, 0)$ because $w_1^* = 0$ and $w_3^* = 0$ by construction.

For every fixed value of λ, PBLP$^1(\lambda)$ is a biobjective linear program. Its weight set is therefore a one-dimensional polytope. However, considering the parameter λ, we can extend this interpretation of the set of the one-dimensional weight set to a higher dimension by a mapping $\mathcal{W}(\text{PBLP}^1(\lambda))$, which is defined below. This extended representation enables us to derive several important theoretical results.

3.5 Definition. For a given λ, we define a *mapping* $\mathcal{W}\big(\text{PBLP}^1(\lambda)\big)$ as,

$$\mathcal{W}\big(\text{PBLP}^1(\lambda)\big) := \left\{ \left(\frac{w_1}{1+w_1\lambda}, \frac{w_2}{1+w_1\lambda}, \frac{w_1\lambda}{1+w_1\lambda} \right) : (w_1, w_2) \in \mathbb{R}^2_{\geqq}, w_1 + w_2 = 1 \right\}. \quad \triangleleft$$

3.6 Proposition. Let $\mathcal{W}\big(\text{PBLP}^1(\lambda)\big)$ and $\mathcal{W}(\text{TOLP})$ be weight sets of PBLP^1 for a fixed value λ and of TOLP, respectively. Then, it holds that

$$\mathcal{W}\big(\text{PBLP}^1(\lambda)\big) \subsetneq \mathcal{W}(\text{TOLP}). \quad \triangleleft$$

Proof. Let $w^* \in \mathcal{W}\big(\text{PBLP}^1(\lambda)\big)$. Then, by the definition of $\mathcal{W}\big(\text{PBLP}^1(\lambda)\big)$, it is

$$w^* = \left(\frac{w_1}{1+w_1\lambda}, \frac{w_2}{1+w_1\lambda}, \frac{w_1\lambda}{1+w_1\lambda} \right)$$

for some $(w_1, w_2) \in \mathbb{R}^2_{\geqq}$ with $w_1 + w_2 = 1$. The sum of the components satisfies

$$\frac{w_1}{1+w_1\lambda} + \frac{w_2}{1+w_1\lambda} + \frac{w_1\lambda}{1+w_1\lambda} = \frac{1+w_1\lambda}{1+w_1\lambda} = 1.$$

Thus, $w^* \in \mathcal{W}(\text{TOLP})$.
To show that $\mathcal{W}\Big(\text{PBLP}^1(\lambda)\Big)$ is a proper subset of $\mathcal{W}(\text{TOLP})$, we consider the particular weight $w' := (0,0,1) \in \mathcal{W}(\text{TOLP})$. Since for any given λ, if we consider the third component of any $w^* \in \mathcal{W}\Big(\text{PBLP}^1(\lambda)\Big)$ it holds that

$$\frac{w_1\lambda}{1+w_1\lambda} \neq 1.$$

Hence, it holds that $w' \notin \mathcal{W}\Big(\text{PBLP}^1(\lambda)\Big)$. □

3.7 Proposition. For all the weight sets of $\text{PBLP}^1(\lambda)$ for $\lambda \geq 0$, it holds that

$$\bigcap_{\lambda\geq 0} \mathcal{W}\Big(\text{PBLP}^1(\lambda)\Big) = \{(0,1,0)\}. \qquad \triangleleft$$

Proof. For an arbitrary $\lambda_i \geq 0$, by the definition of $\mathcal{W}\left(\text{PBLP}^1(\lambda_i)\right)$, we have

$$\mathcal{W}\left(\text{PBLP}^1(\lambda_i)\right) := \left\{\frac{1}{1+w_1\lambda_i}(w_1, w_2, w_1\lambda_i) : (w_1, w_2,) \in \mathbb{R}^2_{\geqq}, w_1 + w_2 = 1\right\}.$$

We first show that the weight $w^* = (0,1,0)$ is an element of $\mathcal{W}\left(\text{PBLP}^1(\lambda_i)\right)$ for all λ_i. Let $\lambda_i \in \mathbb{R}_{\geq}$ be arbitrary and choose $w_1^* = 0$ and $w_2^* = 1$. Then, we get

$$w^* = \left(\frac{0}{1+0\lambda_i}, \frac{1}{1+0\lambda_i}, \frac{0}{1+0\lambda_i}\right) = (0,1,0) \in \mathcal{W}(\text{PBLP}^1(\lambda_i)).$$

Therefore, $w^* \in \mathcal{W}\Big(\text{PBLP}^1(\lambda_i)\Big)$.
Next, we show that there is no other weight shared by any two weight sets of PBLP^1. Let $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ be two arbitrary parameter values, such that $\lambda_1 \neq \lambda_2$. Let $\mathcal{W}\Big(\text{PBLP}^1(\lambda_1)\Big)$ and $\mathcal{W}\Big(\text{PBLP}^1(\lambda_2)\Big)$ be weight sets of $\text{PBLP}^1(\lambda_1)$ and $\text{PBLP}^1(\lambda_2)$, respectively.
Consider a weight $w^* \in \mathcal{W}\Big(\text{PBLP}^1(\lambda_1)\Big)$ and a weight $\tilde{w} \in \mathcal{W}\Big(\text{PBLP}^1(\lambda_2)\Big)$. By the definition of $\mathcal{W}\Big(\text{PBLP}^1(\lambda_1)\Big)$ and $\mathcal{W}\Big(\text{PBLP}^1(\lambda_2)\Big)$, we have

$$w^* = \left(\frac{w_1'}{1+w_1'\lambda_1}, \frac{w_2'}{1+w_1'\lambda_1}, \frac{w_1'\lambda_1}{1+w_1'\lambda_1}\right) \text{where } w_1' + w_2' = 1$$

and

$$\tilde{w} = \left(\frac{w_1''}{1+w_1''\lambda_2}, \frac{w_2''}{1+w_1''\lambda_2}, \frac{w_1''\lambda_2}{1+w_1''\lambda_2}\right) \text{where } w_1'' + w_2'' = 1.$$

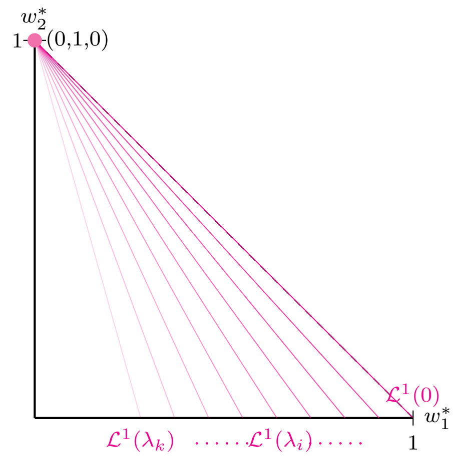


Figure 3.2: An illustration of the family of line segments $\mathcal{L}^1_\lambda$ for PBLP[1] in the weight set of TOLP, where $\mathcal{L}^1(\lambda_i) = \mathcal{L}_{\mathcal{W}(\mathrm{PBLP}^1(\lambda_i))}$ for $\lambda_i \in \{0, \dots, \lambda_k\}$. The line segments $\mathcal{L}^1(\lambda)$ represent weight sets of PBLP[1](λ) for different values of λ.

If we equate the first components of w^* and $\tilde{w}$, we obtain

$$\frac{w_1'}{1 + w_1'\lambda_1} = \frac{w_1''}{1 + w_1''\lambda_2}.$$

Then the components are equal if

$$w_1' = \frac{w_1''}{1 + w_1''\lambda_2}(1 + w_1'\lambda_1). \tag{3.4}$$

Also, the third component of w^* is

$$w_3^* = \frac{w_1'\lambda_1}{1 + w_1'\lambda_1}.$$

We substitute w_1' from Equation (3.4) in w_3^* to get

$$\begin{aligned} w_3^* &= \frac{\frac{w_1''}{1+w_1''\lambda_2}(1 + w_1'\lambda_1)}{1 + w_1'\lambda_1}\lambda_1 \\ &= \frac{w_1''\lambda_1}{1 + w_1''\lambda_2} \\ &\neq \frac{w_1''\lambda_2}{1 + w_1''\lambda_2} = \tilde{w}_3 \end{aligned}$$

because $\lambda_1 \neq \lambda_2$. Therefore, the third components are not equal. □

Therefore, the weight $(0, 1, 0)$ is the only weight shared by all the weight sets of $\mathrm{PBLP}^1(\lambda)$ for all parameter values $\lambda \geq 0$.

3.8 Corollary. Let $\mathcal{W}\big(\mathrm{PBLP}^1(\lambda)\big)$ be the weight set of $\mathrm{PBLP}^1(\lambda)$ and $\mathcal{W}(\mathrm{TOLP})$ be the weight set of the corresponding TOLP. Then the union of all the weight sets of $\mathrm{PBLP}^1(\lambda)$ for every $\lambda \geq 0$ is defined as

$$\bigcup_{\lambda \geq 0} \mathcal{W}\big(\mathrm{PBLP}^1(\lambda)\big) = \{w^* \in \mathcal{W}(\mathrm{TOLP}) : w_1^* \neq 0\} \cup \{(0,1,0)\}.$$

Proof. From Proposition 3.6, it follows that the weight sets of $\mathrm{PBLP}^1(\lambda)$ for every $\lambda \geq 0$ is a subset of $\mathcal{W}(\mathrm{TOLP})$, so the union of all these weight sets will also be a subset of $\mathcal{W}(\mathrm{TOLP})$. Using Theorem 3.3, for any weight in $\mathcal{W}(\mathrm{TOLP})$ except for the weights with $w_1^* = 0$, there exists some λ and $w \in \mathbb{R}^2_{\geqq}$ for $\mathrm{PBLP}^1(\lambda)$. However, by Proposition 3.7, the weight $(0,1,0)$ is the only weight shared by the weight sets of $\mathrm{PBLP}^1(\lambda)$. Therefore, the union of weight sets of $\mathrm{PBLP}^1(\lambda)$ is equivalent to all the weights in $\mathcal{W}(\mathrm{TOLP})$ except for the weights where $w_1^* = 0$ but includes the weight $(0,1,0)$. □

Consequently, Corollary 3.8 implies that if we vary λ in $\mathrm{PBLP}^1(\lambda)$ and find all the corresponding weight sets $\mathcal{W}\big(\mathrm{PBLP}^1(\lambda)\big)$ in $\mathbb{R}^3$, we obtain a nearly complete weight set decomposition of the associated TOLP except for the weights with $w_1^* = 0$ due to the reason mentioned in Theorem 3.3.

Next, we want to characterize the projection of $\mathcal{W}(\mathrm{PBLP}^1(\lambda))$ in $\mathbb{R}^2$. Since the third component of a weight in $\mathcal{W}(\mathrm{PBLP}^1(\lambda))$ can be determined by the first two components, the projection reduces the entire weight set to a 2-dimensional interpretation. Using the definition of $\mathcal{W}\big(\mathrm{PBLP}^1(\lambda)\big)$ for any parameter $\lambda \geq 0$, the third component of any arbitrary weight $w^* \in \mathcal{W}\big(\mathrm{PBLP}^1(\lambda)\big)$ can be written as

$$w_3^* = \lambda w_1^*.$$

Since $w_1^* + w_2^* + w_3^* = 1$, we get

$$1 - (w_1^* + w_2^*) = \lambda w_1^*.$$

This is equivalent to

$$w_2^* = 1 - w_1^*(1+\lambda).$$

This means that every weight $w^* \in \mathcal{W}\big(\mathrm{PBLP}^1(\lambda)\big)$ which is contained in $\mathcal{W}(\mathrm{TOLP})$ satisfies the condition $w_2^* = 1 - w_1^*(1+\lambda)$. This condition can be interpreted as an equation of a line in $\mathbb{R}^2$. Consequently, we can define the projection of weight set $\mathcal{W}\big(\mathrm{PBLP}^1(\lambda)\big)$ in $\mathbb{R}^2$ as a line segment.

3.9 Definition. The *projection of the weight set* $\mathcal{W}\big(\mathrm{PBLP}^1(\lambda)\big)$ in $\mathbb{R}^2$ can be defined as the line segment

$$\mathcal{L}_{\mathcal{W}(\mathrm{PBLP}^1(\lambda))} = \left\{(w_1^*, w_2^*) \mid w^* \in \mathcal{W}(\mathrm{TOLP}),\ w_2^* = 1 - w_1^*(1+\lambda)\right\}. \qquad \triangleleft$$

3.10 Remark. From the definition of $\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^1(\lambda)\right)}$, we observe the following:

(i) The slope m_λ of the line segment $\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^1(\lambda)\right)}$ is given by

$$m_\lambda = -(1+\lambda).$$

(ii) The end points of $\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^1(\lambda)\right)}$ are $(0,1)$ and $\left(\frac{1}{1+\lambda}, 0\right)$. ◁

Remark 3.10.(i) is a direct result of finding the slope of the line segment $\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^1(\lambda)\right)}$ and Remark 3.10.(ii) represents the w_1^*-intercept and w_2^*- intercept of $\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^1(\lambda)\right)}$.

We illustrate a family of line segments that represents weight sets $\mathcal{W}\left(\text{PBLP}^1(\lambda)\right)$ for different values of λ in Figure 3.2. Note, that we use the projection of $w^* \in \mathcal{W}(\text{TOLP})$ onto $(w_1^*, w_2^*) \in \mathbb{R}^2_{\geqq}$ in our analysis.

3.2 Case II: Same parametric dependency on both objectives

We now consider the problem PBLP^2 having a non-negative parameter $\lambda \geq 0$ in both objectives.
For a fixed value of λ, the weighted sum scalarization of $\text{PBLP}^2(\lambda)$ is

$$\min_{x \in X} w_1(c_1x + \lambda d_1x) + w_2(c_2x + \lambda d_1x) \qquad (\text{WS}(\text{PBLP}^2(\lambda), w))$$

where $w \in \mathbb{R}^2_{\geqq}$ and $w_1 + w_2 = 1$.
We reformulate this problem and obtain

$$\min_{x \in X} w_1c_1x + w_2c_2x + (w_1 + w_2)\lambda d_1x.$$

Using $w_2 = 1 - w_1$ this is equivalent to

$$\min_{x \in X} w_1c_1x + (1-w_1)c_2x + \lambda d_1x.$$

It can be interpreted as a weighted sum scalarization problem of the TOLP with a weight vector $(w_1, 1-w_1, \lambda)$. Since $w_1 + w_2 = 1$, it holds that

$$w_1 + w_2 + \lambda = 1 + \lambda \geq 1.$$

We normalize the weights and get

$$\min_{x \in X} \frac{w_1}{1+\lambda}c_1x + \frac{w_2}{1+\lambda}c_2x + \frac{\lambda}{1+\lambda}d_1x$$

which is a particular case of $\mathrm{WS}(\mathrm{TOLP}, w^*)$ with the corresponding weight vector

$$w^* = \left(\frac{w_1}{1+\lambda}, \frac{w_2}{1+\lambda}, \frac{\lambda}{1+\lambda}\right)^\top \in \mathcal{W}(\mathrm{TOLP}). \tag{3.5}$$

We establish the equivalence between the optimal solutions of the weighted sum scalarization problem of PBLP^2 and the weighted sum scalarization problem of the corresponding TOLP in the following theorem.

3.11 Theorem. A feasible solution x^* is optimal for $\mathrm{WS}(\mathrm{TOLP}, w^*)$ with non-negative weights w_1^*, w_2^*, w_3^*, where $w_3^* \neq 1$ if and only if there exists a parameter value $\lambda \geq 0$ and non-negative weights w_1, w_2, such that x^* is optimal for $\mathrm{WS}\big(\mathrm{PBLP}^2(\lambda), w\big)$. ◁

Proof. We first show that if x^* is optimal for $\mathrm{WS}(\mathrm{TOLP}, w^*)$, then it follows that there exists some $\lambda \geq 0$ and non-negative weights w_1, w_2 such that x^* is optimal for $\mathrm{WS}\big(\mathrm{PBLP}^2(\lambda), w\big)$. Let x^* be optimal for $\mathrm{WS}(\mathrm{TOLP}, w^*)$.
Note that $w_3^* \neq 1$ and we define

$$\begin{aligned} \lambda &:= \frac{w_3^*}{1-w_3^*}, \\ w_1 &:= \frac{w_1^*}{1-w_3^*}, \\ w_2 &:= \frac{w_2^*}{1-w_3^*}. \end{aligned} \tag{3.6}$$

Suppose x^* is not optimal for $\mathrm{WS}\big(\mathrm{PBLP}^2(\lambda), w\big)$, i. e., there exists x' that is optimal for $\mathrm{WS}\big(\mathrm{PBLP}^2(\lambda), w\big)$ where $x' \neq x^*$ such that,

$$w_1 c_1 x' + w_2 c_2 x' + \lambda d_1 x' < w_1 c_1 x^* + w_2 c_2 x^* + \lambda d_1 x^*.$$

We plug in Equation (3.6) to get

$$\frac{w_1^*}{1-w_3^*} c_1 x' + \frac{w_2^*}{1-w_3^*} c_2 x' + \frac{w_3^*}{1-w_3^*} d_1 x' < \frac{w_1^*}{1-w_3^*} c_1 x^* + \frac{w_2^*}{1-w_3^*} c_2 x^* + \frac{w_3^*}{1-w_3^*} d_1 x^*.$$

Multiplying by $(1 - w_3^*)$, this is equivalent to

$$w_1^* c_1 x' + w_2^* c_2 x' + w_3^* d_1 x' < w_1^* c_1 x^* + w_2^* c_2 x^* + w_3^* d_1 x^*.$$

This leads to the contradiction that x^* is optimal for $\mathrm{WS}(\mathrm{TOLP}, w^*)$.
Conversely, let x^* be optimal for $\mathrm{WS}\big(\mathrm{PBLP}^2(\lambda), w\big)$ with non-negative weights w_1, w_2 and for some non-negative λ.

We define w_1^*, w_2^*, w_3^* using Equation (3.5),

$$\begin{aligned} w_1^* &:= w_1, \\ w_2^* &:= w_2, \\ w_3^* &:= \lambda. \end{aligned} \tag{3.7}$$

We do not normalize the weight for simplicity.
Suppose x^* is not optimal for $\mathrm{WS}(\mathrm{TOLP}, w^*)$, i. e., there exists x' that is optimal for $\mathrm{WS}(\mathrm{TOLP}, w^*)$ where $x' \neq x^*$ such that,

$$w_1^* c_1 x' + w_2^* c_2 x' + w_3^* d_1 x' < w_1^* c_1 x^* + w_2^* c_2 x^* + w_3^* d_1 x^*.$$

We plug in Equation (3.7) to get

$$w_1 c_1 x' + w_2 c_2 x' + \lambda(1) d_1 x' < w_1 c_1 x^* + w_2 c_2 x^* + \lambda w_1 + w_2 d_1 x^*.$$

Substituting $1 = w_1 + w_2$, we get

$$w_1 c_1 x' + w_2 c_2 x' + \lambda(w_1 + w_2) d_1 x' < w_1 c_1 x^* + w_2 c_2 x^* + \lambda(w_1 + w_2) d_1 x^*.$$

This is equivalent to

$$w_1(c_1 x' + \lambda d_1 x') + w_2(c_2 x' + \lambda d_1 x') < w_1(c_1 x^* + \lambda d_1 x^*) + w_2(c_2 x^* + \lambda d_1 x^*).$$

This leads to a contradiction that x^* is optimal for $\mathrm{WS}\big(\mathrm{PBLP}^2(\lambda), w\big)$. □

We now address the case $w_3^* = 1$ as we assume $w_3^* \neq 1$ in Theorem 3.11. Based on the construction of the proof, the parameter λ is undefined if $w_3^* = 1$. The limit of w_3^* approaching 1 is equivalent to the parameter λ approaching ∞ in the weighted sum scalarization of PBLP^2. Additionally, the one-to-one correspondence of Theorem 3.11 does not hold: there exist $w^* \in \mathbb{R}^3_{\geqq}$ with $w_3^* = 1$ and an optimal solution x^* for $\mathrm{WS}(\mathrm{TOLP}, w^*)$ such that there is no $\lambda \geq 0$ and $w \in \mathbb{R}^2_{\geqq}$ where x^* is optimal for $\mathrm{WS}(\mathrm{PBLP}^2(\lambda), w)$.

This can be illustrated by the following example.

3.12 Example. Let $X \subseteq \mathbb{R}^3$ and $X = \mathrm{conv}(\{(1,1,1), (0,0,1)\})$.
Let $c_1 = (1,0,0), c_2 = (0,1,0), d_1 = (0,0,1)$.
Then for $x = (x_1, x_2, x_3) \in X$, it holds that $c_1 x = x_1$, $c_2 x = x_2$ and $d_1 x = x_3$ and are the objective functions of TOLP. Let $w^* = (0,0,1)$. Then the solutions $(1,1,1)$ and $(0,0,1)$ are both optimal solutions of $\mathrm{WS}(\mathrm{TOLP}, w^*)$ with a value of 1. However, consider the problem $\mathrm{WS}(\mathrm{PBLP}^2(\lambda), w)$. For every value of $\lambda \geq 0$ and $w \in \mathbb{R}^2_{\geqq}$, with $w_1 \geq 0$, it holds that $(0,0,1)$ is a better solution than $(1,1,1)$ for $\mathrm{WS}(\mathrm{PBLP}^2(\lambda), w)$ because the value of $\mathrm{WS}(\mathrm{PBLP}^2(\lambda), w)$ for $(0,0,1)$ is

$$w_1((0) + (\lambda)(1)) + w_2((0) + (\lambda)(1)) = w_1\lambda + w_2\lambda = \lambda$$

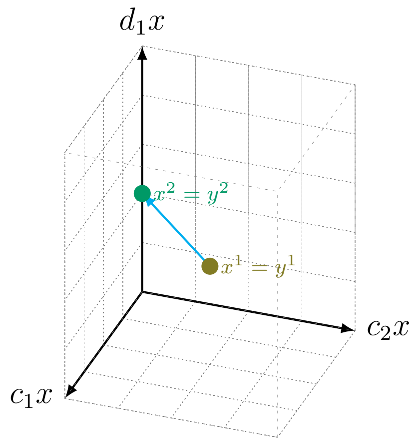


Figure 3.3: Illustration of the nondominated image set of the TOLP in Example 3.12 where $Y_N(\text{TOLP}) = \text{conv}(\{(1,1,1),(0,0,1)\})$

which is less than the solution value for $(1,1,1)$, i. e.

$$w_1((1)+\lambda(1)) + w_2((1)+(\lambda)(1)) = w_1(1+\lambda) + w_2(1+\lambda) = 1+\lambda.$$

Therefore, the solution $(1,1,1)$ is never optimal for $\text{WS}(\text{PBLP}^2(\lambda), w)$. ◁

As in the case of PBLP^1, we extend the weight set of $\text{PBLP}^2(\lambda)$ to a higher dimension using the mapping $\mathcal{W}(\text{PBLP}^2(\lambda))$, which is defined below.

3.13 Definition. For a fixed λ, we define a *mapping* $\mathcal{W}\big(\text{PBLP}^2(\lambda)\big)$ as,

$$\mathcal{W}\big(\text{PBLP}^2(\lambda)\big) := \left\{\left(\frac{w_1}{1+\lambda}, \frac{w_2}{1+\lambda}, \frac{\lambda}{1+\lambda}\right) : (w_1, w_2) \in \mathbb{R}^2_{\geqq}, w_1 + w_2 = 1\right\}. \quad ◁$$

3.14 Proposition. Let $\mathcal{W}\big(\text{PBLP}^2(\lambda)\big)$ and $\mathcal{W}(\text{TOLP})$ be weight sets of PBLP^2 for a fixed λ and the corresponding TOLP, respectively, then

$$\mathcal{W}\big(\text{PBLP}^2(\lambda)\big) \subsetneq \mathcal{W}(\text{TOLP}). \quad ◁$$

Proof. Let $w^* \in \mathcal{W}\big(\text{PBLP}^2(\lambda)\big)$. Then, by the definition of $\mathcal{W}\big(\text{PBLP}^2(\lambda)\big)$,

$$w^* = \left(\frac{w_1}{1+\lambda}, \frac{w_2}{1+\lambda}, \frac{\lambda}{1+\lambda}\right)$$

for some $(w_1, w_2) \in \mathbb{R}^2_{\geqq}$ with $w_1 + w_2 = 1$. Since the sum of the components,

$$\frac{w_1}{1+\lambda} + \frac{w_2}{1+\lambda} + \frac{\lambda}{1+\lambda} = 1.$$

Thus, $w^* \in \mathcal{W}(\text{TOLP})$.
Consider a particular weight $w^* = (0,0,1) \in \mathcal{W}(\text{TOLP})$. For any given value of $\lambda > 0$, if we consider the third component of a weight in $\mathcal{W}\big(\text{PBLP}^2(\lambda)\big)$ it holds that

$$\frac{\lambda}{1+\lambda} \neq 1.$$

Therefore, $w^* \notin \mathcal{W}\big(\text{PBLP}^2(\lambda)\big)$. □

3.15 Proposition. For all the weight sets of $\text{PBLP}^2(\lambda)$ for $\lambda \geq 0$, it holds that

$$\bigcap_{\lambda \geq 0} \mathcal{W}\big(\text{PBLP}^2(\lambda)\big) = \varnothing. \qquad \triangleleft$$

Proof. Let $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ be two arbitrary parameter values, such that $\lambda_1 \neq \lambda_2$. Let $\mathcal{W}\big(\text{PBLP}^2(\lambda_1)\big)$ and $\mathcal{W}\big(\text{PBLP}^2(\lambda_2)\big)$ be weight sets of $\text{PBLP}^2(\lambda_1)$ and $\text{PBLP}^2(\lambda_2)$, respectively. Consider, a weight $w^* \in \mathcal{W}\big(\text{PBLP}^2(\lambda_1)\big)$ and a weight $\tilde{w} \in \mathcal{W}\big(\text{PBLP}^2(\lambda_2)\big)$. By the definition of $\mathcal{W}\big(\text{PBLP}^2(\lambda_1)\big)$ and $\mathcal{W}\big(\text{PBLP}^2(\lambda_2)\big)$, we have

$$w^* = \left(\frac{w'_1}{1+\lambda_1}, \frac{w'_2}{1+\lambda_1}, \frac{\lambda_1}{1+\lambda_1}\right) \text{ where } w'_1 + w'_2 = 1$$

and

$$\tilde{w} = \left(\frac{w''_1}{1+\lambda_2}, \frac{w''_2}{1+\lambda_2}, \frac{\lambda_2}{1+\lambda_2}\right) \text{ where } w''_1 + w''_2 = 1.$$

By comparing the third components of w^* and $\tilde{w}$, we get

$$\frac{\lambda_1}{1+\lambda_1} \neq \frac{\lambda_2}{1+\lambda_2}$$

since $\lambda_1 \neq \lambda_2$. □

Therefore, there is no common weight shared by the weight sets of $\text{PBLP}^2(\lambda)$ for all parameter values $\lambda \geq 0$.

3.16 Corollary. Let $\mathcal{W}\big(\text{PBLP}^2(\lambda)\big)$ and $\mathcal{W}(\text{TOLP})$ be the weight sets of $\text{PBLP}^2(\lambda)$ and the corresponding TOLP, respectively. Then the union of all the weight sets of $\text{PBLP}^2(\lambda)$ for every $\lambda \geq 0$ is defined as

$$\bigcup_{\lambda \geq 0} \mathcal{W}\big(\text{PBLP}^2(\lambda)\big) = \mathcal{W}(\text{TOLP}) \setminus \{(0,0,1)\}.$$

Proof. From Proposition 3.14, the weight set of $\text{PBLP}^2(\lambda)$ for every $\lambda \geq 0$ are a subset of $\mathcal{W}(\text{TOLP})$, so the union of all these weight sets will also be a subset of $\mathcal{W}(\text{TOLP})$. Using Theorem 3.11, for any weight in $\mathcal{W}(\text{TOLP})$ except for the weight $(0,0,1)$, there exists some λ and $w \in \mathbb{R}^2_{\geqq}$ for $\text{PBLP}^2(\lambda)$. Therefore, the union of weight sets of $\text{PBLP}^2(\lambda)$ is equivalent to all the weights in $\mathcal{W}(\text{TOLP})$ except for the weight $(0,0,1)$. □

More precisely, Corollary 3.16 implies that if we vary λ in $\text{PBLP}^2(\lambda)$ and project all the corresponding weight sets $\mathcal{W}\big(\text{PBLP}^2(\lambda)\big)$ in $\mathbb{R}^3$, we obtain almost the entire weight set of the associated TOLP except for the weight $(0,0,1)$ due to the reason mentioned in Theorem 3.11.

Next, we want to characterize the projection of $\mathcal{W}(\text{PBLP}^2(\lambda))$ in $\mathbb{R}^2$. Since the third component of a weight in $\mathcal{W}(\text{PBLP}^2(\lambda))$ can be determined by the first two components, the projection reduces the entire weight set to a 2-dimensional interpretation. Using the definition of $\mathcal{W}\big(\text{PBLP}^2(\lambda)\big)$ for any parameter $\lambda \geq 0$, the third component of any arbitrary weight $w^* \in \mathcal{W}\big(\text{PBLP}^2(\lambda)\big)$ is

$$w_3^* = \frac{\lambda}{1+\lambda}.$$

Since $w_1^* + w_2^* + w_3^* = 1$, we get

$$1 - (w_1^* + w_2^*) = \frac{\lambda}{1+\lambda}.$$

This is equivalent to

$$w_1^* + w_2^* = \frac{1}{1+\lambda}.$$

This means every weight in $\mathcal{W}\big(\text{PBLP}^2(\lambda)\big)$ which is contained in $\mathcal{W}(\text{TOLP})$ satisfies the condition $w_1^* + w_2^* = \frac{1}{1+\lambda}$. This condition can be interpreted as an equation of a line in $\mathbb{R}^2$. Consequently, we can define the projection of a weight set $\mathcal{W}\big(\text{PBLP}^2(\lambda)\big)$ as a line segment in $\mathcal{W}(\text{TOLP})$.

3.17 Definition. The *projection of the weight set* $\mathcal{W}\big(\text{PBLP}^2(\lambda)\big)$ in $\mathbb{R}^2$ can be defined by a line segment

$$\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^2(\lambda)\right)} = \left\{(w_1^*, w_2^*) \mid w^* \in \mathcal{W}(\text{TOLP}),\ w_1^* + w_2^* = \frac{1}{1+\lambda}\right\}. \qquad \triangleleft$$

3.18 Remark. From the definition of $\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^2(\lambda)\right)}$, we observe the following:

(i) The slope m_λ of the line segment $\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^2(\lambda)\right)}$ is given by

$$m_\lambda = -1.$$

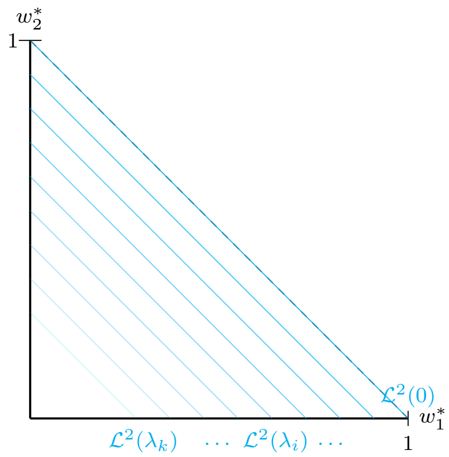


Figure 3.4: An illustration of the family of line segments $\mathcal{L}^2(\lambda)$ for PBLP^2 in the weight set of TOLP, where $\mathcal{L}^2(\lambda_i) = \mathcal{L}_{\mathcal{W}(\text{PBLP}^2(\lambda_i))}$ for $\lambda_i \in \{0, \ldots, \lambda_k\}$. The line segments $\mathcal{L}^2(\lambda)$ represent weight sets of $\text{PBLP}^2(\lambda)$ for different values of λ.

(ii) The end points of $\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^2(\lambda)\right)}$ are $\left(0, \frac{1}{1+\lambda}\right)$ and $\left(\frac{1}{1+\lambda}, 0\right)$. ◁

Remark 3.18.(i) is a direct result of finding the slope of the line segment $\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^2(\lambda)\right)}$. Remark 3.18.(ii) represents the w_1^*-intercept and w_2^*- intercept of $\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^2(\lambda)\right)}$.

We illustrate a family of line segments that represents weight sets $\mathcal{W}\left(\text{PBLP}^2(\lambda)\right)$ for different values of λ in Figure 3.4.

3.2.1 Illustrative examples and further results

In this section, we present results related to both PBLP^1 and PBLP^2 and use examples to illustrate the behaviour of the solution sets.

Every solution set $S(\text{TOLP})$ for TOLP contains a subset of solutions such that there is a bijection between this subset and the set of extreme nondominated images of TOLP. At the same time, the weight set $\mathcal{W}(\text{TOLP})$ can be decomposed into full dimensional weight set components such that there is a bijection between these weight set components and $Y_{\text{EN}}(\text{TOLP})$ (cf. [PGE10]). By Definitions 3.9 and 3.17, for a fixed value of $\lambda \geq 0$, the weight set $\mathcal{W}(\text{PBLP}^j(\lambda))$ is a line segment that lies in $\mathcal{W}(\text{TOLP})$. This line segment intersects some (full-dimensional) weight set components. For each component $\mathcal{W}(y)$, there are three possibilities: $\mathcal{W}(\text{PBLP}^j(\lambda))$ intersects $\mathcal{W}(y)$ either in a single vertex or along an edge, or it passes through the interior of $\mathcal{W}(y)$. The entire segment $\mathcal{W}(\text{PBLP}^j(\lambda))$ can then be decomposed into such intersections (cf. [PGE10]). A solution set for $\text{PBLP}^j(\lambda)$ can be obtained by using all solutions from $S(\text{TOLP})$ where the corresponding weight set component is intersected by $\mathcal{W}(\text{PBLP}^j(\lambda))$. Therefore, $S(\text{TOLP})$ is also a solution set for PBLP^j.

At the same time, Theorems 3.3 and 3.11 imply that, for every full dimensional weight set component $\mathcal{W}(y)$ of an extreme nondominated image $y \in Y_{\text{EN}}(\text{TOLP})$, there is at least one value $\lambda \geq 0$ such that the line segment $\mathcal{W}(\text{PBLP}^j(\lambda))$ intersects the interior of $\mathcal{W}(y)$. Then, a solution set for $\text{PBLP}^j(\lambda)$ must contain at least one solution x such that $(c_1 x, c_2 x, d_1 x)^\top = y$ (cf. [PGE10]). Therefore, every solution set for PBLP^j also contains a solution set for TOLP.

Since every solution set for TOLP contains a solution set for PBLP^j, we can make the following statement regarding minimal solution sets:

3.19 Proposition. A set $S(\text{TOLP}) \subseteq X$ is a minimal solution set for TOLP if and only if $S(\text{TOLP})$ is a minimal solution set for PBLP^j. ◁

An illustration of the weight sets of the two parametric biobjective linear programs PBLP^j $j = 1, 2$ and the weight set of its corresponding TOLP is shown for Example 3.20 in Figure 3.6 and in Figure 3.5, respectively.

3.20 Example. Consider the following PBLP^1 with a non-negative parameter $\lambda \geq 0$:

$$\begin{aligned} \min \quad & \begin{pmatrix} -3x_1 - x_2 + \lambda(x_1 + x_2) \\ x_1 - 2x_2 \end{pmatrix} \\ \text{s. t.} \quad & x \in X := \left\{ x \in \mathbb{R}^2_{\geqq} : 3x_1 + 2x_2 \geq 6; x_1 \leq 10; x_2 \leq 3 \right\}, \end{aligned}$$

and let us also consider a linear PBLP^2:

$$\begin{aligned} \min \quad & \begin{pmatrix} -3x_1 - x_2 + \lambda(x_1 + x_2) \\ x_1 - 2x_2 + \lambda(x_1 + x_2) \end{pmatrix} \\ \text{s.t.} \quad & x \in X \end{aligned}$$

and the corresponding TOLP which is related to both parametric biobjective problems:

$$\begin{aligned} \min \quad & (-3x_1 - x_2, x_1 - 2x_2, x_1 + x_2)^\top \\ \text{s.t.} \quad & x \in X. \end{aligned}$$

The weight set decomposition of the TOLP is composed of four weight set components, $\mathcal{W}(y^i), i = 1, \ldots, 4$ each corresponding to the extreme nondominated images y^1, y^2, y^3, and y^4 as shown in Figure 3.5. The weight sets of $\text{PBLP}^1(\lambda)$ and $\text{PBLP}^2(\lambda)$ for the parameter values $\lambda = 0, 1, 2$ are shown in Figures 3.6a and 3.6b, respectively. ◁

A minimal solution set for the parametric biobjective linear program PBLP^j is defined over the entire parameter set. We are particularly interested in the critical parameter values where a minimal solution set of $\text{PBLP}^j(\lambda)$ changes as λ varies. These values are called *breakpoints*.

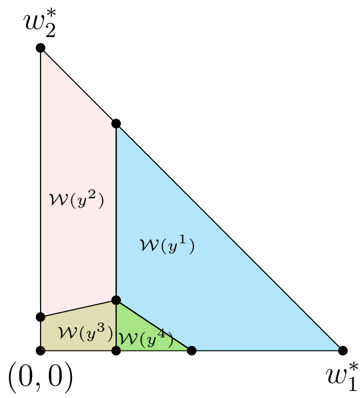
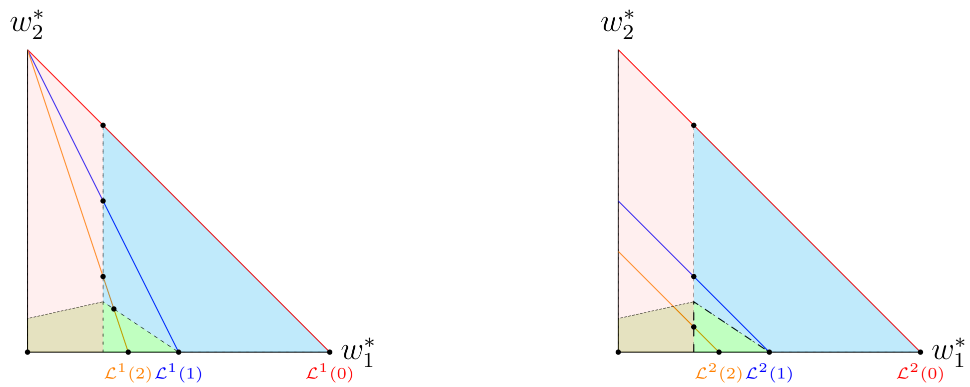


Figure 3.5: Weight set of TOLP with four weight set components



(a) Line segments $\mathcal{L}_{\mathcal{W}(\mathrm{PBLP}^1(\lambda))}$ of $\mathrm{PBLP}^1(\lambda)$, denoted by $\mathcal{L}^1(\lambda)$.

(b) Line segments $\mathcal{L}_{\mathcal{W}(\mathrm{PBLP}^2(\lambda))}$ of $\mathrm{PBLP}^2(\lambda)$, denoted by $\mathcal{L}^2(\lambda)$.

Figure 3.6: An illustration of weight sets of $\mathrm{PBLP}^j(\lambda)$ for parameter values, $\lambda = 0, 1, 2$ in the weight set of TOLP.

3.21 Definition. More formally, for a given PBLP^j, some parameter value $\lambda \geq 0$ is called a breakpoint if, for a small $\varepsilon > 0$,

$$S\left(\mathrm{PBLP}^j(\lambda + \varepsilon)\right) \neq S\left(\mathrm{PBLP}^j(\lambda - \varepsilon)\right),$$

where $S\left(\mathrm{PBLP}^j(\lambda)\right)$ denotes some minimal solution set at parameter λ. ◁

Therefore, the solution to parametric biobjective linear programs PBLP^j consists of the following:

(i) a minimal solution set S for PBLP^j,
(ii) a set of breakpoints in the parameter set and
(iii) parameter intervals for each solution $x \in S$.

We characterize breakpoints of PBLP^j with the help of extreme weights in $\mathcal{W}(\mathrm{TOLP})$.

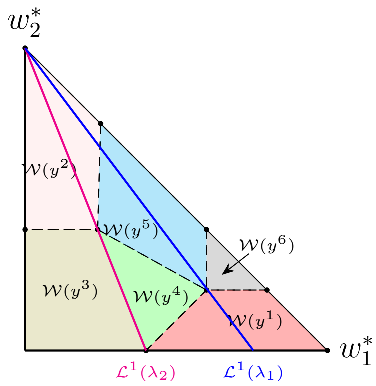


Figure 3.7: Illustration of the different ways breakpoints can behave in PBLP^j. For example: A solution set at the breakpoint λ_1, $S(\text{PBLP}^1(\lambda_1)) = \{x^1, x^5, x^2\}$ is different from $S\left(\text{PBLP}^1(\lambda_1 - \varepsilon)\right) = \{x^1, x^6, x^5, x^2\}$ and $S\left(\text{PBLP}^1(\lambda_1 + \varepsilon)\right) = \{x^1, x^4, x^5, x^2\}$, while the breakpoint λ_2 shows the case of a tie where $S\left(\text{PBLP}^1(\lambda_2)\right) = \{x^4, x^2\}$ and $S\left(\text{PBLP}^1(\lambda_2)\right) = \{x^3, x^2\}$ as well. Line segments $\mathcal{L}^1(\lambda) = \mathcal{L}_{\mathcal{W}(\text{PBLP}^1(\lambda))}$ represents the weight set of $\text{PBLP}^1(\lambda)$ at the two breakpoints $\lambda \in \{\lambda_1, \lambda_2\}$. Note that feasible solutions such as x_1, x_2, x_3, and x_4 map to the extreme nondominated images, y_1, y_2, y_3, and y_4, respectively.

3.22 Proposition. If λ_i is a breakpoint of a PBLP^j with a corresponding weight set $\mathcal{W}\left(\text{PBLP}^j(\lambda_i)\right)$, there exists at least one extreme weight in $\mathcal{W}(\text{TOLP})$ on the intersection of $\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^j(\lambda_i)\right)}$ and $\mathcal{W}(\text{TOLP})$. ◁

Proof. The proof uses the following definition. Let $\lambda_i \geq 0$ be the parameter, $Y_{\text{EN}}(\text{TOLP})$ denote the set of extreme nondominated images of TOLP, and $\mathcal{W}(\text{PBLP}^j(\lambda_i))$ be the weight set of $\text{PBLP}^j(\lambda_i)$. We define

$$\nu(\lambda_i) \coloneqq \left\{\mathcal{W}(y) : y \in Y_{\text{EN}}(\text{TOLP}), \text{int}(\mathcal{W}(y)) \cap \mathcal{W}\left(\text{PBLP}^j(\lambda_i)\right) \neq \varnothing, j = 1, 2\right\}.$$

In particular, the set $\nu(\lambda_i)$ consists of weight set components of $\mathcal{W}(\text{TOLP})$ that are intersected by the weight set $\mathcal{W}(\text{PBLP}^j(\lambda_i))$ passing through their interior.
Let λ_i and λ_{i+1} be two consecutive breakpoints. We consider both cases separately.

Case $j = 1$: By Proposition 3.7, it is trivial that the extreme weight $(0, 1, 0) \in \mathcal{W}(\text{TOLP})$ is contained in $\mathcal{W}\left(\text{PBLP}^1(\lambda)\right)$ for all $\lambda \geq 0$, whether λ is a breakpoint or not. However, this proposition still holds for PBLP^1 when we exclude $(0, 1, 0)$ from the weight sets of $\mathcal{W}\left(\text{PBLP}^1(\lambda)\right)$.

By the definition of a breakpoint, we have $S\left(\text{PBLP}^1(\lambda_i)\right) \neq S\left(\text{PBLP}^1(\lambda_i + \varepsilon)\right)$. Suppose, for contradiction, that no extreme weight of $\mathcal{W}(\text{TOLP})$ lies on the intersection of the line segment $\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^1(\lambda_i)\right)} \setminus \{(0, 1)\}$ and $\mathcal{W}(\text{TOLP})$. Then the line segment $\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^1(\lambda_i)\right)} \setminus \{(0, 1)\}$ passes entirely through the relative interiors of weight set components in $\nu(\lambda_i)$. By the definition of relative interior and the compactness of $\nu(\lambda_i)$, there exists $\delta > 0$ such that

$$\nu(\lambda_i + \delta) = \nu(\lambda_i - \delta) = \nu(\lambda_i).$$

This implies that the line segments $\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^2(\lambda)\right)}$ for $\lambda = \{i - \delta, i, i + \delta\}$ passes through same set of weight set components. Therefore,

$$S\Big(\text{PBLP}^1(\lambda_i + \delta)\Big) = S\Big(\text{PBLP}^1(\lambda_i - \delta)\Big) = S\Big(\text{PBLP}^1(\lambda_i)\Big).$$

This contradicts λ_i being a breakpoint.

Case $j = 2$: By Proposition 3.15, all weight sets $\mathcal{W}\Big(\text{PBLP}^2(\lambda)\Big)$ are unique for $\lambda \geq 0$. The definition of a breakpoint gives $S\Big(\text{PBLP}^2(\lambda_i)\Big) \neq S\Big(\text{PBLP}^2(\lambda_i + \varepsilon)\Big)$.

Suppose, for contradiction, that no extreme weight of $\mathcal{W}(\text{TOLP})$ lies on the intersection of $\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^2(\lambda_i)\right)}$ and $\mathcal{W}(\text{TOLP})$. Then $\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^2(\lambda_i)\right)}$ passes through relative interiors of weight set components in $\nu(\lambda_i)$. By the relative interior properties and compactness of $\nu(\lambda_i)$, there exists $\delta > 0$ such that

$$\nu(\lambda_i + \delta) = \nu(\lambda_i - \delta) = \nu(\lambda_i).$$

This implies that the line segments $\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^2(\lambda)\right)}$ for $\lambda = \{i - \delta, i, i + \delta\}$ pass through the same set of weight set components. Therefore,

$$S(\text{PBLP}^2(\lambda_i + \delta)) = S(\text{PBLP}^2(\lambda_i - \delta)) = S(\text{PBLP}^2(\lambda_i)).$$

This contradicts λ_i being a breakpoint. □

As a consequence of Proposition 3.22, the number of breakpoints in PBLP^j is bounded by the number of extreme weights in $\mathcal{W}(\text{TOLP})$.

Note that not every extreme weight in $\mathcal{W}(\text{TOLP})$ leads to a unique breakpoint. For a given breakpoint λ_i, there might be several extreme weights on the line segment $\mathcal{L}_{\mathcal{W}(\text{PBLP}^j(\lambda_i))}$. We demonstrate this in Example 3.23.

In PBLP^j, breakpoints can exhibit behaviour not observed in single-objective parametric optimization. Certain breakpoints have a unique minimal solution set of their own, which differs from the sets immediately preceding and following it. This happens when the solutions leaving and entering a minimal solution set correspond to extreme nondominated images that are still nondominated but not extreme points for $\text{PBLP}^j(\lambda)$ (see Figure 3.11). As a result, the parameter set is subdivided into a set of intervals and/or unique parameter values, each corresponding to a unique minimal solution set, respectively.

Another special case here is that of a *tie* in solution sets; we define two solution sets to be a *tie* if the corresponding breakpoint serves as both the upper bound of an interval

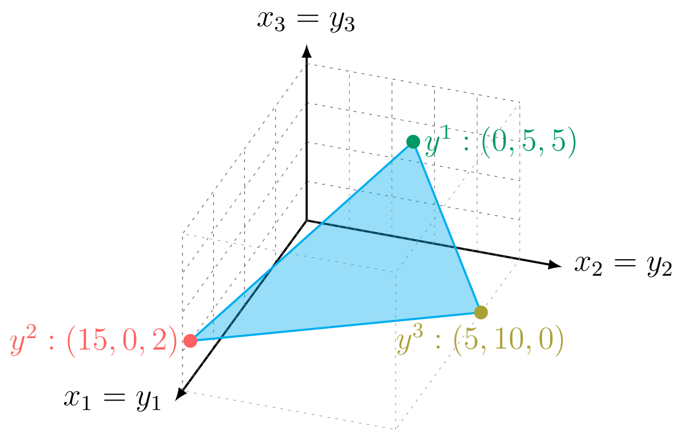


Figure 3.8: Illustration of the nondominated image set of the triobjective problem

and the lower bound of the next consecutive interval. In such cases, we include the breakpoint in the preceding interval and exclude it from the following one. These various breakpoints are visualized in Figure 3.7.

3.23 Example. Consider the following triobjective linear program,

$$\begin{aligned} \min \quad & (x_1, x_2, x_3)^\top \\ \text{s.t.} \quad & x \in X \coloneqq \{x \in \mathbb{R}^3_{\geqq} : \; 2x_1 + 3x_2 + 5x_3 \geq 40, \\ & \qquad 2x_1 + 15x_2 - 15x_3 \geq 0, \\ & \qquad 2x_1 - x_2 + x_3 \geq 0, \\ & \qquad 2x_1 - x_2 - 15x_3 \leq 0\} \end{aligned}$$

and also consider the corresponding PBLP^2 with a non-negative parameter λ,

$$\begin{aligned} \min \quad & (x_1 + \lambda x_3, x_2 + \lambda x_3)^\top \\ \text{s.t.} \quad & x \in X. \end{aligned}$$

The nondominated image set of the problem is a convex hull of three extreme nondominated images, $y^1 = (5, 10, 0)$, $y^2 = (0, 5, 5)$, and $y^3 = (15, 0, 2)$ as shown in Figure 3.8 and has a minimal solution set $S = \{x^1, x^2, x^3\}$ such that $y^1 = x^1, y^2 = x^2$ and $y^3 = x^3$.$\triangleleft$

The weight set of TOLP, Example 3.23 and the line segment representing the weight set of the corresponding $\text{PBLP}^2(\lambda)$, where $\lambda = 1$ are shown in Figure 3.9. We observe that from Figure 3.9 we have two different extreme weights $w^1 = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$ and $w^2 = \left(\frac{1}{5}, \frac{3}{10}, \frac{1}{2}\right)$ which correspond to the same breakpoint $\lambda = 1$. This is also due to the special case of a tie in the solution sets of $\text{PBLP}^2(\lambda)$ before and after the breakpoint i. e. $S(\text{PBLP}^2(1)) = \{x^1, x^2\}$ and as well as $S(\text{PBLP}^2(1)) = \{x^2, x^3\}$.

Since the nondominated image set of $\text{PBLP}^j(\lambda)$ is changing with respect to λ, we define $y^i(\lambda)$ as the extreme nondominated image of $\text{PBLP}^j(\lambda)$ that corresponds to a solution $x^i \in S$.

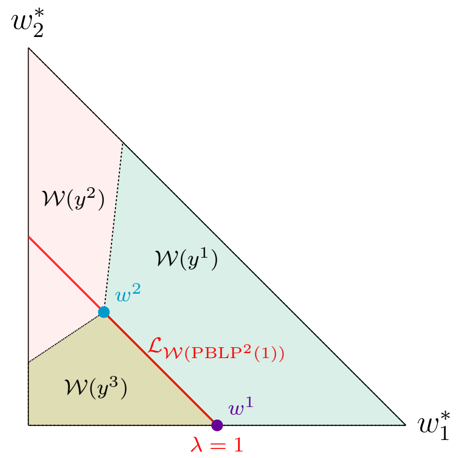


Figure 3.9: The line segment $\mathcal{L}_{\mathcal{W}(\text{PBLP}^2(1))}$ of $\text{PBLP}^2(\lambda)$ at $\lambda = 1$ intersects $\mathcal{W}(\text{TOLP})$ at two extreme weights w' and w''.

The case of a tie in solution sets at a breakpoint λ in $\text{PBLP}^j(\lambda)$ implies that the extreme nondominated images $y^2(\lambda)$ and $y^1(\lambda)$ asymptotically converge to a common nondominated image i. e. $y^2(\lambda) = y^1(\lambda)$. We illustrate this by using the image set of the corresponding biobjective linear program $\text{PBLP}^2(\lambda)$ with varying λ, $\lambda := \{0, 1, 2\}$. It has an image set with the objective functions $y_1 = x_1 + \lambda x_3$ and $y_2 = x_2 + \lambda x_3$ that changes continuously with the variation of parameter λ as shown in Figure 3.10. Note that the two extreme nondominated images $y^2(1)$ and $y^1(1)$ converge to the same point, i.e., $y^2(1) = y^1(1)$ at $\lambda = 1$.

3.24 Corollary. The number of breakpoints is finite. ◁

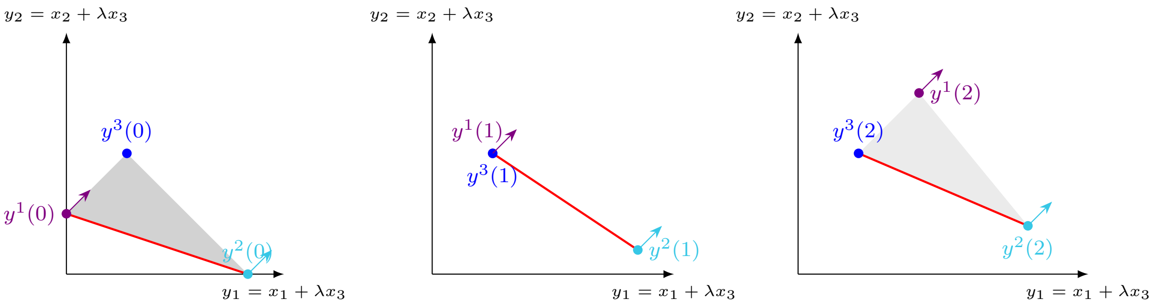
Proof. From Proposition 3.22, we know that the number of breakpoints is bounded by the number of extreme weights in the weight set of TOLP. Moreover, the number of extreme weights in $\mathcal{W}(\text{TOLP})$ is finite due to the finiteness of the extreme points of all weight set components. Thus, the number of breakpoints is finite. □

An analogous case of PBLP^1 arises when the second objective is parametric which yields the following parametric program:

$$\min_{x \in X} \begin{pmatrix} c_1 x \\ c_2 x + \lambda d_1 x \end{pmatrix}. \qquad (\text{PBLP}^1_2)$$

By similar argument, all the results for PBLP^1 hold for the above case and the line segments corresponding to the weight set of the biobjective linear program $\text{PBLP}^1_2(\lambda)$ that intersects $\mathcal{W}(\text{TOLP})$ is shown in Figure 3.12.

As we are interested in finding breakpoints that mark a change in a minimal solution set of $\text{PBLP}^j(\lambda)$, we can use parameter intervals of each solution to determine such breakpoints. We use the construction of the parameter value, $\lambda := \frac{w_3^*}{w_1^*}$ for $\text{PBLP}^1(\lambda)$ from



(a) $\lambda = 0$ **(b)** $\lambda = 1$ **(c)** $\lambda = 2$

Figure 3.10: An illustration of change in the nondominated set for $\text{PBLP}^2(\lambda)$. When $\lambda = 0$ the images $y^2(0)$ and $y^3(0)$ are the extreme nondominated images for $\text{PBLP}^2(0)$. When $\lambda = 1$, the images $y^2(1)$ and $y^1(1)$ are equivalent and $Y_{\text{EN}}(\text{PBLP}^2(1)) = \{y^2(1), y^3(1)\} = \{y^1(1), y^3(1)\}$. This leads to the case of a tie at the breakpoint $\lambda = 1$. When $\lambda = 2$, the image $y^1(2)$ is no longer an extreme nondominated image for $\text{PBLP}^2(2)$ and $Y_{\text{EN}}(\text{PBLP}^2(2)) = \{y^1(2), y^3(2)\}$. The arrows on the images show the direction of the change in y^i with respect to change in λ.

Equation (3.2). The parameter interval of a solution $x \in S(\text{TOLP})$ can be determined using the weight set component of the corresponding $y \in Y_{\text{EN}}(\text{TOLP})$.

3.25 Proposition. Let $x \in S(\text{TOLP})$ be a solution and $y = (c_1 x, c_2 x, d_1 x)^\top \in Y_{\text{EN}}$ be its corresponding extreme nondominated image in TOLP. The *parameter intervals* of x for the parametric problems PBLP^1 and PBLP^2 are

$$\mathcal{I}^1(x) = \left\{ \lambda : \lambda = \frac{1 - w_1^* - w_2^*}{w_1^*}, w^* \in \mathcal{W}(y) \right\}$$

and

$$\mathcal{I}^2(x) = \left\{ \lambda : \lambda = \frac{1 - w_1^* - w_2^*}{1 - w_1^*}, w^* \in \mathcal{W}(y) \right\},$$

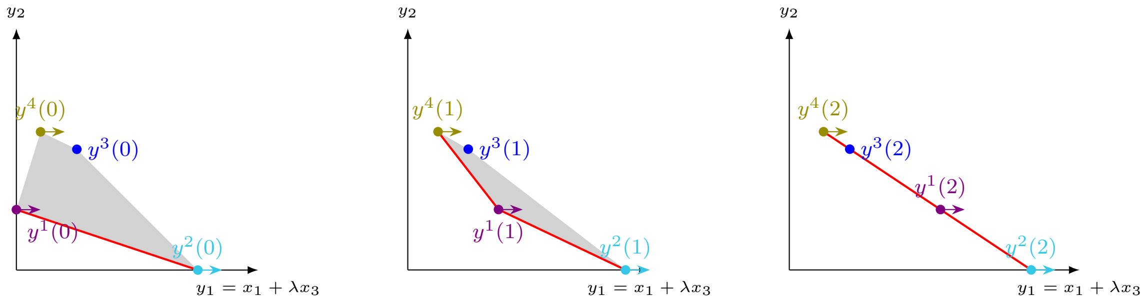
respectively. ◁

Proof. By Theorem 3.3, there exists a parameter $\lambda \geq 0$ and weight $w \in \mathbb{R}^2_{\geq}$ such that x is optimal for $\text{WS}\big(\text{PBLP}^1(\lambda), w\big)$. Since all the weights in $w^* \in \mathcal{W}(y)$ corresponds to a solution $x \in S(\text{TOLP})$(see the proof of Proposition 3.19), we use the construction $\lambda := \frac{w_3^*}{w_1^*}$ from Theorem 3.3 to find parameter values that correspond to the solution x in the parametric problem. Since $w_1^* + w_2^* + w_3^* = 1$, we get

$$\lambda = \frac{1 - w_1^* - w_2^*}{w_1^*}.$$

Therefore, we get the parameter interval $\mathcal{I}^1(x)$ of the solution x for PBLP^1.

By Theorem 3.11, and using similar arguments we get parameter interval of x for $\mathcal{I}^2(x)$ for PBLP^2. □



(a) $\lambda = 0$ **(b)** $\lambda = \frac{3}{2}$ **(c)** $\lambda = \frac{5}{2}$

Figure 3.11: An illustration of change in the nondominated set for $\text{PBLP}^1(\lambda)$ in Example 3.20. When $\lambda = 0$ the images $y^1(0)$ and $y^2(0)$ are the extreme nondominated images for $\text{PBLP}^1(0)$. When $\lambda = \frac{3}{2}$, $Y_{\text{EN}}(\text{PBLP}^1(1)) = \{y^1(1), y^2(1), y^3(1)\}$. Then if $\lambda = \frac{5}{2}$ the images $y^1(2)$ and $y^3(2)$ are no longer extreme nondominated images for $\text{PBLP}^1(2)$ but are still nondominated and $Y_{\text{EN}}(\text{PBLP}^1(2)) = \{y^2(2), y^4(2)\}$. This leads to a unique solution set of $\text{PBLP}(\lambda)$ at the breakpoint $\lambda = \frac{5}{2}$.

In terms of the parameter set, we illustrate the subdivision of the parameter set into a collection of parameter intervals that correspond to an efficient solution of a minimal solution set of PBLP^1 in Figure 3.13a. Analogously, we also illustrate parameter intervals for PBLP^1_2 as $\mathcal{I}^1_2(x)$ in Figure 3.13b.

Furthermore, we can use breakpoints of PBLP^j to decompose the parameter set into intervals or parameter value such that each interval or value correspond to a unique solution set.

We propose two algorithms, the Breakpoints Enumeration Algorithm and the Adapted Weight Set Algorithm to find the breakpoints and the parameter intervals for every solution in a minimal solution set in Chapter 6.

3.3 Parametric triobjective linear program

In this section, we consider a parametric triobjective linear program with three objective functions dependent on a single parameter $\lambda \in \mathbb{R}_{\geq}$.

3.26 Definition (Parametric Triobjective Linear Program). Let $\lambda \in \mathbb{R}_{\geqq}$ be a parameter. A *parametric triobjective linear program* is defined as

$$\begin{aligned} \min \quad & \begin{pmatrix} c_1x + \lambda d_1 x \\ c_2 x \\ c_3 x \end{pmatrix} \\ \text{s.t.} \quad & Ax \geqq b, \\ & x \geqq 0, \end{aligned} \tag{PTLP}$$

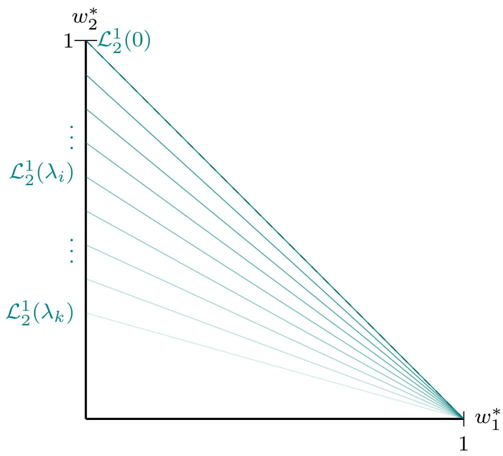


Figure 3.12: An illustration of the family of line segments $\mathcal{L}_2^1(\lambda)$ for PBLP_2^1 in the weight set of TOLP, where $\mathcal{L}_2^1(\lambda) = \mathcal{L}_{\mathcal{W}(\text{PBLP}_2^1(\lambda))}$ for $\lambda \in \{0, \dots, \lambda_k\}$. The line segments $\mathcal{L}_2^1(\lambda)$ represent weight sets of $\text{PBLP}_2^1(\lambda)$ for different values of λ.

where $c_i, i = 1, 2, 3$ and d_1 are row vectors in $\mathbb{R}^n$, $A \in \mathbb{Q}^{m \times n}$, $m, n \in \mathbb{N} \setminus \{0\}$, and $b \in \mathbb{Q}^m$. Again, the set $X := \{x \in \mathbb{R}^n : Ax \geqq b, x \geqq 0\}$ denotes the feasible set. $\triangleleft$

The problem PTLP, for some fixed λ, is a non-parametric triobjective linear program and we denote it by PTLP(λ). Furthermore, we denote the set of extreme nondominated images of PTLP(λ) by $Y_{\text{EN}}(\text{PTLP}(\lambda))$ and a corresponding minimal solution set by $S\,(\text{PTLP}(\lambda))$. Our analysis relates to a minimal solution set that corresponds to the set of extreme nondominated images of PTLP(λ). Therefore, a minimal solution set of PTLP equates to a set that contains efficient solutions of PTLP(λ) for each fixed value of $\lambda \geq 0$. We denote this set by S(PTLP).

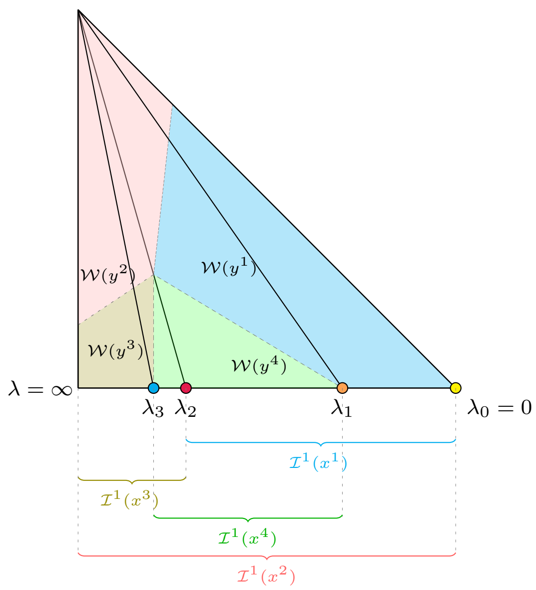
3.27 Definition. A *solution set* $S(\text{PTLP}) \subseteq X$ of PTLP is a set such that for every $\lambda \geq 0$, S(PTLP) contains, as a subset, a solution set for the triobjective problem PTLP(λ). It is called *minimal* if, additionally, there is no other solution set $S' \subseteq X$ for PTLP with $|S'| < |S(\text{PTLP})|$. $\triangleleft$

We relate PTLP to a four-objective linear program using the weighted sum scalarization. To this end, we consider the four-objective linear problem with the same objective functions c_1x, c_2x, c_3x, d_1x, i. e.

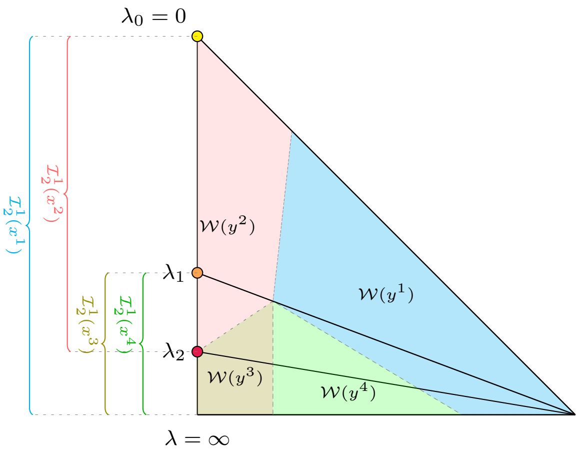
$$\begin{aligned} \min \quad & (c_1x, c_2x, c_3x, d_1x)^\top \\ \text{s. t.} \quad & x \in X \end{aligned} \tag{4-OLP}$$

and denote the set of extreme nondominated images of 4-OLP by Y_{EN}(4-OLP). We denote a minimal solution set that corresponds to the set Y_{EN}(4-OLP) by S(4-OLP). We define the weight set $\mathcal{W}$(4-OLP) of 4-OLP as

$$\mathcal{W}(\text{4-OLP}) := \left\{ w^* \in \mathbb{R}^4_{\geqq} : \sum_{i=1}^{4} w_i^* = 1 \right\}.$$



(a) Parameter intervals, $\mathcal{I}^1(x_i)$ for each efficient solution in S with respect to PBLP^1



(b) Parameter intervals, $\mathcal{I}_2^1(x_i)$ for each efficient solution in S with respect to PBLP$_2^1$

Figure 3.13: An illustration of the subdivision of the parameter set into parameter intervals for a solution set of the PBLP instances in Example 3.20. Note that the intervals depicted here are with respect to the weight set component of $\mathcal{W}$(TOLP) and the actual interval is the corresponding parameter values of the component.

The weighted sum scalarization of 4-OLP with a normalized weight $w^* \in \mathcal{W}(\text{4-OLP})$ is

$$\min_{x \in X} w_1^* c_1 x + w_2^* c_2 x + w_3^* c_3 x + w_4^* d_1 x. \qquad (\text{WS}(\text{4-OLP}, w^*))$$

We approach the problem by applying the weighted sum scalarization to the parametric triobjective linear program PTLP and formally characterize its relationship to the corresponding 4-OLP in our results. For a fixed value of λ, the weighted sum scalarization of PTLP(λ) is

$$\min_{x \in X} w_1(c_1 x + \lambda d_1 x) + w_2 c_2 x + w_3 c_3 x \qquad (\text{WS}(\text{PTLP}(\lambda), w))$$

where $w \in \mathbb{R}^3_{\geqq}$ and $w_1 + w_2 + w_3 = 1$.
We reformulate this problem using $w_3 = 1 - w_1 - w_2$ and obtain

$$\min_{x \in X} w_1 c_1 x + w_2 c_2 x + (1 - w_1 - w_2)\, c_3 x + w_1 \lambda d_1 x.$$

The problem can be interpreted as a weighted sum scalarization of the 4-OLP with weight vector $(w_1, w_2, 1 - w_1 - w_2, w_1\lambda)$. It holds that

$$w_1 + w_1\lambda + w_2 + (1 - w_1 - w_2) = 1 + w_1\lambda \geq 1.$$

We normalize the weights and get

$$\min_{x \in X} \frac{w_1}{1 + w_1\lambda}(c_1 x) + \frac{w_2}{1 + w_1\lambda}(c_2 x) + \frac{w_3}{1 + w_1\lambda}(c_3 x) + \frac{w_1\lambda}{1 + w_1\lambda}(d_1 x),$$

which is a particular case of WS(4-OLP, w^*) with the corresponding weight vector

$$w^* = \left(\frac{w_1}{1 + w_1\lambda}, \frac{w_2}{1 + w_1\lambda}, \frac{w_3}{1 + w_1\lambda}, \frac{w_1\lambda}{1 + w_1\lambda}\right)^\top \in \mathcal{W}(\text{4-OLP}). \qquad (3.8)$$

We now present our main result, which establishes the equivalence between the efficient solutions of the problem PTLP and the efficient solutions of the corresponding four-objective linear program.

3.28 Theorem. A feasible solution x^* is optimal for WS(4-OLP, w^*) with non-negative weight w^* where $w_1^* > 0$ if and only if there exist a parameter $\lambda \geq 0$ and non-negative weight w where $w_1 > 0$ such that x^* is optimal for WS(PTLP(λ), w). ◁

Proof. This proof is analogous to the proofs of Theorems 3.3 and 3.11. Here, note that $w_1^* > 0$ and we define

$$\begin{aligned} \lambda &:= \frac{w_4^*}{w_1^*}, \\ w_1 &:= \frac{w_1^*}{w_1^* + w_2^* + w_3^*}, \end{aligned} \qquad (3.9)$$

$$w_2 := \frac{w_2^*}{w_1^* + w_2^* + w_3^*},$$
$$w_3 := \frac{w_3^*}{w_1^* + w_2^* + w_3^*}$$

in the first part of the proof and using Equation (3.8), we define a non-negative weight w^*

$$\begin{aligned} w_1^* &:= w_1, \\ w_2^* &:= w_2, \\ w_3^* &:= w_3, \\ w_4^* &:= w_1\lambda \end{aligned} \tag{3.10}$$

in the converse case. We do not normalize for simplicity. □

Observe that we assume $w_1^* > 0$ in Theorem 3.28. Based on the construction in the proof, the parameter λ is undefined if $w_1^* = 0$, and w_1^* approaching 0 is equivalent to the parameter λ approaching ∞. And indeed, the one-to-one correspondence from Theorem 3.28 does not hold: there exist $w^* \in \mathbb{R}^4_{\geq}$ with $w_1^* = 0$ and an optimal solution x^* for WS(4-OLP, w^*) such that there is no $\lambda \geq 0$ and $w \in \mathbb{R}^3_{\geqq}$ where x^* is optimal for WS(PTLP(λ), w). This can be illustrated by the following example.

3.29 Example. Let $X \subseteq \mathbb{R}^4$ where $X = \text{conv}(\{(1,1,1,1),(0,0,1,1),(0,0,0,2)\})$. Let the objective functions of the 4-OLP be $c_1x = x_1$, $c_2x = x_2$, $c_3x = x_3$ and $d_1x = x_4$. For the weight $w^* = (0,0,0,1)$, the solutions $(1,1,1,1)$ and $(0,0,1,1)$ are both optimal solutions of WS(4-OLP, w^*) with a value of 1. However, we now consider WS(PTLP(λ), w). For every value of $\lambda \geq 0$ and $w \in \mathbb{R}^3_{\geqq}$ with $w_1 > 0$, the solution $(0,0,1,1)$ is a better solution than $(1,1,1,1)$ for WS(PTLP(λ), w) because the value of WS(PTLP(λ), w) for $(0,0,1,1)$ is

$$w_1((0) + (\lambda)(1)) + w_2(0) + w_3(1) = w_1\lambda + w_3$$

which is less than the value for $(1,1,1,1)$, i. e.

$$w_1((1) + \lambda(1)) + w_2(1) + w_3(1) = w_1 + w_2 + (w_1\lambda + w_3).$$

Additionally, for $w_1 = 0$, the solution $(0,0,0,2)$ is optimal for WS(PTLP(λ), w) with a value of 0 because for $(1,1,1,1)$ and $(0,0,1,1)$, the solution values are $w_2 + w_3 > 0$ and $w_3 \geq 0$, respectively.
Therefore, the solution $(1,1,1,1)$ is never optimal for WS(PTLP(λ), w). ◁

In the converse case, for any given parameter value $\lambda \geq 0$ and a weight $w \in \mathbb{R}^3_{\geqq}$ with $w_1 = 0$ for WS(PTLP(λ), w), we can see that it is possible to construct a weight $w^* \in \mathbb{R}^4_{\geq}$ for WS(4-OLP, w^*) using Equation (3.9). In fact, an optimal solution x

for $\mathrm{WS}(\mathrm{PTLP}(\lambda), w)$ with a weight $w \in \mathbb{R}^3_{\geqq}$ such that $w_1 = 0$ remains optimal for $\mathrm{WS}(4\text{-OLP}, w^*)$ with the weight $w^* = (0, w_2, w_2, 0)$ because $w_1^* = 0$ and $w_4^* = 0$ by construction.

As stated above, for every fixed value of λ, $\mathrm{PTLP}(\lambda)$ is a triobjective linear program. Its projected weight set is a two-dimensional polytope. However, considering the parameter λ, we can extend this interpretation of the set of the two-dimensional weight set to a higher dimension by the mapping $\mathcal{W}(\mathrm{PTLP}(\lambda))$, which is defined below.

3.30 Definition. For a given λ, we define a *mapping* $\mathcal{W}(\mathrm{PTLP}(\lambda))$ as,

$$\mathcal{W}(\mathrm{PTLP}(\lambda)) := \left\{ \frac{1}{1 + w_1\lambda} (w_1, w_2, w_3, w_1\lambda) : (w_1, w_2, w_3) \in \mathbb{R}^3_{\geqq}, \sum_{i=1}^{3} w_i = 1 \right\}. \quad \triangleleft$$

We can now characterize the weight set of $\mathcal{W}(\mathrm{PTLP}(\lambda))$ with respect to the weight set of $\mathcal{W}(4\text{-OLP})$.

3.31 Proposition. Let $\mathcal{W}(\mathrm{PTLP}(\lambda))$ and $\mathcal{W}(4\text{-OLP})$ be weight sets of PTLP for a fixed value λ and of 4-OLP, respectively. Then, it holds that

$$\mathcal{W}(\mathrm{PTLP}(\lambda)) \subsetneq \mathcal{W}(4\text{-OLP}). \quad \triangleleft$$

Proof. Let $w^* \in \mathcal{W}(\mathrm{PTLP}(\lambda))$. Then, by the definition of $\mathcal{W}(\mathrm{PTLP}(\lambda))$, it is

$$w^* = \frac{1}{1 + w_1\lambda}(w_1 + w_2 + w_3 + w_1\lambda)$$

for some $(w_1, w_2, w_3) \in \mathbb{R}^3_{\geqq}$ with $w_1 + w_2 + w_3 = 1$. The sum of the components satisfies

$$\frac{w_1}{1 + w_1\lambda} + \frac{w_2}{1 + w_1\lambda} + \frac{w_3}{1 + w_1\lambda} + \frac{w_1\lambda}{1 + w_1\lambda} = \frac{1 + w_1\lambda}{1 + w_1\lambda} = 1,$$

and, thus $w^* \in \mathcal{W}(4\text{-OLP})$.
To show that $\mathcal{W}(\mathrm{PTLP}(\lambda))$ is a proper subset of $\mathcal{W}(4\text{-OLP})$, we consider the particular weight $w' := (0, 0, 0, 1) \in \mathcal{W}(4\text{-OLP})$. Since for any given λ, if we consider the fourth component of any $w^* \in \mathcal{W}(\mathrm{PTLP}(\lambda))$ it holds that

$$\frac{w_1\lambda}{1 + w_1\lambda} \neq 1.$$

Therefore, it holds that $w' \notin \mathcal{W}(\mathrm{PTLP}(\lambda))$. $\square$

3.32 Proposition. For all the weight sets of $\mathrm{PTLP}(\lambda)$ for $\lambda \geq 0$, it holds that

$$\bigcap_{\lambda \geq 0} \mathcal{W}(\mathrm{PTLP}(\lambda)) = \{w^* \in \mathcal{W}(4\text{-OLP}) : w_2^* + w_3^* = 1\}. \quad \triangleleft$$

Proof. For an arbitrary $\lambda_i \geq 0$, by the definition of $\mathcal{W}\,(\text{PTLP}(\lambda_i))$, we have

$$\mathcal{W}\,(\text{PTLP}(\lambda_i)) := \left\{ \frac{1}{1+w_1\lambda_i}\,(w_1, w_2, w_3, w_1\lambda_i) : (w_1, w_2, w_3) \in \mathbb{R}^3_{\geq}, \sum_{i=1}^{3} w_i = 1 \right\}.$$

We first show that a weight $w^* \in \mathcal{W}(\text{4-OLP})$ such that $w_2^* + w_3^* = 1$ is an element of $\mathcal{W}\,(\text{PTLP}(\lambda_i))$.
Since $w_2^* + w_3^* = 1$, we have $w_1^* = w_4^* = 0$. This implies,

$$\begin{aligned} w^* &= (0, w_2^*, w_3^*, 0) \\ &= \left(\frac{0}{1+0\lambda_i}, \frac{w_2^*}{1+0\lambda_i}, \frac{w_3^*}{1+0\lambda_i}, \frac{0}{1+0\lambda_i} \right). \end{aligned}$$

Therefore, $w^* \in \mathcal{W}(\text{PTLP}(\lambda_i))$.
Next, we show that there is no other weight shared by any two weight sets of PTLP except for the weights that satisfy $w_2^* + w_3^* = 1$.
Let $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ be two arbitrary parameter values, such that $\lambda_1 \neq \lambda_2$. Let $\mathcal{W}(\text{PTLP}(\lambda_1))$ and $\mathcal{W}(\text{PTLP}(\lambda_2))$ be weight sets of $\text{PTLP}(\lambda_1)$ and $\text{PTLP}(\lambda_2)$, respectively.
Consider a weight $w^* \in \mathcal{W}(\text{PTLP}\lambda_1))$ and a weight $\tilde{w} \in \mathcal{W}(\text{PTLP}(\lambda_2))$.
By the definition of $\mathcal{W}(\text{PTLP}(\lambda_1))$ and $\mathcal{W}(\text{PTLP}(\lambda_2))$, we have

$$w^* = \left(\frac{w_1'}{1+w_1'\lambda_1}, \frac{w_2'}{1+w_1'\lambda_1}, \frac{w_3'}{1+w_1\lambda_1}, \frac{w_1'\lambda_1}{1+w_1'\lambda_1} \right) \text{where } w_1' + w_2' + w_3' = 1$$

and

$$\tilde{w} = \left(\frac{w_1''}{1+w_1''\lambda_2}, \frac{w_2''}{1+w_1''\lambda_2}, \frac{w_3''}{1+w_1\lambda_1} \frac{w_1''\lambda_2}{1+w_1''\lambda_2} \right) \text{where } w_1'' + w_2'' + w_3'' = 1.$$

If we equate the first components of w^* and $\tilde{w}$, we obtain

$$\frac{w_1'}{1+w_1'\lambda_1} = \frac{w_1''}{1+w_1''\lambda_2}.$$

Then, the components are equal if

$$w_1' = \frac{w_1''}{1+w_1''\lambda_2}(1+w_1'\lambda_1). \tag{3.11}$$

Also, the fourth component of w^* is

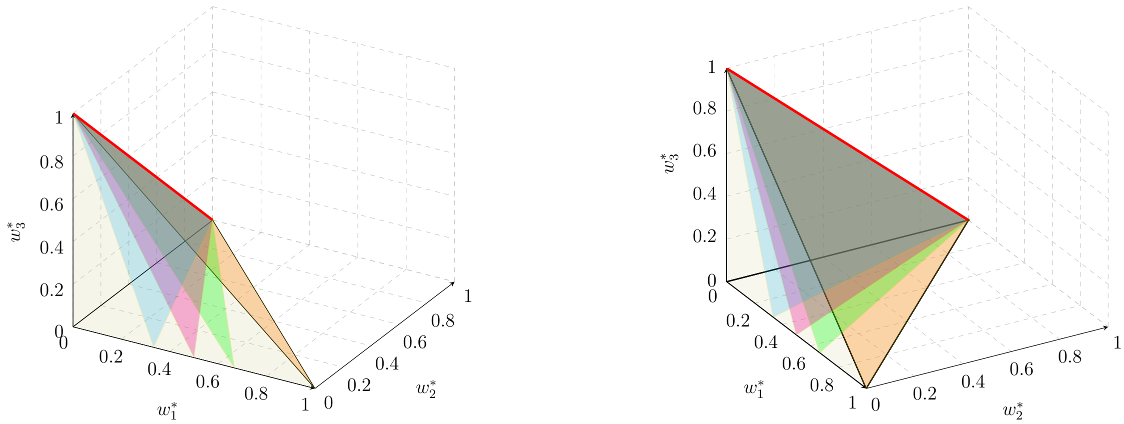
$$w_4^* = \frac{w_1'\lambda_1}{1+w_1'\lambda_1}.$$

We substitute w_1' from Equation (3.11) in w_4^* to get

$$\begin{aligned} w_4^* &= \frac{\frac{w_1''}{1+w_1''\lambda_2}(1+w_1'\lambda_1)}{1+w_1'\lambda_1}\lambda_1 \\ &= \frac{w_1''\lambda_1}{1+w_1''\lambda_2} \\ &\neq \frac{w_1''\lambda_2}{1+w_1''\lambda_2} = \tilde{w}_4 \end{aligned}$$

because $\lambda_1 \neq \lambda_2$. □

In other words, the intersection of all the weight sets of PTLP(λ) for all parameter values $\lambda \geq 0$ is the line segment, defined by $w_2^* + w_3^* = 1$ in $\mathcal{W}(\text{4-OLP}, w^*)$, as shown in Figure 3.14a.



(a) Weight sets $\mathcal{W}(\text{PTLP}(\lambda))$ for different values of λ

(b) Weight sets from a different angle

Figure 3.14: An illustration of weight sets of PTLP(λ) represented by the plane segments $\mathcal{P}(\lambda)$ for parameter values $\lambda = \{0, 0.5, 1, 2\}$ in the weight set of 4-OLP. The line segment marked in red is the intersection of all the weight sets $\mathcal{W}(\text{PTLP}(\lambda))$.

3.33 Corollary. Let $\mathcal{W}(\text{PTLP}(\lambda))$ be the weight set of PTLP(λ) and $\mathcal{W}$(4-OLP) be the weight set of the corresponding 4-OLP. Then the union of all the weight sets of PTLP(λ) for every $\lambda \geq 0$ is defined as

$$\bigcup_{\lambda \geq 0} \mathcal{W}(\text{PTLP}(\lambda)) = \{w^* \in \mathcal{W}(\text{4-OLP}) : w_1^* \neq 0\} \cup \{w^* \in \mathcal{W}(\text{4-OLP}) : w_2^* + w_3^* = 1\}\,. \triangleleft$$

Proof. From Proposition 3.31 we get that the weight sets of PTLP(λ) for every $\lambda \geq 0$ are a subset of $\mathcal{W}$(4-OLP), so the union of all these weight sets is also a subset of $\mathcal{W}$(TOLP). Using Theorem 3.28, for any weight in $\mathcal{W}$(4-OLP) except for the weights with $w_1^* = 0$, there exists some λ and $w \in \mathbb{R}^3_{\geqq}$ for PTLP(λ). However, by Proposition 3.32, the weights in the line segment $w_2^* + w_3^* = 1$ are the only weights shared by weight sets of PTLP(λ). Therefore, the union of weight sets of PTLP(λ) is equivalent to all the weights in $\mathcal{W}$(4-OLP) except for the weights where $w_1^* = 0$ but includes the line segment $w_2^* + w_3^* = 1$. □

More precisely, Corollary 3.33 implies that if we vary λ in PTLP(λ) and project all the corresponding weight sets $\mathcal{W}$(PTLP(λ)) in $\mathbb{R}^3$, we obtain a nearly complete weight set decomposition of the associated 4-OLP except for the weights with $w_1^* = 0$ due to the reason mentioned in Theorem 3.28.

Next, we want to characterize the projection of $\mathcal{W}$(PTLP(λ)) in $\mathbb{R}^3$. Since the fourth component of a weight in $\mathcal{W}$(PTLP(λ)) can be determined by the first three components, the projection reduces the entire weight set to a 3-dimensional interpretation. Using the definition of $\mathcal{W}$(PTLP(λ)), the third component of any arbitrary weight $w^* \in \mathcal{W}$(PTLP(λ)) can be written as

$$w_4^* = \lambda w_1^*.$$

Since $\sum_{i=1}^4 w_i^* = 1$, we get

$$1 - (w_1^* + w_2^* + w_3^*) = \lambda w_1^*.$$

This is equivalent to

$$w_3^* = 1 - w_1^*(1 + \lambda) - w_2^*.$$

This means every weight $w^* \in \mathcal{W}$(PTLP(λ)) which is contained in $\mathcal{W}$(TOLP) satisfies the condition $w_3^* = 1 - w_1^*(1+\lambda) - w_2^*$. This condition can be interpreted as an equation of a plane in $\mathbb{R}^3$. Consequently, we can define the weight set $\mathcal{W}$(PTLP(λ)) as a plane segment in $\mathcal{W}$(4-OLP).

3.34 Definition. The *projection of the weight set* $\mathcal{W}$(PTLP(λ)) in $\mathbb{R}^3$ can be defined as the plane segment,

$$\mathcal{P}_{\mathcal{W}(\mathrm{PTLP}(\lambda))} = \left\{(w_1^*, w_2^*, w_3^*) \mid w^* \in \mathcal{W}(\text{4-OLP}),\ w_3^* = 1 - w_1^*(1+\lambda) - w_2^*\right\}. \quad \triangleleft$$

3.35 Remark. From the definition of $\mathcal{P}_{\mathcal{W}(\mathrm{PTLP}(\lambda))}$ we observe that the vertices of the plane segment, $\mathcal{P}_{\mathcal{W}(\mathrm{PTLP})(\lambda)}$ are $(0,1,0)$, $(0,0,1)$ and $\left(\frac{1}{1+\lambda}, 0, 0\right)$. ◁

Remark 3.35 characterizes the plane segment $\mathcal{P}_{\mathcal{W}(\mathrm{PTLP}(\lambda))}$ by its intercepts with the w_1^*-, w_2^*-, and w_3^*-axes.

A visualization of these plane segments $\mathcal{W}$(PTLP(λ)) for different values of λ is illustrated in Figure 3.14. Note that we use the projection of $w^* \in \mathcal{W}(\text{4-OLP}, w^*)$ onto $(w_1^*, w_2^*, w_3^*) \in \mathbb{R}^3_{\geqq}$ in our analysis.

3.3.1 Illustrative examples and further results

Every solution set S(4-OLP) for 4-OLP consists of a subset of solutions such that there is a bijection between the subset and the extreme nondominated images Y_{EN} of 4-OLP. At the same time, the weight set $\mathcal{W}$(4-OLP) can be decomposed into full dimensional weight set components such that there is a bijection between these weight set components and Y_{EN}(4-OLP) [PGE10]. By Definition 3.34, for a fixed value of $\lambda \geq 0$, the weight set $\mathcal{W}(\text{PTLP}(\lambda))$ is a plane segment in $\mathcal{W}$(4-OLP). This plane segment intersects some full-dimensional weight set components. For each component $\mathcal{W}(y)$, there are four possibilities: $\mathcal{W}(\text{PTLP}(\lambda))$ intersects $\mathcal{W}(y)$ either in a single vertex, along an edge, along a facet, or it passes through the interior of $\mathcal{W}(y)$. The entire segment $\mathcal{W}(\text{PTLP}(\lambda))$ can then be decomposed into such intersections of facets, edges, vertices (cf. [PGE10]). A solution set for PTLP(λ) can be obtained by using all solutions from S(4-OLP) where the corresponding weight set component is intersected. Therefore, S(4-OLP) is also a solution set for PTLP.

At the same time, Theorem 3.28 implies that, for every full dimensional weight set component $\mathcal{W}(y)$ of an extreme nondominated image $y \in Y_{\text{EN}}$(4-OLP), there is at least one value $\lambda \geq 0$ such that the plane $\mathcal{W}(\text{PTLP}(\lambda))$ intersects the interior of $\mathcal{W}(y)$. Then, a solution set for PTLP(λ) must contain at least one solution x such that $(c_1x, c_2x, c_3x, d_1x)^\top = y$ [PGE10]. Therefore, every solution set for PTLP also contains a solution set for 4-OLP.

Since every solution set for 4-OLP contains a solution set for PTLP, we can make the following statement regarding minimal solution sets:

3.36 Corollary. A set $S(\text{4-OLP}) \subseteq X$ is a minimal solution set for 4-OLP if and only if S(4-OLP) is a minimal solution set for PTLP. ◁

An illustration of the weight sets of the parametric triobjective linear program PTLP and the weight set of its corresponding 4-OLP is shown below.

3.37 Example. Consider an instance of 4-OLP:

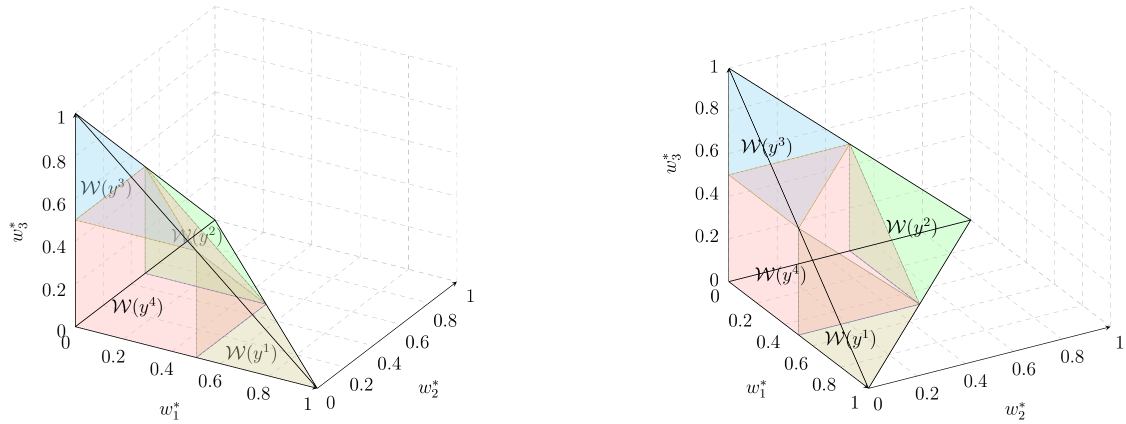
$$\begin{aligned} \min \quad & (x_1, x_2, x_3, x_4)^\top \\ \text{s. t.} \quad & x \in X. \end{aligned}$$

and the corresponding PTLP with a non-negative parameter $\lambda \geq 0$:

$$\begin{aligned} \min \quad & \begin{pmatrix} x_1 + \lambda x_4 \\ x_2 \\ x_3 \end{pmatrix} \\ \text{s. t.} \quad & x \in X, \end{aligned}$$

and the set $X := \operatorname{conv}\{x^1, x^2, x^3, x^4\} + \mathbb{R}^n_{\geqq}$ is the feasible set in $\mathbb{R}^n_{\geqq}$. For this instance, consider that we have a set of nondominated extreme points $Y_{\text{EN}}(\text{4-OLP}) := \{y^1, y^2, y^3, y^4\}$.

Then the weight set decomposition of the 4-OLP is composed of four weight set components, each corresponding to the extreme nondominated images y^1, y^2, y^3, and y^4 as shown in Figure 3.15. ◁



(a) Weight set of a 4-OLP instance

(b) $\mathcal{W}$(4-OLP) from a different angle

Figure 3.15: Weight set decomposition of the 4-OLP instance composed of four weight set components corresponding to $Y_{\text{EN}} := \{y^1, y^2, y^3, y^4\}$

Since solving a parametric triobjective linear program is understood as finding a minimal solution set for all values of λ, it is sufficient to focus on the parameter values where a minimal solution set of the TOLP PTLP(λ) changes as λ varies, known as breakpoints. For a given PTLP, some parameter value $\lambda \geq 0$ is called a breakpoint if, for a small $\varepsilon > 0$,

$$S(\text{PTLP}(\lambda + \varepsilon)) \neq S(\text{PTLP}(\lambda - \varepsilon)).$$

In PTLP, breakpoints have similar characteristics to breakpoints in parametric biobjective linear programs such as certain breakpoints that are associated with a unique minimal solution set differ from the sets adjacent to it and also, *tie* breakpoints. This happens when the solutions leaving and entering the minimal solution set correspond to the extreme nondominated images that are still nondominated but not extreme points for PTLP(λ). In case of a tie, we include the breakpoint in the preceding interval and exclude it from the following one. As a result, the parameter set is divided into a set of parameter intervals and/or unique parameter values, each corresponding to a minimal solution set, respectively.

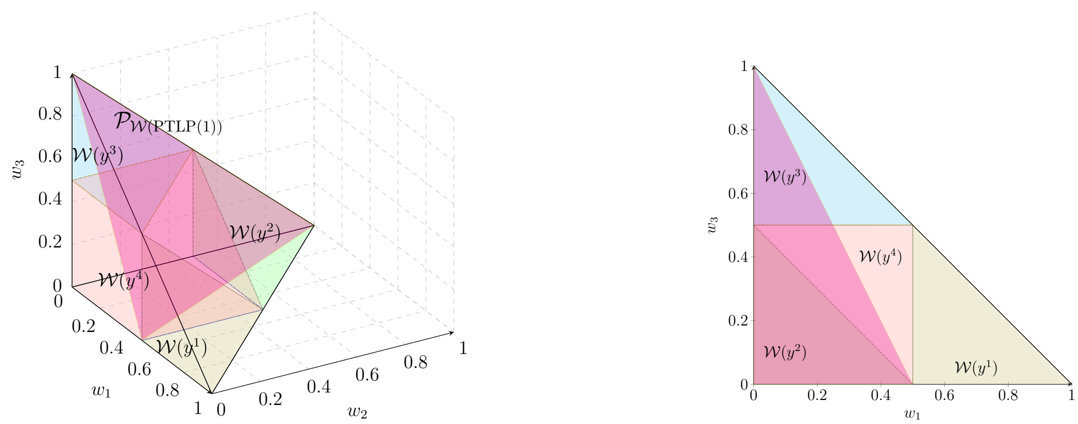
3.38 Proposition. If λ_i is a breakpoint of a PTLP with a corresponding weight set $\mathcal{W}(\text{PTLP}(\lambda_i))$, there exists at least one extreme weight in $\mathcal{W}$(4-OLP) on the intersection of $\mathcal{P}_{\mathcal{W}(\text{PTLP}(\lambda_i))}$ and $\mathcal{W}$(4-OLP). ◁

Proof. The proof is analogous to Case 1 of Proposition 3.22 but instead of $\nu(\lambda)$ we use $\tilde{\nu}$ which is defined as follows. Let $\lambda \geq 0$ be the parameter. Let Y_{EN}(4-OLP) denote the set of extreme nondominated images of 4-OLP and $\mathcal{W}(\text{PTLP}(\lambda))$ be the weight set of PTLP. We define a set of weight set components intersected by a plane segment $\mathcal{P}_{\mathcal{W}(\mathrm{PTLP}(\lambda))}$ by passing through their relative interior as follows

$$\tilde{\nu}(\lambda) := \left\{\mathcal{W}(y) : y \in Y_{\mathrm{EN}}(\text{4-OLP}), \mathrm{int}(\mathcal{W}(y)) \cap \mathcal{P}_{\mathcal{W}(\mathrm{PTLP}(\lambda))} \neq \varnothing\right\}. \qquad \square$$

3.39 Corollary. The number of breakpoints is finite. ◁

Proof. The proof is analogous to Corollary 3.24. □



(a) The plane segment $\mathcal{P}_{\mathcal{W}(\mathrm{PTLP}(1))}$ intersects weight set components $\mathcal{W}(y^2)$, $\mathcal{W}(y^3)$ and $\mathcal{W}(y^4)$.

(b) A different perspective of the plane segment $\mathcal{P}_{\mathcal{W}(\mathrm{PTLP}(1))}$ (view along w_2^* axis).

Figure 3.16: An illustration of the weight set $\mathcal{W}(\mathrm{PTLP}(\lambda))$ for $\lambda = 1$ intersecting some of the weight set components of $\mathcal{W}(\text{4-OLP})$ associated to the instance in Example 3.37. We observe that there is only one breakpoint i. e. $\lambda = 1$ in this instance. A solution set at the breakpoint, $S(\mathrm{PTLP}^1(1)) = \{x^1, x^4, x^2\}$ is different from $S\big(\mathrm{PTLP}^1(1-\varepsilon)\big) = \{x^1, x^3, x^4\}$ and $S\big(\mathrm{PTLP}^1(1+\varepsilon)\big) = \{x^2, x^3, x^4\}$. Note that the feasible solutions x_1, x_2, x_3, and x_4 map to the extreme nondominated images y_1, y_2, y_3, and y_4, respectively.

As we are interested in finding breakpoints that mark a change in a minimal solution set of PTLP(λ), we can use parameter intervals of each solution to determine such breakpoints. The parameter interval of a solution $x \in S(\text{4-OLP})$ can be determined using the weights in a weight set component of the corresponding $y \in Y_{\mathrm{EN}}(\text{4-OLP})$.

3.40 Proposition. Let $x \in S(\text{4-OLP})$ be a solution and $y = (c_1x, c_2x, c_3x, d_2x)^\top \in Y_{\mathrm{EN}}$ be its corresponding extreme nondominated image in 4-OLP. The *parameter interval* of x for the parametric problem PTLP is given by

$$\mathcal{I}^1(x) := \left\{\lambda : \lambda = \frac{1 - w_1^* - w_2^* - w_3^*}{w_1^*}, w^* \in \mathcal{W}(y)\right\}. \qquad ◁$$

Proof. The proof is analogous to the proof of Proposition 3.25. Here we use Theorem 3.28 and the construction of the parameter value, $\lambda := \frac{w_4^*}{w_1^*}$ from Equation 3.9 where $w^* \in \mathcal{W}(y)$. □

In order to find the breakpoints and the parameter intervals for every solution in S(4-OLP), we extend the two algorithms proposed for PBLP^j to parametric triobjective linear program in Section 6.1.2 of Chapter 6.

3.4 Parametric multi-objective linear program

This section is a direct generalization of the case PBLP^1 to multi-objective linear programs. We use the same approach of weighted sum scalarization of these problems.

3.41 Definition (Parametric Multi-objective Linear Program). Let $\lambda \geq 0$ be a non-negative parameter. A *parametric multi-objective linear program* is defined as

$$\begin{aligned} \min \quad & Cx + \lambda Dx \\ \text{s. t.} \quad & Ax \geqq b, \\ & \;x \geqq 0, \end{aligned} \tag{PkLP}$$

where $C, D \in \mathbb{Q}^{k\times n}$ consist of rows $c_i, d_i = 1, \ldots, k$ such that $d_i = 0$, for all $i = 2, \ldots, k$, $A \in \mathbb{Q}^{m\times n}$, $m \in \mathbb{N} \setminus \{0\}$, $b \in \mathbb{Q}^m$ and the set $X := \{x \in \mathbb{R}^n : Ax \geqq b, x \geqq 0\}$ is again the feasible set. ◁

We relate PkLP to a $(k + 1)$-objective linear program using the weighted sum scalarization. To this end, we consider the $(k + 1)$-objective linear program with the same objective functions $c_1x, c_2x, \ldots, c_kx,\ d_1x$, i. e.

$$\begin{aligned} \min \quad & (c_1x, c_2x, \ldots, c_kx, d_1x,)^\top \\ \text{s. t.} \quad & x \in X \end{aligned} \tag{(k+1)-OLP}$$

and denote the set of extreme nondominated images of $(k+1)$-OLP by $Y_{\text{EN}}((k+1)\text{-OLP})$. We denote a minimal solution set that corresponds to the set $Y_{\text{EN}}((k+1)\text{-OLP})$ by S_{k+1}. We define the weight set $\mathcal{W}((k + 1)\text{-OLP})$ of $(k + 1)$-OLP as

$$\mathcal{W}((k + 1)\text{-OLP}) := \left\{ w^* \in \mathbb{R}^{k+1}_{\geqq} : \sum_{i=1}^{k+1} w_i^* = 1 \right\}.$$

The weighted sum scalarization of $(k + 1)$-OLP with a normalized weight $w^* \in \mathcal{W}((k + 1)\text{-OLP})$ is

$$\min_{\text{x}\in\text{X}} \sum_{1}^{k} w_i^* c_i x + w_{k+1}^* d_1 x. \tag{WS((k+1)-OLP, w*)}$$

We apply the weighted sum scalarization to the parametric k-objective linear program PkLP and formally characterize its relationship to the corresponding $(k+1)$-OLP-objective linear program in our results. For a fixed value of λ, the weighted sum scalarization of PkLP is

$$\min_{x \in X} w_1(c_1x + \lambda d_1x) + w_2c_2x + \cdots + w_kc_kx \qquad (\mathrm{WS}(\mathrm{PkLP}(\lambda), w))$$

where $w \in \mathbb{R}^k_{\geq}$ and $\sum_{i=1}^k w_i = 1$.
We reformulate this problem using $w_k = 1 - \sum_{i=1}^{k-1} w_i$ and obtain

$$\min_{x \in X} w_1c_1x + w_2c_2x + \cdots + \left(1 - \sum_{i=1}^{k-1} w_i\right) c_kx + w_1\lambda d_1x.$$

The problem can be interpreted as a weighted sum scalarization of the $(k+1)$-OLP with weight vector $\left(w_1, w_2, \ldots, 1 - \sum_{i=1}^{k-1}, w_1\lambda\right)$. It holds that

$$w_1 + w_1\lambda + w_2 + \cdots + 1 - \sum_{i=1}^{k-1} = 1 + w_1\lambda \geq 1.$$

We normalize the weights and get

$$\min_{x \in X} \frac{w_1}{1+\lambda w_1}(c_1x) + \frac{w_2}{1+\lambda w_1}(c_2x) + \cdots + \frac{w_k}{1+\lambda w_1}(c_kx) + \frac{w_1\lambda}{1+\lambda w_1}(d_1x).$$

This can be considered as a special case of the weighted sum scalarization of $(k+1)$-OLP with the corresponding weight vector

$$w^* = \left(\frac{w_1}{1+\lambda w_1}, \ldots, \frac{w_k}{1+\lambda w_1}, \frac{w_1\lambda}{1+\lambda w_1}\right)^\top \in \mathcal{W}((k+1)\text{-OLP}). \qquad (3.12)$$

We now state our main result relating the efficient solutions of parametric k-objective linear program to the efficient solutions of the corresponding $(k+1)$-objective linear program.

3.42 Theorem. A feasible solution x^* is optimal for $\mathrm{WS}((k+1)\text{-OLP}, w^*)$ with non-negative weights $w_1^*, w_2^*, \ldots, w_{k+1}^*$ where $w_1^* > 0$ if and only if there exists a parameter $\lambda \geq 0$ and non-negative weights $w_1, w_2, \ldots, w_k$ with $w_1 > 0$ such that x^* is optimal for $\mathrm{WS}(\mathrm{PkLP}(\lambda), w)$. ◁

Proof. This proof is analogous to the proofs of Theorem 3.3. Note that $w_1^* > 0$ and here, we define

$$\lambda := \frac{w_{k+1}^*}{w_1^*},$$

$$\begin{aligned} w_1 &:= \frac{w_1^*}{\sum_1^k w_i^*}, \\ &\vdots \\ w_k &:= \frac{w_k^*}{\sum_1^k w_i^*} \end{aligned} \tag{3.13}$$

in the first part of the proof and we define $w_1^*, w_2^*, \ldots, w_{k+1}^*$ using Equation (3.12),

$$\begin{aligned} w_1^* &:= w_1 \\ &\vdots \\ w_k^* &:= w_k \\ w_{k+1}^* &:= w_1\lambda \end{aligned} \tag{3.14}$$

in the converse case. We do not normalize for simplicity. □

Here again, observe that we assume $w_1^* > 0$ in Theorem 3.42. However, this does not affect the results for the same reason stated in Theorems 3.3 and 3.28.

We can also establish the result that relates the weight sets of PkLP(λ) to the weight set of $(k+1)$-OLP. Since for a fixed value of λ, PkLP(λ) is a k-objective linear program, its projected weight set is a $k-1$ dimensional polytope. However, if we consider the parameter λ then we can extend this $k-1$ dimensional weight set to a higher dimension by the mapping $\mathcal{W}(\text{PkLP}(\lambda))$, which is defined below.

3.43 Definition. For a given λ, we define a *mapping* $\mathcal{W}(\text{PkLP}(\lambda))$ as

$$\mathcal{W}(\text{PkLP}(\lambda)) := \left\{ \frac{1}{1+w_1\lambda}(w_1, \ldots w_k, w_1\lambda) : (w_1, \ldots, w_k) \in \mathbb{R}^k_{\geqq}, \sum_{i=1}^k w_i = 1 \right\}. \quad \lhd$$

We can now characterize the weight set of $\mathcal{W}(\text{PkLP}(\lambda))$ with respect to the weight set of $\mathcal{W}((k+1)\text{-OLP})$.

3.44 Proposition. Let $\mathcal{W}(\text{PkLP}(\lambda))$ and $\mathcal{W}((k+1)\text{-OLP})$ be weight sets of PkLP for a fixed value λ and of $(k+1)$-OLP, respectively. Then, it holds

$$\mathcal{W}(\text{PkLP}(\lambda)) \subsetneq \mathcal{W}((k+1)\text{-OLP}). \quad \lhd$$

Proof. The proof is analogous to the proof of Proposition 3.31.
To show that $\mathcal{W}(\text{PkLP}(\lambda))$ is a proper subset of $\mathcal{W}((k+1)\text{-OLP})$, we consider the particular weight $w' := (0, \ldots, 0, 1) \in \mathcal{W}((k+1)\text{-OLP})$ and show that the weight $w' \notin \mathcal{W}((k+1)\text{-OLP})$. □

3.45 Proposition. For all the weight sets of PkLP(λ) for $\lambda \geq 0$, it holds that

$$\bigcap_{\lambda \geq 0} \mathcal{W}(\text{PkLP}(\lambda)) = \left\{ w^* \in \mathcal{W}((k+1)\text{-OLP}) : \sum_{i=2}^{k} w_i^* = 1 \right\}. \quad \triangleleft$$

Proof. For any arbitrary $\lambda_i \geq 0$, by the definition of $\mathcal{W}(\text{PkLP}(\lambda_i))$, we have

$$\mathcal{W}(\text{PkLP}(\lambda_i)) := \left\{ \frac{1}{1+w_1\lambda_i}(w_1, \ldots, w_k, w_1\lambda_i) : (w_1, \ldots, w_k) \in \mathbb{R}^k_{\geq}, \sum_{i=2}^{k} w_i = 1 \right\}.$$

We first show that a weight $w^* \in \mathcal{W}((k+1)\text{-OLP})$ such that $\sum_{i=2}^{k} w_i^* = 1$ is an element of $\mathcal{W}(\text{PkLP}(\lambda_i))$.
Since $\sum_{i=2}^{k} w_i^* = 1$ we have $w_1^* = w_{k+1}^* = 0$. This implies,

$$\begin{aligned} w^* &= (0, w_2^*, \ldots, w_k^*, 0) \\ &= \left(\frac{0}{1+0\lambda_i}, \frac{w_2^*}{1+0\lambda_i}, \ldots, \frac{w_k^*}{1+0\lambda_i}, \frac{0}{1+0\lambda_i} \right). \end{aligned}$$

Therefore, $w^* \in \mathcal{W}(\text{PkLP}(\lambda_i))$.
It is left to show that there is no other weight shared by any two weight sets of $\mathcal{W}(\text{PkLP}(\lambda))$. This can be done analogously to the proof of Proposition 3.32. □

Based on the definition of the weight sets of PkLP(λ), for any $w^* \in \mathcal{W}(\text{PkLP}(\lambda))$, we observe that it holds, $w_{k+1}^* = w_1^*\lambda$. Since the $(k+1)^{\text{th}}$ component of a weight in $\mathcal{W}(\text{PkLP}(\lambda))$ can be determined by the first k components, the projection reduces the entire weight set to a k-dimensional interpretation. Since $w_{k+1}^* = 1 - \sum_{i=1}^{k} w_i^*$ we obtain the following remark.

3.46 Remark. Let $\mathcal{W}(\text{PkLP}(\lambda))$ be the weight set of PkLP(λ) and $\mathcal{W}((k+1)\text{-OLP})$ be the weight set of the corresponding $(k+1)$-OLP. Then, the *projection of the weight set* $\mathcal{W}(\text{PkLP}(\lambda))$ in $\mathbb{R}^k$ can be defined as a k-dimensional polytope,

$$\mathcal{Q}_{\mathcal{W}(\text{PkLP})(\lambda)} = \left\{ (w_1^*, \ldots, w_k^*) \mid w^* \in \mathcal{W}(\text{PkLP}(\lambda)),\ w_1^*(1+\lambda) + \sum_{i=2}^{k} w_i^* = 1 \right\}. \quad \triangleleft$$

Finally, we relate a solution set of PkLP to a solution set of $(k+1)$-OLP. Every solution set S_{k+1} of $(k+1)$-OLP consists of a subset of solutions such that there is a one-to-one correspondence between S_{k+1} and the extreme nondominated images of $(k+1)$-OLP. Meanwhile, the weight set $\mathcal{W}((k+1)\text{-OLP})$ can be decomposed into full dimensional weight set components such that there is a bijection between these weight set components and $Y_{\text{EN}}((k+1)\text{-OLP})$ [PGE10]. By Remark 3.46, for a fixed value of $\lambda \geq 0$, the weight set $\mathcal{W}(\text{PkLP}(\lambda))$ is a k-dimensional polytope, $\mathcal{Q}_{\mathcal{W}(\text{PkLP})(\lambda)}$ that lies in $\mathcal{W}((k+1)\text{-OLP})$. This polytope intersects some of weight set components. For each component $\mathcal{W}(y)$, there are different ways $\mathcal{W}(\text{PkLP}(\lambda))$ intersects $\mathcal{W}(y)$. Then the entire polytope $\mathcal{W}(\text{PkLP}(\lambda))$ can be decomposed into such intersections of vertices, edges, and facets (cf. [PGE10]). A solution set for PkLP(λ) can be obtained by using all solutions from S_{k+1} where the corresponding weight set component is intersected. Therefore, S_{k+1} is also a solution set for PkLP.

At the same time, Theorem 3.42 implies that, for every full dimensional weight set component $\mathcal{W}(y)$ of an extreme nondominated image $y \in Y_{\text{EN}}((k+1)\text{-OLP})$, there is at least one value $\lambda \geq 0$ such that the plane $\mathcal{W}(\text{PkLP}(\lambda))$ intersects the interior of $\mathcal{W}(y)$. Then, a solution set for PkLP(λ) must contain at least one solution x such that $(c_1 x, \ldots, c_k x, d_1 x)^\top = y$ [PGE10]. Therefore, every solution set for PkLP also contains a solution set for $(k+1)$-OLP.

Since every solution set for $(k+1)$-OLP contains a solution set for PkLP, we can make the following statement regarding minimal solution sets:

3.47 Corollary. A set $S_{k+1} \subseteq X$ is a minimal solution set for $(k+1)$-OLP if and only if S_{k+1} is a minimal solution set for PkLP. ◁

Thus, we have shown that a minimal solution set of parametric k-objective linear program is also a minimal solution set for the related multi-objective linear program.

CHAPTER 4

Special parametric biobjective linear program

In this chapter, we explore a special parametric biobjective linear program that is closely related to the problem PBLP^2 in Chapter 3 as it has same parametric dependency in both objectives. However the parametric objectives are different. It is a distinct problem because it is only relatable to a four-objective linear program, even though it involves only one parameter.

4.1 Same parameter in different parametric objectives

More precisely, we consider the parametric problem

$$\min_{x \in X} \begin{pmatrix} c_1 x + \lambda d_1 x \\ c_2 x + \lambda d_2 x \end{pmatrix} \tag{PBLP^3}$$

with a non-negative parameter $\lambda \geq 0$. We denote this problem by PBLP^3 in order to maintain notational consistency with the single parametric cases in Chapter 3.

The problem PBLP^3, for some fixed λ, is a non-parametric biobjective linear program and we denote it by PBLP$^3(\lambda)$. Furthermore, we denote its set of extreme nondominated images by $Y_{\text{EN}}\text{PBLP}^3(\lambda)$ and a corresponding minimal solution set by $S(\text{PBLP}^3(\lambda))$. As in the previous chapter, we analyse a minimal solution set that corresponds to extreme nondominated images of PBLP$^3(\lambda)$. Therefore, a minimal solution set of PBLP^3 equates to a set that contains efficient solutions of PBLP$^3(\lambda)$ for each fixed value of $\lambda \geq 0$. We denote this set by $S(\text{PBLP}^3)$.

4.1 Definition. A *solution set* $S(\text{PBLP}^3) \subseteq X$ of PBLP^3 is a set such that for every $\lambda \geq 0$, $S(\text{PBLP}^3)$ contains, as a subset, a solution set for the biobjective linear program PBLP$^3(\lambda)$. It is called *minimal* if, additionally, there is no other solution set $S' \subseteq X$ for PBLP^3 with $|S'| < \left|S(\text{PBLP}^3)\right|$. ◁

We relate PBLP^3 to the corresponding four-objective linear program with the same objective functions c_1x, c_2x, d_1x, d_2x, i. e.

$$\begin{aligned} \min \quad & (c_1x, c_2x, d_1x, d_2x)^\top \\ \text{s. t.} \quad & x \in X \end{aligned} \tag{4-OLP}$$

and denote the set of extreme nondominated images of 4-OLP by Y_{EN}(4-OLP). We denote a minimal solution set that corresponds to the set Y_{EN}(4-OLP) by S_4. We define the weight set $\mathcal{W}$(4-OLP) of 4-OLP as

$$\mathcal{W}(\text{4-OLP}) := \left\{ w^* \in \mathbb{R}^4_{\geqq} : \sum_{i=1}^{4} w_i^* = 1 \right\}.$$

The weighted sum scalarization of 4-OLP with a normalized weight $w^* \in \mathcal{W}$(4-OLP) is

$$\min_{\text{x}\in\text{X}} \; w_1^* c_1 x + w_2^* c_2 x + w_3^* d_1 x + w_4^* d_2 x. \tag{WS(4-OLP, w^*)}$$

We approach the problem by applying the weighted sum scalarization to PBLP$^3(\lambda)$ and formally characterize its relation to the corresponding 4-OLP.

For a fixed value of λ, the weighted sum scalarization of PBLP$^3(\lambda)$ is

$$\min_{\text{x}\in\text{X}} \; w_1(c_1x + \lambda d_1 x) + w_2(c_2 x + \lambda d_2 x) \tag{WS(PBLP$^3(\lambda), w$)}$$

where $w \in \mathbb{R}^2_{\geqq}$ and $w_1 + w_2 = 1$.
We reformulate this problem using $w_2 = 1 - w_1$ and obtain

$$\min_{\text{x}\in\text{X}} \; w_1 c_1 x + (1 - w_1) c_2 x + w_1 \lambda d_1 x + (1 - w_1)\lambda d_2 x.$$

The problem can be interpreted as a weighted sum scalarization of the 4-OLP with the weight vector $(w_1, 1 - w_1, w_1\lambda, (1 - w_1)\lambda)$. It holds that

$$w_1 + (1 - w_1) + w_1\lambda + (1 - w_1)\lambda = 1 + \lambda \geq 1.$$

We normalize the weights and get

$$\min_{\text{x}\in\text{X}} \; \frac{w_1}{1+\lambda}(c_1x) + \frac{w_2}{1+\lambda}(c_2x) + \frac{w_1\lambda}{1+\lambda}(d_1x) + \frac{w_2\lambda}{1+\lambda}(d_2x).$$

which is a particular case of WS(4-OLP, w^*) with the corresponding weight vector

$$w^* = \left(\frac{w_1}{1+\lambda}, \frac{w_2}{1+\lambda}, \frac{w_1\lambda}{1+\lambda}, \frac{w_2\lambda}{1+\lambda} \right)^\top \in \mathcal{W}(\text{4-OLP}). \tag{4.1}$$

We now establish the relation between the efficient solutions of the problem PBLP^3 and the efficient solutions of the corresponding 4-OLP.

4.2 Theorem. If there exists a parameter $\lambda \geq 0$ and non-negative weights w_1, w_2, such that x^* is optimal for $\mathrm{WS}(\mathrm{PBLP}^3(\lambda), w)$ then there exists non-negative weight w^* such that x^* is optimal for $\mathrm{WS}(\text{4-OLP}, w^*)$. ◁

Proof. We show that x^* is optimal for $\mathrm{WS}\left(\mathrm{PBLP}^3(\lambda), w\right)$ implying that there exists non-negative weight w^* such that x^* is optimal for $\mathrm{WS}\left(\text{4-OLP}, w^*\right)$.
Let x^* is optimal for $\mathrm{WS}\left(\mathrm{PBLP}^3(\lambda), w\right)$. We define $w_1^*, \ldots, w_4^*$ using Equation (4.1),

$$\begin{aligned} w_1^* &:= w_1, \\ w_2^* &:= w_2, \\ w_3^* &:= w_1\lambda \\ w_4^* &:= w_2\lambda. \end{aligned} \tag{4.2}$$

We do not normalize for simplicity. Suppose x^* is not optimal for $\mathrm{WS}\left(\text{4-OLP}, w^*\right)$. Then there exists some x' that is feasible for $\mathrm{WS}\left(\text{4-OLP}, w^*\right)$ where $x' \neq x^*$ such that

$$w_1^* c_1 x' + w_2^* c_2 x' + w_3^* d_1 x' + w_4^* d_2 x' < w_1^* c_1 x^* + w_2^* c_2 x^* + w_3^* d_1 x^* + w_4^* d_2 x^*.$$

We plug in Equation (4.2) to get

$$w_1 c_1 x' + w_2 c_2 x' + w_1 \lambda d_1 x' + w_2 \lambda d_2 x' < w_1 c_1 x^* + w_2 c_2 x^* + w_1 \lambda d_1 x^* + w_2 \lambda d_2 x^*.$$

This is equivalent to

$$w_1(c_1 x' + \lambda d_1 x') + w_2(c_2 x' + \lambda d_2 x') < w_1(c_1 x^* + \lambda d_1 x^*) + w_2(c_2 x^* + \lambda d_2 x^*).$$

This leads to a contradiction that x^* is optimal for $\mathrm{WS}\left(\mathrm{PBLP}^3(\lambda), w\right)$. □

Conversely, given a weight $w^* \in \mathcal{W}(\text{4-OLP})$ for $\mathrm{WS}(\text{4-OLP}, w^*)$, we can express a weight $w \in \mathbb{R}^2_{\geqq}$ and a parameter λ for $\mathrm{WS}\left(\mathrm{PBLP}^3(\lambda), w\right)$ in terms of the weight w^*. We, therefore, use Equation (4.2) to define

$$\begin{aligned} w_1 &= w_1^*, \\ w_2 &= w_2^*, \\ \lambda &= \frac{w_3^*}{w_1^*} = \frac{w_4^*}{w_2^*} \end{aligned} \tag{4.3}$$

where $w_1^* > 0$ and $w_2^* > 0$. We can exclude the weights $w_1^* = 0$ and $w_2^* = 0$ due to the same reason as in Theorem 3.3 and thus it does not affect the results.

However, the converse of Theorem 4.2 does not hold because there exists some weight $w^* \in \mathcal{W}(\text{4-OLP})$ and an optimal solution x^* for $\text{WS}(\text{4-OLP}, w^*)$ such that there is no $\lambda \geq 0$ and weight $w \in \mathbb{R}^2_{\geqq}$ such that x^* is optimal for $\text{WS}(\text{PBLP}^3(\lambda), w)$.

We use an example to show this.

4.3 Example. Let x^* be optimal for $\text{WS}(\text{4-OLP}, w^*)$ with $w^* \in \mathcal{W}(\text{4-OLP})$, such that $w_1^* = \frac{1}{4}; w_2^* = \frac{1}{2}; w_3^* = \frac{1}{4}; w_4^* = 0$.

Given a weight $w^* \in \mathcal{W}(\text{4-OLP})$, we construct λ, w_1 and w_2 for $\text{WS}(\text{PBLP}^3(\lambda), w)$ using Equation (4.3). We get

$$\begin{aligned} w_1 &= \frac{1}{4}, \\ w_2 &= \frac{1}{2}, \\ \lambda &= \frac{w_3^*}{w_1^*} = \frac{\frac{1}{4}}{\frac{1}{4}} = 1 \\ \neq \lambda &= \frac{w_4^*}{w_2^*} = \frac{0}{\frac{1}{2}} = 0. \end{aligned}$$

Therefore, x^* is not optimal for $\text{WS}(\text{PBLP}^3(\lambda), w)$ since there exists no λ that satisfy $\frac{w_3^*}{w_1^*} = \frac{w_4^*}{w_2^*}$. ◁

For every fixed value of λ, $\text{PBLP}^3(\lambda)$ is a biobjective linear program and its weight set is a one-dimensional polytope. We embed the weight set of $\text{PBLP}^3(\lambda)$ from one dimension to three dimension using the mapping $\mathcal{W}\left(\text{PBLP}^3(\lambda)\right)$ which is defined below.

4.4 Definition. For a fixed λ, we define a *mapping* $\mathcal{W}\left(\text{PBLP}^3(\lambda)\right)$,

$$\mathcal{W}\left(\text{PBLP}^3(\lambda)\right) := \left\{\left(\frac{w_1}{1+\lambda}, \frac{w_2}{1+\lambda}, \frac{w_1\lambda}{1+\lambda}, \frac{w_2\lambda}{1+\lambda}\right) : (w_1, w_2) \in \mathbb{R}^2_{\geq}, w_1 + w_2 = 1\right\}. \quad ◁$$

Next, we characterize the extended structure of the weight sets of $\text{WS}(\text{PBLP}^3(\lambda), w)$ in terms of the weight set of 4-OLP.

4.5 Proposition. Let $\mathcal{W}\left(\text{PBLP}^3(\lambda)\right)$ and $\mathcal{W}(\text{4-OLP})$ be weight sets of PBLP^3 for a fixed λ and the corresponding $\mathcal{W}(\text{4-OLP})$, respectively, then

$$\mathcal{W}(\text{PBLP}^3(\lambda)) \subsetneq \mathcal{W}(\text{4-OLP}). \quad ◁$$

Proof. Let $w^* \in \mathcal{W}\left(\mathrm{PBLP}^3(\lambda)\right)$. Then by the definition of $\mathcal{W}\left(\mathrm{PBLP}^3(\lambda)\right)$, it is

$$w^* = \left(\frac{w_1}{1+\lambda}, \frac{w_2}{1+\lambda}, \frac{w_1\lambda}{1+\lambda}, \frac{w_2\lambda}{1+\lambda}\right).$$

The sum of the components satisfies

$$\frac{w_1}{1+\lambda} + \frac{w_2}{1+\lambda} + \frac{w_1\lambda}{1+\lambda} + \frac{w_2\lambda}{1+\lambda} = 1,$$

and thus $w^* \in \mathcal{W}(\text{4-OLP})$.
To show that $\mathcal{W}(\mathrm{PBLP}^3(\lambda))$ is a proper subset of $\mathcal{W}(\text{4-OLP})$, we consider the particular weight $w' := (0,0,0,1) \in \mathcal{W}(\text{4-OLP})$. Since for any given $\lambda > 0$, the fourth component of any $w^* \in \mathcal{W}(\mathrm{PBLP}^3(\lambda))$ satisfies

$$\frac{w_2\lambda}{1+\lambda} \neq 1.$$

Hence, its holds that $w' \notin \mathcal{W}(\mathrm{PBLP}^3(\lambda))$. □

4.6 Proposition. For all the weight sets of $\mathrm{PBLP}^3(\lambda)$ for $\lambda \geq 0$, it holds that

$$\bigcap_{\lambda \geq 0} \mathcal{W}\Big(\mathrm{PBLP}^3(\lambda)\Big) = \varnothing. \quad \triangleleft$$

Proof. Let $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ be two arbitrary parameter values, such that $\lambda_1 \neq \lambda_2$. Let $\mathcal{W}\Big(\mathrm{PBLP}^3(\lambda_1)\Big)$ and $\mathcal{W}\Big(\mathrm{PBLP}^3(\lambda_2)\Big)$ be weight sets of $\mathrm{PBLP}^3(\lambda_1)$ and $\mathrm{PBLP}^3(\lambda_2)$, respectively. Consider a weight $w^* \in \mathcal{W}\Big(\mathrm{PBLP}^3(\lambda_1)\Big)$ and a weight $\tilde{w} \in \mathcal{W}\Big(\mathrm{PBLP}^3(\lambda_2)\Big)$. By the definition of $\mathcal{W}\Big(\mathrm{PBLP}^3(\lambda_1)\Big)$ and $\mathcal{W}\Big(\mathrm{PBLP}^3(\lambda_2)\Big)$, we have

$$w^* = \left(\frac{w_1'}{1+\lambda_1}, \frac{w_2'}{1+\lambda_1}, \frac{w_1'\lambda_1}{1+\lambda_1}, \frac{w_2'\lambda_1}{1+\lambda_1}\right) \text{ where } w_1' + w_2' = 1$$

and

$$\tilde{w} = \left(\frac{w_1''}{1+\lambda_2}, \frac{w_2''}{1+\lambda_2}, \frac{w_2''\lambda_2}{1+\lambda_2}, \frac{w_2''\lambda_2}{1+\lambda_2}\right) \text{ where } w_1'' + w_2'' = 1.$$

If we compare the first components of w^* and $\tilde{w}$, i. e. $\frac{w_1'}{1+\lambda_1}$ and $\frac{w_1''}{1+\lambda_2}$, then the components are equal if

$$w_1' = \frac{w_1''(1+\lambda_1)}{1+\lambda_2}. \tag{4.4}$$

Simultaneously, the third component of w^* is

$$w_3^* = \frac{w_1'\lambda_1}{1+\lambda_1}.$$

We plug in w_1' in Equation (4.4) to get

$$\begin{aligned} w_3^* &= \frac{\left(\frac{w_1''(1+\lambda_1)}{1+\lambda_2}\right)\lambda_1}{1+\lambda_1} \\ &= \left(\frac{w_1''}{1+\lambda_2}\right)\lambda_1 \neq \left(\frac{w_1''}{1+\lambda_2}\right)\lambda_2 = \tilde{w}_3 \end{aligned}$$

because $\lambda_1 \neq \lambda_2$. □

In other words, the weight sets of $\text{PBLP}^3(\lambda)$ for all $\lambda \geq 0$ do not share any weights in common.

4.7 Proposition. For the union of all the weight sets $\mathcal{W}(\text{PBLP}^3(\lambda))$ for all $\lambda \geq 0$ it holds that:

(i) $\{w^* \in \mathcal{W}(\text{4-OLP}) : w_1^* + w_3^* = 1 \text{ and } w_3^* \neq 1\} \subset \bigcup\limits_{\lambda \geq 0} \mathcal{W}\left(\text{PBLP}^3(\lambda)\right)$.

(ii) $\{w^* \in \mathcal{W}(\text{4-OLP}) : w_1^* = 0, w_3^* = 0 \text{ and } w_2^* \neq 0\} \subset \bigcup\limits_{\lambda \geq 0} \mathcal{W}\left(\text{PBLP}^3(\lambda)\right)$. ◁

Proof. For an arbitrary $\lambda \geq 0$, by the definition of $\mathcal{W}\left(\text{PBLP}^3(\lambda)\right)$, we have

$$\mathcal{W}\left(\text{PBLP}^3(\lambda)\right) := \left\{\left(\frac{w_1}{1+\lambda}, \frac{w_2}{1+\lambda}, \frac{w_1\lambda}{1+\lambda}, \frac{w_2\lambda}{1+\lambda}\right) : (w_1, w_2) \in \mathbb{R}^2_{\geq}, w_1 + w_2 = 1\right\}.$$

Let w^* and $\tilde{w}$ be weights in $\mathcal{W}(\text{4-OLP})$.

For Proposition 4.7.(i), we show that a weight $w^* \in \mathcal{W}(\text{4-OLP})$ that satisfy $w_1^* + w_3^* = 1$ is an element of $\mathcal{W}(\text{PBLP}^3(\lambda))$ for some λ. Let $\lambda \in \mathbb{R}_{\geq}$ be arbitrary and choose $w_2^* = 0$ and $w_4^* = 0$. Then, we get

$$w^* = \left(\frac{1}{1+\lambda}, \frac{0}{1+\lambda}, \frac{(1)\lambda}{1+\lambda}, \frac{(0)\lambda}{1+\lambda},\right)$$

where $w_1 + w_2 = 1 + 0 = 1$. Thus, it holds that $w^* \in \mathcal{W}(\text{PBLP}^3(\lambda))$.

Similarly, for Proposition 4.7.(ii), we show that $\tilde{w} \in \mathcal{W}(\text{4-OLP})$ that satisfy $\tilde{w}_1 = \tilde{w}_3 = 0$ is an element of $\mathcal{W}(\text{PBLP}^3(\lambda))$ for some λ. Let $\lambda \in \mathbb{R}_{\geq}$ be arbitrary and choose $\tilde{w}_1 = 0$ and $\tilde{w}_3 = 0$. Then we get

$$\tilde{w} = \left(\frac{0}{1+\lambda}, \frac{1}{1+\lambda}, \frac{(0)\lambda}{1+\lambda}, \frac{(1)\lambda}{1+\lambda}\right).$$

where $w_1 + w_2 = 0 + 1 = 1$. Thus, $\tilde{w} \in \mathcal{W}(\text{PBLP}^3(\lambda))$. □

All these weights in $\mathcal{W}$(4-OLP) that meets the requirement $w_1^* + w_3^* = 1$ and $\tilde{w}_1 = \tilde{w}_3 = 0$ are illustrated in Figure 4.1.

Next, we show that the union of weight sets of $\text{PBLP}^3(\lambda)$ is a strict subset of $\mathcal{W}$(4-OLP).

4.8 Corollary. Let $\mathcal{W}\left(\text{PBLP}^3(\lambda)\right)$ be the weight set of $\text{PBLP}^3(\lambda)$ and $\mathcal{W}$(4-OLP) be the weight set of the corresponding 4-OLP. Then

$$\bigcup_{\lambda \geq 0} \mathcal{W}\left(\text{PBLP}^3(\lambda)\right) = \left\{w^* \in \mathcal{W}(\text{4-OLP}) \mid \frac{w_3^*}{w_1^*} = \frac{w_4^*}{w_2^*} \text{ with } w_1^* \neq 0, w_2^* \neq 0\right\}$$
$$\cup \{w^* \in \mathcal{W}(\text{4-OLP}) : w_1^* = 0, w_3^* = 0\}$$
$$\cup \{w^* \in \mathcal{W}(\text{4-OLP}) : w_1^* + w_3^* = 1\} \qquad \triangleleft$$

Proof. From Proposition 4.5, the weight set of $\text{PBLP}^3(\lambda)$ for every $\lambda \geq 0$ is a subset of $\mathcal{W}$(4-OLP), so the union of all these weight sets will also be a subset of $\mathcal{W}$(4-OLP). Using Equation (4.3), there exists some λ and $w \in \mathbb{R}^2_{\geqq}$ for $\text{PBLP}^3(\lambda)$ only for those weights $w^* \in \mathcal{W}$(4-OLP) that satisfy $\frac{w_3^*}{w_1^*} = \frac{w_4^*}{w_2^*}$ and $w_1^* > 0$ and $w_2^* > 0$. However, by Proposition 4.7 there exists weights in the union of $\mathcal{W}(\text{PBLP}^3(\lambda))$ that satisfy $w_1^* + w_3^* = 1$ and $w_1^* = 0, w_3^* = 0$, respectively. Therefore, the union of weight sets of $\text{PBLP}^3(\lambda)$ is equivalent to a subset of $\mathcal{W}$(4-OLP) as described above. □

This implies that if we vary λ in $\text{PBLP}^3(\lambda)$ then the union of all the weight sets $\mathcal{W}\left(\text{PBLP}^3(\lambda)\right)$ in $\mathbb{R}^3$ is a strict subset of the associated $\mathcal{W}$(4-OLP) due to Proposition 4.8.

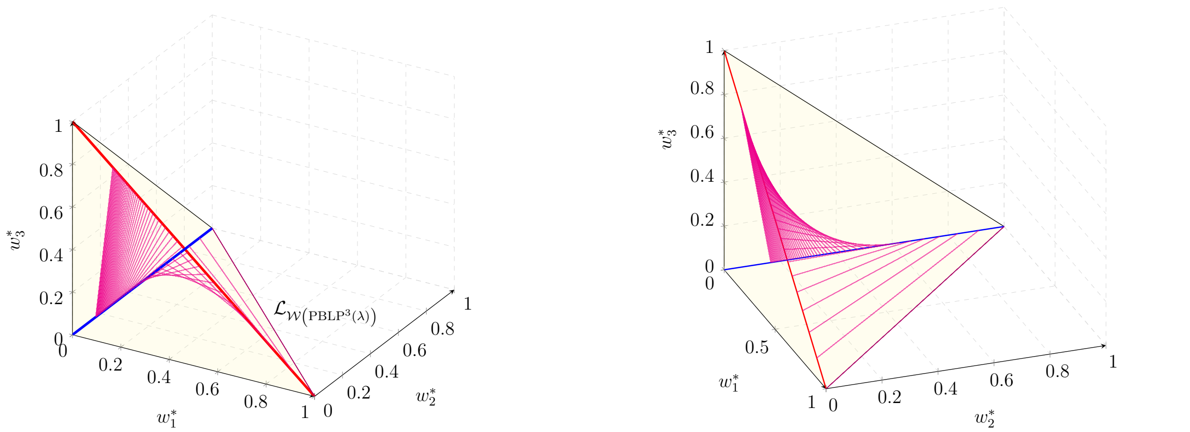
Next, we want to characterize the projection of $\mathcal{W}(\text{PBLP}^3(\lambda))$ in $\mathbb{R}^3$. Since the third and forth components of a weight in $\mathcal{W}(\text{PBLP}^3(\lambda))$ can be determined by the first and second components, respectively, the projection reduces the entire weight set to a 3-dimensional interpretation. Using the definition of $\mathcal{W}\left(\text{PBLP}^3(\lambda)\right)$ (cf. Definition 4.4) for any parameter $\lambda \geq 0$, the third and component of any arbitrary weight $w^* \in \mathcal{W}\left(\text{PBLP}^3(\lambda)\right)$ is

$$w_3^* = w_1^* \lambda$$

and

$$w_4^* = w_2^* \lambda.$$

This means every weight in $\mathcal{W}\left(\text{PBLP}^3(\lambda)\right)$ which is contained in $\mathcal{W}$(TOLP) satisfies the condition $w_3^* = w_1^* \lambda$ and $w_4^* = w_2^* \lambda$.



(a) The weight sets $\mathcal{W}(\text{PBLP}^3(\lambda))$ for varying λ within the weight set of 4-OLP.

(b) The weight sets $\mathcal{W}(\text{PBLP}^3(\lambda))$ from a different angle

Figure 4.1: An illustration of the family of line segments representing weight sets of $\text{PBLP}^3(\lambda)$ for different values of λ that forms a subset of $\mathcal{W}$(4-OLP). The highlighted red line segment, $w_1^* + w_3^* = 1$ and the blue line segment, $w_1^* = w_3^* = 0$ are both part of the subset. The red line segment comprises of weights in $\mathcal{W}$(4-OLP) that satisfy $w_1^* + w_3^* = 1$. The blue line segment comprises of weights in $\mathcal{W}$(4-OLP) that satisfy $w_1^* + w_3^* = 1$.

4.9 Definition. The *projection of the weight set* $\mathcal{W}\left(\text{PBLP}^3(\lambda)\right)$ in $\mathbb{R}^3$ can be defined as the line segment

$$\mathcal{L}_{\mathcal{W}\left(\text{PBLP}^3(\lambda)\right)} = \left\{(w_1^*, w_2^*, w_3^*) \mid w^* \in \mathcal{W}(\text{4-OLP}),\ \frac{w_3^*}{w_1^*} = \frac{1 - w_1^* - w_2^* - w_3^*}{w_1^*} = \lambda\right\}. \quad \triangleleft$$

A visualization of the weight set of $\text{PBLP}^3(\lambda)$ being a subset of weight set of the corresponding 4-OLP is shown in Figure 4.1. The family of line segments $\mathcal{L}_{\mathcal{W}(\text{PBLP}^3(\lambda))}$ representing weight sets of $\text{PBLP}^3(\lambda)$ for varying λ are illustrated in Figure 4.1. We observe that this family of line segments forms only a surface-like intersection in $\mathcal{W}$(4-OLP). This implies that the family of line segments for the parametric problem may fail to intersect some of the weight set components of $\mathcal{W}$(4-OLP) as shown in Figure 4.2.

4.1.1 Illustrative examples and further results

Every solution set S_4 for 4-OLP contains a subset of solutions such that there is a bijection between this subset and the extreme nondominated images Y_{EN} of 4-OLP. At the same time, the weight set $\mathcal{W}$(4-OLP) can be decomposed into full dimensional weight set components such that there is a bijection between these weight set components and Y_{EN}(4-OLP) (cf. [PGE10]). By Definition 4.9 for a fixed value of $\lambda \geq 0$, the weight set $\mathcal{W}(\text{PBLP}^3(\lambda))$ is a line segment that lies in $\mathcal{W}$(4-OLP). This line segment intersects some full-dimensional weight set components. For each component $\mathcal{W}(y)$ that are intersected, there are four possibilities: $\mathcal{W}(\text{PBLP}^3(\lambda))$ intersects $\mathcal{W}(y)$ either in a single

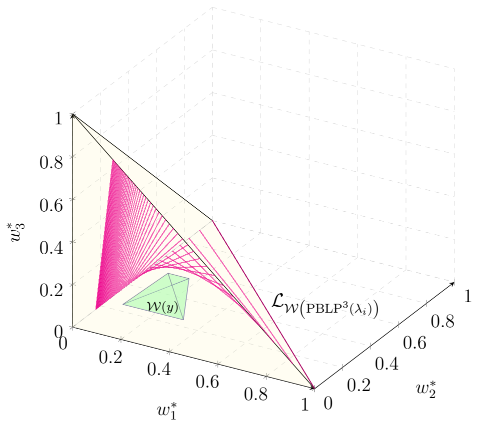


Figure 4.2: An illustration of the family of line segments $\mathcal{L}_{\mathcal{W}(\mathrm{PBLP}^3(\lambda))}$ that fails to intersect the weight set component $\mathcal{W}(y)$ of 4-OLP.

vertex, along an edge, along a facet, or it passes through the interior of $\mathcal{W}(y)$. The entire segment $\mathcal{W}(\mathrm{PBLP}^3(\lambda))$ can then be decomposed into such intersections (cf. [PGE10]). A solution set for $\mathrm{PBLP}^3(\lambda)$ can be obtained by using all solutions from S_4 where the corresponding weight set component is intersected by $\mathcal{W}(\mathrm{PBLP}^3(\lambda))$. Therefore, a subset of S_4 is also a solution set for PBLP^3.

However, since the converse of Theorem 4.2 does not hold as shown in Example 4.3, it holds that, for some full dimensional weight set component $\mathcal{W}(y)$ of an extreme nondominated image $y \in Y_{\mathrm{EN}}(4\text{-OLP})$, there exists no $\lambda \geq 0$ such that the line segment $\mathcal{W}(\mathrm{PBLP}^3(\lambda))$ intersects $\mathcal{W}(y)$. Then, a solution set for $\mathrm{PBLP}^3(\lambda)$ may not have a solution x such that $(c_1x, c_2x, d_1x, d_2x)^\top = y$. Therefore, it is possible that a solution set for PBLP^3 is a subset of a solution set for 4-OLP.

Since every solution set for 4-OLP contains a solution set for PBLP^3, we can make the following statement regarding minimal solution sets:

4.10 Proposition. If the sets $S(\mathrm{PBLP}^3) \subseteq X$ and $S_4 \subseteq X$ are minimal solution sets of PBLP^3 and 4-OLP, respectively, then

$$|S(\mathrm{PBLP}^3)| \leq |S_4|. \qquad \triangleleft$$

We observe that solving the parametric problem $\mathrm{PBLP}^3(\lambda)$ for all $\lambda \geq 0$ does not necessarily yield a minimal solution set of the 4-OLP. However, solving 4-OLP will yield a superset of a solution set of PBLP^3 as shown in Figure 4.3

As in the cases PBLP^1 and PBLP^2, we are also interested in breakpoints in the parameter set for PBLP^3 where a minimal solution set of $\mathrm{PBLP}^3(\lambda)$ changes as λ varies.

4.11 Definition. For a given PBLP^3, some parameter value $\lambda \geq 0$ is called a breakpoint if, for a small $\varepsilon > 0$,

$$S\left(\mathrm{PBLP}^3(\lambda + \varepsilon)\right) \neq S\left(\mathrm{PBLP}^3(\lambda - \varepsilon)\right),$$

where $S\left(\mathrm{PBLP}^3(\lambda)\right)$ is a minimal solution set at parameter λ. ◁

Therefore, the solution to the problem PBLP^3 consists of the following:

(i) a minimal solution set $S(\mathrm{PBLP}^3)$ for PBLP^3,
(ii) a set of breakpoints in the parameter set and
(iii) parameter intervals of each solution in $S(\mathrm{PBLP}^3)$.

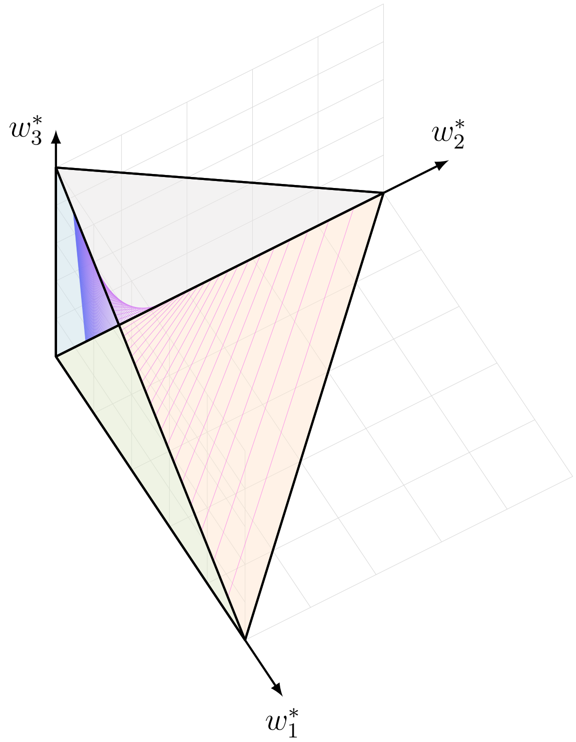
4.12 Proposition. If λ_i is a breakpoint with a corresponding weight set $\mathcal{W}(\mathrm{PBLP}^3(\lambda_i))$, then there exists at least one weight on a face of some weight set component in $\mathcal{W}$(4-OLP) on the intersection of $\mathcal{L}_{\mathcal{W}(\mathrm{PBLP}^3(\lambda_i))}$ and $\mathcal{W}$(4-OLP). ◁

Proof. The proof is analogous to Proposition 3.32. □

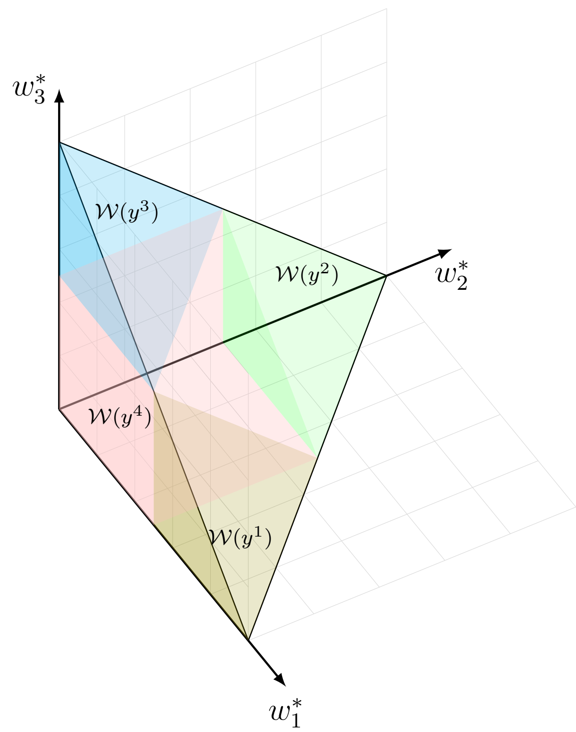
Next, we use Example 3.37 to illustrate the weight sets of PBLP^3 for varying λ in the weight set of the corresponding 4-OLP (see Figure 4.3). For this particular example, we observe that the weight sets of the parametric problem intersects all the weight set components of $\mathcal{W}$(4-OLP) at least once. Therefore, a solution set of the 4-OLP is also a solution set of the parametric problem.

Note, that due to Proposition 4.12, we can use weight set components of $\mathcal{W}$(4-OLP) to determine breakpoints and parameter intervals for the problem PBLP^3. We use all the weights of a weight set component that satisfy Equation (4.3) to compute parameter λ and verify whether it is a breakpoint or not. Due to the construction of λ, we only consider $w^* \in \mathcal{W}$(4-OLP) where $w_1^* > 0$ and $w_2^* > 0$. Moreover, such weights with $w_1^* = 0$ and $w_2^* = 0$ imply that the corresponding parameter $\lambda \to \infty$.

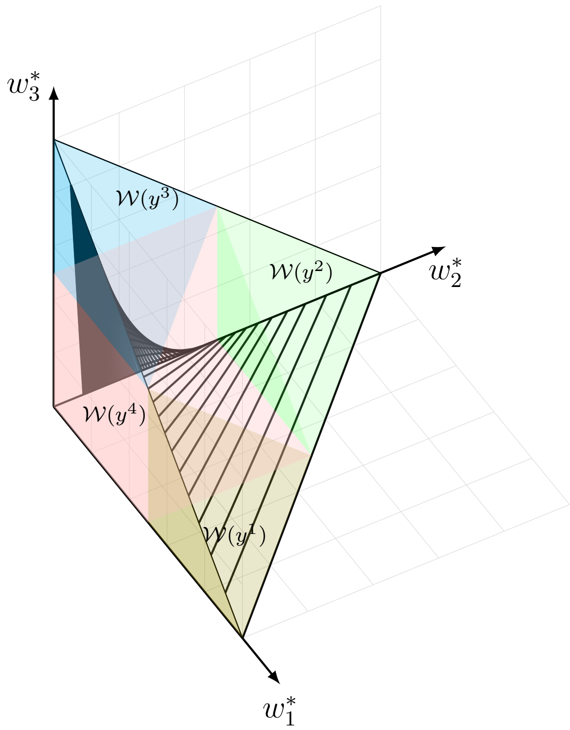
Furthermore, we can use breakpoints of PBLP^3 to decompose the parameter set into intervals or parameter value such that each interval or value correspond to a unique solution set.



(a) Weight sets of PBLP$^3(\lambda)$ with varying λ



(b) Weight set of the 4-OLP instance in Example 3.37



(c) Weight sets of PBLP$^3(\lambda)$ in the weight set of 4-OLP

Figure 4.3: An illustration of the weight sets of PBLP$^3(\lambda)$ in the weight set of 4-OLP of Example 3.37.

Chapter 5

Multi-parametric biobjective linear programs

In this chapter, we extend our analysis to parametric linear programs with more than one parameter. We presented two formulations of the multi-parametric program in the introduction, and here, we consider the one with parametric objective functions. We mainly work with biparametric biobjective linear programs with two different non-negative parameters, λ and μ, λ in the first objective and μ in the second.

Our focus lies in two specific cases of biparametric biobjective linear programs. In the first case, we consider different parametric dependency in both objectives with the same parametric objective, and in the second case we consider different parametric dependency with different parametric objectives. More precisely, we consider the following cases.

Case I : Same parametric objectives

$$\min_{\mathrm{x}\in\mathrm{X}} \begin{pmatrix} c_1x + \lambda d_1 x \\ c_2 x + \mu d_1 x \end{pmatrix} \tag{BBLP^1}$$

Case II : Different parametric objectives

$$\min_{\mathrm{x}\in\mathrm{X}} \begin{pmatrix} c_1x + \lambda d_1 x \\ c_2 x + \mu d_2 x \end{pmatrix} \tag{BBLP^2}$$

The two cases are followed by the generalization of the biparametric problem BBLP^2 to multi-parametric biobjective problems with different parametric dependency and different parametric objectives.

We build upon the same framework as in previous chapters, i. e. we relate the parametric problems to the corresponding multi-objective problems and use the weighted sum scalarization to analyse these problems. This helps us in relating the solution sets of the two problems and characterizing the parameter set of the biparametric biobjective linear programs with respect to the weight set of the corresponding multi-objective linear programs.

The problem BBLP^j, $j = 1, 2$, for some fixed λ and some fixed μ, is a non-parametric biobjective linear program and we denote it by $\mathrm{BBLP}^j(\lambda, \mu)$. Furthermore, we denote the set of extreme nondominated images of $\mathrm{BBLP}^j(\lambda,\mu)$ by $Y_{\mathrm{EN}}\left(\mathrm{BBLP}^j(\lambda,\mu)\right)$ and its corresponding minimal solution set by $S\left(\mathrm{BBLP}^j(\lambda,\mu)\right)$.

Our analysis relates to a minimal solution set that corresponds to extreme nondominated images of $\mathrm{BBLP}^j(\lambda,\mu)$ for every non-negative λ and μ. Therefore, a minimal solution set of BBLP^j equates to a set that contains efficient solutions of $\mathrm{BBLP}^j(\lambda,\mu)$ for each pair of parameters $(\lambda,\mu)\in\mathbb{R}^2_{\geqq}$. We denote this set by $S(\mathrm{BBLP}^j)$.

5.1 Definition. A *solution set* $S(\mathrm{BBLP}^j)\subseteq X$ of BBLP^j is a set such that for every $(\lambda,\mu)\in\mathbb{R}^2_{\geqq}$, $S(\mathrm{BBLP}^j)$ contains, as a subset, a solution set for the biobjective program $\mathrm{BBLP}^j(\lambda,\mu)$. It is called *minimal* if, additionally, there is no other solution set $S'\subseteq X$ for BBLP^j with $|S'|<\left|S(\mathrm{BBLP}^j)\right|$. ◁

A minimal solution set for the biparametric biobjective linear program BBLP^j is defined over the entire parameter set in $\mathbb{R}^2_{\geqq}$. We are particularly interested in the critical regions $\mathcal{R}^j(x)\subseteq\mathbb{R}^2_{\geqq}$ corresponding to a solution x in a minimal solution set of BBLP^j (cf. Definition 2.6). Therefore, the solution to biparametric biobjective linear programs BBLP^j consists of the following;

(i) a minimal solution set $S(\mathrm{BBLP}^j)$ for BBLP^j and
(ii) critical regions $\mathcal{R}^j(x)$ with respect to each $x\in S(\mathrm{BBLP}^j)$ in the parameter set.

5.1 Case I : Same parametric objectives

We approach the problem by applying the weighted sum scalarization to the biparametric biobjective linear program BBLP^1 and formally characterize its relation to the corresponding triobjective linear program defined as TOLP in Chapter 3.

For a fixed value of $\lambda\geq 0$ and $\mu\geq 0$, the weighted sum scalarization of $\mathrm{BBLP}^1(\lambda,\mu)$ is

$$\min_{x\in X} w_1\left(c_1x+\lambda d_1x\right)+w_2\left(c_2x+\mu d_1x\right) \qquad (\mathrm{WS}(\mathrm{BBLP}^1(\lambda,\mu),w))$$

where $w\in\mathbb{R}^2_{\geqq}$ and $w_1+w_2=1$. We reformulate this problem using $w_2=1-w_1$ and obtain

$$\min_{x\in X} w_1c_1x+(1-w_1)c_2x+(w_1\lambda+(1-w_1)\mu)d_1x.$$

The problem can be interpreted as a weighted sum scalarization of the TOLP with weight vector $(w_1,1-w_1,w_1\lambda+(1-w_1)\mu)$. It holds that

$$w_1+1-w_1+w_1\lambda+(1-w_1)\mu=1+w_1\lambda+(1-w_1)\mu\geq 1.$$

We normalize the weights and get

$$\min_{x\in X} \frac{w_1}{1+w_1\lambda+w_2\mu}(c_1x) + \frac{w_2}{1+w_1\lambda+w_2\mu}(c_2x) + \frac{w_1\lambda+w_2\mu}{1+w_1\lambda+w_2\mu}(d_1x),$$

which is a particular case of WS(TOLP, w^*) with the corresponding weight vector

$$w^* = \frac{1}{1+w_1\lambda+w_2\mu}\left(w_1, w_2, w_1\lambda+w_2\mu\right)^\top \in \mathcal{W}(\text{TOLP}). \tag{5.1}$$

We present a result relating the efficient solutions of WS(BBLP$^1(\lambda,\mu), w$) and the efficient solutions of the corresponding WS(TOLP, w^*).

5.2 Theorem. If there exists two parameters $\lambda \geq 0, \mu \geq 0$ and non-negative weights w_1, w_2 where $w_1, w_2 > 0$, such that x^* is optimal for WS $\left(\text{BBLP}^1(\lambda,\mu), w\right)$ then x^* is optimal for WS(TOLP, w^*) for some non-negative weights w_1^*, w_2^*, and w_3^*. $\triangleleft$

Proof. Let x^* be optimal for WS(BBLP$^1(\lambda,\mu), w$), with non-negative weights w_1, w_2, and for some parameter values $\lambda \geq 0$ and $\mu \geq 0$.
We define

$$\begin{aligned} w_1^* &:= w_1, \\ w_2^* &:= w_2, \\ w_3^* &:= w_1\lambda + w_2\mu \end{aligned} \tag{5.2}$$

using Equation (5.1). For simplicity, we do not normalize. Suppose x^* is not optimal for WS(TOLP, w^*), i.e. there exists x' which is optimal for WS(TOLP, w^*) where $x' \neq x^*$ such that,

$$w_1^*c_1x' + w_2^*c_2x' + w_3^*d_1x' < w_1^*c_1x^* + w_2^*c_2x^* + w_3^*d_1x^*.$$

We plug in Equation (5.2) to get

$$w_1c_1x' + w_2c_2x' + (w_1\lambda+w_2\mu)\,d_1x' < w_1c_1x^* + w_2c_2x^* + (w_1\lambda+w_2\mu)d_1x^*.$$

This is equivalent to

$$w_1c_1x' + w_2c_2x' + w_1\lambda d_1x' + w_2\mu d_1x' < w_1c_1x^* + w_2c_2x^* + w_1\lambda d_1x^* + w_2\mu d_1x^*.$$

This can be reformulated to

$$w_1\left(c_1x' + \lambda d_1x'\right) + w_2\left(c_2x' + \mu d_1x'\right) < w_1\Big(c_1x^* + \lambda d_1x^*\Big) + w_2\left(c_2x^* + \mu d_1x^*\right).$$

This leads to a contradiction that x^* is optimal for WS(BBLP$^1(\lambda,\mu), w$). $\square$

Conversely, given a weighted sum scalarization WS(TOLP, w^*), we can express the parameters of WS $\left(\text{BBLP}^1(\lambda,\mu), w\right)$ in terms of the weights $w^* \in \mathcal{W}(\text{TOLP})$ as a function of both λ and μ. In particular, we observe that a weight w^* in Equation (5.1) depends on two parameters λ and μ where $w_3^* = w_1^*\lambda + w_2^*\mu$. This creates a one-dimensional family of (λ, μ) pairs corresponding to the same w^*. In order to obtain a unique mapping, we fix either λ or μ.

As a result, the parameter value can be determined from a weight $w^* \in \mathcal{W}(\text{TOLP})$ as follows:

(i) For fixed λ:

$$\mu = \frac{w_3^* - w_1^*\lambda}{w_2^*} \tag{5.3}$$

(ii) For fixed μ:

$$\lambda = \frac{w_3^* - w_2^*\mu}{w_1^*} \tag{5.4}$$

where $w_1^* > 0$ and $w_2^* > 0$. We can exclude the weights with $w_1^* = 0$ and $w_2^* = 0$ due to the same reason stated in Theorem 3.3.

As $\text{BBLP}^1(\lambda, \mu)$ constitutes a biobjective linear problem, its associated weight set forms a one-dimensional polytope. However, by considering the parameters λ and μ, we can extend this interpretation of the one-dimensional weight set to a higher dimension through the mapping $\mathcal{W}\big(\text{BBLP}^1(\lambda, \mu)\big)$ which is defined below.

5.3 Definition. For a given λ and μ, we define a *mapping* $\mathcal{W}\big(\text{BBLP}^1(\lambda, \mu)\big)$ as,

$$\mathcal{W}\big(\text{BBLP}^1(\lambda, \mu)\big) := \left\{ \frac{1}{1 + w_1\lambda + w_2\mu}(w_1, w_2, w_1\lambda + w_2\mu) : w \in \mathbb{R}^2_{\geq}, w_1 + w_2 = 1 \right\}.$$

This extended representation enables us to derive several important theoretical results.

5.4 Proposition. Let $\mathcal{W}\left(\text{BBLP}^1(\lambda, \mu)\right)$ and $\mathcal{W}(\text{TOLP})$ be the weight sets of BBLP^1 for a fixed λ and a fixed μ and TOLP, respectively. Then,

$$\mathcal{W}\big(\text{BBLP}^1(\lambda, \mu)\big) \subsetneq \mathcal{W}(\text{TOLP}).$$ ◁

Proof. Let $w^* \in \mathcal{W}\big(\text{BBLP}^1(\lambda, \mu)\big)$. Then, by the definition of $\mathcal{W}\big(\text{BBLP}^1(\lambda, \mu)\big)$, it is

$$w^* = \left(\frac{w_1}{1 + w_1\lambda + w_2\mu}, \frac{w_2}{1 + w_1\lambda + w_2\mu}, \frac{w_1\lambda + w_2\mu}{1 + w_1\lambda + w_2\mu} \right)$$

for some $w \in \mathbb{R}^2_{\geqq}$ with $w_1 + w_2 = 1$. Since the sum of the components satisfies

$$\frac{w_1}{1 + w_1\lambda + w_2\mu} + \frac{w_2}{1 + w_1\lambda + w_2\mu} + \frac{w_1\lambda + w_2\mu}{1 + w_1\lambda + w_2\mu} = 1,$$

it holds that $w^* \in \mathcal{W}(\text{TOLP})$.
To show $\mathcal{W}\Big(\text{BBLP}^1(\lambda,\mu)\Big)$ is a proper subset of $\mathcal{W}(\text{TOLP})$, we consider a particular weight $w^* = (0,0,1) \in \mathcal{W}(\text{TOLP})$. Since for any given λ and μ, the third component of a weight in $\mathcal{W}(\text{BBLP}(\lambda,\mu))$,

$$\frac{w_1\lambda + w_2\mu}{1 + w_1\lambda + w_2\mu} \neq 1.$$

Therefore, it holds that $w^* \notin \mathcal{W}\Big(\text{BBLP}^1(\lambda,\mu)\Big)$. □

Next, we want to characterize the projection of $\mathcal{W}(\text{BBLP}^1(\lambda,\mu))$ in $\mathbb{R}^2$. Since the third component of a weight in $\mathcal{W}(\text{BBLP}^1(\lambda,\mu))$ can be determined by the first two components, the projection reduces the entire weight set to a 2-dimensional interpretation. Using the definition of $\mathcal{W}\Big(\text{BBLP}^1(\lambda,\mu)\Big)$ for parameters $\lambda \geq 0$ and $\mu \geq 0$, the third component of any arbitrary weight $w^* \in \mathcal{W}\Big(\text{BBLP}^1(\lambda,\mu)\Big)$ is

$$w_3^* = w_1^*\lambda + w_2^*\mu.$$

Since $w_1^* + w_2^* + w_3^* = 1$, we get

$$1 - (w_1^* + w_2^*) = w_1^*\lambda + w_2^*\mu.$$

This is equivalent to

$$w_1^*(1+\lambda) + w_2^*(1+\mu) = 1.$$

This means every weight in $\mathcal{W}\Big(\text{BBLP}^1(\lambda,\mu)\Big)$ which is contained in $\mathcal{W}(\text{TOLP})$ satisfies the condition $w_1^*(1+\lambda)+w_2^*(1+\mu) = 1$. This condition can be interpreted as an equation of a line segment in $\mathbb{R}^2$. Consequently, we can define the weight set $\mathcal{W}\Big(\text{BBLP}^1(\lambda,\mu)\Big)$ as a line segment in $\mathcal{W}(\text{TOLP})$.

5.5 Definition. The *projection of the weight set* $\mathcal{W}(\text{BBLP}^1(\lambda,\mu))$ in $\mathbb{R}^2_{\geqq}$ can be defined as the line segment,

$$\mathcal{L}_{\mathcal{W}\left(\text{BBLP}^1(\lambda,\mu)\right)} = \left\{(w_1^*, w_2^*) \mid w^* \in \mathcal{W}(\text{TOLP}),\ w_1^*(1+\lambda) + w_2^*(1+\mu) = 1\right\}. \qquad \triangleleft$$

5.6 Remark. From the definition of $\mathcal{L}_{\mathcal{W}(\text{BBLP}^1(\lambda,\mu))}$, we observe the following:

(i) The slope $m_{\lambda\mu}$ of the line segment $\mathcal{L}_{\mathcal{W}(\text{BBLP}^1(\lambda,\mu))}$ is given by

$$m_{\lambda\mu} = \frac{-\lambda - 1}{1+\mu}.$$

(ii) The end points of the line segment $\mathcal{L}_{\mathcal{W}(\text{BBLP}^1(\lambda,\mu))}$ are $(0, \frac{1}{1+\mu})$ and $\left(\frac{1}{1+\lambda}, 0\right)$. ◁

Remark 5.6.(i) is a straightforward slope of the line segment and Remark 5.6.(ii) represents the w_1^*-intercept and w_2^*-intercept of the line segment $\mathcal{L}_{\mathcal{W}(\text{BBLP}^1(\lambda,\mu))}$.

5.7 Proposition. For two distinct pair of parameter values (λ_1, μ_1) and (λ_2, μ_2) such that $\lambda_1 \neq \lambda_2$, and $\mu_1 \neq \mu_2$, it holds that

$$\mathcal{L}_{\mathcal{W}(\text{BBLP}^1(\lambda_1,\mu_1))} \cap \mathcal{L}_{\mathcal{W}(\text{BBLP}^1(\lambda_2,\mu_2))} = \{\mathbf{w}\} \text{ or } \varnothing$$

where $\mathbf{w} \in \mathcal{W}(\text{TOLP})$ is an intersection point of the line segments $\mathcal{L}_{\mathcal{W}(\text{BBLP}^1(\lambda_1,\mu_1))}$ and $\mathcal{L}_{\mathcal{W}(\text{BBLP}^1(\lambda_2,\mu_2))}$. ◁

Proof. The proof follows from the definition of $\mathcal{L}_{\mathcal{W}(\text{BBLP}^1(\lambda,\mu))}$ that,

$$\mathcal{L}_{\mathcal{W}(\text{BBLP}^1(\lambda_1,\mu_1))} = \{(w_1^*, w_2^*) \mid w^* \in \mathcal{W}(\text{TOLP}),\ w_1^*(1+\lambda_1) + w_2^*(1+\mu_1) = 1\}$$

and

$$\mathcal{L}_{\mathcal{W}(\text{BBLP}^1(\lambda_2,\mu_2))} = \{(w_1^*, w_2^*) \mid w^* \in \mathcal{W}(\text{TOLP}),\ w_1^*(1+\lambda_2) + w_2^*(1+\mu_2) = 1\}.$$

The end points of $\mathcal{L}_{\mathcal{W}(\text{BBLP}^1(\lambda_1,\mu_1))}$ are $\left(0, \frac{1}{1+\mu_1}\right)$ and $\left(\frac{1}{1+\lambda_1}, 0\right)$. Similarly, the end points of $\mathcal{L}_{\mathcal{W}(\text{BBLP}^1(\lambda_2,\mu_2))}$ are $\left(0, \frac{1}{1+\mu_2}\right)$ and $\left(\frac{1}{1+\lambda_2}, 0\right)$. Since $\lambda_1 \neq \lambda_2$ and $\mu_1 \neq \mu_2$, either the line segments do not intersect in $\mathcal{W}(\text{TOLP})$ or they intersect at a single point $\mathbf{w} \in \mathcal{W}(\text{TOLP})$. □

The problem BBLP^1 can be further divided into two sub-problems where one parameter is varying and the other is fixed. We denote them as $\text{BBLP}^1_\mu(\lambda)$ where the parameter λ is varying and μ is fixed and, as $\text{BBLP}^1_\lambda(\mu)$ where the parameter μ is varying and λ is fixed. We examine how the solution sets change when we fix λ and vary $\mu \geq 0$, and vice versa. Therefore, we want to find minimal solution sets for $\text{BBLP}^1_\mu(\lambda)$ and $\text{BBLP}^1_\lambda(\mu)$, denoted by $S\left(\text{BBLP}^1_\mu(\lambda)\right)$ and $S\left(\text{BBLP}^1_\lambda(\mu)\right)$, respectively. Since the two sub-problems are analogous, we focus only on one of them in detail.

5.8 Definition. A *solution set* $S\left(\text{BBLP}^1_{\mu^*}(\lambda)\right) \subseteq X$ of BBLP^1_λ is a set such that for every $\lambda \geq 0$ and a fixed $\mu^* \geq 0$, it contains, as a subset, a minimal solution set for the BOLP $\text{BBLP}^1_{\mu^*}(\lambda)$). It is called *minimal* if, additionally, there is no other solution set $S' \subseteq X$ for BBLP^1_λ with $|S'| < \left|S\left(\text{BBLP}^1_{\mu^*}(\lambda)\right)\right|$. ◁

We now establish the connection between the problem BBLP^1 to the problem PBLP^1 in Chapter 3.

5.9 Corollary. A feasible solution x^* is optimal for $\text{WS}(\text{TOLP}, w^*)$ with non-negative weights w_1^*, w_2^*, w_3^* where $w_1^* > 0$ if and only if there exists a parameter $\lambda \geq 0$ and $\mu = 0$ and non-negative weights w_1, w_2 where $w_1 > 0$ such that x^* is optimal for $\text{WS}\left(\text{BBLP}^1_0(\lambda), w\right)$. ◁

(a) $\mathcal{W}(\text{BBLP}^1_0(\lambda))$ for all $\lambda \geq 0$ **(b)** $\mathcal{W}(\text{BBLP}^1_0(\mu))$ for all $\mu \geq 0$

Figure 5.1: An illustration of the weight sets of BBLP^1 when one of the parameter values is fixed to 0 and the other parameter is varying.

Proof. We can observe that for a fixed parameter $\mu = 0$, the parametric problem BBLP^1_λ is reduced to PBLP^1,

$$\min_{\mathrm{x} \in \mathrm{X}} \begin{pmatrix} c_1 x + \lambda d_1 x \\ c_2 x \end{pmatrix}.$$

The result on optimality follows from Theorem 3.3 □

Since the biparametric problem reduces to the single parametric problem, we now restate the result with regard to the minimal solution set of the biparametric problem.

5.10 Corollary. A set $S(\text{TOLP}) \subseteq X$ is a minimal solution set for TOLP if and only if $S(\text{TOLP})$ is a minimal solution set for $\text{BBLP}^1_{\mu^*}(\lambda)$ for a fixed $\mu^* = 0$. ◁

Proof. By Corollary 5.9 and Proposition 3.19. □

5.11 Remark. Let $\mathcal{W}(\text{BBLP}^1_{\mu^*}(\lambda))$ be the weight sets of $\text{BBLP}^1_{\mu^*}(\lambda)$ where μ is fixed and λ varying. If $\mu^* = 0$, then the union of such weight sets is

$$\bigcup_{\lambda_i \geq 0} \mathcal{W}(\text{BBLP}^1_0(\lambda_i)) = \{w^* \in \mathcal{W}(\text{TOLP}) : w_1^* \neq 0\} \cup \{(0, 1, 0)\}.$$

Remarks 5.11 follows from the fact that when the parameter $\mu = 0$ the problem reduces to PBLP^1 (see Corollary 3.8). A visualization of this case is shown in Figure 5.1a.

We now examine a special structural behaviour of the weight sets of the parametric problem when one of the parameter is fixed to a positive non-zero value. First, we show this in terms of the weight sets of $\text{BBLP}^1_\mu(\lambda)$.

5.12 Proposition. Let $\mathcal{W}\left(\text{BBLP}^1_{\mu^*}(\lambda)\right)$ be the weight set of $\text{BBLP}^1_{\mu^*}(\lambda)$ and $\mathcal{W}(\text{TOLP})$ be the weight set of the corresponding TOLP. Then, for a fixed $\mu^* > 0$ it holds that

$$\bigcup_{\lambda_i \geq 0} \mathcal{W}(\text{BBLP}^1_{\mu^*}(\lambda_i)) \subsetneq \mathcal{W}(\text{TOLP}). \quad ◁$$

Proof. From Proposition 5.4, all the weight sets of $\mathrm{BBLP}^1_{\mu^*}(\lambda)$ is a subset of $\mathcal{W}(\mathrm{TOLP})$. Next we show that the weight $(0,1,0) \in \mathcal{W}(\mathrm{TOLP})$ is not in $\mathcal{W}(\mathrm{BBLP}^1_{\mu^*}(\lambda))$ for all $\lambda \geq 0$. For a fixed $\mu^* > 0$, the family of line segments corresponding to the weight sets of $\mathrm{BBLP}^1_{\mu^*}(\lambda)$ for all $\lambda \geq 0$ is given by,

$$\mathcal{L}_{\mathcal{W}(\mathrm{BBLP}^1_{\mu^*}(\lambda))} := \left\{ (w_1^*, w_2^*) \mid w^* \in \mathcal{W}(\mathrm{TOLP}),\ w_2^* = \frac{-(1+\lambda)}{1+\mu^*} w_1^* + \frac{1}{1+\mu^*} \right\}.$$

Then for any given $\lambda \geq 0$,

$$w_2^* = \underbrace{\frac{-(1+\lambda)}{1+\mu^*}}_{\leq 0} w_1^* + \frac{1}{1+\mu^*} \leq \frac{1}{1+\mu^*} < 1$$

since $\mu^* > 0$. Therefore, the weight $(0,1,0) \notin \mathcal{L}_{\mathcal{W}\left(\mathrm{BBLP}^1_{\mu^*}(\lambda)\right)}$ for any λ. □

As a result of Proposition 5.12, we observe that for a fixed $\mu^* > 0$, the corresponding weight sets $\mathcal{W}(\mathrm{BBLP}^1_{\mu^*}(\lambda))$ in $\mathcal{W}(\mathrm{TOLP})$ has a restricted w_2^* entry i. e. $w_2^* \in (0, \frac{1}{1+\mu^*}]$ whereas $w_1^* \in (0,1]$. Therefore, we provide an exact description of the union of all weight sets for $\mathrm{BBLP}^1_{\mu^*}(\lambda)$. For $\mu^* > 0$, the union of weight sets is given by

$$\bigcup_{\lambda_i \geq 0} \mathcal{W}(\mathrm{BBLP}^1_{\mu^*}(\lambda_i)) = \left\{ w^* \in \mathcal{W}(\mathrm{TOLP}) : w_2^* \leq \frac{1}{1+\mu^*}(-w_1^* + 1), w_1^* \neq 0 \right\} \cup \left\{ \left(0, \frac{1}{1+\mu^*}, 0\right) \right\}.$$

An illustration of the line segments of this behaviour is shown in Figure 5.2.

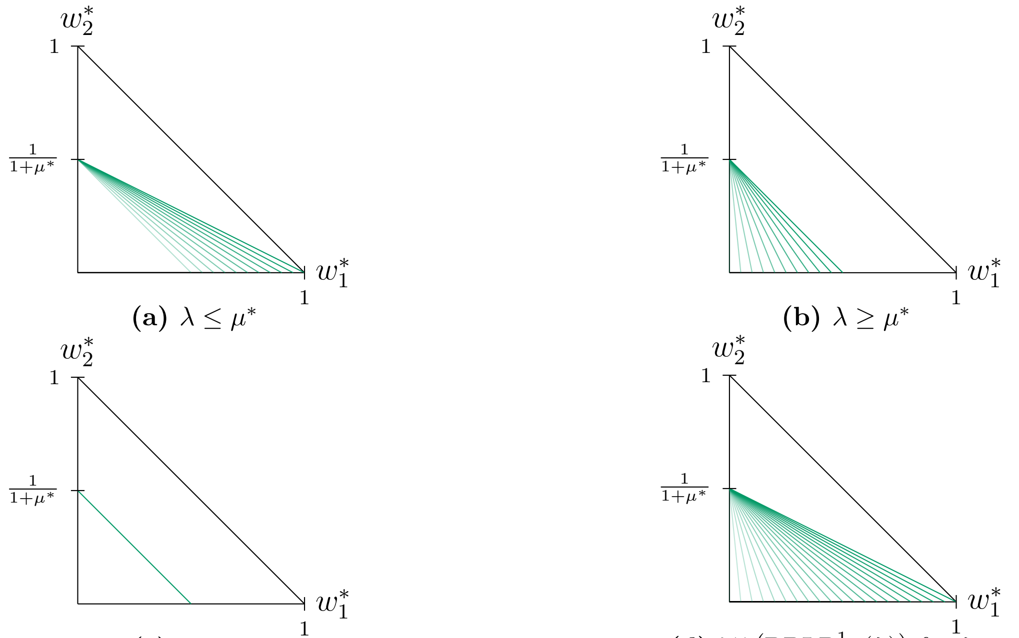
5.13 Remark. Furthermore, using Proposition 5.12 and the description of weight sets of BBLP^1_λ, for $\mu^* > 0$ and $\mu' > 0$ such that $\mu^* < \mu'$, we observe that

$$\bigcup_{\lambda_i \geq 0} \mathcal{W}(\mathrm{BBLP}^1_{\mu'}(\lambda_i)) \subsetneq \bigcup_{\lambda_i \geq 0} \mathcal{W}(\mathrm{BBLP}^1_{\mu^*}(\lambda_i)). \qquad \triangleleft$$

Consequently, the area covered by the family of line segments $\mathcal{W}(\mathrm{BBLP}^1_{\mu'}(\lambda))$ in the weight set of TOLP is contained in the area covered by the family of line segments $\mathcal{W}(\mathrm{BBLP}^1_{\mu^*}(\lambda))$ if $\mu^* < \mu'$.

Similarly, we now present analogous results for the sub-problem $\mathrm{BBLP}^1_\lambda(\mu)$. First, we can fix $\lambda = 0$ and vary parameter μ to relate the problem $\mathrm{BBLP}^1_0(\mu)$ to PBLP^1_2.

5.14 Corollary. A feasible solution x^* is optimal for $\mathrm{WS}(\mathrm{TOLP}, w^*)$ with non-negative weights w_1^*, w_2^*, w_3^* where $w_2^* > 0$ if and only if there exists a parameter $\mu \geq 0$ and $\lambda = 0$ and non-negative weights w_1, w_2 where $w_2 > 0$ such that x^* is optimal for $\mathrm{WS}\left(\mathrm{BBLP}^1_0(\mu), w\right)$. ◁



(a) $\lambda \leq \mu^*$ **(b)** $\lambda \geq \mu^*$ **(c)** $\lambda = \mu^*$ **(d)** $\mathcal{W}\left(\mathrm{BBLP}^1_{\mu^*}(\lambda)\right)$ for $\lambda \geq 0$

Figure 5.2: An illustration of the family of line segments $\mathcal{L}_{\mathcal{W}\left(\mathrm{BBLP}^1_{\mu^*}(\lambda)\right)}$ for a fixed parameter $\mu^* > 0$ and varying parameter λ in the weight set of TOLP.

.

Proof. We can observe that for a fixed parameter $\lambda = 0$, the parametric problem $\mathrm{BBLP}^1_0(\mu)$ is reduced to PBLP^1_2 i. e.

$$\min_{\mathrm{x}\in\mathrm{X}} \begin{pmatrix} c_1 x \\ c_2 x + \mu d_1 x \end{pmatrix}.$$

This proof is analogous to the proof of Theorem 3.3. Here, we define

$$\begin{aligned} w_1 &:= \frac{w_1^*}{1 - w_3^*}, \\ w_2 &:= \frac{w_2^*}{1 - w_3^*}, \\ \mu &:= \frac{w_3^*}{w_2^*} \end{aligned} \tag{5.5}$$

for the first part of the proof and we define

$$\begin{aligned} w_1^* &:= w_1, \\ w_2^* &:= w_2, \\ w_3^* &:= \frac{w_2 \mu}{1 + w_2 \mu} \end{aligned} \tag{5.6}$$

in the converse case. □

5.15 Corollary. A set $S(\mathrm{TOLP}) \subseteq X$ is a minimal solution set for TOLP if and only if $S(\text{TOLP})$ is a minimal solution set for $\text{BBLP}_0^1(\mu)$ for a fixed $\lambda = 0$. ◁

Proof. By Corollary 5.14 and Proposition 3.19 □

5.16 Remark. Let $\mathcal{W}(\text{BBLP}^1_{\lambda^*}(\mu))$ be the weight sets of $\text{BBLP}^1_{\lambda^*}(\mu)$ where λ^* is fixed and μ varying. If $\lambda^* = 0$, then the union of such weight sets is

$$\bigcup_{\mu_i \geq 0} \mathcal{W}(\text{BBLP}^1_0(\mu_i) = \{w^* \in \mathcal{W}(\text{TOLP}) : w_2^* \neq 0\} \cup \{(1,0,0)\}$$

Remarks 5.16 follows analogously from Remark 5.11 and Corollary 3.8. A visualization of the union of such weight sets is shown in Figure 5.1b.

5.17 Proposition. Let $\mathcal{W}\left(\text{BBLP}^1_\lambda(\mu)\right)$ be the weight set of $\text{BBLP}^1_\lambda(\mu)$ and $\mathcal{W}(\text{TOLP})$ be the weight set of the corresponding TOLP. Then, for a fixed $\lambda^* > 0$,

$$\bigcup_{\mu_i \geq 0} \mathcal{W}(\text{BBLP}^1_{\lambda^*}(\mu_i)) \subsetneq \mathcal{W}(\text{TOLP}). \quad ◁$$

Proof. The proof is analogous to the proof of Proposition 5.12 where we show that the weight $(1,0,0) \in \mathcal{W}(\text{TOLP})$ but $(1,0,0) \notin \mathcal{L}_{\mathcal{W}\left(\text{BBLP}^1_{\lambda^*}(\mu)\right)}$ for any $\mu \geq 0$. □

Similarly, from Proposition 5.17, we observe that for a fixed $\lambda^* > 0$, the corresponding weight sets $\mathcal{W}(\text{BBLP}^1_{\lambda^*}(\mu))$ in $\mathcal{W}(\text{TOLP})$ has a restricted w_1^* i. e. $w_1^* \in (0, \frac{1}{1+\lambda^*}]$ whereas $w_2^* \in (0,1]$.

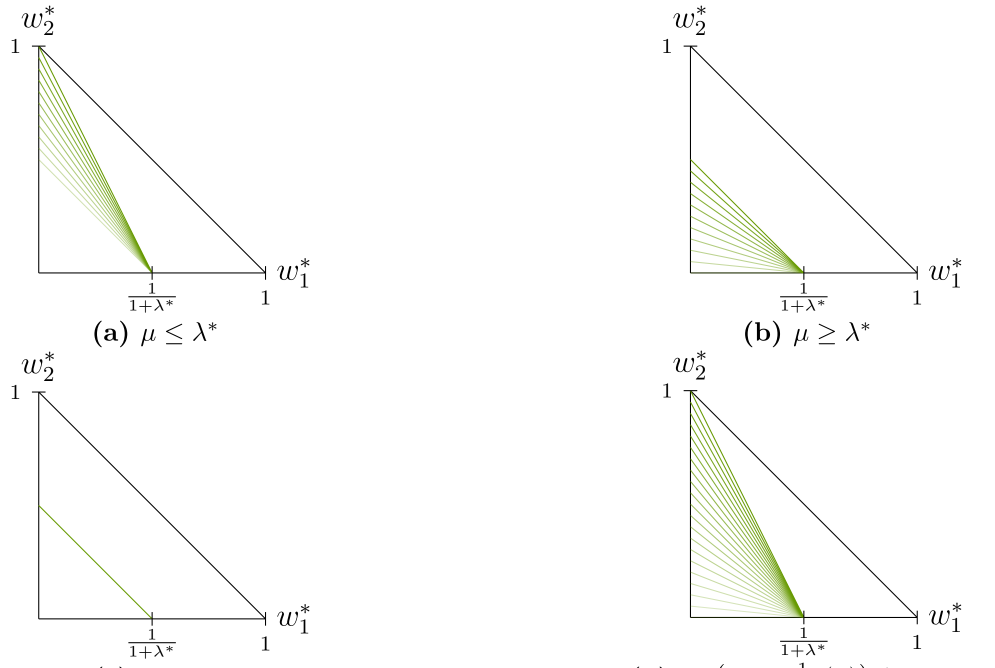
Therefore, we provide an exact description of the union of all weight sets for $\text{BBLP}^1_\lambda(\mu)$. For $\lambda^* > 0$, the union of weight sets is

$$\bigcup_{\mu_i \geq 0} \mathcal{W}(\text{BBLP}^1_{\lambda^*}(\mu_i)) =$$
$$\left\{w^* \in \mathcal{W}(\text{TOLP}) : w_1^* \leq \frac{1}{1+\lambda^*}(-w_2^*+1), w_2^* \neq 0\right\} \cup \left\{\left(\frac{1}{1+\lambda^*}, 0, 0\right)\right\}.$$

Furthermore, using Proposition 5.17 and the description of weight sets for $\text{BBLP}^1_\lambda(\mu)$, for $\lambda^* > 0$ and $\lambda' > 0$ such that $\lambda^* < \lambda'$, we observe that

$$\bigcup_{\mu_i \geq 0} \mathcal{W}(\text{BBLP}^1_{\lambda'}(\mu_i)) \subsetneq \bigcup_{\mu_i \geq 0} \mathcal{W}(\text{BBLP}^1_{\lambda^*}(\mu_i)).$$

Due to the behaviour of weight sets of $\text{BBLP}^1_\mu(\lambda)$ and $\text{BBLP}^1_\lambda(\mu)$ with restricted parameter values we observe that it is possible to get an incomplete minimal solution set for the problem.



(a) $\mu \leq \lambda^*$ **(b)** $\mu \geq \lambda^*$ **(c)** $\mu = \lambda^*$ **(d)** $\mathcal{W}\left(\text{BBLP}^1_{\lambda^*}(\mu)\right)$ for $\mu \geq 0$

Figure 5.3: An illustration of the family of line segments $\mathcal{L}_{\mathcal{W}(\text{BBLP}^1_{\lambda^*}((\mu))}$ for a fixed parameter $\lambda^* > 0$ and varying parameter μ on the weight set of TOLP.

5.1.1 Further results and illustrative examples

Since every solution set $S(\text{TOLP})$ for TOLP contains a subset of solutions such that there is a bijection between this subset and the extreme nondominated images Y_{EN} of TOLP. At the same time, the weight set $\mathcal{W}(\text{TOLP})$ can be decomposed into full dimensional weight set components such that there is a bijection between these weight set components and $Y_{\text{EN}}(\text{TOLP})$ (cf. [PGE10]). By Propositions 5.6 (i), for fixed values of $\lambda \geq 0$ and $\mu \geq 0$, the weight set $\mathcal{W}\left(\text{BBLP}^1(\lambda,\mu)\right)$ is a line segment that lies in $\mathcal{W}(\text{TOLP})$. This line segment intersects some (full-dimensional) weight set components. For each component $\mathcal{W}(y)$, there are three possibilities: $\mathcal{W}(\text{BBLP}^1(\lambda,\mu))$ intersects $\mathcal{W}(y)$ either in a single vertex, along an edge, or it passes through the interior of $\mathcal{W}(y)$. The entire segment $\mathcal{W}\left(\text{BBLP}^1(\lambda,\mu)\right)$ can then be decomposed into such intersections (cf. [PGE10]). A solution set for $\text{BBLP}^1(\lambda,\mu)$ can be obtained by using all solutions from $S(\text{TOLP})$ where the corresponding weight set component is intersected by $\mathcal{W}\left(\text{BBLP}^1(\lambda,\mu)\right)$. However, due to Propositions 5.12 and 5.17 the line segments of $\mathcal{W}\left(\text{BBLP}^1_\mu(\lambda)\right)$ and $\mathcal{W}\left(\text{BBLP}^1_\lambda(\mu)\right)$ do not intersect the entire $\mathcal{W}(\text{TOLP})$. As a result, the family of line segments might fail to intersect some of the weight set components of $\mathcal{W}(\text{TOLP})$. Therefore, in general, it is possible that the solution set obtained for $\text{BBLP}^1_\mu(\lambda)$ and $\text{BBLP}^1_\lambda(\mu)$ with a restricted parameter μ and λ, respectively, is a subset of $S(\text{TOLP})$.

Since every solution set for TOLP contains a solution set for BBLP^1 as a subset, we can make the following statements regarding minimal solution sets:

5.18 Proposition. Let $S\left(\mathrm{BBLP}^1_\mu(\lambda)\right)$ and $S\left(\mathrm{BBLP}^1_\lambda(\mu)\right)$ be minimal solution sets of $\mathrm{BBLP}^1_\mu(\lambda)$ for a fixed μ, and $\mathrm{BBLP}^1_\lambda(\mu)$ for a fixed λ, respectively. Let $S(\mathrm{TOLP})$ be a minimal solution set of the corresponding TOLP. Then,

(i) $|S\left(\mathrm{BBLP}^1_\mu(\lambda)\right)| \subseteq |S(\mathrm{TOLP})|$ and

(ii) $|S\left(\mathrm{BBLP}^1_\lambda(\mu)\right)| \subseteq |S(\mathrm{TOLP})|$. ◁

From Corollaries 5.10 and 5.15, we observe that we can find a minimal solution set for the problem BBLP^1 by solving either of the two cases i. e. $\mathrm{BBLP}^1_0(\lambda)$ or $\mathrm{BBLP}^1_0(\mu)$ for all values of λ and μ, respectively. Consequently, a minimal solution set of BBLP^1 can be obtained by computing a minimal solution set of TOLP.

However, we are also interested in finding the critical regions $\mathcal{R}^1(x)$, i. e. for a solution $x \in S(\mathrm{TOLP})$, we find the set of all combinations of the parameters (λ, μ) such that x is efficient for $\mathrm{BBLP}^1(\lambda, \mu)$ whenever $(\lambda, \mu) \in \mathcal{R}^1(x)$. In order to find these critical regions we use the weight set of the BOLP $\mathrm{BBLP}^1(\lambda, \mu)$ and its connection to the weight set of the corresponding TOLP.

Using the fact that the weight set of $\mathrm{BBLP}^1(\lambda, \mu)$ for a fixed μ and a fixed λ is a subset of the weight set of the corresponding TOLP, we can decompose the parameter set for the parametric problem into critical regions that correspond to a solution $x \in S(\mathrm{TOLP})$.

5.19 Definition. Given a weight set component $\mathcal{W}(y) \subset \mathcal{W}(\mathrm{TOLP})$ and its set of extreme weights $E_{\mathcal{W}(y)}$. We define two subsets of $E_{\mathcal{W}(y)}$ as follows:

$$\mathcal{A}_1(y) = E_{\mathcal{W}(y)} \cap \{w^* : \nexists\ \bar{w} \in \mathcal{W}(y) \text{ such that } (w_1^*, w_2^*) \leq (\bar{w}_1, \bar{w}_2)\}$$
$$\mathcal{A}_2(y) = E_{\mathcal{W}(y)} \cap \{w^* : \nexists\ \bar{w} \in \mathcal{W}(y) \text{ such that } (w_1^*, w_2^*) \geq (\bar{w}_1, \bar{w}_2)\}\,. \quad ◁$$

We can visualize the elements of sets $\mathcal{A}_1(y)$ and $\mathcal{A}_2(y)$ in Figure 5.4.

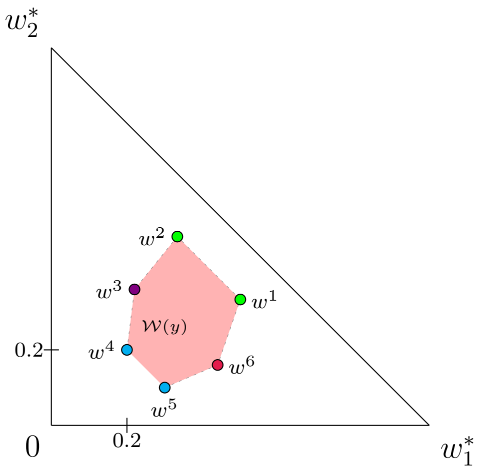
In order to explain the structure of the critical region of a solution we first define line segments in the parameter set using Equation 5.1.

5.20 Definition. Let w^* be a weight in $\mathcal{W}(\mathrm{TOLP})$. The *line segment* in the parameter set corresponding to the weight w^* is defined as:

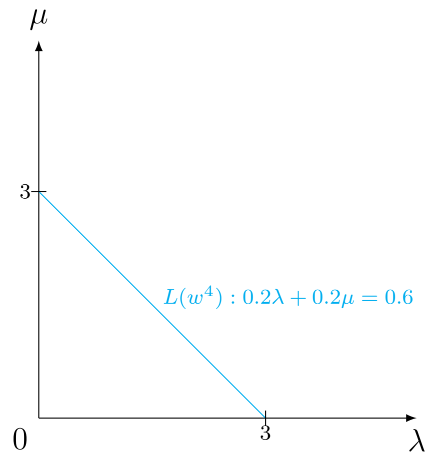
$$L(w^*) := \left\{(\lambda, \mu) \in \mathbb{R}^2_{\geqq} : w_1^*\lambda + w_2^*\mu = w_3^*\right\}. \quad ◁$$

It is a line segment because of Equations (5.3) and (5.4). Essentially, a weight in $\mathcal{W}(\mathrm{TOLP})$ corresponds to two parameters in $\mathrm{BBLP}^1(\lambda, \mu)$. This implies that a weight $(w_1^*, w_2^*, w_3^*) \in \mathcal{W}(\mathrm{TOLP})$ corresponds to a line segment in the parameter set of BBLP^1 as shown in Figure 5.4b.

We now define the parameter region that will correspond to a solution in a minimal solution set.



(a) The extreme weights of $\mathcal{W}(y)$ marked in green and blue belongs to sets $\mathcal{A}_1(y)$ and $\mathcal{A}_2(y)$, respectively.



(b) The line segment $L(w^4)$ in the parameter set corresponds to the weight $w^4 \in \mathcal{W}(y)$ in Figure 5.4a.

Figure 5.4: An illustration of sets $\mathcal{A}_1(y) = \{w^1, w^2\}$ and $\mathcal{A}_2(y) = \{w^4, w^5\}$ of a weight set component $\mathcal{W}(y)$ and a visualization of the line segment $L(w^4)$ in the parameter set that corresponds to the weight $w^4 = (0.2, 0, 2, 0.6) \in \mathcal{W}(y)$.

5.21 Definition. Let $x \in S(\text{TOLP})$ be a solution and $y = (c_1x, c_2x, d_1x)^\top \in Y_{\text{EN}}$ be its corresponding extreme nondominated image in TOLP. The *critical region* of x for the parametric problem BBLP^1 is defined as:

$$\mathcal{R}^1(x) := \{(\lambda, \mu) : (\lambda, \mu) \in L(w^*), w^* \in \mathcal{W}(y)\}. \qquad \triangleleft$$

In other words, it is a set of all parameter combinations $(\lambda, \mu) \in \mathbb{R}^2_{\geqq}$ for which x remains optimal for BBLP^1.

Our goal is to describe the critical region, so we show that the critical region is between the lower and upper envelope of piecewise linear functions. Hereby, we define the two functions as follows.

5.22 Definition. The *lower envelope* of a critical region is the minimum of a finite collection of piecewise linear functions, i. e.

$$E_\ell(y) := \max\{\mu : \exists\, w^* \in \mathcal{A}_1(y) : (\lambda, \mu) \in L(w^*) \text{ for some } \lambda \geq 0\}$$

where $\mathcal{A}_1(y) \subset E_{\mathcal{W}(y)}$. Similarly, the *upper envelope* of a critical region is the maximum of a finite collection of piecewise linear functions, i. e.

$$E_u(y) := \min\{\mu : \exists\, w^* \in \mathcal{A}_2(y) : (\lambda, \mu) \in L(w^*) \text{ for some } \lambda \geq 0\}$$

where $\mathcal{A}_2(y) \subset E_{\mathcal{W}(y)}$. $\triangleleft$

5.23 Lemma. Let weight w' be a convex combination of any two weights $w_1, w_2 \in \mathcal{A}_1(y)$. The corresponding line segments of weights w', w_1 and w_2 in the parameter set are $L(w')$, $L(w_1)$, and $L(w_2)$, respectively. Then for a fixed $\lambda^* \geq 0$ there exists some $\mu \in E_\ell(y)$ such that

$$\mu' \geq \mu. \qquad \triangleleft$$

Proof. Given that the weight w' is a convex combination of any two weights $w_1, w_2 \in \mathcal{A}_1(y)$,

$$w' = \alpha w^1 + (1-\alpha) w^2$$

for some $\alpha \in [0,1]$ and $w^1, w^2 \in \mathcal{A}_1(y)$. More precisely,

$$w' = \left(\alpha w_1^1 + (1-\alpha) w_1^2, \alpha w_2^1 + (1-\alpha) w_2^2, \alpha w_3^1 + (1-\alpha) w_3^2\right). \tag{5.7}$$

The corresponding line segments $L(w')$, $L(w^1)$, and $L(w^2)$ of weights w', w^1 and w^2, respectively, are

$$\begin{aligned} L(w')&\colon w_1'\lambda + w_2'\mu = w_3', \\ L(w_1)&\colon w_1^1\lambda + w_2^1\mu = w_3^1 \text{ and} \\ L(w_2)&\colon w_1^2\lambda + w_2^2\mu = w_3^2 \end{aligned}$$

such that both the line segments $L(w^1)$ and $L(w^2)$ are part of the lower envelope in the parameter set.
For a fixed λ^*, the corresponding μ values in $L(w')$, $L(w^1)$ and $L(w^2)$ are

$$\begin{aligned} \mu' &= \frac{w_3' - w_1'\lambda^*}{w_2'} \geq 0, \\ \mu^1 &= \frac{w_3^1 - w_1^1\lambda^*}{w_2^1} \geq 0, \text{and} \\ \mu^2 &= \frac{w_3^2 - w_1^2\lambda^*}{w_2^2} \geq 0 \end{aligned}$$

respectively. Substituting w' from Equation (5.7) in μ', we get

$$\mu' = \frac{(\alpha w_3^1 + (1-\alpha) w_3^2) - (\alpha w_1^1 + (1-\alpha) w_1^2)\,\lambda^*}{\alpha w_2^1 + (1-\alpha) w_2^2}.$$

This is equivalent to

$$\mu' = \frac{\alpha\,(w_3^1 - w_1^1\lambda^*) + (1-\alpha)\,(w_3^2 - w_1^2\lambda^*)}{\alpha w_2^1 + (1-\alpha) w_2^2}.$$

For $\alpha \in [0,1]$, it holds that

$$\frac{w_3^2 - w_1^2\lambda^*}{w_2^2} \leq \frac{\alpha\,(w_3^1 - w_1^1\lambda^*) + (1-\alpha)\,(w_3^2 - w_1^2\lambda^*)}{\alpha w_2^1 + (1-\alpha) w_2^2} \leq \frac{w_3^1 - w_1^1\lambda^*}{w_2^1}.$$

This implies $\mu^2 \geq \mu' \geq \mu^1$ where $\mu^1 \in E_\ell(y)$. □

5.24 Lemma. Let weight w' be a convex combination of any two weights $w_1, w_2 \in \mathcal{A}_2(y)$. The corresponding line segments of weights w', w_1 and w_2 are $L(w')$, $L(w^1)$, and $L(w^2)$, respectively. Then for a fixed λ^* there exists some $\mu \in E_u(y)$ such that

$$\mu' \leq \mu. \qquad \triangleleft$$

Proof. The proof is analogous to the proof of Lemma 5.23. □

5.25 Theorem. A region $\mathcal{R}^1(x) \in \mathbb{R}^2_{\geqq}$ is a critical region of $x \in S(\text{TOLP})$ and $y = (c_1x, c_2x, d_1x)^\top \in Y_{\text{EN}}(\text{TOLP})$ if and only if it is equivalent to the area between the lower envelope $E_\ell(y)$ and the upper envelope $E_u(y)$ i. e.

$$\mathcal{R}^1(x) = \left\{(\lambda, \mu) \in \mathbb{R}^2_{\geqq} : w^* \in \mathcal{W}(y), E_\ell(y) \leq \mu \leq E_u(y)\right\}. \qquad \triangleleft$$

Proof. We prove this theorem in two parts; first, if a point is an element of $\mathcal{R}^1(x)$ then it lies between the two envelopes and second, if a point lies between the two envelopes then it is in $\mathcal{R}^1(x)$.

First, we show that a point in the critical region lies above the lower envelope and later we show that it lies below the upper envelope.

Let $(\lambda^*, \mu^*) \in \mathcal{R}^1(x)$. There exist a weight $w^* \in \mathcal{W}(y)$ such that $w_1^*\lambda^* + w_2^*\mu^* = w_3^*$. Given the line segment $L(w^*)\colon w_1^*\lambda^* + w_2^*\mu^* = w_3^*$, for a fixed λ^* it holds that

$$\mu^* = \frac{w_3^* - w_1^*\lambda^*}{w_2^*} \geq 0 \tag{5.8}$$

where $w_2^* > 0$.
Since $w^* \in \mathcal{W}(y)$, there exists a weight $w' \geq w^*$ such that w' is a convex combination of two weights $w^1, w^2 \in \mathcal{A}_1(y)$. The corresponding line segment $L(w')$ of weight w' is $L(w')\colon w_1'\lambda + w_2'\mu = w_3'$ and for a fixed λ^* in $L(w')$, we get

$$\mu' = \frac{w_3' - w_1'\lambda^*}{w_2'} \geq 0.$$

Therefore, the following component-wise comparison between w' and w^* holds:

$$\begin{aligned} w_1' &\geq w_1^* \\ w_2' &\geq w_2^* \\ w_3' &\leq w_3^*. \end{aligned}$$

We substitute $w_3^* = 1 - w_1^* - w_2^*$ in μ^* to get

$$\mu^* = \frac{1 - w_2^* - w_1^*(1 + \lambda^*)}{w_2^*} \geq \frac{1 - w_2' - w_1'(1 + \lambda^*)}{w_2'} = \mu'$$

since $w'_1 \geq w^*_1$ and $w'_2 \geq w^*_2$. From Lemma 5.23, we have $\mu' \geq \mu_\ell$ such that $\mu_\ell \in E_\ell(y)$ is a point in the lower envelope. Therefore,

$$\mu^* \geq \mu' \geq \mu_\ell.$$

Similarly, since $w^* \in \mathcal{W}(y)$ there exists a weight $w'' \leq w^*$ such that w'' is a convex combination of two weights $w^1, w^2 \in \mathcal{A}_2(y)$. The corresponding line segment $L(w'')$ of weight w'' is $L(w'')$: $w''_1\lambda + w''_2\mu = w''_3$ and for a fixed λ^* in $L(w'')$, we get

$$\mu'' = \frac{w''_3 - w''_1\lambda^*}{w''_2} \geq 0.$$

Therefore, the following component-wise comparison between w'' and w^* holds:

$$\begin{aligned} w''_1 &\leq w^*_1 \\ w''_2 &\leq w^*_2 \\ w''_3 &\geq w^*_3. \end{aligned}$$

We substitute $w^*_3 = 1 - w^*_1 - w^*_2$ in μ^* to get

$$\mu^* = \frac{1 - w^*_2 - w^*_1(1+\lambda^*)}{w^*_2} \leq \frac{1 - w''_2 - w''_1(1+\lambda^*)}{w''_2} = \mu''$$

since $w''_1 \leq w^*_1$ and $w''_2 \leq w^*_2$. From Lemma 5.24, we have $\mu'' \leq \mu_u$ such that $\mu_u \in E_u(y)$ is a point in the upper envelope. Therefore,

$$\mu^* \leq \mu'' \leq \mu_u.$$

Conversely, given a point (λ^*, μ^*) that lies in between the two envelopes. Then, there exists some $\mu' \in E_\ell(y)$ and $\mu'' \in E_u(y)$ such that

$$\mu' \leq \mu^* \leq \mu'' \tag{5.9}$$

for some $\lambda \geq 0$. By the definition of the two envelopes, there exists some $w' \in \mathcal{A}_1(y)$ and $w'' \in \mathcal{A}_2(y)$.

Due to Equation (5.9), there is some w^* such that

$$w' \geq w^* \geq w''.$$

Therefore, $w^* \in \mathcal{W}(y)$ due to convexity.

Suppose, for contradiction, $(\lambda^*, \mu^*) \notin \mathcal{R}^1(x)$ then there is some weight $w^* \in \mathcal{W}(\text{TOLP})$ such that the line segment $\mathcal{L}_{\mathcal{W}(\text{BBLP}^1(\lambda^*,\mu^*))} : w^*_1(1+\lambda^*) + w^*_2(1+\mu^*) = 1$ does not pass through $\mathcal{W}(y)$. As a result the weight $w^* \notin \mathcal{W}(y)$.

This leads to contradiction that $w^* \in \mathcal{W}(y)$. □

In other words, we can use a subset of extreme weights of a weight set component in $\mathcal{W}(\text{TOLP})$ to determine the critical region of a solution. Because this subset of extreme weights determine the lower envelope and the upper envelope of piecewise linear functions of a critical region in the parameter set. We visualize a weight set component and the corresponding critical region of the solution in Figures 5.5 and 5.6 respectively.

From the results above, we can observe that the number of line segments required to define a critical region is bounded by the number of vertices of the weight set component.

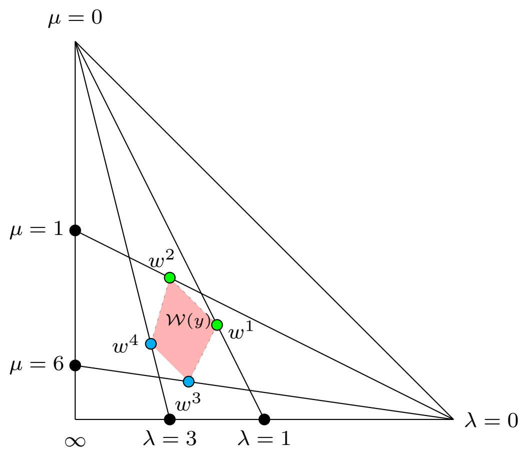


Figure 5.5: An illustration of a weight set component $\mathcal{W}(y)$ and the corresponding boundary points of the parameter interval $\mathcal{I}^1(x)$ with respect to problems PBLP^1 and PBLP^1_2. Extreme weights of $\mathcal{W}(y)$ marked in green and blue are in sets $\mathcal{A}_1(y)$ and $\mathcal{A}_2(y)$, respectively.

5.26 Corollary. If the sets $\mathcal{A}_1(y)$ and $\mathcal{A}_2(y)$ of $\mathcal{W}(y)$ are singleton sets then critical region $\mathcal{R}^1(x)$ is convex. ◁

Proof. Given, $\mathcal{A}_1(y) = \{w'\}$ and $\mathcal{A}_2(y) = \{w''\}$. Then,

$$E_\ell(y) = \{\mu : (\lambda, \mu) \in L(w')\}\,,$$
$$E_u(y) = \{\mu : (\lambda, \mu) \in L(w'')\}\,.$$

Then critical region is determined by the line segments $L(w')$ and $L(w'')$ such that .

$$\mathcal{R}^1(x) = \left\{(\lambda, \mu) \in \mathbb{R}^2_{\geqq} \mid \frac{w'_3 - w'_1\lambda}{w''_2} \leq \mu \leq \frac{w''_3 - w''_1\lambda}{w''_2}\right\}.$$

Since the intersection of two half spaces is convex, $\mathcal{R}^1(x)$ is convex. □

Although in general, critical regions for solutions of BBLP^1 are not convex.

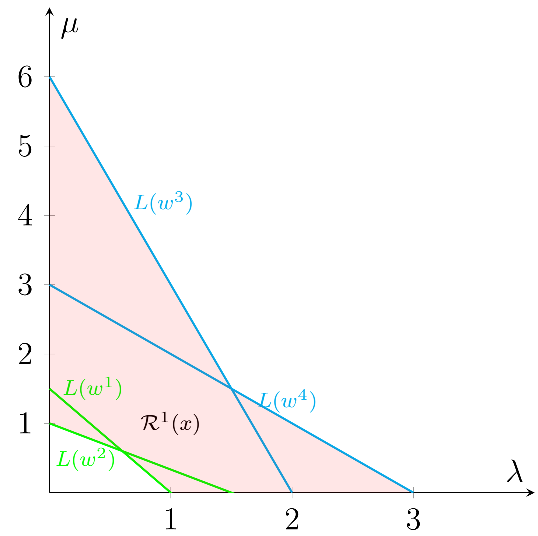


Figure 5.6: An illustration of the line segments $L(w^1)$, $L(w^2)$, $L(w^3)$ and $L(w^4)$ in the parameter set corresponding to extreme weights w^1, w^2, w^3, w^4 in Figure 5.5. The line segments in green and blue makes up the upper envelope and lower envelope of $\mathcal{R}^1(x)$, respectively.

Next, we illustrate the critical regions of a solution set of the biparametric biobjective linear program for Example 3.4. Consider the following BBLP[1] with non-negative parameters $\lambda \geq 0$ and $\mu \geq 0$:

$$\begin{aligned} \min \quad & \begin{pmatrix} -3x_1 - x_2 + \lambda(x_1 + x_2) \\ x_1 - 2x_2 + \mu(x_1 + x_2) \end{pmatrix} \\ \text{s. t.} \quad & x \in X := \left\{ x \in \mathbb{R}^2_{\geqq} : 3x_1 + 2x_2 \geq 6; x_1 \leq 10; x_2 \leq 3 \right\}, \end{aligned}$$

and the corresponding TOLP which is related to the biparametric biobjective problem:

$$\begin{aligned} \min \quad & (-3x_1 - x_2, x_1 - 2x_2, x_1 + x_2)^\top \\ \text{s. t.} \quad & x \in X. \end{aligned}$$

The weight set decomposition of the TOLP is composed of weight set components $\mathcal{W}(y^i)$, $i = 1, \dots 4$. There is a corresponding minimal solution set $S(\text{TOLP})$ to $Y_{\text{EN}}(\text{TOLP})$. The parameter intervals for $x \in S(\text{TOLP})$ in case of $\text{BBLP}^1_\lambda(0)$ and $\text{BBLP}^1_\mu(0)$ are shown in Figure 5.7. The critical regions of corresponding solutions in a minimal solution set $S(\text{TOLP})$ is shown in Figure 5.8.

5.2 Case II : Different parametric objectives

In this section we will focus on biobjective linear problems with two different parameters.

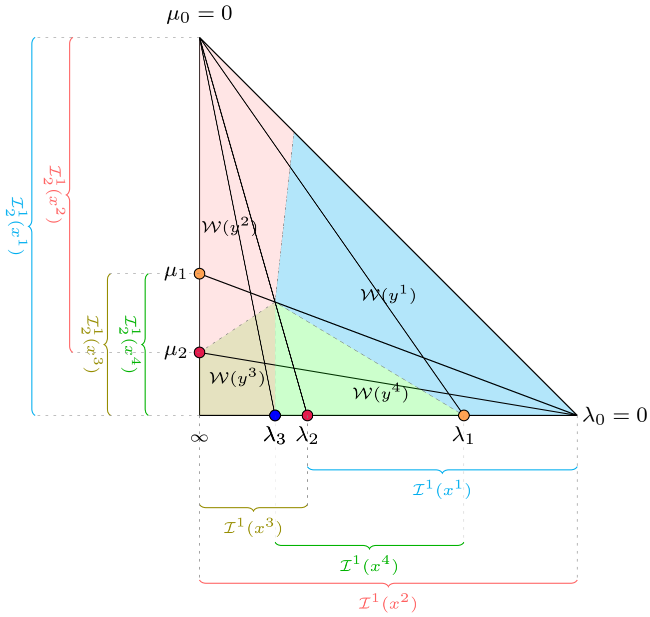


Figure 5.7: The weight set of TOLP with parameter intervals for every $x \in S(\text{TOLP})$ with respect to $\text{BBLP}_0^1(\lambda)$ as well as $\text{BBLP}_0^1(\mu)$.

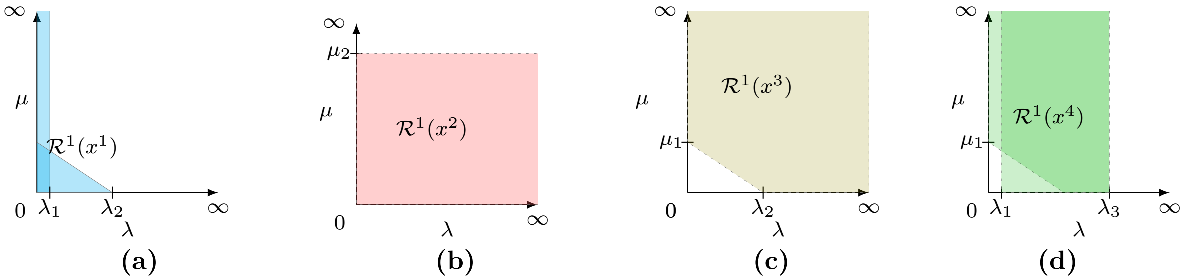


Figure 5.8: The critical regions for each solution in $S(\text{TOLP}) \coloneqq \{x^1, x^2, x^3, x^4\}$ with respect to BBLP^1. Note, that in the critical region $\mathcal{R}^1(x^1)$, parameter interval for λ is $[0, \lambda_2]$ when $\mu = 0$ which is also denoted as $\mathcal{I}^1(x^1)$ in Figure 5.7. All the critical regions are unbounded for this example because the weight set components in $\mathcal{W}(\text{TOLP})$ contains weights with either $w_1^* = 0$ or $w_2^* = 0$.

The problem BBLP^2, for some fixed λ and μ, is a non-parametric biobjective linear program and we denote it by $\text{BBLP}^2(\lambda,\mu)$. We denote the set of extreme nondominated images of $\text{BBLP}^2(\lambda,\mu)$ by $Y_{\text{EN}}\big(\text{BBLP}^2(\lambda,\mu)\big)$ and its corresponding minimal solution set by $S(\text{BBLP}^2(\lambda,\mu))$.

Our analysis relates to a minimal solution set that corresponds to extreme nondominated images of $\text{BBLP}^2(\lambda,\mu)$. Therefore, we are interested in identifying a union of minimal solution sets of $\text{BBLP}^2(\lambda,\mu)$ for every possible combinations of λ and μ. We denote it by $S(\text{BBLP}^2)$.

5.27 Definition. A *solution set* $S(\text{BBLP}^2) \subseteq X$ of BBLP^2 is a set such that for a fixed $\lambda \geq 0$ and a fixed $\mu \geq 0$, $S(\text{BBLP}^2)$ contains, as a subset, a minimal solution set for the BOLP $\text{BBLP}^2(\lambda,\mu)$. It is called *minimal* if, additionally, there is no other solution set $S'(\text{BBLP}^2) \subseteq X$ for BBLP^2 with $\left|S'(\text{BBLP}^2)\right| < \left|S(\text{BBLP}^2)\right|$. ◁

We will relate BBLP^2 to the corresponding four-objective linear program defined as 4-OLP in Chapter 4. Hereby, we consider the four-objective linear problem with the same objective functions c_1x, c_2x, d_1x, and d_2x. The set of extreme nondominated images of 4-OLP is denoted by $Y_{\text{EN}}(\text{4-OLP})$ and a minimal solution set of 4-OLP is denoted by S_4. We use the weight set $\mathcal{W}(\text{4-OLP})$ of 4-OLP and its weighted sum scalarization with normalized weight $w^* \in \mathcal{W}(\text{4-OLP})$ is $\text{WS}(\text{4-OLP}, w^*)$.

We use the weighted sum scalarization of the parametric biobjective linear program BBLP^2 and formally characterize its relation to the corresponding 4-OLP in our results.

For a fixed value of λ and μ, the weighted sum scalarization of BBLP^2 is

$$\min_{x\in X} w_1(c_1x + \lambda d_1x) + w_2(c_2x + \mu d_2x) \qquad (\text{WS}(\text{BBLP}^2(\lambda,\mu), w))$$

where $w \in \mathbb{R}^2_{\geqq}$ and $w_1 + w_2 = 1$. We reformulate this problem and obtain

$$\min_{x\in X} w_1c_1x + w_1\lambda d_1x + w_2c_2x + w_2\mu d_2x.$$

The problem can be interpreted as a weighted sum scalarization problem of the 4-OLP. It holds that,

$$w_1 + w_2 + w_1\lambda + w_2\mu = 1 + w_1\lambda + w_2\mu \geq 1.$$

We normalize the weights and get

$$\min_{x\in X} \frac{w_1c_1x}{1+w_1\lambda+w_2\mu} + \frac{w_2c_2x}{1+w_1\lambda+w_2\mu} + \frac{w_1\lambda d_1x}{1+w_1\lambda+w_2\mu} + \frac{w_2\mu d_2x}{1+w_1\lambda+w_2\mu}. \qquad (5.10)$$

This is a special case of the weighted sum scalarization of 4-OLP with normalized weight vector

$$w^* = \left(\frac{1}{1 + w_1\lambda + w_2\mu}(w_1, w_2, w_1\lambda, w_2\mu)\right)^\top \in \mathcal{W}(\text{4-OLP}). \tag{5.11}$$

We now establish the relation between the efficient solutions of the problem BBLP^2 and the efficient solutions of the corresponding 4-OLP is formalized in the following theorem.

5.28 Theorem. A feasible solution x^* is optimal for WS(4-OLP, w^*) with non-negative weight w^* where $w_1^* > 0$ and $w_2^* > 0$ if and only if there exists parameters $\lambda, \mu \geq 0$ and non-negative weights w_1, w_2 where $w_1 > 0$ and $w_2 > 0$ such that x^* is optimal for WS(BBLP$^2(\lambda, \mu), w$). ◁

Proof. We first show that x^* is optimal for WS(4-OLP, w^*) implying that there exists $\lambda, \mu \geq 0$ and weights $w_1 > 0$, $w_2 > 0$ such that x^* is optimal for WS$\left(\text{BBLP}^2(\lambda, \mu), w\right)$. Let x^* be optimal for WS(4-OLP, w^*).
Note that $w_1^* > 0$ and $w_2^* > 0$ and we define

$$\begin{aligned} \lambda &:= \frac{w_3^*}{w_1^*} \\ \mu &:= \frac{w_4^*}{w_2^*} \\ w_1 &:= \frac{w_1^*}{w_1^* + w_2^*} \\ w_2 &:= \frac{w_2^*}{w_1^* + w_2^*}. \end{aligned} \tag{5.12}$$

Suppose x^* is not optimal for WS$\left(\text{BBLP}^2(\lambda, \mu), w\right)$. Then there exists some x' which is feasible for WS$\left(\text{BBLP}^2(\lambda, \mu), w\right)$ where $x' \neq x^*$ such that,

$$w_1(c_1x' + \lambda d_1x') + w_2(c_2x' + \mu d_2x') < w_1(c_1x^* + \lambda d_1x^*) + w_2(c_2x^* + \mu d_2x^*).$$

We plug in Equation (5.12) to get

$$\begin{aligned} &\frac{w_1^*}{w_1^* + w_2^*}\left(c_1x' + \frac{w_3^*}{w_1^*}d_1x'\right) + \frac{w_2^*}{w_1^* + w_2^*}\left(c_2x' + \frac{w_4^*}{w_2^*}d_2x'\right) \\ &\qquad < \frac{w_1^*}{w_1^* + w_2^*}\left(c_1x^* + \frac{w_3^*}{w_1^*}\, d_1x^*\right) + \frac{w_2^*}{w_1^* + w_2^*}\left(c_2x^* + \frac{w_4^*}{w_2^*}d_2x^*\right). \end{aligned}$$

This is equivalent to

$$w_1^*c_1x' + w_2^*c_2x' + w_3^*d_1x' + w_4^*d_2x' < w_1^*c_1x^* + w_2^*c_2x^* + w_3^*d_1x^* + w_4^*d_2x^*.$$

This leads to a contradiction that x^* is optimal for $\text{WS}(\text{4-OLP}, w^*)$.
Conversely, let x^* be optimal for $\text{WS}\left(\text{BBLP}^2(\lambda, \mu), w\right)$ with non-negative weights w_1, w_2 and for some non-negative λ and μ.
We define $w_1^*, \ldots, w_4^*$ using Equation (5.11),

$$\begin{aligned} w_1^* &:= w_1 \\ w_2^* &:= w_2 \\ w_3^* &:= w_1\lambda \\ w_4^* &:= w_2\mu \end{aligned} \tag{5.13}$$

Suppose x^* is not optimal for $\text{WS}(\text{4-OLP}, w^*)$. Then there exists x' that is feasible for $\text{WS}(\text{4-OLP}, w^*)$ where $x' \neq x^*$ such that

$$w_1^* c_1 x' + w_2^* c_2 x' + w_3^* d_1 x' + w_4^* d_2 x' < w_1^* c_1 x^* + w_2^* c_2 x^* + w_3^* d_1 x^*.$$

We plug in Equation (5.13) to get

$$w_1 c_1 x' + w_2 c_2 x' + w_1 \lambda d_1 x' + w_2 \mu d_2 x' < w_1 c_1 x^* + w_2 c_2 x^* + w_1 \lambda d_1 x^* + w_2 \mu d_2 x^*.$$

This is equivalent to

$$w_1(c_1 x' + \lambda d_1 x') + w_2(c_2 x' + \mu d_2 x') < w_1(c_1 x^* + \lambda d_1 x^*) + w_2(c_2 x^* + \mu d_2 x^*).$$

This leads to a contradiction that x^* is optimal for $\text{WS}\left(\text{BBLP}^2(\lambda, \mu), w\right)$. □

Observe that we assume $w_1^* > 0$ and $w_2^* > 0$ in Theorem 5.28. Based on the construction in the proof, the parameters λ and μ are undefined if $w_1^* = 0$ and $w_2^* = 0$ respectively. In fact w_1^* and w_2^* approaching 0 is equivalent to the parameters λ and μ approaching ∞ respectively. And indeed, the one-to-one correspondence from Theorem 5.28 does not hold:

- there exist $w^* \in \mathbb{R}^4_{\geqq}$ with $w_1^* = 0$ and an optimal solution x^* for $\text{WS}(\text{4-OLP}, w^*)$ such that there is no $\lambda \geq 0$ and $w \in \mathbb{R}^2_{\geqq}$ where x^* is optimal for $\text{WS}(\text{BBLP}^2(\lambda, \mu), w)$,
- there exist $w^* \in \mathbb{R}^4_{\geqq}$ with $w_2^* = 0$ and an optimal solution x^* for $\text{WS}(\text{4-OLP}, w^*)$ such that there is no $\mu \geq 0$ and $w \in \mathbb{R}^2_{\geqq}$ where x^* is optimal for $\text{WS}(\text{BBLP}^2(\lambda, \mu), w)$.

This can be illustrated by the following example.

5.29 Example. Let $X \subseteq \mathbb{R}^4$ and $X = \text{conv}(\{(1,1,1,1), (0,0,1,1), (0,0,0,1)\})$. Let $c_1 = (1,0,0,0), c_2 = (0,1,0,0), d_1 = (0,0,1,0), d_2 = (0,0,0,1)$.
Then for $x = (x_1, x_2, x_3, x_4) \in X$, it holds that $c_1 x = x_1$, $c_2 x = x_2$, $d_1 x = x_3$ and $d_2 x = x_4$ and are the objective functions of 4-OLP.
Let $w^* = (0,0,0,1)$, then all three solutions $(1,1,1,1), (0,1,1,1)$ and $(0,0,0,1)$ are optimal for $\text{WS}(\text{4-OLP}, w^*)$ with a value of 1. However, consider now the problem $\text{WS}\left(\text{BBLP}^2(\lambda, \mu), w\right)$. For every value of $\lambda \geq 0$, $\mu \geq 0$ and $w \in \mathbb{R}^2_{\geqq}$ with $w_1 > 0$,

the solution $(0,0,0,1)$ is a better than $(0,0,1,1)$ and $(1,1,1,1)$ because the value of $\mathrm{WS}(\mathrm{BBLP}^2(\lambda,\mu),w)$ for $(0,0,0,1)$ is

$$w_1((0)+(\lambda)(0))+w_2((0)+\mu(1))=w_2\mu$$

which is less than the values for $(0,0,1,1)$ and $(1,1,1,1)$, i. e.

$$w_1((0)+(\lambda)(1))+w_2((0)+\mu(1))=w_1\lambda+w_2\mu$$

and

$$w_1((1)+\lambda(1))+w_2(1)+w_3(1)=w_1(1+\lambda)+w_2(1+\mu),$$

respectively. Additionally, for $w_1=0$, it holds $w_2=1$ and the solutions $(0,0,0,1)$ and $(0,0,1,1)$ are optimal for $\mathrm{WS}(\mathrm{BBLP}^2(\lambda,\mu),w)$ with a value of μ while the solution $(1,1,1,1)$ have a value of $1+\mu$. Therefore, the solution $(1,1,1,1)$ is never optimal for $\mathrm{WS}(\mathrm{BBLP}^2(\lambda,\mu),w)$. ◁

In the converse case, for any given parameter value $\lambda\geq 0$ and $\mu\geq 0$ and a weight $w\in\mathbb{R}^2_{\geqq}$ with $w_1=0$ or $w_2=0$ for $\mathrm{WS}(\mathrm{BBLP}^2(\lambda,\mu),w)$, we can see that it is possible to construct a weight $w^*\in\mathbb{R}^4_{\geq}$ for $\mathrm{WS}(4\text{-OLP},w^*)$ using Equation (5.13). In fact, an optimal solution x for $\mathrm{WS}\left(\mathrm{BBLP}^2(\lambda,\mu),w\right)$ with a weight $w\in\mathbb{R}^2_{\geqq}$ such that $w_1=0$ remains optimal for $\mathrm{WS}(4\text{-OLP},w^*)$ with the weight $w^*=(0,w_2,0,w_2\mu)$ because $w_1^*=0$ and $w_3^*=0$ by construction. And an optimal solution x for $\mathrm{WS}\left(\mathrm{BBLP}^2(\lambda,\mu),w\right)$ with a weight $w\in\mathbb{R}^2_{\geqq}$ such that $w_2=0$ remains optimal for $\mathrm{WS}(4\text{-OLP},w^*)$ with the weight $w^*=(w_1,0,w_1\lambda,0)$ because $w_2^*=0$ and $w_4^*=0$ by construction.

Here again, for every fixed pair of (λ,μ), $\mathrm{BBLP}^2(\lambda,\mu)$ is a biobjective linear program. Its weight set is therefore a one-dimensional polytope. We embed the one-dimensional weight set of $\mathrm{BBLP}^2(\lambda,\mu)$ in the three dimensional weight set $\mathcal{W}(4\text{-OLP})$ by using the mapping $\mathcal{W}(\mathrm{BBLP}^2(\lambda,\mu))$.

5.30 Definition. For a fixed λ and a fixed μ, we define a *mapping* $\mathcal{W}\left(\mathrm{BBLP}^2(\lambda,\mu)\right)$,

$$\mathcal{W}\left(\mathrm{BBLP}^2(\lambda,\mu)\right):=\left\{\frac{1}{1+w_1\lambda+w_2\mu}\left(w_1,w_2,w_1\lambda,w_2\mu\right):w\in\mathbb{R}^2_{\geq},w_1+w_2=1\right\}.$$

We now examine the structure of the weight sets of $\mathrm{BBLP}^2(\lambda,\mu)$ when one of the parameter is fixed and the other parameter is varying within the weight set of 4-OLP.

5.31 Proposition. Let $\mathcal{W}\left(\mathrm{BBLP}^2(\lambda,\mu)\right)$ and $\mathcal{W}(4\text{-OLP})$ be weight sets of BBLP^2 for fixed λ,μ and the corresponding 4-OLP, respectively. It holds that

$$\mathcal{W}\left(\mathrm{BBLP}^2(\lambda,\mu)\right)\subsetneq\mathcal{W}(4\text{-OLP}).$$ ◁

Proof. Let $w^* \in \mathcal{W}\left(\mathrm{BBLP}^2(\lambda,\mu)\right)$. Then, by the definition of $\mathcal{W}\left(\mathrm{BBLP}^2(\lambda,\mu)\right)$,

$$w^* = \frac{1}{1 + w_1\lambda + w_2\mu}\left(w_1, w_2, w_1\lambda, w_2\mu\right)$$

for some $w \in \mathbb{R}^2_{\geqq}$ with $w_1 + w_2 = 1$. The sum of the components satisfy

$$\frac{w_1}{1 + w_1\lambda + w_2\mu} + \frac{w_2}{1 + w_1\lambda + w_2\mu} + \frac{w_1\lambda}{1 + w_1\lambda + w_2\mu} + \frac{w_2\mu}{1 + w_1\lambda + w_2\mu} = 1.$$

and thus $w^* \in \mathcal{W}(\text{4-OLP})$.
To prove that $\mathcal{W}(\mathrm{BBLP}^2(\lambda,\mu))$ is a proper subset of $\mathcal{W}(\text{4-OLP})$, it suffices to note that the weight $w' := (0,0,0,1)$ lies in $\mathcal{W}(\text{4-OLP})$ but not in $\mathcal{W}(\mathrm{BBLP}^2(\lambda,\mu))$. Since for any given λ and μ, the fourth component of any $w^* \in \mathcal{W}\left(\mathrm{BBLP}^2(\lambda,\mu)\right)$ satisfies

$$\frac{w_2\mu}{1 + w_1\lambda + w_2\mu} \neq 1,$$

therefore $w' \notin \mathcal{W}\left(\mathrm{BBLP}(\lambda,\mu)\right)$. □

5.32 Proposition. Let $\mathcal{W}\left(\mathrm{BBLP}^2(\lambda,\mu)\right)$ be the weight set of $\mathrm{BBLP}^2(\lambda,\mu)$ and a weight $w^* = \left(\frac{1}{1+\lambda^*}, 0, \frac{\lambda^*}{1+\lambda^*}, 0\right) \in \mathcal{W}(\text{4-OLP})$. Then, for a fixed $\lambda^* > 0$

$$\bigcap_{\mu \geq 0} \mathcal{W}(\mathrm{BBLP}^2(\lambda^*,\mu)) = \{w^*\}\,. \qquad \triangleleft$$

Proof. For an arbitrary $\mu_i \geq 0$, by the definition of $\mathcal{W}\left(\mathrm{BBLP}^2(\lambda^*,\mu_i)\right)$, we have

$$\mathcal{W}\left(\mathrm{BBLP}^2(\lambda^*,\mu_i)\right) := \left\{\frac{1}{1 + w_1\lambda^* + w_2\mu_i}\left(w_1, w_2, w_1\lambda^*, w_2\mu_i\right) : w \in \mathbb{R}^2_{\geq}, w_1 + w_2 = 1\right\}.$$

We first show that the weight $w^* = \left(\frac{1}{1+\lambda^*}, 0, \frac{\lambda^*}{1+\lambda^*}\right)$ is an element of $\mathcal{W}\left(\mathrm{BBLP}^2(\lambda^*,\mu_i)\right)$ for all μ_i. Since $w_2^* = 0$ and $w_4^* = 0$ in the weight w^*, we have

$$\begin{aligned} w^* &= \left(\frac{1}{1 + \lambda^*(1) + \mu_i(0)}, \frac{0}{1 + \lambda^*(1) + \mu_i(0)}, \frac{(1)\lambda^*}{1 + \lambda^*(1) + \mu_i(0)}, \frac{0}{1 + \lambda^*(1) + \mu_i(0)}\right) \\ &= \left(\frac{1}{1 + \lambda^*(1) + \mu_i(0)}(1, 0, \lambda^*, 0)\right). \end{aligned}$$

Therefore, $w^* \in \mathcal{W}\left(\mathrm{BBLP}^2(\lambda^*,\mu_i)\right)$.
Next we show that there is no common weight shared by any two weight sets of $\mathcal{W}\left(\mathrm{BBLP}^2(\lambda^*,\mu_i)\right)$.

Let $\mu_1 \geq 0$ and $\mu_2 \geq 0$ be two arbitrary parameter values, such that $\mu_1 \neq \mu_2$. Let $\mathcal{W}\big(\text{BBLP}^2(\lambda^*, \mu_1)\big)$ and $\mathcal{W}\big(\text{BBLP}^2(\lambda^*, \mu_2)\big)$ be weight sets of $\text{BBLP}^2(\lambda^*, \mu_1)$ and $\text{BBLP}^2(\lambda^*, \mu_2)$, respectively. Consider a weight $\hat{w} \in \mathcal{W}\big(\text{BBLP}^2(\lambda^*, \mu_1)\big)$ and a weight $\tilde{w} \in \mathcal{W}\big(\text{BBLP}^2(\lambda^*, \mu_2)\big)$. By the definition of the weight sets, we have

$$\hat{w} = \left(\frac{1}{1 + w_1\lambda^* + w_2\mu_1}\,(w_1', w_2', w_1'\lambda^*, w_2'\mu_1)\right) \text{ where } w_1' + w_2' = 1$$

and

$$\tilde{w} = \left(\frac{1}{1 + w_1\lambda^* + w_2\mu_2}\,(w_1'', w_2'', w_1''\lambda^*, w_2''\mu_1)\right) \text{ where } w_1'' + w_2'' = 1.$$

If we compare the first components of $\hat{w}$ and $\tilde{w}$, i. e. $\frac{w_1'}{1+w_1'\lambda^*+w_2'\mu_1}$ and $\frac{w_1''}{1+w_1''\lambda^*+w_2''\mu_2}$, then the components are equal if

$$w_1' = \frac{w_1''}{1 + w_1''\lambda^* + w_2''\mu_2}(1 + w_1'\lambda^* + w_2'\mu_1). \tag{5.14}$$

Simultaneously, the third component of $\hat{w}$ is

$$\hat{w}_3 = \frac{w_1'\lambda^*}{1 + w_1'\lambda^* + w_2'\mu_1}.$$

We plug w_1' from Equation (5.14) in w_3^* to get

$$\begin{aligned}
\hat{w}_3 &= \frac{\frac{w_1''}{1+w_1''\lambda^*+w_2''\mu_2}(1 + w_1'\lambda^* + w_2'\mu_1)}{1 + w_1'\lambda^* + w_2'\mu_1}\lambda_1 \\
&= \frac{w_1''\lambda_1}{1 + w_1''\lambda^* + w_2''\mu_2} \\
&\neq \frac{w_1''\lambda_2}{1 + w_1''\lambda^* + w_2''\mu_2} = \tilde{w}_3
\end{aligned}$$

because $\mu_1 \neq \mu_2$. □

Hence, the weight $w^* = \left(\frac{1}{1+\lambda^*}, 0, \frac{\lambda}{1+\lambda^*}, 0\right)$ is the only weight shared by all the weight sets of $\text{BBLP}^2(\lambda^*, \mu)$ for all parameter values $\mu \geq 0$. And all these shared weights of $\bigcap\limits_{\mu_i \geq 0} \mathcal{W}(\text{BBLP}^2(\lambda^*, \mu_i))$ for all $\lambda \geq 0$ make up the line segment marked in green in Figure 5.9.

Similarly, we show that the weight $\left(0, \frac{1}{1+\mu^*}, 0, \frac{\mu^*}{1+\mu^*}\right)$ is the only weight shared by the weight sets of $\text{BBLP}^2(\lambda, \mu)$ for a fixed μ^* and varying λ.

5.33 Proposition. Let $\mathcal{W}\left(\text{BBLP}^2(\lambda,\mu)\right)$ be the weight set of $\text{BBLP}^2(\lambda,\mu)$ and a weight $w^* = \left(0, \frac{1}{1+\mu^*}, 0, \frac{\mu^*}{1+\mu^*}\right) \in \mathcal{W}(\text{4-OLP})$. Then, for a fixed $\mu^* > 0$

$$\bigcap_{\lambda \geq 0} \mathcal{W}(\text{BBLP}^2(\lambda,\mu^*)) = \{w^*\}\,. \qquad \triangleleft$$

Proof. The proof is analogous to the proof of Proposition 5.32. Here, we first show that the weight $w^* = \left(0, \frac{1}{1+\mu^*}, 0, \frac{\mu^*}{1+\mu^*}\right)$ is the only weight shared by the weight sets of $\mathcal{W}\left(\text{BBLP}^2(\lambda,\mu^*)\right)$ for all $\lambda \geq 0$ with a fixed μ^*. □

All the shared weights from $\bigcap_{\lambda\geq 0} \mathcal{W}(\text{BBLP}^2(\lambda,\mu^*))$ for all $\mu \geq 0$ make up the line segment (w_2^*- axis) highlighted in blue in Figure 5.9.

5.34 Proposition. For the union of all the weight sets $\mathcal{W}(\text{BBLP}^2(\lambda,\mu))$ for all $(\lambda,\mu) \in \mathbb{R}^2_{\geqq}$ it holds that:

(i) $\{w^* \in \mathcal{W}(\text{4-OLP}) : w_1^* + w_3^* = 1 \text{ and } w_3^* \neq 1\} \subset \bigcup_{\mu\geq 0}\bigcup_{\lambda\geq 0} \mathcal{W}\left(\text{BBLP}^2(\lambda,\mu)\right)$

(ii) $\{w^* \in \mathcal{W}(\text{4-OLP}) : w_1^* = 0, w_3^* = 0 \text{ and } w_2^* \neq 0\} \subset \bigcup_{\mu\geq 0}\bigcup_{\lambda\geq 0} \mathcal{W}\left(\text{BBLP}^2(\lambda,\mu)\right)$. ◁

Proof. The proof for Propositions 5.34.(i) and 5.34.(ii) follows from Propositions 5.32 and 5.33, respectively. □

5.35 Proposition. For two pairs of parameter values (λ_1,μ_1) and (λ_2,μ_2), if $\lambda_1 \neq \lambda_2$ and $\mu_1 \neq \mu_2$

$$\mathcal{W}\left(\text{BBLP}^2(\lambda_1,\mu_1)\right) \cap \mathcal{W}\left(\text{BBLP}^2(\lambda_2,\mu_2)\right) = \varnothing. \qquad \triangleleft$$

Proof. The proof is analogous to the proof of Proposition 3.7. □

5.36 Corollary. Let $\mathcal{W}\left(\text{BBLP}^2(\lambda,\mu)\right)$ and $\mathcal{W}(\text{4-OLP})$ be the weight sets of $\text{BBLP}^2(\lambda)$ and the corresponding 4-OLP, respectively. Then the union of the family of weight sets for all $(\lambda,\mu) \in \mathbb{R}^2_{\geqq}$ is defined as:

$$\begin{aligned}\bigcup_{\mu\geq 0}\bigcup_{\lambda\geq 0} \mathcal{W}\left(\text{BBLP}^2(\lambda,\mu)\right) =& \{w^* \in \mathcal{W}(\text{4-OLP}) : w_1^* \neq 0, w_2^* \neq 0\}\\ &\cup \{w^* \in \mathcal{W}(\text{4-OLP}) : w_1^* = 0, w_3^* = 0\}\\ &\cup \{w^* \in \mathcal{W}(\text{4-OLP}) : w_1^* + w_3^* = 1\}. \qquad \triangleleft\end{aligned}$$

Proof. From Proposition 5.33, the weight set of $\text{BBLP}^2(\lambda, \mu)$ for every pair $(\lambda, \mu) \in \mathbb{R}^2_{\geqq}$ is a subset of $\mathcal{W}$(4-OLP), so the union of all these weight sets will also be a subset of $\mathcal{W}$(TOLP). Using Theorem 5.28, there exists some λ, some μ and $w \in \mathbb{R}^2_{\geqq}$ for $\text{BBLP}^2(\lambda, \mu)$ for all weights $w^* \in \mathcal{W}$(4-OLP) where $w_1^* > 0$ and $w_2^* > 0$. However, by Proposition 5.34 the weights in $\mathcal{W}$(4-OLP) that satisfy $w_1^* + w_3^* = 1$ and $w_1^* = 0, w_3^* = 0$, respectively are in the union of all weight sets of $\text{BBLP}^2(\lambda, \mu)$. □

Corollary 5.36 implies that if we vary λ and μ in $\text{BBLP}^2(\lambda, \mu)$ and project all the corresponding weight sets $\mathcal{W}\left(\text{BBLP}^2(\lambda, \mu)\right)$ in $\mathbb{R}^3$, we obtain a nearly complete weight set decomposition of the associated 4-OLP except for the weights with $w_1^* = 0$ and $w_2^* = 0$ due to the reason mentioned in Theorem 5.28. And since $w_1^* = 0$ and $w_2^* = 0$ results in $w_3^* = 0$ and $w_4^* = 0$, respectively, we also miss out the facets of $\mathcal{W}$(4-OLP) on the $w_1^* - w_3^*$ plane and $w_2^* - w_3^*$ plane except for the w_2^*-axis and the line segment $w_1^* + w_3^* = 1$.

Next, we want to characterize the projection of $\mathcal{W}(\text{BBLP}^2(\lambda, \mu))$ in $\mathbb{R}^3$. Since the third and fourth components of a weight in $\mathcal{W}(\text{BBLP}^2(\lambda, \mu))$ can be determined by the first and second components, respectively, the projection reduces the entire weight set to a 3-dimensional interpretation. Using the definition of $\mathcal{W}\left(\text{BBLP}^2(\lambda, \mu)\right)$ (cf. 4.4) for some $(\lambda, \mu) \in \mathbb{R}^2_{\geqq}0$, the third component of any arbitrary weight $w^* \in \mathcal{W}\left(\text{BBLP}^2(\lambda, \mu)\right)$ is

$$w_3^* = w_1^* \lambda.$$

Since $w_3^* = 1 - (w_1^* + w_2^* + w_4^*)$, we get

$$1 - (w_1^* + w_2^* + w_4^*) = w_1^* \lambda.$$

Using $w_4^* = \mu w_2^*$, this is equivalent to

$$1 - w_1^* - w_2^* - \mu w_2^* = w_1^* \lambda.$$

Therefore

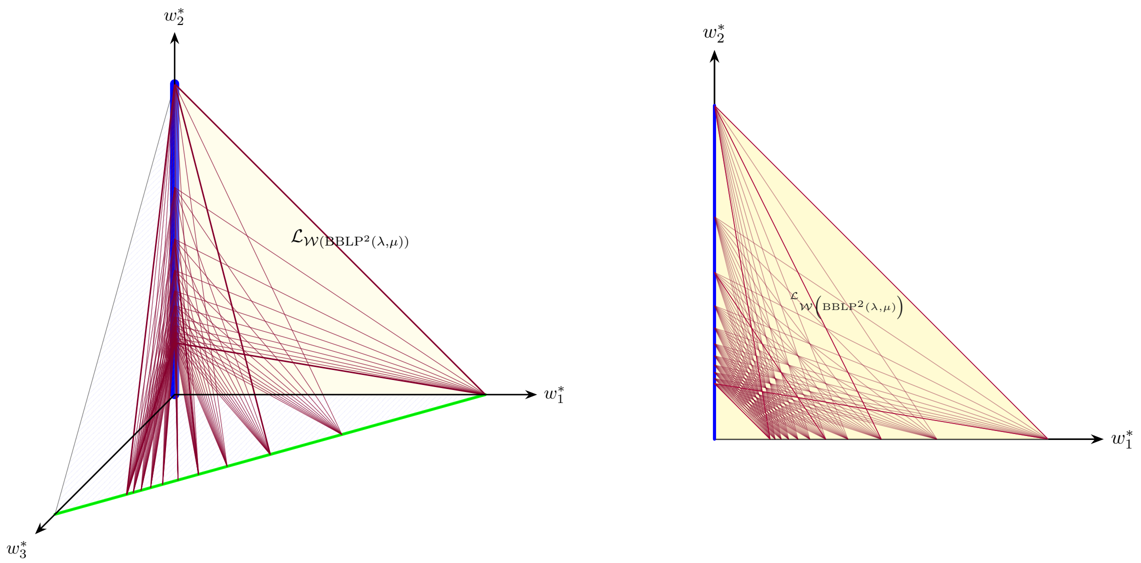
$$(1 + \lambda) w_1^* + (1 + \mu) w_2^* = 1.$$

This means every weight in $\mathcal{W}\left(\text{BBLP}^2(\lambda, \mu)\right)$ which is contained in $\mathcal{W}$(4-OLP) satisfies the condition $(1 + \lambda) w_1^* + (1 + \mu) w_2^* = 1$. This condition can be interpreted as an equation of a line in $\mathbb{R}^3$. Consequently, we can define the projection of weight set $\mathcal{W}\left(\text{BBLP}^2(\lambda, \mu)\right)$ in $\mathbb{R}^3$ as a line segment.

5.37 Definition. The *projection of the weight set* $\mathcal{W}\left(\text{BBLP}^2(\lambda, \mu)\right)$ in $\mathbb{R}^3_{\geqq}$ can be defined as the line segment,

$$\mathcal{L}_{\mathcal{W}\left(\text{BBLP}^2(\lambda,\mu)\right)} := \{(w_1^*, w_2^*, w_3^*) \mid w^* \in \mathcal{W}(\text{4-OLP}),\ (1 + \lambda) w_1^* + (1 + \mu) w_2^* = 1\}$$

embedded in $\mathcal{W}$(4-OLP). ◁



(a) A family of line segments representing weight sets of $\mathrm{BBLP}^2(\lambda,\mu)$ in a $\mathcal{W}$(4-OLP)

(b) View of the line segments from the direction of w_3^* axis and towards $w_1^* - w_2^*$ plane

Figure 5.9: An illustration of weight sets of $\mathrm{BBLP}^2(\lambda,\mu)$ on the weight set of 4-OLP with varying λ and μ. The facets of $\mathcal{W}$(4-OLP) marked with blue dashed lines are not part of the weight sets of $\mathrm{BBLP}^2(\lambda,\mu)$ except for the line segment $w_1^* + w_3^* = 1$ (due to Proposition 5.32) and the w_2^*-axis (due to Proposition 5.33) marked in green and blue, respectively.

The end points of the line segment $\mathcal{L}_{\mathcal{W}\left(\mathrm{BBLP}^2(\lambda,\mu)\right)}$ are $\left(0, \frac{1}{1+\mu}, 0\right)$ and $\left(\frac{1}{1+\lambda}, 0, \frac{\lambda}{1+\lambda}\right)$.

The line segments representing the weight sets of $\mathrm{BBLP}^2(\lambda,\mu)$ for varying λ and μ are illustrated in Figure 5.9.

As illustrated in Figure 5.9 the union of weight sets of $\mathrm{BBLP}^2(\lambda,\mu)$ for all λ and μ is indeed a subset of the weight set of the corresponding 4-OLP. Just like the case of BBLP^1, if we vary λ and μ in $\mathrm{BBLP}^2(\lambda,\mu)$, we get a near complete knowledge of $\mathcal{W}$(4-OLP).

5.2.1 Further results and illustrative examples

Every solution set S_4 for 4-OLP consists of solutions such that there is a bijection between S_4 and the extreme nondominated images Y_{EN}(4-OLP). At the same time, the weight set $\mathcal{W}$(4-OLP) can be decomposed into full dimensional weight set components such that there is a bijection between these components and Y_{EN}(4-OLP) [PGE10]. By Definition 5.37, for fixed values of $\lambda \geq 0$ and $\mu \geq 0$, the weight set $\mathcal{W}\left(\mathrm{BBLP}^2(\lambda,\mu)\right)$ is a line segment that is embedded in $\mathcal{W}$(4-OLP). This line segment intersects some (full-dimensional) weight set components. For each component $\mathcal{W}(y)$, there are four possibilities: $\mathcal{W}\left(\mathrm{BBLP}^2(\lambda,\mu)\right)$ intersects $\mathcal{W}(y)$ either in a single vertex, along an edge, along a facet or it passes through the interior of $\mathcal{W}(y)$. The entire segment $\mathcal{W}\left(\mathrm{BBLP}^2(\lambda,\mu)\right)$ can then be decomposed into such intersections (cf. [PGE10]). A solution set for $\mathrm{BBLP}^2(\lambda,\mu)$ can be obtained by using all solutions from S_4 where the corresponding weight set component is intersected. Therefore, S_4 is also a solution set for BBLP^2.

At the same time, Theorem 5.28 implies that, for every full dimensional weight set component $\mathcal{W}(y)$ of an extreme nondominated image $y \in Y_{\mathrm{EN}}(4\text{-OLP})$, there is at least one value $\lambda \geq 0$ and one value $\mu \geq 0$ such that the line segment $\mathcal{L}_{\mathcal{W}\left(\mathrm{BBLP}^2(\lambda,\mu)\right)}$ intersects the interior of $\mathcal{W}(y)$. Then, a solution set for BOLP $\mathrm{BBLP}^2(\lambda,\mu)$ must contain at least one solution x such that $(c_1x, c_2x, d_1x, d_2x)^\top = y$ [PGE10]. Therefore, every solution set $S(\mathrm{BBLP}^2)$ for the biparametric problem BBLP^2 also contains a solution set for 4-OLP.

Since every solution set for 4-OLP contains a solution set for BBLP^2, we can make the following statement regarding minimal solution sets:

5.38 Proposition. A set $S_4 \subseteq X$ is a minimal solution set for 4-OLP if and only if it is a minimal solution set for BBLP^2. ◁

Next, we are interested in finding regions in the parameter set which corresponds to each of the solutions in a minimal solution set. The idea is to find this region by using the weight set components of the $\mathcal{W}(4\text{-OLP})$.

5.39 Definition. Let $x \in S_4$ be a solution and $y \in Y_{\mathrm{EN}}(4\text{-OLP})$ be its corresponding extreme nondominated image. The *critical region* of x, denoted $\mathcal{R}^2(x)$ is defined as:

$$\mathcal{R}^2(x) \coloneqq \left\{(\lambda,\mu) \in \mathbb{R}^2_{\geqq} : \lambda = \frac{w_3^*}{w_1^*},\ \mu = \frac{w_4^*}{w_2^*},\ w^* \in \mathcal{W}(y)\right\}. \qquad \triangleleft$$

In other words, $\mathcal{R}^2(x)$ contains all combination of parameters (λ,μ) for which $x \in S_4$ remains optimal for $\mathrm{BBLP}^2(\lambda,\mu)$.

5.40 Proposition. A critical region $\mathcal{R}^2(x)$ is path connected. ◁

Proof. Let $\pi : \mathcal{W}(y) \to \mathcal{R}^2(x)$ be a mapping such that $\pi(w^*) = \left(\frac{w_3^*}{w_1^*}, \frac{w_4^*}{w_2^*}\right)$ where $w^* \in \mathcal{W}(y)$. Since a weight set component $\mathcal{W}(y)$ is a polytope, it is path-connected. By the definition of $\mathcal{R}^2(x)$, we know the mapping π is continuous. If we take any two points, $w', w'' \in \mathcal{W}(y)$ such that $\pi(w') = r'$, $\pi(w'') = r''$, then since $\mathcal{W}(y)$ is path-connected, there exists a continuous path $p : [0,1] \to \mathcal{W}(y)$ such that $p(0) = w'$ and $p(1) = w''$. Then, $\pi \circ p : [0,1] \to \mathcal{R}^2(x)$ is a continuous path from r' to r'' in $\mathcal{R}^2(x)$. Therefore, $\mathcal{R}^2(x)$ is path-connected.

We further note that the critical region for the solutions in BBLP^1 is path-connected because it is a region between two envelopes of linear functions. This follows from the result that the critical region is precisely the set bounded by two envelopes of linear functions, which consequently forms a connected set. Next, we show an illustration of the critical regions for solutions in a minimal solution set of the biparametric biobjective linear programs for an instance.

5.41 Example. Consider the following linear BBLP^2 with a non-negative parameters $\lambda, \mu \geq 0$:

$$\begin{aligned} \min \quad & \begin{pmatrix} x_1 + \lambda x_3 \\ x_2 + \mu x_4 \end{pmatrix} \\ \text{s. t.} \quad & x \in X, \end{aligned}$$

where the row vectors c_1, c_2, d_1 and d_2 are the unit vectors e^i and $i = 1, \ldots, 4$, respectively and the set $X \coloneqq \operatorname{conv}\{x^1, x^2, x^3, x^4\} + \mathbb{R}^n_{\geqq}$ is the feasible set in $\mathbb{R}^n_{\geqq}$. The corresponding 4-OLP which is related to biparametric biobjective linear program is:

$$\begin{aligned} \min \quad & (x_1, x_2, x_3, x_4)^\top \\ \text{s. t.} \quad & x \in X. \end{aligned} \qquad \triangleleft$$

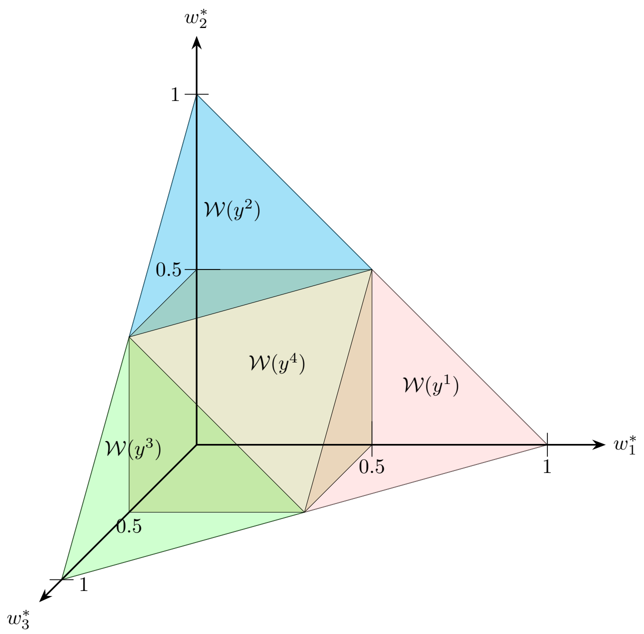
Given an instance of $Y_{\text{EN}}(\text{4-OLP}) \coloneqq \{y^1, y^2, y^3, y^4\}$ for Example 5.41 and it's corresponding minimal solution set $S_4 \coloneqq \{x^1, x^2, x^3, x^4\}$. The weight set decomposition of the 4-OLP is composed of four weight set components, each corresponding to the extreme nondominated images y^1, y^2, y^3, and y^4 as shown in Figure 5.10. Since the weight set of the example is simple, we can use Definition 5.39 to determine the critical region where we map the vertices and edges of each weight set component of $\mathcal{W}(\text{4-OLP})$ on the parameter set in $\mathbb{R}^2$. We find the corresponding parameter values using $\lambda = \frac{w_3^*}{w_1^*}$ and $\mu = \frac{w_4^*}{w_2^*}$. Thus, the critical regions for solutions in a minimal solution set S_4 with respect to BBLP^2 is illustrated in Figure 5.11.

However, when the weight set of a 4-OLP has weight set components with several facets or more then the critical regions corresponding to the solution gets complicated.

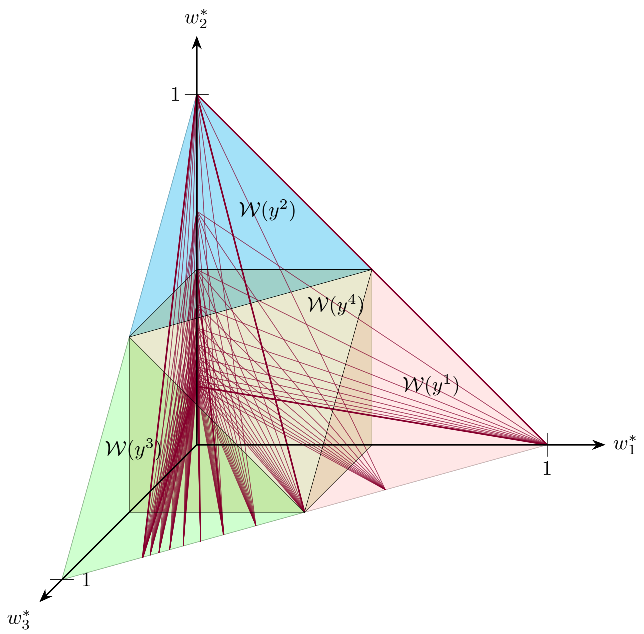
5.3 Multi-parametric biobjective linear program

The analysis of the biparametric problem BBLP^2 provides the basis for extending the problem to a broader class of multi-parametric biobjective programs, characterized by varying parametric dependencies and differently parameterized objectives.

We consider a multi-parametric biobjective linear program with k different non-negative parameters in the first objective and l different non-negative parameters in the second



(a) Weight set decomposition for the 4-OLP instance, which is composed of four weight set components corresponding to $Y_{\text{EN}} := \{y_1, y_2, y_3, y_4\}$.



(b) Weight sets of the BOLP $\text{BBLP}^2(\lambda, \mu)$ for varying combinations of λ and μ in the $\mathcal{W}$(4-OLP)

Figure 5.10: An illustration of the weight sets of $\text{BBLP}^2(\lambda, \mu)$ in the weight set of a 4-OLP with respect to Example 5.41.

objective where $c_1, c_2, d_1, \dots d_{k+l}$, are the objective coefficients,

$$\min_{x \in X} \begin{pmatrix} c_1 x + \lambda_1 d_1 x + \dots + \lambda_k d_k x \\ c_2 x + \lambda_{k+1} d_{k+1} x + \dots + \lambda_{k+l} d_{k+l} x \end{pmatrix}$$

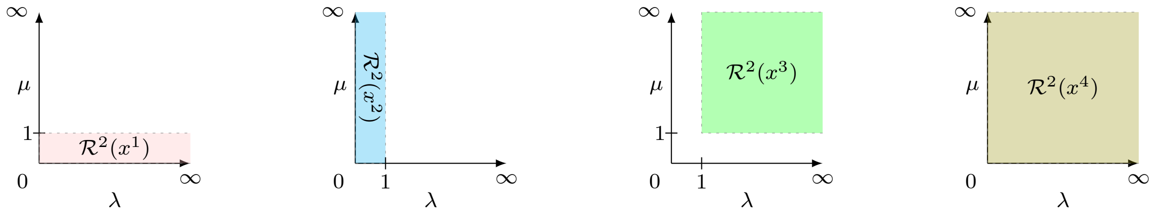


Figure 5.11: Critical regions for each solution in $S_4 \coloneqq \{x^1, x^2, x^3, x^4\}$ with respect to BBLP^2

can be rewritten as

$$\min_{x \in X} \begin{pmatrix} c_1 x + \sum_{i=1}^{k} \lambda_i d_i x \\ c_2 x + \sum_{j=k+1}^{k+l} \lambda_j d_j x \end{pmatrix}. \tag{MBLP}$$

The problem MBLP, for some fixed parameters λ_i where $i = 1, \ldots, k + l$, is a non-parametric biobjective linear program, and we denote it by MBLP(λ). Furthermore, we denote the set of extreme non-dominated images of MBLP(λ) by Y_{EN}(MBLP(λ)) and a corresponding minimal solution set by S(MBLP(λ)).

We use the same approach as in the previous case, by relating MBLP to the corresponding $(k + l + 2)$-objective linear program using the weighted sum scalarization. To this end, we consider the $(k + l + 2)$-objective linear problem with the objective functions c_1x, c_2x, and d_ix where $i = 1, \ldots, k + l$, i. e.

$$\begin{aligned} \min \quad & (c_1 x, c_2 x, d_1 x, \ldots, d_{k+l} x)^\top \\ \text{s. t.} \quad & x \in X, \end{aligned} \tag{$(k+l+2)$-OLP}$$

and denote its set of extreme nondominated images by $Y_{\text{EN}}((k+l+2)\text{-OLP})$. We define the weight set $\mathcal{W}((k + l + 2)\text{-OLP})$ of $(k + l + 2)$-OLP as

$$\mathcal{W}((k + l + 2)\text{-OLP}) \coloneqq \left\{ w^* \in \mathbb{R}^{k+l+2}_{\geqq} : \sum_{i=1}^{k+l+2} w_i^* = 1 \right\}.$$

The weighted sum scalarization of $(k+l+2)$-OLP with normalized weights $w_1^*, \ldots, w_{k+l+2}^*$,

$$\min_{x \in X} w_1^* c_1 x + w_2^* c_2 x + \sum_{i=3}^{k+l+2} w_i^* d_{i-2} x. \tag{WS($(k+l+2)$-OLP, w^*)}$$

We now apply the weighted sum scalarization to the parametric problem MBLP to relate it to corresponding $(k + l + 2)$-OLP.

The weighted sum scalarization of MBLP(λ) is

$$\min_{x \in X} w_1 \left(c_1 x + \sum_{i=1}^{k} \lambda_i d_i x \right) + w_2 \left(c_2 x + \sum_{j=k+1}^{k+l} \lambda_j d_j x \right) \tag{WS(MBLP(λ), w)}$$

where $w \in \mathbb{R}^2_{\geq}$ and $w_1 + w_2 = 1$. We reformulate this problem and obtain

$$\min_{x \in X} w_1 c_1 x + w_1 \sum_{i=1}^{k} \lambda_i d_i x + w_2 c_2 x + w_2 \sum_{j=k+1}^{k+l} \lambda_j d_j x.$$

The problem can be interpreted as a weighted sum scalarization of $(k+l+2)$-OLP. It holds that

$$w_1 + w_2 + w_1 \sum_{i=1}^{k} \lambda_i + w_2 \sum_{j=k+1}^{k+l} \lambda_j = 1 + w_1 \sum_{i=1}^{k} \lambda_i + w_2 \sum_{j=k+1}^{k+l} \lambda_j \geq 1.$$

We normalize the weights and get

$$\min_{x \in X} \frac{w_1}{\mathcal{S}} c_1 x + \frac{w_2}{\mathcal{S}} c_2 x + \frac{w_1}{\mathcal{S}} \sum_{i=1}^{k} \lambda_i d_i x + \frac{w_2}{\mathcal{S}} \sum_{j=k+1}^{k+l} \lambda_j d_j x$$

where $\mathcal{S} = 1 + w_1 \sum_{i=1}^{k} \lambda_i + w_2 \sum_{j=k+1}^{k+l} \lambda_j$, which is a special case of WS($(k+l+2)$-OLP$, w^*$) with corresponding weights

$$w^* := \left(\frac{w_1}{\mathcal{S}}, \frac{w_2}{\mathcal{S}}, \frac{w_1 \lambda_1}{\mathcal{S}}, \ldots, \frac{w_1 \lambda_k}{\mathcal{S}}, \frac{w_2 \lambda_{k+1}}{\mathcal{S}}, \ldots, \frac{w_2 \lambda_{k+l}}{\mathcal{S}} \right). \tag{5.15}$$

We now present an important result on the equivalence of the efficient solutions of the problem MBLP(λ) and the efficient solutions of $(k+l+2)$-OLP.

5.42 Theorem. A feasible solution x^* is optimal for WS($(k+l+2)$-OLP$, w^*$) with non-negative weights $w_1^*, \ldots, w_{k+l+2}^*$, where $w_1^*, w_2^* > 0$ if and only if there exist parameters $\lambda_1, \ldots \lambda_{k+l} \geq 0$ and and non-negative weights w_1, w_2 where $w_1, w_2 > 0$ such that x^* is optimal for WS(MBLP(λ)$, w$). ◁

Proof. We first show that x^* is optimal for WS($(k+l+2)$-OLP$, w^*$), implying that there exist $\lambda_1, \ldots \lambda_{k+l} \geq 0$ and non-negative weights w_1, w_2 such that x^* is optimal for WS(MBLP(λ)$, w$). Let x^* be optimal for WS($(k+l+2)$-OLP$, w^*$).
Note that $w_1^* > 0$ and $w_2^* > 0$ and we define

$$\begin{aligned} w_1 &:= w_1^* \\ w_2 &:= w_2^* \\ \lambda_1 &:= \frac{w_3^*}{w_1^*} \\ &\vdots \\ \lambda_k &:= \frac{w_{k+2}^*}{w_1^*} \\ \lambda_{k+1} &:= \frac{w_{k+3}^*}{w_2^*} \end{aligned} \tag{5.16}$$

$$\vdots$$
$$\lambda_{k+l} \coloneqq \frac{w^*_{k+l+2}}{w^*_2}.$$

Suppose x^* is not optimal for WS(MBLP(λ), w). Then there exists some x' that is feasible for WS(MBLP(λ), w) where $x' \neq x^*$ such that

$$w_1(c_1x' + \sum_{i=1}^{k} \lambda_i d_i x') + w_2(c_2x' + \sum_{j=k+1}^{k+l} \lambda_j d_j x') < w_1(c_1x + \sum_{i=1}^{k} \lambda_i d_i x^*) + w_2(c_2x + \sum_{j=k+1}^{k+l} \lambda_j d_j x^*).$$

We plug in Equation (5.16) to get

$$w^*_1(c_1x' + \frac{w^*_3}{w^*_1}d_1x' + \dots + \frac{w^*_{k+2}}{w^*_1}d_kx') + w^*_2(c_2x' + \frac{w^*_{k+3}}{w^*_2}d_{k+1}x' + \dots + \frac{w^*_{k+l+2}}{w^*_2}d_{k+l}x')$$
$$< w^*_1(c_1x^* + \frac{w^*_3}{w^*_1}d_1x^* + \dots + \frac{w^*_{k+2}}{w^*_1}d_kx^*) + w^*_2(c_2x^* + \frac{w^*_{k+3}}{w^*_2}d_{k+1}x^* + \dots + \frac{w^*_{k+l+2}}{w^*_2}d_{k+l}x^*).$$

This leads to

$$w^*_1c_1x' + w^*_2c_2x' + \sum_{i=3}^{k+l+2} w^*_i d_{i-2}x' < w^*_1c_1x^* + w^*_2c_2x^* + \sum_{i=3}^{k+l+2} w^*_i d_{i-2}x^*.$$

This leads to a contradiction that x^* is optimal for WS($(k+l+2)$-OLP, w^*).
Conversely, let x^* be optimal for WS(MBLP(λ), w), with non-negative weights w_1, w_2 and for some non-negative $\lambda_1, \dots, \lambda_{k+l}$.
We define $w^*_1, \dots, w^*_{k+l+2}$ using Equation (5.15),

$$\begin{aligned} w^*_1 &\coloneqq w_1 \\ w^*_2 &\coloneqq w_2 \\ w^*_3 &\coloneqq w_1\lambda_1 \\ &\vdots \\ w^*_{k+2} &\coloneqq w_1\lambda_k \\ w^*_{k+3} &\coloneqq w_2\lambda_{k+1} \\ &\vdots \\ w^*_{k+l+2} &\coloneqq w_2\lambda_{k+l}. \end{aligned} \tag{5.17}$$

Suppose x^* is not optimal for WS($(k+l+2)$-OLP, w^*), i.e. there exists x' which is optimal for $\text{WS}((k+l+2)\text{-OLP}, w^*)$ where $x' \neq x^*$ such that,

$$w_1^* c_1 x' + w_2^* c_2 x' + \sum_{i=3}^{k+l+2} w_i^* d_{i-2} x' < w_1^* c_1 x^* + w_2^* c_2 x^* + \sum_{i=3}^{k+l+2} w_i^* d_{i-2} x^*.$$

We plug in Equation (5.17) to get

$$w_1 c_1 x' + w_2 c_2 x' + w_1 \lambda_1 d_1 x' + \cdots + w_1 \lambda_k d_k x' + w_2 \lambda_{k+1} d_{k+1} x' + \cdots + w_2 \lambda_{k+l} d_{k+l} x'$$

$$< w_1 c_1 x^* + w_2 c_2 x^* + w_1 \lambda_1 d_1 x^* + \cdots + w_1 \lambda_k d_k x^* + w_2 \lambda_{k+1} d_{k+1} x^* + \cdots + w_2 \lambda_{k+l} d_{k+l} x^*.$$

This is equivalent to

$$w_1 c_1 x' + w_1 \sum_{i=1}^{k} \lambda_i d_i x' + w_2 c_2 x + w_2 \sum_{j=k+1}^{k+l} \lambda_j d_j x'$$

$$< w_1 c_1 x^* + w_1 \sum_{i=1}^{k} \lambda_i d_i x^* + w_2 c_2 x^* + w_2 \sum_{j=k+1}^{k+l} \lambda_j d_j x^*.$$

This leads to a contradiction that x^* is optimal for $\text{WS}(\text{MBLP}(\lambda), w)$. □

We can also extend the results on the weight set of the biparametric biobjective linear program to the multi-parametric biobjective linear program using similar arguments. As the weight set for a multi-parametric biobjective linear program for a fixed parameter vector is always one dimensional, we observe that the union of these weight sets for every combination of the parameter vectors will obtain almost the entire weight set of the associated multi-objective linear program. We have already shown this for the biparametric biobjective problem. Using the same argument as in Proposition 5.38 and Theorem 5.42 we can make the following statement for general multi-parametric biobjective linear programs.

5.43 Proposition. A set $S_{k+l+2} \subseteq X$ is a minimal solution set for (k+l+2)-OLP if and only if S_{k+l+2} is a minimal solution set for MBLP. ◁

This means we can obtain a minimal solution set of multi-parametric biobjective linear program by computing a minimal solution set of the corresponding MOLP. Similarly, we define a critical region for every solution in the minimal solution set of a MBLP as follows.

5.44 Definition. Let $x \in S_{k+l+2}$ be a solution and $y = (c_1 x, c_2 x, d_1 x, \ldots, d_{k+l} x)^\top \in Y_{\text{EN}}((k+l+2)\text{-OLP})$ be its corresponding extreme nondominated image. The *critical* *region* of x, denoted $\mathcal{R}^{k+l}(x)$ is defined as:

$$\mathcal{R}^{k+l}(x) := \Bigg\{(\lambda_1,\dots,\lambda_{k+l}) \in \mathbb{R}^{k+l}_{\geqq} : \lambda_1 = \frac{w_3^*}{w_1^*},\ \dots,\ \lambda_k = \frac{w_{k+2}^*}{w_1^*},$$
$$\lambda_{k+1} = \frac{w_{k+3}^*}{w_2^*},\ \dots,\ \lambda_{k+l} = \frac{w_{k+l+2}^*}{w_2^*},\ w^* \in \mathcal{W}(y)\Bigg\}. \quad \triangleleft$$

In other words, $\mathcal{R}^{k+l}(x)$ contains all combinations of parameter vectors $(\lambda_1,\dots,\lambda_{k+l})$ for which $x \in S_{k+l+2}$ remains efficient for MBLP$(\lambda_1,\dots,\lambda_{k+l})$.

However, critical regions are more complicated in a higher dimensional parameter set. Therefore, a structural inference of a critical region corresponding to the weight set component of MOLP is difficult to be established.

Chapter 6

Algorithms for breakpoints and critical regions

In this chapter, we propose algorithms to solve the parametric problems described in Chapters 3 and 5. As shown in Proposition 3.22, and Corollaries 5.10 and 5.15, a minimal solution set of PBLP^j and BBLP^1 can be obtained by computing a minimal solution set of TOLP. Similarly, from Proposition 5.38 a minimal solution set of BBLP^2 can be obtained by computing a minimal solution set of 4-OLP. Thus, established algorithms from multi-objective optimization can be used directly to compute minimal solution sets for PBLP^j, as well as BBLP^1. However, when solving PBLP^j, we are also interested in the set of breakpoints in the parameter set. And in the case of biparametric problems BBLP^j, we want to compute critical regions in the parameter set.

We propose two approaches to compute the breakpoints of PBLP^j. The first approach uses a minimal solution set $S(\text{TOLP})$ that is computed by any existing algorithm for TOLP. For each solution x in $S(\text{TOLP})$, two linear programs are solved to determine the parameter interval consisting of all values of λ for which x is in a minimal solution set of $\text{PBLP}^j(\lambda)$. The union of the interval boundaries of all solutions in $S(\text{TOLP})$ is the set of breakpoints. In contrast, the second approach requires the usage of an algorithm for TOLP that computes its weight set decomposition by design (for example, [PGE10]). Then, the aforementioned intervals can be computed without additional overhead.

6.1 Breakpoint enumeration algorithm

In this algorithm, we want to find the set of breakpoints of PBLP^1 and PBLP^2 using the weight set components of all $y \in Y_{\text{EN}}$ from the corresponding triobjective linear program. The basic idea of the algorithm is to first use some existing algorithm to compute $Y_{\text{EN}}(\text{TOLP})$. Then, for every extreme nondominated image y, we look at its weight set component and compute its boundaries to find the parameter intervals with respect to an efficient solution of PBLP^j.

Every weight set component $\mathcal{W}(y)$ corresponds to a solution $x \in S(\text{TOLP})$ such that x is optimal for $\text{PBLP}^j(\lambda)$ for some $\lambda \geq 0$ (see the proof of Proposition 3.19). We have defined in Definitions 3.9 and 3.17, that for every parameter value $\lambda \geq 0$, the weight sets of $\text{PBLP}^j(\lambda)$ are line segments intersecting $\mathcal{W}(\text{TOLP})$. Moreover, each weight set component of $\mathcal{W}(\text{TOLP})$ is intersected by at least one line segment. In particular, every weight set component is bounded by two specific line segments that mark its start and end. Since we can compute the parameter values of these specific line segments using two extreme weights of $\mathcal{W}(y)$ we can make the following statement.

6.1 Proposition. Let $\mathcal{I}^j(x)$ be the parameter interval of a solution $x \in S(\text{TOLP})$ where $S(\text{TOLP})$ is a minimal solution set of TOLP. Let λ_ℓ and λ_u correspond to the two specific line segments that mark the start and end of bounding the weight set component $\mathcal{W}(y)$ where $y = (c_1 x, c_2 x, d_1 x)^\top \in Y_{\text{EN}}(\text{TOLP})$. Then it holds that

$$\mathcal{I}^j(x) = [\lambda_\ell, \lambda_u]. \qquad \triangleleft$$

This parameter interval consists of the parameter values for which the corresponding solution $x \in S(\text{TOLP})$ of y is optimal for $\text{PBLP}^j(\lambda)$ where $\lambda \in [\lambda_\ell, \lambda_u]$.

For each weight set component $\mathcal{W}(y)$, we are interested in finding these two bounding line segments and their corresponding parameter values λ_ℓ and λ_u. The two bounding line segments define the interval $[\lambda_\ell, \lambda_u]$ where λ_ℓ is the minimum and λ_u is the maximum parameter value, as shown in Figure 6.1.

More precisely, for each weight set component $\mathcal{W}(y)$ of $\mathcal{W}(\text{TOLP})$, we solve the following program for PBLP^1:

$$\begin{aligned} \min \quad & \lambda \\ \text{s.t.} \quad & w \in \mathcal{W}(y), \\ & w \in \mathcal{W}(\text{PBLP}^1(\lambda)), \\ & w \in \mathbb{R}^3_{\geqq} \setminus \{(0,1,0)\} \end{aligned} \qquad (P^1_{\text{wsc}}(\lambda))$$

to determine the lower parameter bound λ_ℓ, and solve the corresponding maximization program to find λ_u. We exclude $(0, 1, 0)$ as an extreme weight of any weight set component because it results in a parameter interval of $[0, \infty)$ for some solutions, which can be misleading as the actual interval is bounded. Instead, for these weight set components, we use other extreme weights to determine the correct parameter interval.

For PBLP^2, we solve the analogous program for each weight set component $\mathcal{W}(y)$:

$$\begin{aligned} \min \quad & \lambda \\ \text{s.t.} \quad & w \in \mathcal{W}(y), \\ & w \in \mathcal{W}(\text{PBLP}^2(\lambda)), \\ & w \in \mathbb{R}^3_{\geqq}. \end{aligned} \qquad (P^2_{\text{wsc}}(\lambda))$$

We formulate the weight set component $\mathcal{W}(y)$ as constraints in $P^j_{\text{wsc}}(\lambda)$ (cf. [Ben98]):

$$\mathcal{W}(y) = \left\{ (w_1, w_2, w_3) : \begin{array}{r} A^\top v - C^\top w \geqq 0 \\ b^\top v - y^\top w = 0 \\ \mathbf{1}^\top w = 1 \\ v \in \mathbb{R}^m_{\geqq}, w \in \mathbb{R}^3_{\geqq} \end{array} \right\} \tag{6.1}$$

where $C \in \mathbb{Q}^{3\times n}$ consists of rows c_1, c_2 and d_1. For notational simplicity in describing the algorithm, we temporarily define a weight set component $\mathcal{W}(y)$ as

$$\mathcal{W}(y) := \left\{ (w_1, w_2) : Pw \geqq q, w \in \mathbb{R}^{m+3}_{\geqq} \right\} \tag{6.2}$$

where $P \in \mathbb{Q}^{n+4\times m+3}$ represents a coefficient matrix and $q \in \mathbb{Q}^{n+4}$ is a right-hand side vector.

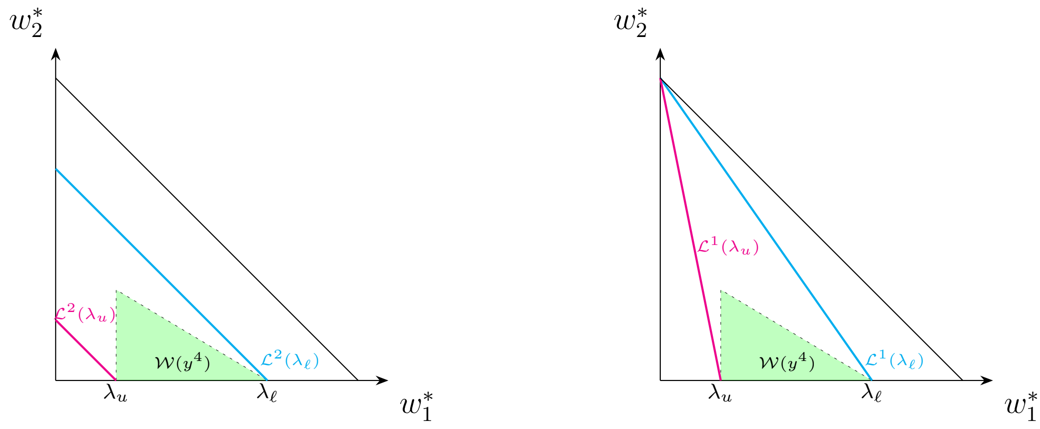
We begin with the problem PBLP^2 because it is easier and will be followed by its adaptation to PBLP^1. For PBLP^2, we substitute the description of the line segment, $\mathcal{L}_{\mathcal{W}(\text{PBLP}^2(\lambda))}$ from Definition 3.17 for $\mathcal{W}(\text{PBLP}^2(\lambda))$ and use the definition of $\mathcal{W}(y)$ from Equation (6.2) into the program $P^2_{\text{wsc}}(\lambda)$ to obtain

$$\begin{aligned} \min \quad & \lambda \\ \text{s.t.} \quad & Pw \geqq q, \\ & w_1 + w_2 = \frac{1}{1+\lambda}, \\ & \lambda \geq 0, \quad w \in \mathbb{R}^{m+3}_{\geqq}. \end{aligned} \tag{$\mathcal{P}^2(\lambda)$}$$

However, we reformulate the above program to the following;

$$\begin{aligned} \max \quad & w_1 + w_2 \\ \text{s.t.} \quad & Pw \geqq q \\ & w \in \mathbb{R}^{m+3}_{\geqq}. \end{aligned} \tag{$\mathcal{P}^2_{\text{wsc}}(\lambda_\ell)$}$$

by incorporating the constraint $w_1 + w_2 = \frac{1}{1+\lambda}$ directly into the objective function, thereby removing the variable λ. This is valid because the function $\frac{1}{1+\lambda}$ is strictly decreasing for $\lambda \geq 0$ which means minimizing λ is equivalent to maximizing $w_1 + w_2$. After finding the maximum value, say $s_{\max}$, the corresponding value of λ_ℓ is calculated as $\lambda_\ell = \frac{1}{s_{\max}} - 1$.
Subsequently, we solve the minimization variant to determine λ_u, as illustrated in Figure 6.1a. We observe that the optimal value of this program can be 0, which occurs because the maximization variant of the program $\mathcal{P}^2(\lambda)$ is unbounded. This implies that the parameter value is approaching infinity. Therefore, we state that the corresponding solution remains optimal over the parameter interval $[\lambda_\ell, \infty)$ in such cases. Similarly for PBLP^1, we substitute the description of the line segment, $\mathcal{L}_{\mathcal{W}(\text{PBLP}^1(\lambda))}$ from Defini-



(a) Line segments $\mathcal{L}^2_{\lambda_\ell}$ and $\mathcal{L}^2_{\lambda_u}$ for PBLP^2

(b) Line segments $\mathcal{L}^1(\lambda_\ell)$ and $\mathcal{L}^1(\lambda_u)$ for PBLP^1

Figure 6.1: An illustration of weight set component $\mathcal{W}(y^1)$ with bounding line segments $\mathcal{L}^j(\lambda_\ell)$ and $\mathcal{L}^j(\lambda_u)$ for PBLP^j, where $\mathcal{L}^j(\lambda) = \mathcal{L}_{\mathcal{W}(\text{PBLP}^j(\lambda))}$ for $\lambda \in \{\lambda_\ell, \lambda_u\}$.

tions 3.9 into the program $P^1_{\text{wsc}}(\lambda)$ to get

$$\begin{aligned} \min \quad & \lambda \\ \text{s.t.} \quad & Pw \geqq q, \\ & w_1(1+\lambda) + w_2 = 1, \\ & (w_1, w_2) \neq (0,1) \\ & w \in \mathbb{R}^{m+3}_{\geqq}, \lambda \geq 0. \end{aligned}$$

In this program, we cannot apply the direct approach used in PBLP^2 because in the constraint $w_1(1+\lambda) + w_2 = 1$, λ is not additively separable from the weight variables. We therefore normalize the constraint by dividing by $2+\lambda$ to obtain:

$$\left(\frac{1+\lambda}{2+\lambda}\right) w_1 + \left(\frac{1}{2+\lambda}\right) w_2 = \frac{1}{2+\lambda}.$$

We then define ℓ_1 and ℓ_2 as

$$\ell_1 := \frac{1+\lambda}{2+\lambda}; \quad \ell_2 := \frac{1}{2+\lambda}$$

and reformulate the problem as

$$\begin{aligned} \min \quad & \ell_1 \\ \text{s.t.} \quad & Pw \geqq q, \\ & \ell_1 w_1 + \ell_2 w_2 = \ell_2, \\ & \ell_1 + \ell_2 = 1, \\ & (w_1, w_2) \neq (0,1), \\ & w \in \mathbb{R}^{m+3}_{\geqq}, \ell \in \mathbb{R}^2_{\geqq}. \end{aligned} \tag{$\mathcal{P}$}$$

Crucially, because $\ell_1 = \left(\frac{1+\lambda}{2+\lambda}\right)$ is a strictly increasing function of $\lambda \geq 0$, the objective of minimizing λ is equivalent to minimizing ℓ_1. However, the constraint $\ell_1 w_1 + \ell_2 w_2 = \ell_2$ in $\mathcal{P}$ remains non-linear, as does the exclusion of $(0, 1, \mathbf{0})$. We overcome these problems by solving the linear program

$$\begin{aligned} \max \quad & \ell_1 \\ \text{s.t.} \quad & P^\top u \geqq \begin{pmatrix} \ell_1 \\ \ell_2 \\ \mathbf{0} \end{pmatrix}, \\ & q^\top u - \ell_2 = 0, \\ & \ell_1 + \ell_2 = 1, \\ & u \in \mathbb{R}^{n+4}_{\leqq}, \ell \in \mathbb{R}^2_{\geqq}, \end{aligned} \qquad (\mathcal{P}^1_{\text{wsc}}(\lambda_\ell))$$

instead of $\mathcal{P}$, where $\mathbf{0}$ is the zero vector with $m+1$ entries.

6.2 Proposition. Let ℓ_1^* be the optimal function value of $\mathcal{P}$. Then ℓ_1^* is also the optimal function value of the linear program $\mathcal{P}^1_{\text{wsc}}(\lambda_\ell)$. ◁

Proof. In this proof we use the following pair of dual linear programming problems

$$\begin{aligned} \max \quad & \ell_1 w_1 + \ell_2 w_2 \\ \text{s.t.} \quad & Pw \geqq q \\ & w \in \mathbb{R}^{m+3}_{\geqq} \end{aligned} \qquad (L_{\max}(\ell))$$

$$\begin{aligned} \min \quad & q^\top u \\ \text{s.t.} \quad & P^\top u \geqq \begin{pmatrix} \ell_1 \\ \ell_2 \\ \mathbf{0} \end{pmatrix} \\ & u \in \mathbb{R}^{n+4}_{\leqq}. \end{aligned} \qquad (L_{\min}(\ell))$$

Let (ℓ^*, w^*) be an optimal solution of $\mathcal{P}$ with solution value ℓ_1^*. The proof is split into two parts; first, we show that there is a corresponding feasible solution (ℓ^*, u^*) of $\mathcal{P}^1_{\text{wsc}}(\lambda_\ell)$. Second, we show that (ℓ^*, u^*) is also an optimal solution of $\mathcal{P}^1_{\text{wsc}}(\lambda_\ell)$.

(i) Feasibility
Since (ℓ^*, w^*) is feasible for $\mathcal{P}$,

$$\ell_1^* w_1^* + \ell_2^* w_2^* = \ell_2^*. \qquad (*)$$

Clearly, w^* is also feasible for $L_{\max}(\ell^*)$ and has an objective function value ℓ_2^*. Suppose w^* is not optimal for $L_{\max}(\ell^*)$, then there exists some w' that is feasible for $L_{\max}(\ell^*)$ and

$$\ell_1^* w_1' + \ell_2^* w_2' > \ell_1^* w_1^* + \ell_2^* w_2^* = \ell_2^*.$$

This implies

$$\ell_1^* w_1' + \ell_2^*(w_2' - 1) > 0.$$

Since $\ell_2^* = 1 - \ell_1^*$,

$$\ell_1^*(1 + w_1' - w_2') + w_2' - 1 > 0. \tag{$**$}$$

Moreover, we have $(w_1, w_2) \neq (0, 1)$ and $w \in \mathbb{R}^{m+3}_{\geqq}$. This means that the terms $1 + w_1' - w_2'$ and $w_2' - 1$ in Inequality ($**$) are positive and negative, respectively. Thus, there exists some $\ell_1' < \ell_1^*$ such that

$$\ell_1^*(1 + w_1' - w_2') + w_2' - 1 > \ell_1'(1 + w_1' - w_2') - w_2' + 1 = 0.$$

Using $\ell_2' = 1 - \ell_1'$, this implies

$$\ell_1' w_1' + \ell_2' w_2' = \ell_2'.$$

This contradicts our assumption that (ℓ^*, w^*) is an optimal solution of $\mathcal{P}$ since (ℓ', w') is also feasible and achieves a better solution value. Thus, w^* is optimal for $L_{\max}(\ell^*)$.
Since $L_{\max}(\ell^*)$ is feasible and has an optimal solution, the dual $L_{\min}(\ell^*)$ is feasible and bounded. Due to strong duality, there exists some u^* that is feasible for $L_{\min}(\ell^*)$ such that

$$q^\top u^* = \ell_1^* w_1^* + \ell_2^* w_2^*.$$

This leads to

$$q^\top u^* = \ell_2^*.$$

Thus, (ℓ^*, u^*) is also feasible for $\mathcal{P}^1_{\text{wsc}}(\lambda_\ell)$.

(ii) Optimality
Assume ℓ_1^* is not the optimal function value of $\mathcal{P}^1_{\text{wsc}}(\lambda_\ell)$. Then, there exists $\tilde{\ell}_1$ such that $\tilde{\ell}_1 > \ell_1^*$. The corresponding optimal solution $(\tilde{\ell}, \tilde{u})$ is feasible for $\mathcal{P}^1_{\text{wsc}}(\lambda_\ell)$ and, in particular, it satisfies

$$q^\top \tilde{u} = \tilde{\ell}_2.$$

Clearly, $\tilde{u}$ is feasible for $L_{\min}(\tilde{\ell})$. At the same time, w^* is feasible for $L_{\max}(\tilde{\ell})$ and due to weak duality,

$$\tilde{\ell} w_1^* + \tilde{\ell}_2 w_2^* \leq q^\top \tilde{u} = \tilde{\ell}_2.$$

However, from Equality (∗), we obtain

$$\ell_1^* w_1^* + \ell_2^* w_2^* = \ell_2^*.$$

Using $\ell_2^* = 1 - \ell_1^*$, we get

$$\ell_1^*(1 + w_1^* - w_2^*) + w_2^* - 1 = 0.$$

Since $\tilde{\ell}_1 > \ell_1^*$ and due to the same reasoning for Inequality (∗∗),

$$\begin{aligned} &\tilde{\ell}_1(1 + w_1^* - w_2^*) + w_2^* - 1 > 0 \\ \implies &\tilde{\ell}_1 w_1^* + \tilde{\ell}_2 w_2^* > \tilde{\ell}_2. \end{aligned}$$

This is not possible due to weak duality, and we get a contradiction. Thus, ℓ_1^* is an optimal function value of $\mathcal{P}^1_{\text{wsc}}(\lambda_\ell)$. □

After finding the maximum value of $\mathcal{P}^1_{\text{wsc}}(\lambda_\ell)$, say ℓ_1^*, the corresponding value of λ_ℓ is calculated using

$$\lambda_\ell = \frac{1 - 2\ell_1^*}{\ell_1^* - 1}. \tag{6.3}$$

Observe that the optimal value of $\ell_1 = 1$ in $\mathcal{P}^1_{\text{wsc}}(\lambda_\ell)$ implies $\lambda \to \infty$ in the parameter set.

Using a similar argument, we solve the following minimization variant of $\mathcal{P}^1_{\text{wsc}}(\lambda_\ell)$ to compute the minimum ℓ_1 that corresponds to the parameter λ_u

$$\begin{aligned} \min \quad & \ell_1 \\ \text{s.t.} \quad & P^\top u \leqq \begin{pmatrix} \ell_1 \\ \ell_2 \\ \mathbf{0} \end{pmatrix}, \\ & q^\top u - \ell_2 = 0, \\ & \ell_1 + \ell_2 = 1, \\ & u \in \mathbb{R}^{n+4}_{\geqq}, \ell \in \mathbb{R}^2_{\geqq}. \end{aligned} \tag{$\mathcal{P}^1_{\text{wsc}}(\lambda_u)$}$$

As we have simplified the weight set component in Equation (6.2), we now incorporate the formal definition of the weight set component into the linear program $\mathcal{P}^1_{\text{wsc}}(\lambda_\ell)$.

Algorithm 6.1: Breakpoint enumeration algorithm

Require: A minimal solution set $S(\text{TOLP})$ that corresponds to the extreme nondominated images Y_{EN} of the corresponding TOLP.

Ensure: For all $x \in S(\text{TOLP})$, a parameter interval $[\lambda_\ell, \lambda_u]$ and a set of breakpoints $\mathcal{B}$.

forall $x \in X$ **do**

  $\lambda_\ell \leftarrow$ Solve $\mathcal{P}^j_{\text{wsc}}(\lambda)$

  $\lambda_u \leftarrow$ Solve the opposite variant of $\mathcal{P}^j_{\text{wsc}}(\lambda)$

  Add $\lambda_\ell^{(k)}$ and $\lambda_u^{(k)}$ to the breakpoint list $\mathcal{B}$

Consequently, we solve the following linear program to find λ_ℓ:

$$\begin{aligned} \max \quad & & \ell_1 & \\ \text{s.t.} \quad & Ax + bx_{\text{opt}} & & \leqq 0, \\ & -Cx - yx_{\text{opt}} + x_w - \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} \ell & & \leqq 0, \\ & x_w - & \ell_2 & = 0, \\ & & \ell_1 + \ell_2 & = 1, \\ & x \in \mathbb{R}^n_{\geqq},\ x_{\text{opt}} \in \mathbb{R},\ x_w \in \mathbb{R},\ \ell \in \mathbb{R}^2_{\geqq}. \end{aligned} \tag{$\mathcal{P}^1_x(\lambda_u)$}$$

This procedure is applied to each solution $x \in S(\text{TOLP})$, generating the corresponding parameter interval $[\lambda_\ell, \lambda_u]$. The union of all interval boundaries constitutes the breakpoints set, with the complete algorithm presented in Algorithm 6.1.

If necessary, the breakpoints can be sorted using the collection of parameter intervals obtained for all $x \in S(\text{TOLP})$. Each resulting parameter interval (or a particular value) in the parameter set corresponds to a unique minimal solution set, with breakpoints marking the critical parameter values at which transitions between solution sets occur. Specialized data structures, for example interval trees, can be used to optimize this sorting and partitioning process (cf. [Ede83]).

6.3 Theorem. The algorithm finds all the breakpoints in the parameter set. ◁

Proof. The maximum and minimum values of the parameter interval dictate when a feasible solution x enters and exits a minimal solution set $S(\text{TOLP})$. Thus, leading to a change in a minimal solution set and serves as breakpoints. □

6.4 Theorem. The algorithm has a running time of $\mathcal{O}(|S|\, T_{\text{ws}})$, where T_{ws} is the running time of the linear program $\mathcal{P}^j_{\text{wsc}}(\lambda)$. ◁

Proof. The minimization and maximization variants of the linear program $\mathcal{P}^j_{\text{wsc}}(\lambda)$ can be solved polynomially in the encoding size of the program PBLP^j, say T_{ws} (cf.[BM15]).

Other evaluations such as computing the corresponding parameter value and sorting also take polynomial time. The algorithm iterates over all the extreme nondominated images in Y_{EN}, thus it has a running time of $\mathcal{O}(|S|\,T_{\text{ws}})$. □

6.1.1 Illustrative example

We now use an instance of PBLP[1] with a parameter λ i. e. Example 3.4 to illustrate the Breakpoint Enumeration Algorithm:

$$\begin{aligned} \min \quad & \begin{pmatrix} -3x_1 - x_2 + \lambda(x_1 + x_2) \\ x_1 - 2x_2 \end{pmatrix} \\ \text{s. t.} \quad & x \in X \coloneqq \{x \in \mathbb{R}^2_{\geqq} : 3x_1 + 2x_2 \geq 6; x_1 \leq 10; x_2 \leq 3\}. \end{aligned}$$

Using an existing multi-objective optimization algorithm we compute the set of extreme nondominated images, $Y_{\text{EN}}(\text{TOLP}) \coloneqq \{y^1, y^2, y^3, y^4\}$ i. e. $y^1 = (-33, 4, 13)$, $y^2 = (-3, -6, 3)$, $y^3 = (-6, 2, 2)$ and $y^4 = (-30, 10, 10)$. A minimal solution set that corresponds to $Y_{\text{EN}}(\text{TOLP})$ is $S \coloneqq \{x^1, x^2, x^3, x^4\}$.

We start with the weight set component $\mathcal{W}(y^1)$ which corresponds to a solution $x \in S(\text{TOLP})$. The parameter interval $\mathcal{I}^1(x^1)$ for x is initialized to $[0, 0]$. We solve the two linear programs $\mathcal{P}^1_{\text{wsc}}(\lambda_u)$ and its minimization variant to find maximum ℓ_1 i. e. $\ell_1^* = \frac{1}{2}$ and minimum ℓ_1 i. e. $\tilde{\ell}_1 = \frac{10}{13}$, respectively. For both maximum ℓ_1 and minimum ℓ_1, we simultaneously compute the corresponding parameter values using Equation (6.3) to get

$$\begin{aligned} \lambda_\ell &= \frac{1 - 2\ell_1^*}{\ell_1^* - 1} = \frac{1 - 2(\frac{1}{2})}{\frac{1}{2} - 1} = 0 \\ \lambda_u &= \frac{1 - 2\tilde{\ell}_1}{\tilde{\ell}_1 - 1} = \frac{1 - 2(\frac{10}{13})}{\frac{10}{13} - 1} = \frac{7}{3}. \end{aligned}$$

Thus, we get the parameter interval $\mathcal{I}(x^1) \coloneqq [0, \frac{7}{3}]$ for the solution x^1.

We iterate this procedure for the remaining weight set components of y^2, y^3 and y^4 and obtain parameter intervals $[0, \infty]$, $[\frac{7}{3}, 0]$, and $[1, 3]$, respectively along with the set of breakpoints $\mathcal{B} \coloneqq \left\{0, 1, \frac{7}{3}, 3\right\}$. Furthermore, with the help of some data structures such as interval trees, we compute unique solution sets shown in the Table 6.1. The weight set decomposition of the associated TOLP of this instance along with the line segments representing weight sets of PBLP[1](λ) at some breakpoints are shown in Figure 6.2.

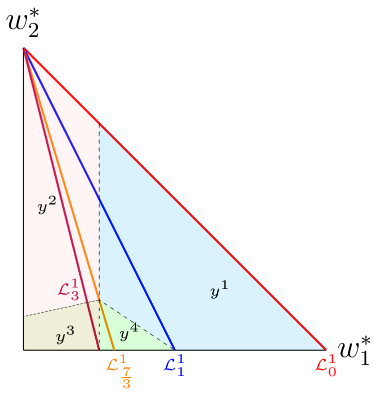


Figure 6.2: An illustration of line segments $\mathcal{L}_\lambda^1$ where $\mathcal{L}_\lambda^1 = \mathcal{L}_{\mathcal{W}(\text{PBLP}^1(\lambda))}$ for $\lambda \in \left\{0, 1, \frac{7}{3}, 3\right\}$ marking a change in a solution set.

Parameter	**Solution sets**
$[0, 1]$	$\{x^1, x^2\}$
$[1, \frac{7}{3}]$	$\{x^1, x^2, x^4\}$
$\frac{7}{3}$	$\{x^2, x^4\}$
$[\frac{7}{3}, 3]$	$\{x^2, x^3, x^4\}$
$[3, \infty)$	$\{x^2, x^3\}$

Table 6.1: Solution sets and their corresponding parameters.

6.1.2 Extension to PTLP

We now extend this algorithm to parametric triobjective linear programs. In the case of PTLP, we use the fact that the weight sets PTLP(λ) for all $\lambda \geq 0$ is a family of plane segments in the weight set of 4-OLP.

We use some existing algorithm to compute Y_{EN}(4-OLP). Then, for every extreme nondominated image $y \in Y_{\text{EN}}$(4-OLP), we look at its weight set component $\mathcal{W}(y)$ and compute its boundaries to find the parameter intervals, $[\lambda_\ell, \lambda_u]$, with respect to an efficient solution of PTLP i. e. $x \in S$(4-OLP).

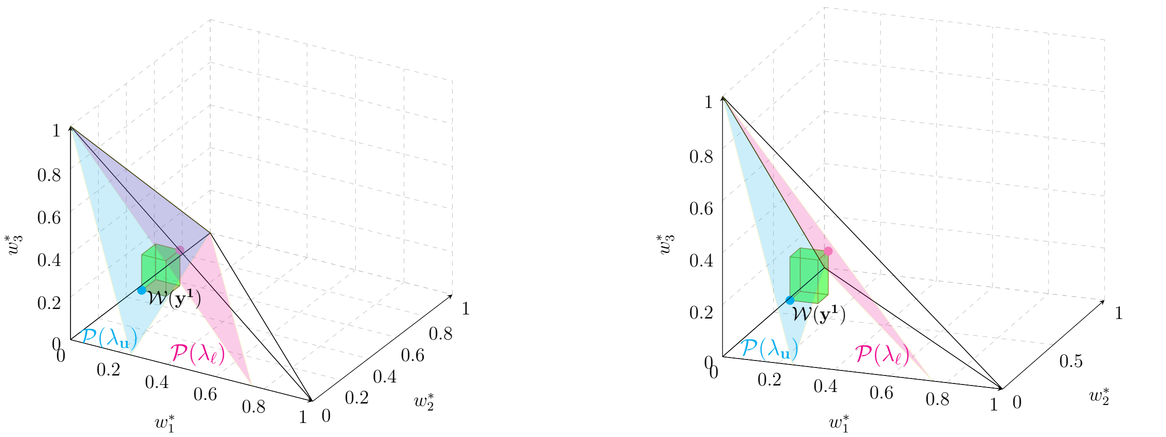
Moreover, each weight set component of $\mathcal{W}$(4-OLP) is intersected by at least one plane segment, defined in Definition 3.34. For each weight set component $\mathcal{W}(y)$, we are interested in finding two bounding plane segments and their corresponding parameter values λ_ℓ and λ_u, as shown in Figure 6.3.

Therefore, for each weight set component $\mathcal{W}(y)$ of $\mathcal{W}$(4-OLP), we solve the following program for PTLP:

$$\begin{aligned} \min \quad & \lambda \\ \text{s.t.} \quad & w \in \mathcal{W}(y), \\ & w \in \mathcal{P}_{\mathcal{W}(\text{PTLP}(\lambda))}, \\ & w_2 + w_3 \neq 1 \end{aligned}$$

to determine the lower parameter bound λ_ℓ, and solve the corresponding maximization program to find λ_u.

For any weight set component, we do not consider the extreme weight which lies on the line segment $w_1^* + w_3^* = 1$ for the same reason provided in the program $P_{\text{wsc}}^1(\lambda)$. But



(a) Plane segments $\mathcal{P}_{\lambda_\ell}$ and$\mathcal{P}_{\lambda_u}$ for PTLP **(b)** Plane segments from a different angle

Figure 6.3: An illustration of plane segments $\mathcal{P}(\lambda_\ell)$ and $\mathcal{P}(\lambda_u)$ for PTLP enclosing the weight set component $\mathcal{W}(y^1)$, where $\mathcal{P}(\lambda) = \mathcal{P}_{\mathcal{W}(\text{PTLP}(\lambda))}$ for $\lambda \in \{\lambda_\ell, \lambda_u\}$.

instead, we use other extreme weights of the weight set component to determine the correct parameter interval.

As formulated in Equation (6.1), we can also express the weight set component $\mathcal{W}(y)$ of $\mathcal{W}$(4-OLP) as constraints in the above program. However, we use a simpler description for the weight set component as

$$\mathcal{W}(y) := \left\{(w_1, w_2, w_3) : Pw \geqq q, w \in \mathbb{R}^{m+4}_{\geqq}\right\} \tag{6.4}$$

where $P \in \mathbb{Q}^{n+4\times m+4}$ represents a coefficient matrix and $q \in \mathbb{Q}^{n+4}$ is a right-hand side vector.

We substitute the description of the plane segment, $\mathcal{P}_{\mathcal{W}(\text{PTLP}(\lambda))}$ from Definition 3.34 into the above program to get

$$\begin{aligned} \min \quad & \lambda \\ \text{s.t.} \quad & Pw \geqq q, \\ & w_1(1+\lambda) + w_2 + w_3 = 1, \\ & w_2 + w_3 \neq 1 \\ & w \in \mathbb{R}^{m+4}_{\geqq}, \lambda \geq 0. \end{aligned} \tag{$\mathcal{P}^3(\lambda)$}$$

Since λ is not additively separable in the constraint $w_1(1+\lambda)+w_2+w_3 = 1$, we therefore normalize the constraint by dividing by $3 + \lambda$ to obtain:

$$\left(\frac{1+\lambda}{3+\lambda}\right) w_1 + \left(\frac{1}{3+\lambda}\right) w_2 + \left(\frac{1}{3+\lambda}\right) w_3 = \frac{1}{3+\lambda}.$$

We then define ℓ_1, ℓ_2 and ℓ_3 as

$$\ell_1 := \frac{1+\lambda}{3+\lambda}; \quad \ell_2 := \frac{1}{3+\lambda}; \quad \ell_3 := \frac{1}{3+\lambda},$$

which implies that $\ell_2 = \ell_3$. We reformulate the problem $\mathcal{P}^3(\lambda)$ as

$$\begin{aligned} \min \quad & \ell_1 \\ \text{s.t.} \quad & Pw \geqq q, \\ & \ell_1 w_1 + \ell_2(w_2 + w_3) = \ell_2, \\ & \ell_1 + 2\ell_2 = 1, \\ & w_2 + w_3 \neq 1 \\ & w \in \mathbb{R}^{m+4}_{\geqq}, \ell \in \mathbb{R}^3_{\geqq}. \end{aligned} \tag{$\mathcal{P}^3$}$$

Crucially, because $\ell_1 = \left(\frac{1+\lambda}{3+\lambda}\right)$ is a strictly increasing function of $\lambda \geq 0$, the objective of minimizing λ is equivalent to minimizing ℓ_1.

However, the constraint $\ell_1 w_1 + \ell_2(w_2 + w_3) = \ell_2$ in $\mathcal{P}^3$ remains non-linear, as does the exclusion of the line segment $w_2 + w_3 = 1$. Therefore, we solve the following linear program

$$\begin{aligned} \max \quad & \ell_1 \\ \text{s.t.} \quad & P^\top u \geqq \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_2 \\ \mathbf{0} \end{pmatrix}, \\ & q^\top u - \ell_2 = 0, \\ & \ell_1 + 2\ell_2 = 1, \\ & u \in \mathbb{R}^{n+4}_{\leqq}, \ell \in \mathbb{R}^3_{\geqq}, \end{aligned} \tag{$\mathcal{P}^3_{\text{wsc}}(\lambda_\ell)$}$$

instead of $\mathcal{P}^3$, where $\mathbf{0}$ is the zero vector with $m+1$ entries.

6.5 Proposition. Let ℓ_1^* be the optimal function value of $\mathcal{P}^3$. Then ℓ_1^* is also the optimal function value of the linear program $\mathcal{P}^3_{\text{wsc}}(\lambda_\ell)$. ◁

Proof. In this proof we use the following pair of dual linear programming problems

$$\begin{array}{lll} \max & \ell_1 w_1 + \ell_2(w_2 + w_3) & \\ \text{s.t.} & Pw \geqq q & (L_{\max}(\ell)) \\ & w \in \mathbb{R}^{m+4}_{\geqq} & \end{array}$$

$$\begin{array}{lll} \min & q^\top u & \\ \text{s.t.} & P^\top u \geqq \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_2 \\ \mathbf{0} \end{pmatrix} & (L_{\min}(\ell)) \\ & u \in \mathbb{R}^{n+4}_{\leqq}. & \end{array}$$

Let (ℓ^*, w^*) be an optimal solution of $\mathcal{P}^3$ with solution value ℓ_1^*. The proof is split into two parts; first, we show that there is a corresponding feasible solution (ℓ^*, u^*) of $\mathcal{P}^3_{\text{wsc}}(\lambda_\ell)$. Second, we show that (ℓ^*, u^*) is also an optimal solution of $\mathcal{P}^3_{\text{wsc}}(\lambda_\ell)$.

(i) Feasibility
Since (ℓ^*, w^*) is feasible for $\mathcal{P}^3$,

$$\ell_1^* w_1^* + \ell_2^*(w_2^* + w_3^*) = \ell_2^*. \tag{$*$}$$

Clearly, w^* is also feasible for $L_{\max}(\ell^*)$ and has an objective function value ℓ_2^*. Suppose w^* is not optimal for $L_{\max}(\ell^*)$, then there exists some w' that is feasible for $L_{\max}(\ell^*)$ and

$$\ell_1^* w_1' + \ell_2^*(w_2' + w_3') > \ell_1^* w_1^* + \ell_2^*(w_2^* + w_3^*) = \ell_2^*.$$

This implies

$$\ell_1^* w_1' + \ell_2^*(w_2' + w_3') - \ell_2^* > 0.$$

Since $\ell_2^* = \frac{1-\ell_1^*}{2}$,

$$\ell_1^* \left(w_1' + \frac{1 - (w_2' + w_3')}{2} \right) + \frac{w_2' + w_3'}{2} - 1 > 0. \tag{$**$}$$

Moreover, we have $w_1 + w_2 < 1$ and $w \in \mathbb{R}^{m+4}_{\geqq}$. This means that the terms $\left(w_1' + \frac{1-(w_2'+w_3')}{2} \right)$ and $\frac{w_2'+w_3'}{2} - 1$ in the Inequality ($**$) are positive and negative, respectively. Thus, there exists some $\ell_1' < \ell_1^*$ such that

$$\begin{aligned} \ell_1^* \left(w_1' + \frac{1 - (w_2' + w_3')}{2} \right) + \frac{w_2' + w_3'}{2} - 1 \\ > \ell_1' \left(w_1' + \frac{1 - (w_2' + w_3')}{2} \right) + \frac{w_2' + w_3'}{2} - 1 = 0. \end{aligned}$$

Using $\ell'_2 = \frac{1-\ell'_1}{2}$, this implies

$$\ell'_1 w'_1 + \ell'_2(w'_2 + w'_3) = \ell'_2.$$

This contradicts our assumption that (ℓ^*, w^*) is an optimal solution of $\mathcal{P}^3$ since (ℓ', w') is also feasible and achieves a better solution value. Thus, w^* is optimal for $L_{\max}(\ell^*)$.
Since $L_{\max}(\ell^*)$ is feasible and has an optimal solution, the dual $L_{\min}(\ell^*)$ is feasible and bounded. Due to strong duality, there exists some u^* that is feasible for $L_{\min}(\ell^*)$ such that

$$q^\top u^* = \ell_1^* w_1^* + \ell_2^*(w_2^* + w_3^*).$$

This leads to

$$q^\top u^* = \ell_2^*.$$

Thus, (ℓ^*, u^*) is also feasible for $\mathcal{P}^3_{\mathrm{wsc}}(\lambda_\ell)$.

(ii) Optimality
Assume ℓ_1^* is not the optimal function value of $\mathcal{P}^3_{\mathrm{wsc}}(\lambda_\ell)$. Then, there exists $\tilde{\ell}_1$ such that $\tilde{\ell}_1 > \ell_1^*$. The corresponding optimal solution $(\tilde{\ell}, \tilde{u})$ is feasible for $\mathcal{P}^3_{\mathrm{wsc}}(\lambda_\ell)$ and, in particular, it satisfies

$$q^\top \tilde{u} = \tilde{\ell}_2.$$

Clearly, $\tilde{u}$ is feasible for $L_{\min}(\tilde{\ell})$. At the same time, w^* is feasible for $L_{\max}(\tilde{\ell})$ and due to weak duality,

$$\tilde{\ell} w_1^* + \tilde{\ell}_2(w_2^* + w_3^*) \leq q^\top \tilde{u} = \tilde{\ell}_2.$$

However, from Equality $(*)$, we obtain

$$\ell_1^* w_1^* + \ell_2^*(w_2^* + w_3^*) = \ell_2^*.$$

Using $\ell_2^* = \frac{1-\ell_1^*}{2}$, we get

$$\ell_1^* \left(w_1^* + \frac{1 - (w_2^* + w_3^*)}{2} \right) + \frac{w_2^* + w_3^*}{2} - 1 = 0.$$

Since $\tilde{\ell}_1 > \ell_1^*$ and due to the same reasoning for Inequality $(**)$,

$$\tilde{\ell}_1 \left(w_1^* + \frac{1 - (w_2^* + w_3^*)}{2} \right) + \frac{w_2^* + w_3^*}{2} - 1 > 0$$
$$\implies \tilde{\ell}_1 w_1^* + \tilde{\ell}_2(w_2^* + w_3^*) > \tilde{\ell}_2.$$

This is not possible due to weak duality, and we get a contradiction. Thus, ℓ_1^* is an optimal function value of $\mathcal{P}^3_{\text{wsc}}(\lambda_\ell)$. □

After finding the maximum value of $\mathcal{P}^3_{\text{wsc}}(\lambda_\ell)$, say ℓ_1^*, the corresponding value of λ_ℓ is calculated as $\lambda_\ell = \frac{1-3\ell_1^*}{\ell_1^*-1}$. Observe that the optimal value of $\ell_1 = 1$ in $\mathcal{P}^3_{\text{wsc}}(\lambda_\ell)$ implies $\lambda \to \infty$ in the parameter set.

Using a similar argument, we solve the following minimization variant of $\mathcal{P}^3_{\text{wsc}}(\lambda_\ell)$ to compute the minimum ℓ_1 that corresponds to the parameter λ_u

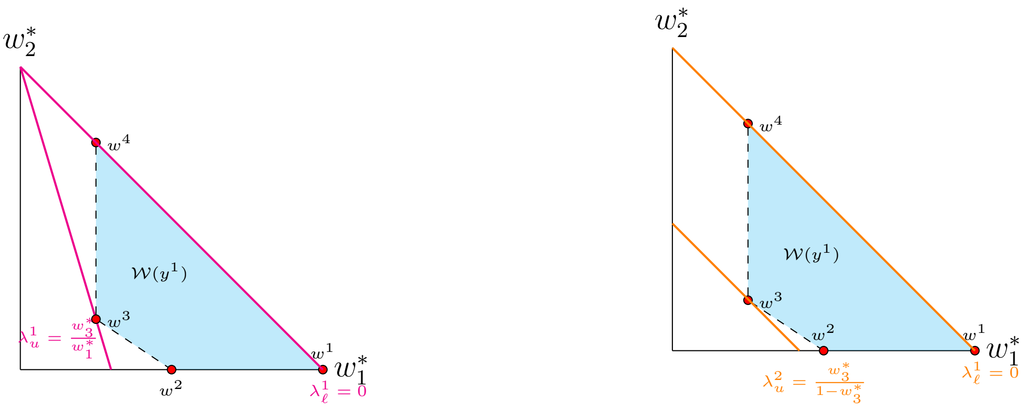
$$\begin{aligned}
\min \quad & \ell_1 \\
\text{s.t.} \quad & P^\top u \leqq \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \mathbf{0} \end{pmatrix}, \\
& q^\top u - \ell_2 = 0, \\
& \ell_1 + 2\ell_2 = 1, \\
& u \in \mathbb{R}^{n+4}_{\geqq}, \ell \in \mathbb{R}^2_{\geqq}.
\end{aligned} \tag{$\mathcal{P}^3_{\text{wsc}}(\lambda_u)$}$$

We iterate this procedure for all the weight set components in $\mathcal{W}$(4-OLP) and compute the set of breakpoints for the parametric problem PTLP and parameter intervals for each solution in S(4-OLP).

6.2 Adapted weight set algorithm

As shown in the Breakpoint Enumeration Algorithm (Algorithm 6.1), the weight set components of $\mathcal{W}$(TOLP) can be used to solve PBLP^j. We now discuss a strategy that directly utilizes weight set decompositions of TOLP to address our problem. There are several algorithms to find extreme nondominated images of a multi-objective linear program which computes weight set decomposition as an auxiliary result (such as in Benson and Sun [BS02], Przybylski et al. [PGE10] and Halffmann et al. [Hal+20]). These algorithms involves verifying extreme weights of the weight set component of an extreme nondominated image of TOLP using weighted sum. We use one such algorithm, which we refer to as the Weight Set Algorithm.

The idea behind the algorithm is as follows: For each extreme weight verified by the Weight Set Algorithm, we calculate the corresponding parameter value. These values are computed using Equations (3.3) and (3.7) for PBLP^1 and PBLP^2, respectively, as shown in Figure 6.4. For each solution $x \in S$, we want to find the parameter interval $[\lambda_l, \lambda_u]$ as described in Proposition 6.1. This interval is initialised using the parameter values of the first two verified vertices. Subsequently, if the corresponding parameter value is greater than the upper bound, the upper bound is updated; if it is less than the



(a) Evaluation of λ^1 associated to $\mathcal{W}(y^1)$ for PBLP^1

(b) Evaluation of λ^2 associated to $\mathcal{W}(y^1)$ for PBLP^1

Figure 6.4: An illustration of evaluation of λ^j using extreme weights (marked in red) of $\mathcal{W}(y^1)$. Extreme weights w^1 and w^4 yield the lower bound λ_ℓ^1 and the extreme weight w^3 yield the upper bound λ_u^1. A solution $x^1 \in S$ is optimal for the parameter intervals $[0, \lambda_u^1]$ in PBLP^1 and $[0, \lambda_u^2]$ in PBLP^2.

lower bound, the lower bound is updated. Note that if the extreme weights obtained by the Weight Set Algorithm have $w_1^* = 0$ and $w_1^* = w_2^* = 0$ for PBLP^1 and PBLP^2, respectively, then the parameter value is set to infinity. Thus, each time the Weight Set Algorithm obtains a weight set component of an extreme non-dominated image $y \in Y_{\text{EN}}(\text{TOLP})$, we also get the parameter interval. Thus, we determine the parameter interval $[\lambda_\ell, \lambda_u]$ within which the corresponding $x \in S(\text{TOLP})$ of y remains optimal.

Parameter value calculations and comparisons with current interval bounds require $\mathcal{O}(1)$ time per operation in addition to the time taken to enumerate each extreme weight of a weight set component. Consequently, this constant-time overhead is asymptotically negligible in the overall weight set decomposition algorithm. However, a notable limitation of this method is its dependence on a weight set decomposition algorithm, which restricts the choice of the underlying multi-objective programming algorithms. Whereas the Breakpoint Enumeration Algorithm 6.1 is efficient in scenarios if the existing algorithm used to compute the set $Y_{\text{EN}}(\text{TOLP})$ is computationally more efficient than a complete weight set decomposition algorithm. Although, in general it has a more significant overhead which is bounded by $\mathcal{O}(|S|\, T_{\text{ws}})$ (see Theorem 6.4).

6.2.1 Extension to PTLP

We extend the Adapted Weight Set Algorithm to parametric triobjective linear programs.

In case of PTLP, we use the fact that the weight sets PTLP(λ) for all $\lambda \geq 0$ is a family of plane segments in the weight set of 4-OLP as shown in Figure 3.14.

The algorithm follows the same approach by using the extreme weights of the weight set components to calculate the parameter intervals for every $x \in S(\text{4-OLP})$. The corresponding parameter values are computed using Equation (3.8) from Theorem 3.28 for PTLP. We repeat this process for all weight set components of $\mathcal{W}(\text{4-OLP})$.

6.3 Algorithm for critical region detection

In this section we propose an algorithm to determine the critical region of a solution for biparametric biobjective linear programs discussed in Chapter 5.

The idea follows from the Adapted Weight Set Algorithm whereby a weight set decomposition algorithm of the related multi-objective linear program will be used. Since such an algorithm computes all the weight set components associated to each extreme nondominated image in $Y_{\text{EN}}(\text{MOLP})$, we can again use the extreme weights of these components to find the critical region. For biparametric biobjective linear programs BBLP^1 and BBLP^2, we compute the weight sets $\mathcal{W}(\text{TOLP})$ and $\mathcal{W}(\text{4-OLP})$, respectively. This approach ensures that all extreme weights of every weight set component of the weight set are identified.

In the case of BBLP^1, for each weight set component $\mathcal{W}(y)$ of $\mathcal{W}(\text{TOLP})$, we only use a subset of extreme weights given by $\mathcal{A}_1(y) \cup \mathcal{A}_2(y)$. We can sort this subset by using the component-wise ordering mentioned in Definition 5.19. As a consequence of Theorem 5.42, the critical region is characterized as the area between two envelopes $E_\ell(y)$ and $E_u(y)$. Therefore, the corresponding line segments $L(w^*)$ where $w^* \in \mathcal{A}_1(y) \cup \mathcal{A}_2(y)$ are used to compute the critical region $\mathcal{R}^1(x)$ for a solution $x \in S(\text{TOLP})$ in the parameter set. We iterate this for every weight set component and find the critical region for every $x \in S(\text{TOLP})$.

And in case of BBLP^2, for each weight set component $\mathcal{W}(y)$ of $\mathcal{W}(\text{4-OLP})$, we use all the extreme weights $w^* \in \mathcal{W}(y)$. We compute the corresponding parameters λ and μ using the following equations:

$$\lambda = \frac{w_3^*}{w_1^*},$$
$$\mu = \frac{w_4^*}{w_2^*}$$

in order to find the critical region $\mathcal{R}^2(x)$. We iterate this for every weight set component till we have the critical regions $\mathcal{R}^2(x)$ for every $x \in S_4$.

Chapter 7

Conclusion

In this thesis, we developed a theoretical framework that relates the solution sets of a class of parametric programs to multi-objective linear programs. This framework uses the structure of the weight sets of these two types of programs. We relied on existing multi-objective programming methods and the weight set decomposition of the multi-objective linear program to solve parametric biobjective linear programs. Our focus was on two broad cases: first, programs with a single parameter, and second, programs with two different parameters. Our approach uses the weighted sum scalarization of parametric biobjective linear programs and their corresponding multi-objective linear programs. As a result yielding bridging the gap between parametric biobjective linear programs and the corresponding multi-objective linear programs.

The first part of the thesis focused on parametric biobjective linear programs with a single parameter and in the second part we investigated biparametric biobjective linear programs with two different parameters. We used the weighted sum scalarization and derived the relation between the weight sets of parametric problems and the weight set of the corresponding multi-objective linear program. Then we established a connection between the weight set of multi-objective linear programs and the structure of the parameter set of parametric problems. We showed the equivalence of minimal solution sets of parametric problems and their corresponding multi-objective linear programs. Using similar arguments, we generalized our findings to parametric multi-objective linear programs. We proposed two algorithms that address two distinct cases of parametric biobjective linear programs. The first algorithm builds on any multi-objective linear programming algorithm and solves linear programs to find breakpoints in the parameter set. For the second algorithm, we adapted a weight set decomposition algorithm specifically to enumerate the breakpoints.

There is a scope for future research in continuation of this thesis by extending it to parametric multi-objective mixed integer programs and multi-objective combinatorial programs. As this thesis only uses weighted sum scalarization, it would be interesting to explore other multi-objective optimization techniques, such as the Budget-Constraint Method, where one objective function is kept and the others are added to the constraints subject to some bound. Adopting a budget-constraint approach to a parametric multi-objective linear program will be comparable to those of an existing multi-parametric linear program with regard to the right-hand sides (cf. [GN72]). However, in this case, the objective function will also be parametrized along with the constraints.

The use of weight set of the four-objective linear program in Chapter 5 resulted in subdivision of the parameter set into critical regions for each solution in a minimal solution set but it would be interesting to further analyze the structure of this critical region. For instance, to figure out whether the critical region is convex or not and if it requires some other conditions. It would be interesting to investigate if we can get a compact representation of the critical region while using only a smaller number of weights from the corresponding weight set component.

The algorithms proposed in Chapter 6 rely on the weight set component and weight set decomposition of the corresponding MOLP, so it would be of interest if there is a new approach to find the set of breakpoints without relying on existing algorithms.

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