On Some Aspects of Investment into High-Yield Bonds

Helen Kovilyanskaya

Vom Fachbereich Mathematik
der Universität Kaiserslautern
zur Verleihung des akademischen Grades
Doktor der Naturwissenschaften
(Doctor rerum naturalium, Dr. rer. nat.)
genehmigte Dissertation.

1.Gutachter: Prof. Dr. R. Korn, Kaiserslautern
2.Gutachter: Prof. Dr. K. Janßen, Düsseldorf

Disputation: 1 März 2005

D386

Contents

List of Symbols

$\mathbb{R}$	set of real numbers
$\mathbb{R}_0 := [0, +\infty)$	set of nonnegative real numbers
$\mathbb{R}_+ := (0, +\infty)$	set of positive real numbers
$\mathbb{N}$	set of natural numbers
$\mathbb{N}_0$	nonnegative numbers
$\mathbb{Z}$	set of integer numbers
$\mathcal{B}(S)$	Borel σ-algebra over $S \subseteq \mathbb{R}$
$\mathbf{N} : \mathbb{R} \to [0, 1]$	standard normal distribution function
δ_x	Dirac measure at point x
$\boldsymbol{\lambda}$	Lebesgue measure
$\mu_1 \ll \mu_2$	measure μ_1 is absolute continuous with respect to measure μ_2
$x \wedge y$	infimum of two real numbers x and y
$x \vee y$	supremum of two real numbers x and y
X^+	positive part of the random variable X
$\mathbb{1}_A$	indicator function of the set A
$\parallel \cdot \parallel_p$	for the random variable X, $\parallel X \parallel_p := (E\lvert X\rvert^p)^{\frac{1}{p}}$ for $p > 0$
$\mathcal{L}^p(\Omega, \mathcal{F}, P)$	the set of all real-valued random variables X on$(\Omega, \mathcal{F}, P)$ such that $\parallel X \parallel_p < \infty,\ p > 0$
$C^\infty(S)$	$:= \{f : S \to R \mid \forall k \in \mathbb{N}_+ \exists f^{(k)}\},\ S \subseteq \mathbb{R}$
[X,Y]	quadratic variation of processes X and Y
X^c, X^d	continuous and discontinuous part of the semimartingale X
$P_1 * P_2$	convolution of distributions P_1 and P_2
$a.s.$	almost surely
A_t	$:= [t, T] \cap [t, \tau)$
$\lfloor \cdot \rfloor$	for $z \in \mathbb{R}$, $\lfloor z \rfloor := \sup\{n \in \mathbb{Z} : n \leq z\}$

Chapter 1

Introduction

In the last decades high-yield (also: defaultable, junk) bonds became an attractive form of investment worldwide. Recently the credit derivatives market (where products linked to dynamics of high-yield bond portfolios are traded) experienced an explosive growth as well. Usually such credit products are relatively long-term investments. As a rule, their profitability depends on coupon payments during the lifetime of the portfolio and on the payment of the principal value at maturity. A typical example is the contract which pays some part of coupon payment in the case it exceeds a predefined benchmark. The current work suggests an approach to managing and evaluation of credit portfolios in incomplete markets.
Investments into credit products often promise a high yield. Return on this kind of investment can be high even in times of economic slow-down when the stock market does not show a significant positive trend and riskless interest rates are low. At the same time it is hard for an investor to estimate and calculate all the risks involved.
Any analysis of a bond profitability can be done only after detailed studying and classification of the borrower, later called the *issuer* of the bond. We concentrate ourselves on public bond issuers rather than on small borrowers such as a private person, household etc. In other words, the wealth or the value of the bond issuer in question is influenced by many small factors. Examples of defaultable bonds we are considering are *corporate* bonds issued by a low-rated firm or *Brady* bonds issued by governments of countries with an emerging economical situation. The word *'firm'* is often used in the current work as a synonym of the word 'issuer'.
The first problem in managing the risk of investment into a 'junk bond' (i.e. a bond with a comparably high probability of default) is that there is usually no perfect hedging for it. Moreover, there is probably no hedge strategy at all if a firm which issues bonds has no shares to be openly traded on the

stock market.
The second aspect is that typically, information about market situation is (at least partially) covered and practically inaccessible to the owner of a credit portfolio. Since empirical studies show that profitability of investment into junk bonds increases when diversification grows, it becomes a common practice for small investors to rely on a subcontracted organization such as fund etc. instead of making more risky direct investment. The following situation is typical:
A subcontracted organization manages a portfolio of defaultable bonds and reports about the state of the portfolio at discrete times. It is inconvenient and causes additional costs to get a continuous flow of information about the performance of the portfolio and each of its parts. Nevertheless, the evaluation of the credit product should be done.
There are two principal approaches which are applicable for credit products. The first one originates from the ideas of Black and Scholes (1973) which were developed later by Merton (1974). This approach models the bond price based on the firm's value. Further contribution to the firm's value setup were made by Black and Cox (1976) whose assumption was that default of the bond happens before its maturity if the firm's value hits some lower boundary. The second approach is concentrated on the idea of default intensity. Starting in the paper of Jarrow and Turnbull (1995) where the default is driven by a Poisson process with constant intensity with known payoff at default, it is developed by Duffie and Singleton (1999) and Madan and Unal (1998). In these papers intensity and recovery get a more sophisticated form. Jarrow, Lando and Turnbull (1997) regard the setup of multiple rating classes. Duffie and Lando (1997) showed that there is a link between the two approaches. Nevertheless, the first approach cannot be combined directly with the second one. Intensity in the usual sense cannot be obtained if information is complete. When the firm's value is observable, the default of the firm's bond is a predictable stopping time and has no intensity as a consequence.
Here we regard the situation when both approaches can be combined due to the incompleteness of the information. It is regarded a situation when the information about the actual values of the firms which issued bonds in question is updated at discrete points of time. This information makes it possible to derive and update the probabilities of default and (in the case they exist) the intensities of the firm's defaults for the next following time interval. This is the time interval between the last time the information has been reported and the next coming time of update. The overview of some widely used models of the firm value process and the derivation of default distribution and default intensity based on the firm value model in some practically important

cases is deduced in Chapter 3.
Chapter 4 considers the pricing of a single defaultable bond. Some important properties of this basic credit product which will be used in Chapter 5 are proved. At the end of this chapter some aspects related to the calibration of defaultable bond are discussed.
Chapter 5 introduces a more complicated structure. This synthetic structure is a credit portfolio. Its building blocks are single defaultable bonds which were considered in the previous chapter 4. The description of each portfolio depends not only on its constituting parts but also on links between them. Theoretically, a portfolio can be made as sophisticated as possible. In the present work some assumptions about the way these blocks were 'glued' together are made. The most attention is paid to the portfolio 'chain of bonds' which constantly contains one bond. In this portfolio a defaulted bond is immediately replaced by a new one. Other credit portfolios can be expressed as a weighted sum of 'chain of bonds' portfolios. Conditions of no-arbitrage and L^p-boundness of the 'chain of bonds' portfolio are given. The corresponding properties are proved.
The 'chain of bonds' portfolio is, in general, not Markov. In Chapter 5 it is constructed an extension of the portfolio process which is Markov. Using this construction the distribution of the face value process is derived. In some cases the distribution was calculated numerically. It is surprising that in some cases the distribution of the firm value process can be expressed with the help of compound Poisson distribution. The theorem which justifies this approach is proved here.
There is a great variety of books devoted to the problem of portfolio optimization such as [26], [27], [36] when investment is in assets free of credit risk. Optimization of a portfolio which consists of defaultable bonds unlike the previous topic, is rarely treated in literature. This problem was considered first by Merton ([34]). Korn and Kraft ([28]) solved the problem of credit portfolio optimization in the case of continuous trading and stochastic interest rate.
Some aspects of stochastic control related to portfolios of defaultable bonds are looked at in Chapter 6. Continuous trading cannot be applied any more since throughout the present work it is assumed that the information is imperfect and updated at discrete times.

Acknowledgments

First of all I would like to thank my supervisor Prof. Dr. R. Korn for all he has taught me, for his valuable suggestions and determination the objectives of my work. I am extremely grateful to Prof. Dr. K. Janßen for creating friendly and inspiring atmosphere, for his attentive correction and discussion

of my work.
I highly appreciate technical support provided by O. Shorokhov and S. Kiselev. Thank you for inspiration and fruitful discussions.
I gratefully mention my former colleagues from 'DBV-Winterthur', especially Dr. M. Meurer, Dr. J. M. Wicke, Dr. O. Schmitt. Thank you for initiating the topic of my research and introducing me into your world.
I thank Prof. Dr. A. Janssen, N. Koval, V. Ostrovski and Prof. Dr. M.I. Portenko for comments, corrections and suggestions to my thesis. At last, but not the least I thank my parents for there enormous love and always being there for me.

Chapter 2

Model Specification

Let us list some definitions and notations which will be often used later on: All the processes are defined on the time interval $[0, T]$ with the fixed maturity time $T < \infty$.
Let $(\Omega, \mathcal{F}, P)$ be a probability space. $(\mathcal{F}_t)_{t\in[0,T]}$ denotes a filtration on it. The filtration $(\mathcal{F}_t)_{t\in[0,T]}$ will be specified later in Chapter 5.
Assume that the riskless interest rate r follows an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted positive bounded process. The corresponding savings account B is given by

$$B_t = \exp\left(\int_0^t r(s)ds\right).$$

Maturity T is assumed to be common for all considered financial products.

Definition 1 *The face (principal) value of a bond* is an amount of money to be paid at maturity time T to the bond holder in the case of no default prior to the maturity time of the bond.

Definition 2 A fixed income investment that has a fixed interest rate or coupon, payable on the *face (principal) value* is called a *bond.*

In order to specify a bond contract completely, its cash flow must be determined. The cash flow assigns to every time $t \in [0, T]$ a total coupon size to be paid at t. Note that an amount of money to be paid at $t = T$ equals the face value of a bond.
Thus, the cash flow normed by the face amount can be associated to a measure of a certain kind. Let $\mathcal{M}([0, T])$ represent the class of finite measures on the Borel σ-algebra $\mathcal{B}([0, T])$ which put the weight 1 to the point T:

$$\mathcal{M}([0,T]) := \{\mu : \mathcal{B}[0,T] \to \mathbb{R}_0 \mid \mu([0,T]) < \infty,\ \mu(\{T\}) = 1\}$$

Let us also denote by μ_d a 'discounted' measure μ. The discounting parameter is the riskless interest rate r, i.e. the measure which is absolutely continuous with respect to μ with the Radon-Nikodym density $\frac{\partial \mu_d}{\partial \mu}$ given by

$$\frac{\partial \mu_d}{\partial \mu}(t) = e^{-\int_0^t r(s)ds} = B_t^{-1},\ t \in [0, T]$$

All discounted measures form a class

$$\mathcal{M}_d([0,T]) := \{\mu_d : d\mu_d(t) = e^{-\int_0^t r(s)ds} d\mu(t),\ \mu \in \mathcal{M}([0,T])\}$$

Regard some examples of bonds and corresponding measures:

Example 1 A zero-coupon bond with maturity T corresponds to the measure $\delta_T \in \mathcal{M}([0,T])$. It is the minimal measure of the set $\mathcal{M}([0,T])$ in the sense that for all $\mu \in \mathcal{M}([0,T])$ it is valid:

$$\delta_T \ll \mu.$$

Example 2 A 2-year bond which pays coupon c at the end of each year corresponds according to definition 1 to the measure $\nu \in \mathcal{M}([0,T])$ given by

$$\nu = c_0 \delta_{T/2} + \delta_T \text{ for } T = 2,$$

where $c_0 = \frac{c}{1+c}$. The corresponding discounted measure ν_d is thus

$$\nu_d = c_0 e^{-\int_0^1 r(s)ds} \delta_1 + e^{-\int_0^2 r(s)ds} \delta_2.$$

The table below shows the total cash flow which should be paid on the bond with principal value x according to the contract:

time t	1	2
payment at t	xc_0	x

Remark. The representation above differs from the common terminology. According to this terminology if it is assumed that if the principal value of the bond is x and the yearly coupon payment is c, the payment measure is of the form $\nu = c\delta_{T/2} + (1+c)\delta_T$. But in the current context the value $(1+c)x$ is taken as the principal value in order to match the definition 1. The representation used here turns out to be more convenient later on.

Example 3 Extending the previous example 2 consider a bond which pays coupon $c_i \in \mathbb{R}_+$ at t_i, where $t_i \in (0,T)$ for $i = 1,\dots,n$. Such a bond corresponds to the measure $\nu^n \in \mathcal{M}([0,T])$ given by

$$\nu^n = \sum_{i=1}^{n} c_i \delta_{t_i} + \delta_T.$$

The discounted measure ν_d^n in this case is

$$\nu_d^n = \sum_{i=1}^{n} c_i e^{-\int_0^{t_i} r(s)ds} \delta_{t_i} + e^{-\int_0^T r(s)ds} \delta_T.$$

When n converges to infinity it might be more comfortable to work with a theoretical approximation. This leads to the notion of continuously payable coupon. For example, consider a constant coupon:

Example 4 A bond with a constant coupon payment c is related according to definition 1 to the measure

$$\mu = c\boldsymbol{\lambda}|_{[0,T]} + \delta_T, \tag{2.1}$$

where $\boldsymbol{\lambda}|_{[0,T]}$ denotes the restriction of the Lebesgue measure $\boldsymbol{\lambda}$ on $\mathcal{B}([0,T])$.

Note that the total discounted payment on the time interval $[0,T]$ announced by the issuer of the bond when the face value is x and the payment measure equals μ can be expressed as

$$x \int_0^T e^{-\int_0^t r(s)ds} d\mu(t) = x\mu_d([0,T])$$

This value can not be taken as a fair bond price. As it is known from practice the announced payments might not happen. In this case we talk about default.

After default all further payments are canceled. In the case of default prior to maturity it is common that the organization which issued a bond pays *recovery* to an investor who purchased a bond of this firm. Let $R \in [0,1)$ denote a *recovery rate*. If the face value of a defaulted bond was x, the total value of recovery is Rx. This payment is a sort of a compensation to an investor for not getting the rest of the money announced by the borrowing firm.

Bonds have different probabilities of default. It depends on the issuer and on the economical situation during the lifetime of a bond.

If there is no chance that the announced payment will not take place, the

bond is called riskless or default-free. A 'real world' approximation of a riskless bond is a bond which was issued by a government (central bank) of a country where the economical situation can be characterized as wealthy and stable.
A bit lower (but still very high!) level of credit worthiness have bonds issued by a wealthy multinational corporation.

Definition 3 A bond which has a positive probability of default is called a *defaultable* (also: junk or high-yield) bond.

It is common to associate issuers of such a bond with relatively small firms that probably have liquidity problems or with governments of countries with emerging state of economy. In the last case bonds are usually referred to as Brady bonds. The lower the level of credit worthiness and the ability to make timely payments of promised interest or principal value by the issuer, the cheaper the price of the bond. In the current work we concentrate mostly on the study of defaultable bonds and related financial products.

In order to model defaults and to give a reasonable definition of the price of a junk bond regard a random variable $\tau : \Omega \to \mathbb{R}_0$ which is a stopping time with respect to the filtration $(\mathcal{F}_t)$. It indicates the default time of a bond.
Denote by Q an equivalent to P martingale measure. This measure is related to the risk-neutral world. In this world discounted firm value processes are Q-martingales. The measure Q along with the real-world measure P will be regarded and defined more precisely in Chapter 3 on page 16. It is risk-neutral in the sense that the value of a firm, which pays no coupons or dividends, discounted by the savings account B, follows an $(\mathcal{F}_t)_{t\in[0,T]}$-martingale under Q. Denote by

$$A_t := [t, T] \cap [0, \tau), \ \ t \in [0, T]$$

the random set on the time interval $[0, T]$ which starts at t and lasts until maturity or default, whichever is smaller.
Now the definition of the bond price can be given:

Definition 4 (see [1], page 10) The fair *price of a defaultable bond* with the payment μ, recovery R and default time τ evaluated at the time $t \in [0, T]$ is defined as

$$E_Q \left(B_t \left(B_\tau^{-1} R \mathbb{1}_{\{\tau \leq T\}} + \mu_d(A_t) \right) | \mathcal{F}_t \right)$$

Definition 4 implies the following bounds for the bond price:

Lemma 1 *Let the constant $r^* \geq 0$, maturity $T \in \mathbb{R}_+$ and the payment measure $\mu \in \mathcal{M}([0,T])$ be fixed. Assume that the riskless interest rate r is an adapted to $(\mathcal{F}_t)_{t\geq 0}$ process bounded on $[0,T]$ from above by r^*, i.e. $r(t) \leq r^*$ for all $t \in [0,T]$ P-a.s. Then for arbitrary $t \in [0,T]$ the price of a defaultable bond as in Definition 4 evaluated at t belongs to the interval $[p_*, p^*] \subseteq \mathbb{R}_+$, with the boundaries equal to*

$$p_* = Re^{-r^*T}, \tag{2.2}$$

$$p^* = R + \mu([0,T])] \tag{2.3}$$

Proof: Due to the inequality

$$p^* \ := \ R + \mu([0,T]) \geq E_Q(Re^{-\int_t^\tau r(s)ds}\mathbb{1}_{\{\tau \leq T\}} + B_t\mu_d(A_t))$$

we obtain the upper bound p^*.
In its turn, using that $1 = \mathbb{1}_{\{\tau\leq T\}} + \mathbb{1}_{\{\tau>T\}}$ and $e^{r^*T} \geq B_T \geq B_s$, for all $0 \leq s \leq T$ the lower bound p_* results from the following considerations:

$$\begin{aligned} p_* \ &:= \ Re^{-r^*T} \leq E_Q\left(B_\tau^{-1}R\mathbb{1}_{\{\tau\leq T\}} + B_T^{-1}\mathbb{1}_{\{\tau>T\}}\right) \\ &\leq \ E_Q(RB_T^{-1}\mathbb{1}_{\{\tau\leq T\}} + \mu_d(A_t)) \end{aligned}$$

□

Remark. The lower bound $p_* = Re^{-rT}$ does not depend on the choice of the measure μ.

Later on, if it is not specified explicitly, the riskless interest rate r is a constant. The discount factor is $B_t = e^{-rt}$ in this case. The simplified version of Definition 4 will be often used:

Definition 5 The *fair price p_t of a defaultable bond* with the payment μ, recovery R and default time τ evaluated at the time $t \in [0,T]$ is defined as

$$p_t := E_Q\left(e^{rt}\left(e^{-r\tau}R\mathbb{1}_{\{\tau\leq T\}} + \mu_d(A_t)\right)|\mathcal{F}_t\right)$$

The bond price takes an especially simple form when the bond is a zero-coupon bond that pays no recovery $R = 0$, and the default time τ is exponentially distributed. In this case $Q(\tau \leq t) = 1 - e^{-\lambda t}$, $\lambda \in \mathbb{R}_+$ and $\mu = \delta_T$, the bond price evaluated at $t = 0$ according to definition 5 equals

$$E_Q(e^{-rT}\mathbb{1}_{\{\tau>T\}}) = e^{-rT}Q(\tau > T) = e^{-(\lambda+r)T}$$

Some additional information related to this example can be found in Chapter 3 on page 35.

Further examples are given in Chapter 4, equation (4.9) from Theorem 17, equation (4.17) from Theorem 21 and equation (4.21). Credit worthiness is modeled by a distribution of default time in every example. Equations listed above give a possibility to see how the price of a risky bond depends on the parameters of the default distribution.

Assume that two bonds have equal prices. The theorem below answers the question: Is it possible to find a connection between their times of default? As Lemma 2 states, there is a connection between distribution functions of the default times. Introduce first some useful notions.
Let S denote the class of signed measures on $[0, T]$, i.e.

$$S := \{\gamma = \gamma_+ - \gamma_- | \gamma_+, \gamma_- : \mathcal{B}([0, T]) \to \mathbb{R}_0 \text{ are measures}\}.$$

As before, γ_d denotes the equivalent to $\gamma \in S$ measure such that $\frac{d\gamma_d(t)}{d\gamma(t)} = e^{-rt}$.
For $\gamma \in S$ denote by $\gamma^\perp := \{f \in C([0, T]) : \int_0^T f d\gamma = 0\}$ the class of continuous functions which are orthogonal to the signed measure γ.
Denote by $F^\tau, F^\theta : \mathbb{R}_0 \to [0, 1]$ the distribution functions of the default times τ, θ correspondingly: $F^\tau(t) = Q(\tau \le t)$, $F^\theta(t) = Q(\theta \le t)$. It is not assumed that the distribution functions F^τ, F^θ are not degenerate, i.e. it is allowed that $\lim_{t\to\infty} F^\tau(t) < 1$ or $\lim_{t\to\infty} F^\theta(t) < 1$. In the remainder of the current chapter we denote by p_0^τ, p_0^θ the bond prices evaluated at the time 0 related to default times τ and θ correspondingly: $p_0^\tau := E_Q\left(e^{-r\tau} R \mathbb{1}_{\{\tau \le T\}} + \mu_d([0, T] \cap [0, \tau))\right)$ and $p_0^\theta := E_Q\left(e^{-r\theta} R \mathbb{1}_{\{\theta \le T\}} + \mu_d([0, T] \cap [0, \theta))\right)$.

Lemma 2 *Let τ and θ be default times of two bonds with the same payment measure $\mu \in \mathcal{M}([0, T])$ and recovery $R \in [0, 1]$.*
Assume that

$$p_0^\tau = p_0^\theta. \tag{2.4}$$

Then

$$(t \mapsto (F^\tau(t) - F^\theta(t))) \in \gamma_d^\perp,$$

where

$$\gamma = \mu - R(\delta_T + r\boldsymbol{\lambda}|_{[0,T]}). \tag{2.5}$$

Proof: Using Definition 5 and equality $Q(\{\tau > T\}) = 1 - F^\tau(T)$, we have

$$\begin{aligned} p_0^\tau &= E_Q\left(e^{-r\tau} R \mathbb{1}_{\{\tau \le T\}} + \mu_d([0, T] \cap [0, \tau))\right) \\ &= E_Q\left(\mu_d([0, T]) \mathbb{1}_{\{\tau > T\}} + (e^{-r\tau} R + \mu_d([0, \tau))) \mathbb{1}_{\{\tau \le T\}}\right) \\ &= (1 - F^\tau(T))\mu_d([0, T]) + \int_0^T (Re^{-rs} + \mu_d([0, s))) dF^\tau(s). \end{aligned}$$

Analogously, we have

$$p_0^\theta = (1 - F^\theta(T))\mu_d([0,T]) + \int_0^T (Re^{-rs} + \mu_d([0,s)))dF^\theta(s).$$

From assumption (2.4) above it follows that

$$\begin{aligned} 0 = p_0^\tau - p_0^\theta &= (F^\theta(T) - F^\tau(T))\mu_d([0,T]) \\ &\quad + \int_0^T (Re^{-rs} + \mu_d([0,s)))d(F^\tau(s) - F^\theta(s)). \end{aligned}$$

Applying the integration by parts formula (see e.g. Appendix B.2)

$$A(T)B(T) - A(0)B(0) = \int_0^T A(s-)dB(s) + \int_0^T B(s)dA(s)$$

for A, B given by $A(s) = Re^{-rs} + \mu_d([0,s])$ and $B(s) = F^\theta(s) - F^\tau(s)$ with $s \in [0,T]$ and taking into account that $F^\tau(0) = F^\theta(0) = 0$, the expression above transforms into

$$\begin{aligned} 0 &= (F^\theta(T) - F^\tau(T))\mu_d([0,T]) \\ &\quad + (Re^{-rT} + \mu_d([0,T]))(F^\tau(T) - F^\theta(T)) \\ &\quad - \int_0^T (F^\tau(t) - F^\theta(t))(-r)Re^{-rt}dt \\ &\quad - \int_0^T (F^\tau(t) - F^\theta(t))d\mu_d(t) \\ &= Re^{-rT}(F^\tau(T) - F^\theta(T)) \\ &\quad + \int_0^T (F^\tau(t) - F^\theta(t))d(rR\boldsymbol{\lambda}(t) - \mu(t))_d \\ &= \int_0^T (F^\theta(t) - F^\tau(t))d\gamma_d(t) \end{aligned}$$

where $\gamma = \mu + R\delta_T + rR\boldsymbol{\lambda}|_{[0,T]}$. This proves the statement of the lemma. □

As it is shown in Lemma 2, if two defaultable bonds have equal prices and payment measures, the distribution functions of default times belong to the same hyperplane. The function which is orthogonal to the hyperplane is given by relation (2.5). Notice that the statement is valid for arbitrary payment measure $\mu \in \mathcal{M}([0,T])$. For $\mu = c\boldsymbol{\lambda}|_{[0,T]} + \delta_T$ the lemma will be used in Chapter 4 in the calibration procedure.

Chapter 3

Default Distribution and Intensity

There is a huge variety of measures which are used to model default probabilities. For example, the exponential distribution with

$$Q(\tau \leq t) = 1 - e^{-\lambda_0 t},\ \lambda \in \mathbb{R}_+,\ t \in \mathbb{R}_0$$

is a natural choice under the intensity paradigm. The popularity of this distribution in the area of evaluation of credit products is of course due to the fact that the intensity of the exponential distribution has the simplest possible form. It is constant prior to default: $\lambda(t) = \lambda_0 \in \mathbb{R}_+$.
A different approach of the evaluation of a credit product is a firm value method. Application of the ideas of the firm value models shows an importance of another type of default distributions. Default time under the firm value paradigm is modeled as a hitting time related to some stochastic process.
The current section has the following structure: it starts by the consideration of models describing the firm value dynamics. In some cases the time of default can be interpreted as the first passage time related to a Brownian Motion. Thus, the distribution of default time can be derived from the theory of Brownian Motion. Some properties of this stopping time are listed. These properties will be often used later on. At the end, some practically relevant distributions of default times are derived from the firm value model. These distributions will be used later in Chapter 4 in order to price defaultable bonds and in Chapter 5 for the formulation of examples of some credit portfolios.

3.1 Models of the Firm Value

Let $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t\geq 0}, Q)$ be a stochastic basis which supports a Brownian motion W. The filtration $(\mathcal{F}(t))_{t\geq 0}$ satisfies the usual conditions.
Assume first that a firm issues a bond and the debt payments related to this bond contract are negligible in the sense that the influence of the coupon and (or) principal payment on the dynamics of the issuer's firm value is small and will not be modeled. The following two models of the firm value are often used in this situation:

3.1.1 'Risk-neutral' Growth.

According to Merton [35], consider the firm's value process V given by the stochastic differential equation

$$dV(t) = V(t)(r(t)dt + \sigma_V dW(t)), \qquad (3.1)$$
$$V(0) = V_0,$$

where the progressively measurable process r denotes the riskless interest rate, the constant $\sigma_V > 0$ is the volatility of the firm's value. The firm value given by equation (3.1) is used in order to price defaultable bonds (Chapter 4) since it corresponds to the risk-neutral measure.

3.1.2 'Real World' Growth.

The firm value grows with the rate $\gamma \in \mathbb{R}$ which differs in general from the riskless interest rate r. It reflects the investor's subjective beliefs about the firm's success or an estimated firm's grows rate. The firm value process V is given by

$$dV(t) = V(t)(\gamma dt + \sigma_V dW(t)), \qquad (3.2)$$
$$V(0) = V_0.$$

It generates the real-world measure P. The process given by (3.2) and as a consequence, the real-world measure plays an important role in Chapter 6 where the question about an optimal investment is studied.
The solution of the equations (3.1) and (3.2) is given by

$$V(t) = V_0 e^{(\gamma - \frac{1}{2}\sigma^2)t + \sigma W(t)} \qquad (3.3)$$

Remark: In the case of equation (3.1) it is set $\gamma := r$

3.1.3 Significant Debt Payment.

The two models of the firm value listed above are used as a good approximation when the running coupon payments of the firm do not have a significant impact on the firm's value (i.e. they do not appear in the equation for the firm value). As it will be seen on page 21 below, for many models of default the distribution of default time can be expressed explicitly if the firm's value dynamics is given by the stochastic differential equations (3.1)-(3.2). This makes it possible to obtain some bond prices under both previously given models.
In the situation when the debt payment influences the dynamics of the firm's value it is more reasonable to consider a more general model. Assume now that the firm's value satisfies the stochastic differential equation

$$dV(t) = rV(t)dt - d\nu(t) + \sigma V(t)dW(t), \tag{3.4}$$

$$V(0-) = V_0.$$

where $V_0 > 0$ and ν is a deterministic measure on $[0, T]$ such that $\nu \ll \mu \in \mathcal{M}([0, T])$.
Equation (3.4) is a generalization of the previously considered case with negligible debt payment. Indeed, equation (3.1) is a partial case of (3.4) if we set $\nu = 0$. Notice that formulae related to the situation when debt payment significantly influences firm value dynamics, can be arbitrarily complicated (see e.g. page 21, Proposition 5). On the other hand, the theory related to (3.1) is well developed and formulae are relatively simple. That is why the models resulting from equations (3.1) and (3.4) are treated separately. Despite its generality, it is possible to find the solution of equation (3.4) explicitly.

Lemma 3 *Let U be a process of bounded variation. For a continuous semimartingale S put $Z := \exp(S - \frac{1}{2}[S, S] - S_0)$. Then the equation*

$$dX = -dU + XdS, \tag{3.5}$$

$$X(0-) = X_0$$

has the unique solution given by

$$X(t) = Z(t)\left(X_0 - \int_0^t Z^{-1}(s)dU(s)\right) \tag{3.6}$$

Proof: Consider the decomposition

$$U = U^c + U^d$$

of U into its continuous U^c and its discontinuous U^d part such that $U^c(0) := 0$. For $t \in \mathbb{R}_0$ define the process F as

$$F(t) = X(t) + Z(t) \int_0^t Z^{-1}(s) dU^d(s).$$

Note that since $dX(t) = -dU(t) + X(t)dS(t)$, for the process F it is valid:

$$\begin{aligned} dF(t) &= dX(t) + d\left(Z(t) \int_0^t Z^{-1}(s) dU^d(s)\right) \\ &= -dU(t) + X(t)dS(t) + dU^d(t) \\ &\quad + \left(Z(t) \int_0^t Z^{-1}(s) dU^d(s)\right) d(S - \frac{1}{2}[S,S])(s) \\ &\quad + \frac{1}{2}\left(Z(t) \int_0^t Z^{-1}(s) dU^d(s)\right) d[S,S](s) \end{aligned}$$

The equality above follows from equation (3.5) used for dX term and differentiation by parts formula applied to $Z(t) \int_0^t \frac{dU^d(s)}{Z(s)}$. Thus,

$$dF(t) = -dU(t) + X(t)dS(t) + dU^d(t) + \left(Z(t) \int_0^t Z^{-1}(s) dU^d(s)\right) dS(t)$$

and the process F satisfies the equation

$$dF = -dU^c + F dS, \tag{3.7}$$

where S is a continuous semimartingale and U^c is a continuous process of bounded variation. Following the procedure as in [24] page 414, consider the representation $F = GZ$ and note that equation (3.7) is now equivalent to

$$d(GZ) = -dU^c + GZ dS.$$

Since $ZdS = dZ$, applying the integration by parts rule for continuous semimartingales $d(GZ) = GdZ + ZdG + d[G,Z]$, we obtain that

$$ZdG + d[G,Z] = -dU^c \tag{3.8}$$

Note that

$$[G,Z] = [G, \int ZdS] = \int Zd[G,S] = [\int ZdG, S] = [-U^c - [G,Z], S].$$

The last equality follows from (3.8). Since U^c is a process of bounded variation, $[U,S] = 0$ and thus $[G,Z] = 0$.
Equation (3.8) turns now into

$$ZdG = -dU^c,$$

which yields $dG = -Z^{-1}dU^c$. Since $X_0 = F(0)$, integration from 0 to t implies that

$$X(t) = F(t) - Z(t)\int_0^t Z^{-1}(s)dU^d(s) = Z(t)(X_0 - \int_0^t Z^{-1}(s)d(U^c(s)+U^d(s)).$$

$\square$

Remark: Expression (3.6) can be written as

$$X(t) = Z(t)\left(X_0 - \sum_{s\le t}\frac{1}{Z(s)}\Delta U_s^d - \int_0^t Z^{-1}(s)dU^c(s)\right).$$

Corollary 4 *The firm value described by equation (3.4) is given by*

$$V(t) = e^{(r-\frac{1}{2}\sigma^2)t+\sigma W(t)}\left(V_0 - \int_0^t e^{(\frac{1}{2}\sigma^2-r)s-\sigma W(s)}d\nu(s)\right) \tag{3.9}$$

Proof: Apply Lemma 3 to the semimartingale $S := rt + \sigma W$ and the process U of bounded variation given by $U(t) := \nu([0,t])$, $t \in \mathbb{R}_0$. $\square$

Remark. Equation (3.3) is easily obtained from Corollary 4 if we set $\nu = 0$.

It is reasonable to define the time of default as the stopping time with respect to the filtration $(\mathcal{F}_t)_{t\ge 0}$ as follows:

$$\tau := \inf\{t : V(t) \le 0\}.$$

Example 5 $\nu = 0$. As was noticed on page 17, this choice of the measure ν corresponds to equation (3.1). Its solution V given by (3.3) is a positive process: $V > 0$ a.s. Thus, it is more natural in this case to consider default time $\tau := \inf\{t : V(t) \le M\}$ for some positive bound M such that $M < V_0$. This approach and its extensions are treated on page 24

Example 6 $\nu = M'\delta_{\{T\}}$, where $M' > 0$ is a constant. This is by complexity the next case following after example 5.

1. Let the default time be chosen as $\tau := \inf\{t : V(t) \le M\}$, $M > 0$. This model is considered later on page 26. Corollary 9 gives default distribution in this case. The bond price is calculated in Theorem 21.

2. According to the classical case treated by Merton, default time corresponds to $\tau := \inf\{t : V(t) \leq 0\}$. The bond can be interpreted as a zero-coupon bond with the total payment M in the case of no default at maturity T.

   (a) If recovery R paid at maturity T is defined as an $\mathcal{F}(T)$-measurable random variable
   $$R = \frac{V(T)\mathbb{1}_{\{V(T)\leq M\}}}{M},$$
   then the bond payment is given by
   $$M - (M - V(T))^+.$$
   This case can be treated according to the Merton's approach. Merton interpreted the bond contract in this case as a linear combination of a constant riskless zero-coupon bond and a European put. Thus, in order to price the bond it is enough to know the price of the corresponding European put.

   (b) If recovery is a constant $R \in (0, 1)$, the bond in fact is a digital put with the random payment at maturity T
   $$M(\mathbb{1}_{\{V(T)\geq M\}} + R\mathbb{1}_{\{V(T)<M\}}) = M(R + (1 - R)\mathbb{1}_{\{V(T)\geq M\}}).$$
   Thus, in order to price the bond it is enough to find the probability $Q(V(T) \geq M)$.

Example 7 $\nu = c\boldsymbol{\lambda}$. In this case there is no simple form of default distribution and bond price. Equation (3.4) is now

$$dV(t) = (rV(t) - c)dt + \sigma V(t)dW(t), \tag{3.10}$$
$$V(0) = V_0.$$

The measure ν is a natural choice if the bond payment corresponds to the measure given by (2.1):
$$\mu = c\boldsymbol{\lambda}|_{[0,T]} + \delta_T.$$
Equation (3.10) is reasonable in particular in the situation when a small firm issues a bond with a constant coupon payment, which negatively influences the growth of the firm's value. The payment of the debt has a big impact on the development of the firm's wealth and must be taken into account. The total rate of the bond payment which is paid by the firm to all its bond

holders is c.
According to Corollary 4, solution of equation (3.10) is given by

$$V(t) = ce^{(r-\sigma^2/2)t+\sigma W(t)}\left(\frac{V_0}{c} - \int_0^t e^{(\sigma^2/2-r)s-\sigma W(s)}ds\right).$$

Proposition 5 *Denote by* $\nu_1 := 2r\sigma^{-2} - 1$ *and* $\nu_2 := 2c\sigma^{-2}V_0^{-1}$*. The default distribution* $Q(\tau \le t)$ *equals*

$$\begin{aligned} Q(\tau \le t) &= 1 - \mathbb{1}_{(-1,\infty)}(\nu_1)\frac{\Gamma(\nu_1,\nu_2)}{\Gamma(\nu_1)} + \mathbb{1}_{(2,\infty)}(\nu_1)e^{-\nu_2} \\ &\times \sum_{m=0}^{\lfloor\frac{\nu_1-2}{2}\rfloor} U'(-m, c_m, \nu_2)\frac{(-1)^m(\nu_2)^{\nu_1-1-m}e^{-\frac{1}{2}(\nu_1-1+m(\nu_1-2-m)\sigma^2 t)}}{m!\Gamma(d_m)(\nu_1 - 1 + m(\nu_1 - 2 - m))} \\ &- \int_0^\infty \frac{2}{\pi(4x^2+\nu_1^2)}\left|\frac{\Gamma(1+ix-\nu_{\frac{1}{2}})}{\Gamma(2ix)}\right|^2 g(\nu_2, x)e^{-\frac{1}{8}(4x^2+\nu_1^2)\sigma^2 t}dx \end{aligned}$$

Here the functions are denoted as follows:
Γ *is the Gamma-function,* $g : \mathbb{R}_+ \times \mathbb{R}_0 \to \mathbb{R}$ *is defined by*

$$g(y, x) := e^{-y}y^{ix+\frac{1}{2}\nu_1}U(1 + ix - \frac{1}{2}\nu_1, 1 + 2ix, y),$$

where U *and* U' *are the confluent hypergeometric functions.*

Proof: see [30], page 363. □

3.2 Distribution of the Default Time

3.2.1 Crossing of Two Geometric Brownian Motions

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$ be a stochastic basis with the usual conditions on filtration.
Fix $\mu_i \in \mathbb{R}$, $\sigma_i > 0$ for $i = 1, 2$ and $x_1 > x_2 \in \mathbb{R}_+$. Regard two processes X_1, X_2 on $(\Omega, \mathcal{F}, P)$ given by the stochastic differential equations

$$dX_i(t) = X_i(t)(\mu_i dt + \sigma_i dW_i(t)), \tag{3.11}$$

$$X_i(0) = x_i > 0$$

and the process

$$Y := \ln \frac{x_1 X_2}{x_2 X_1}$$

for $i = 1, 2$.
Here W_i for $i = 1, 2$ are assumed to be Brownian motions on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with the correlation coefficient $\rho \in (-1, 1)$, i.e. $[W_1, W_2]_t = \rho t$. Equation (3.11) is a particular case of stochastic differential equation (3.4). By Lemma 3 the solution of equation (3.11) is given by the 'variation of constants' formula (see for example, [27] p. 313)

$$X_i(t) = x_i e^{(\mu_i - \frac{1}{2}\sigma_i^2)t + \sigma_i W_i(t)} \tag{3.12}$$

for $t \geq 0$.
The process Y is a Brownian motion with drift. The first time when the processes X_1 and X_2 cross, can be expressed as the hitting time related to the process Y. In this way the results of the previous section can be applied.

Lemma 6 *1. The process Y satisfies*

$$Y(t) = \nu t + \sigma W(t) \tag{3.13}$$

where $W = \frac{\sigma_2 W_2 - \sigma_1 W_1}{\sigma}$ is a standard Brownian Motion and the parameters ν, σ are given by the relations:

$$\nu = \mu_2 - \mu_1 - \frac{1}{2}(\sigma_2^2 - \sigma_1^2),$$

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2.$$

2. The stopping time

$$\theta = \inf\{t : X_1(t) \leq X_2(t)\}$$

coincides with the stopping time

$$\tau_Y^a = \inf\{t : Y(t) \geq a\},$$

where $a := \ln \frac{x_1}{x_2}$.

Proof: Using the form (3.12) of the solution of the equations (3.11) yields

$$\begin{aligned} Y(t) &= \ln \left(\frac{x_1 x_2 e^{(\mu_2 - \frac{1}{2}\sigma_2^2)t + \sigma_2 W_2(t)}}{x_1 x_2 e^{(\mu_1 - \frac{1}{2}\sigma_1^2)t + \sigma_1 W_1(t)}} \right) \\ &= \left(\mu_2 - \mu_1 - \frac{1}{2}(\sigma_2^2 - \sigma_1^2) \right) t + \sigma_2 W_2(t) - \sigma_1 W_1(t) \\ &= \nu t + \sigma W(t). \end{aligned}$$

Notice that if $W_1,\ W_2$ are correlated with coefficient ρ, the processes $\frac{\sigma_2 W_2 - \sigma_1 W_1}{\sigma}$ is a Brownian motion itself iff σ is given by

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2.$$

This proves the part 1) of the lemma.
Since $Y(0) = 0 < \ln \frac{x_1}{x_2}$ the inequality

$$Y(t) = \ln \frac{x_1 X_2(t)}{x_2 X_1(t)} \geq \ln \frac{x_1}{x_2}$$

is equivalent to

$$X_2(t) \geq X_1(t) \text{ for } x_1 > x_2.$$

Thus, the random sets $\{t : X_1(t) \leq X_2(t)\}$ and $\{t : Y(t) \geq a\}$ coincide. The same is automatically valid for their infimum. This implies the statement 2) of the lemma. □

Time of default τ will be regarded as a moment when a firm's value (which is represented by the process X_1) reaches a lower bound (represented by X_2) of some special form:

$$\tau := \inf\{t : X_1(t) \leq X_2(t)\}.$$

Let the firm's value process $V := X_1$ be determined by one of the equations (3.1)-(3.2), i.e. put $\mu_1 := r$ (or $\mu_1 := \gamma$) and $\sigma_1 := \sigma_V$.
Let the bounding process X_2 be modeled as the solution of the stochastic differential equation

$$dX_2(t) = X_2(t)(\mu_2 dt + \sigma_2 dW_2(t)),\ X_2(0) = M.$$

The bound a and the parameters of the process Y (equation (3.13)) are now as follows

$$\begin{array}{rcl} a & = & \ln \frac{V(0)}{M}, \\ \nu & = & \mu_2 - r + \frac{1}{2}(\sigma_V^2 - \sigma_2^2), \\ \sigma^2 & = & \sigma_V^2 + \sigma_2^2 - 2\rho\sigma_V\sigma_2. \end{array} \tag{3.14}$$

Such a bound can be used when the liabilities of a firm have stochastic nature or when the bounding process plays a role of a benchmark which itself is a random process. For example, in order to measure performance of the investment, the process X_2 may represent a price of a stock, index etc. Using lemma 46 the distributions of the stopping time τ is calculated explicitly in the following corollary:

Corollary 7 *Let functions* $h_1, h_2 : \mathbb{R}_0 \to \mathbb{R}$ *be determined by*

$$h_1(s) := \frac{a - \nu s}{\sigma\sqrt{s}} \quad \text{and} \quad h_2(s) := \frac{a + \nu s}{\sigma\sqrt{s}}.$$

The parameters a, ν *and* σ *are defined according to (3.14).*

1. *The distribution function* F *of the default time* τ *is given by*

$$F(s) = P(\tau \leq s) = N(h_1(s)) + e^{a2\sigma_V^{-2}\nu} N(h_2(s)). \tag{3.15}$$

2. *The corresponding density function* f *is given by*

$$f(s) = \frac{a}{\sigma\sqrt{2\pi s^3}} e^{-h_1^2(s)/2}. \tag{3.16}$$

3. *For the fixed parameters* r, σ *and fixed time* $t > 0$*, the default probability*

$$\begin{array}{rcl} P^t_{r,\sigma} : (0, +\infty) & \to & [0,1] \\ a & \mapsto & F(t) \end{array}$$

considered as a function of the parameter a *is a monotone decreasing function.*

Proof: Applying Lemma 6 with the substitutions

$$\tilde{W} = Y(t), \;\; a = \ln\frac{V_0}{M}$$

we get the statements a) and b). Statement c) follows from Proposition 49. □

Example 8 Constant bound

The 'classical' way to represent the default time is to model it as the first time when the firm value is below some predefined constant bound. In the terms of the process X_2 this situation corresponds to

$$X_2(t) = M$$

for all $t > 0$ and for some $M > 0$. In this case the stopping time τ_Y^a is defined by the following bound a and the parameters ν, σ of the process Y from the equation (3.13):

$$\begin{array}{rcl} a & = & \ln\frac{V(0)}{M}, \\ \nu & = & -r + \frac{1}{2}\sigma_V^2, \\ \sigma & = & \sigma_V. \end{array} \tag{3.17}$$

This is a widely used model of corporate default. The constant M represents the critical level of firm assets.

Example 9 Exponentially increasing bound
Modeling of the default as the first time when the firm value crosses some exponential bound seems to be more realistic than the modeling which uses the constant bound. The first factor which can be incorporated in the case of the exponential bound is the inflation. The second factor is a possible coupon payment or growing firm's liabilities. Nevertheless, according to corollary 7 this bound is treated similarly to the constant one.
The process X_2 is deterministic and has the following time dependence:

$$X_2(t) = Me^{\mu_2 t}$$

for $t > 0$.
The parameters defining the stopping time τ_Y^a are then

$$\begin{aligned} a &= \ln \tfrac{V(0)}{M}, \\ \nu &= \mu_2 - r + \tfrac{1}{2}\sigma_V^2, \\ \sigma &= \sigma_V. \end{aligned} \tag{3.18}$$

The model is especially useful if it is considered the probability of default in the long term perspective.

3.2.2 Incomplete Accounting Information: the Initial Value of the Boundary is Random

This case represents the situation when the firm value cannot be observed directly. It is due to Duffie, Lando ([8]). They assumed that the observed firm value process X_1 delivers, in general, noisy and (or) delayed information about the firm's assets. Default happens when the real firm value hits some bound $M \in \mathbb{R}_+$ predetermined by the owners of the firm. According to [8], the firm value process is a geometric Brownian motion given by

$$dV(t) = V(t)(rdt + \sigma_V dW(t)),$$

$$V(0) = V_0.$$

Let the random variable U be normally distributed with mean $E(U)$ and variance $var(U)$. With the help of the random variable U it is modeled the imperfectness of the accounting information. The observed firm value V' is given by

$$V'(t) = e^U V(t) = V_0 \exp\left(U + \left(r - \frac{1}{2}\sigma_V^2\right) t + \sigma_V W(t)\right). \tag{3.19}$$

The observed initial value of the firm is thus $V'(0) = V'_0 = V_0 e^U$.
It is assumed that the real firm value is known to the owners of the firm as well as the critical value M at which the firm should be liquidated. The default time previously defined as $\tau = \inf\{t \geq 0 : V(t) \leq M\}$ can be equivalently represented as

$$\begin{aligned}\tau &= \inf\{t \geq 0 : V'(t) \leq Me^U\} \\ &= \inf\left\{t \geq 0 : \ln V'_0 + \left(r - \frac{1}{2}\sigma_V^2\right)t + \sigma W(t) \leq \ln M + U\right\}.\end{aligned}$$

Consider the case of imperfect accounting information in the framework of the current Chapter. As before, the firm value process is given by (3.1), The bounding process X_2 is constant with random initial value

$$X_2(t) = Me^U \text{ for all } t \geq 0.$$

The random initial value Me^U of the process X_2 is lognormally distributed. The bound a (which is a random variable now) and the parameters of the process Y are given by:

$$\begin{aligned}a &= \ln \tfrac{V(0)}{M} - U, \\ \nu &= -r + \tfrac{1}{2}\sigma_V^2, \\ \sigma &= \sigma_V.\end{aligned} \tag{3.20}$$

Thus, the default time is the first time when the firm value enters a random area. The currently described model significantly differs from the models listed in Section 3.2.1. The density and the intensity of default converge to zero when the maturity T approaches 0 (i.e. for short-term bonds). This is not the case if parameters are given by (3.20). In this case default density and intensity are bounded away from zero near 0. For more details see [8].

3.2.3 Constant Bound with a Jump at Maturity

This is a slight modification of the case considered in 3.2.1 which is caused by economical reasons. From the praxis it is known that many bonds suddenly default exactly at maturity time. This can be easily explained. Indeed, since the interest on the bond is usually much less than its principal amount, the company can pay the interest during the lifetime of the bond and meets financial problems only at maturity when the amount to be paid is unusually high and not available for the firm.
Considering this, the previous case can be extended. Define now two constants $M' \geq M > 0$ and set

$$X_2(t) = M\mathbb{1}_{[0,T)}(t) + M'\mathbb{1}_{[T,+\infty)}(t) \tag{3.21}$$

Let the firm value process follow the geometric Brownian motion given by the SDE

$$dX_1(t) = X_1(t)(rdt + \sigma_V dW(t))$$

with $X_1(0) = V_0$. According to this model the default time is defined as

$$\tau := \inf\{t : X_1(t) \leq X_2(t)\}.$$

The parameters of the process Y in (3.13) are as in Section 3.2.1. But the border a is generally time dependent and not continuous any more.

$$\begin{aligned} a(t) &= \ln \tfrac{V_0}{M} + \ln \tfrac{M}{M'} \mathbb{1}_{[T,\infty)}(t), \\ \nu &= \tfrac{1}{2}\sigma_V^2 - r, \\ \sigma &= \sigma_V. \end{aligned} \tag{3.22}$$

We consider the *running maximum* process M_Y of Y on $(\Omega, \mathcal{F}, P)$ defined as

$$M_Y(t) := \max_{0 \leq s \leq t} Y(s).$$

Note that by considerations as in the proof of Lemma 6 the random set $\{\tau > T\} = \{Y(t) < a(t),\ \forall t \leq T\} \in \mathcal{F}(T)$ coincides now with the set $\{Y(T) < \ln \frac{V_0}{M'},\ M_Y(T) < \ln \frac{V_0}{M}\}$. This will be used in the proof of Lemma 8. Notice that the default time defined above can be equivalently described as the first time when the firm which has significant debt payment (see 3.1.3) crosses a constant bound. The equation (3.4) which describes the firm value process can be written as follows

$$dV(t) = rV(t)dt - d((M' - M)\delta_T) + \sigma V dW(t).$$

Here the measure $(M' - M)\delta_T \ll \delta_T$. The solution of equation (3.4) provided by Corollary 4 has the simple form

$$\begin{aligned} V(t) &= e^{(r-\frac{1}{2}\sigma^2)t+\sigma W(t)}(V(0) + \int_0^t e^{-(r-\frac{1}{2}\sigma^2)s-\sigma W(s)}(M' - M)d\delta_T(s)) \\ &= V(0)e^{(r-\frac{1}{2}\sigma^2)t+\sigma W(t)} + (M' - M)\delta_T(t). \end{aligned}$$

The default time can be equivalently defined as $\tau := \inf\{t : V(t) \leq M\}$. The distribution of τ can be determined by the means of Lemma 8.

Lemma 8 *The probability $Q(\tau > T)$ is given by*

$$Q(\tau > T) = N\left(\frac{\ln \frac{V_0}{M'} + \nu T}{\sigma\sqrt{T}}\right) - \left(\frac{V_0}{M}\right)^{2\mu\sigma^{-2}} N\left(\frac{\ln \frac{M^2}{V_0 M'} + \nu T}{\sigma\sqrt{T}}\right).$$

Proof: Regard the probability measure $\tilde{P}$ on $(\Omega, \mathcal{F})$ such that $\frac{Y}{\sigma}$ which is written as

$$\frac{Y}{\sigma}(t) = \frac{\nu}{\sigma}t + W(t)$$

is a standard Brownian motion on $(\Omega, \mathcal{F}, \tilde{P})$. The probability $\tilde{P}(\frac{Y}{\sigma} \in dw, \frac{M_Y(T)}{\sigma} \in dm)$ is known. According to [25], page 95, it equals

$$\begin{aligned} \tilde{P}\left(\tfrac{Y}{\sigma}(T) \in dw, \tfrac{M_Y(T)}{\sigma} \in dm\right) &= \mathbb{1}_{\mathbb{R}_+}(m-w)\mathbb{1}_{\mathbb{R}_0}(m)\tfrac{2(2m-w)}{\sqrt{2\pi T^3}} \\ &\quad \times \exp\left(-\tfrac{(2m-w)^2}{2T}\right) dmdw. \end{aligned} \tag{3.23}$$

Return to the initial probability space $(\Omega, \mathcal{F}, Q)$ and the standard Brownian motion W on it

$$W(t) = \frac{Y}{\sigma}(t) - \frac{\nu}{\sigma}t.$$

Since $\frac{\nu}{\sigma}$ is a constant, Novikov's condition

$$E\left(\exp\left(\frac{1}{2}\int_0^T \frac{\nu^2}{\sigma^2}\right) dt\right) = e^{\frac{\nu^2}{2\sigma^2}T} < \infty$$

is trivially satisfied and the process

$$Z(t) = e^{\frac{\nu}{\sigma}\frac{Y}{\sigma}(t) - \frac{1}{2}\frac{\nu^2}{\sigma^2}t}$$

is a martingale with respect to Q. Thus, the Girsanov transformation on $[0, T]$ (see, for example, [25] page 191) can be applied. The probability $Q(Y(T) < \ln\frac{V_0}{M'}, M_Y(T) < \ln\frac{V_0}{M})$ equals to

$$Q\left(\frac{Y}{\sigma}(T) < \frac{\ln\frac{V_0}{M'}}{\sigma}, \frac{M_Y(T)}{\sigma} < \frac{\ln\frac{V_0}{M}}{\sigma}\right) = E_{\tilde{P}}\left(\mathbb{1}_{\{\frac{Y}{\sigma}(T) < \frac{\ln\frac{V_0}{M'}}{\sigma}, \frac{M_Y(T)}{\sigma} < \frac{\ln\frac{V_0}{M}}{\sigma}\}} Z(T)\right).$$

In order to simplify notations, denote by $\mu := \frac{\nu}{\sigma}$, $b' := \frac{1}{\sigma}\ln\frac{V_0}{M'}$, $b := \frac{1}{\sigma}\ln\frac{V_0}{M}$. Note the condition $M' \geq M$ implies that $b' < b$. The probability $Q(Y(T) < \ln\frac{V_0}{M'}, M_Y(T) < \ln\frac{V_0}{M})$ now equals the expectation

$$\begin{aligned} I &:= E_{\tilde{P}}\left(\mathbb{1}_{\{\frac{Y}{\sigma}(T) < b', \frac{M_Y(T)}{\sigma} < b\}} \exp\left(\mu\frac{Y}{\sigma}(T) - \frac{1}{2}\mu^2 T\right)\right) \\ &= \int_A \exp\left(\mu w - \frac{1}{2}\mu^2 T\right) \tilde{P}\left(\frac{Y}{\sigma}(T) \in dw, \frac{M_Y(T)}{\sigma} \in dm\right), \end{aligned}$$

where the set $A \subset \mathbb{R}^2$ is given by

$$A = \{(w, m) \in \mathbb{R}^2 : w \leq m,\ 0 \leq m \leq b,\ w \leq b'\}.$$

Represent the expectation as the sum of two integrals:

$$I := I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \int_{-\infty}^{0\wedge b'} \frac{\exp\left(\mu w - \frac{1}{2}\mu^2 T\right)}{\sqrt{2\pi T}} \left(\int_0^b \frac{2(2m-w)}{T} \exp\left(-\frac{(2m-w)^2}{2T}\right) dm \right) dw \\ &= \int_{-\infty}^{0\wedge b'} \frac{\exp\left(\mu w - \frac{1}{2}\mu^2 T\right)}{\sqrt{2\pi T}} \left(\exp\left(-\frac{w^2}{2T}\right) - \exp\left(-\frac{(w-2b)^2}{2T}\right)\right) dw \\ &= \int_{-\infty}^{0\wedge b'} \frac{1}{\sqrt{2\pi T}} \left(\exp\left(-\frac{(w-\mu T)^2}{2T}\right) - e^{2b\mu} \exp\left(-\frac{(w-(2b+T\mu))^2}{2T}\right)\right) dw \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_{0\wedge b'}^{b\wedge b'} \frac{\exp\left(\mu w - \frac{1}{2}\mu^2 T\right)}{\sqrt{2\pi T}} \left(\int_{w\vee 0}^{b} \frac{2(2m-w)}{T} \exp\left(-\frac{(2m-w)^2}{2T}\right) dm \right) dw \\ &= \int_{0\wedge b'}^{b'} \frac{\exp\left(\mu w - \frac{1}{2}\mu^2 T\right)}{\sqrt{2\pi T}} \left(\exp\left(-\frac{w^2}{2T}\right) - \exp\left(-\frac{(w-2b)^2}{2T}\right)\right) dw \\ &= \int_{0\wedge b'}^{b'} \frac{1}{\sqrt{2\pi T}} \left(\exp\left(-\frac{(w-\mu T)^2}{2T}\right) - e^{2b\mu} \exp\left(-\frac{(w-(2b+T\mu))^2}{2T}\right)\right) dw. \end{aligned}$$

The second assertion for both I_1 and I_2 follows after the change of variable

$$u := \frac{(2m-w)^2}{2T},\ du = \frac{2(2m-w)}{T} dm,$$

which turns the limits of integration in both cases into $\frac{w^2}{2T}$ (the lower limit) and to $\frac{(w-2b)^2}{2T}$ (the upper limit).
Now the sum of I_1 and I_2 can be found

$$\begin{aligned} I &= \int_{-\infty}^{b'} \frac{1}{\sqrt{2\pi T}} \left(\exp\left(-\frac{(w-\mu T)^2}{2T}\right) - e^{2b\mu} \exp\left(-\frac{(w-(2b+T\mu))^2}{2T}\right)\right) dw \\ &= N\left(\frac{b'-\mu T}{\sqrt{T}}\right) - e^{2b\mu} N\left(\frac{b'-2b-\mu T}{\sqrt{T}}\right). \end{aligned}$$

Inserting b, b' and μ gives the statement of the lemma. $\Box$

Now the distribution of the default time can be easily obtained:

Corollary 9 *Assume that the firm value $V = X_1$ and the bound X_2 are chosen according to (3.1) and (3.21) correspondingly. The default distribution under this assumptions is given by*

$$Q(\tau \le t) = N\left(\frac{\ln\frac{M}{V_0} + (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) + \left(\frac{M}{V_0}\right)^{2r\sigma^{-2}-1} N\left(\frac{\ln\frac{M}{V_0} - (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right),$$

for $t \in [0, T)$.

$$\begin{aligned} Q(\tau \le T) &= N\left(\frac{\ln\frac{M'}{V_0} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\ &\quad + \left(\frac{M}{V_0}\right)^{2r\sigma^{-2}-1} N\left(\frac{\ln\frac{M^2}{V_0 M'} - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right). \end{aligned}$$

The probability of default at maturity is

$$\begin{aligned} Q(\tau = T) &= N\left(\frac{\ln\frac{M'}{V_0} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln\frac{M}{V_0} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\ &\quad + \left(\frac{M}{V_0}\right)^{2r\sigma^{-2}-1} \left(N\left(\frac{\ln\frac{M^2}{V_0 M'} - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\right. \\ &\quad \left. - N\left(\frac{\ln\frac{M}{V_0} - (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{T}}\right)\right). \end{aligned}$$

Proof: The distribution of default time on $[0, T)$ clearly coincides with the classical one (3.16) as in Corollary 7. The probability

$$Q(\tau \le T) = 1 - Q(\tau > T)$$

is found with the help of Lemma 8 $\Box$

Remark: It can be seen that if $M = M'$ the default probability from Corollary 9 obtained above coincides with the classical one (3.15) from Corollary 7.

Figure 3.1 displays the probability of default at maturity depending on the quotient $\frac{M'}{M}$ in the case of three different volatilities (the left picture) and

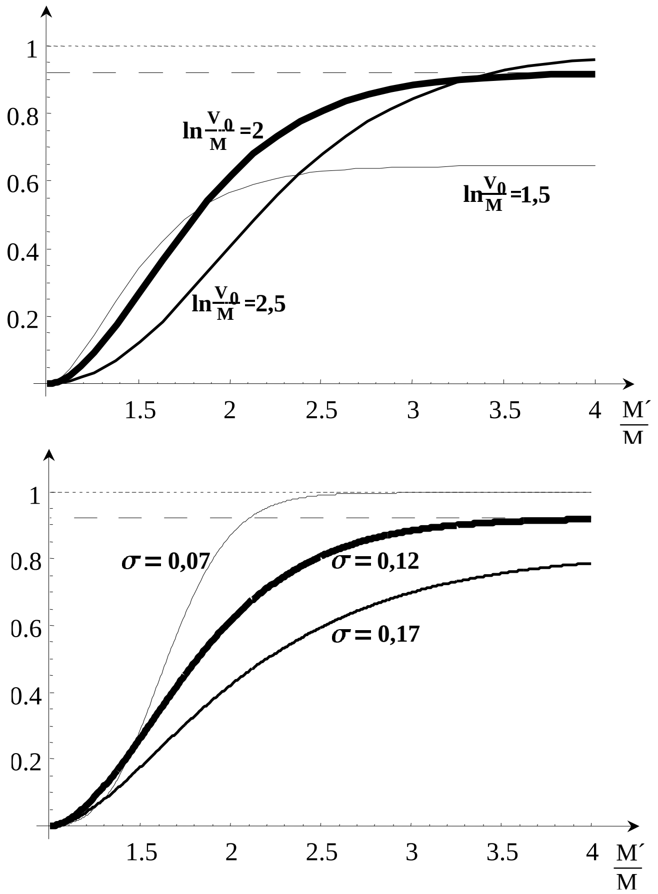

Figure 3.1: The probability $Q(\tau = T)$ of a default at maturity depending on the quotient $\frac{M'}{M}$ for three different parameters a (the picture on the top) and three different volatilities σ (the picture on the bottom).

three different initial firm values (the picture on the right). The parameters are chosen as follows:
time to maturity $T = 7$ years, the riskless interest rate $r = 0,03$. The reference curve on both pictures corresponds to the parameters $\sigma = 0,12$ and $a = 0,7$ ($\frac{V_0}{M} = 2$). When the quotient $\frac{M'}{M}$ converges to infinity, the probability of default at the time T tends to $1 - Q(\tau < T)$. The lower dashed line represents this value related to the reference curve.
The thick line on the left picture is related to the volatility $\sigma = 0,17$ and the thin line is related to $\sigma = 0,07$, the parameter $a = 0,7$ equals to the one for the reference curve.
On the right picture it is shown the probabilities when the volatility $\sigma = 0,12$ is fixed. The thick line corresponds to the parameter $a = 0,83$ ($\frac{V_0}{M} = 2,5$ in

this case) and the thin one corresponds to $a = 0,41$ ($\frac{V_0}{M} = 1,5$).
It can be seen on the figure, that the probability of default at maturity approaches the limit $\lim_{\frac{M'}{M}\to\infty} Q(\tau = T) = 1 - Q(\tau < T)$ relatively fast, i.e. for relatively small values of quotient $\frac{M'}{M}$. But the graphics which correspond to different parameters σ or a can cross, what means that the probability $Q(\tau = T)$ is not a monotonic function of neither the parameter σ nor the parameter a.

3.2.4 Discussion of Default Models

The proposed representation of default as the first time when the firm value process enters some random boundary includes all possible default distributions. It seems that some restrictions on the class of possible firm values as well as on the class of possible (random) boundaries should be specified. Nevertheless, there is still no agreement in the literature which class of processes appropriately describes dynamics of the firm value and which boundaries can be regarded. These aspects need a more detailed research and a closer look on empirical data.
The model considered in Section 3.2.1 which defines default time as the time of the first crossing of some constant boundary by geometric Brownian motion possesses clear economic interpretation and mathematical beauty. But it can be hardly accepted by practitioners. Their first argument is that just before maturity the probability of default suddenly drastically increases. This effect can be explained by a constant boundary on $[0, T)$ which has a jump at maturity T as it was considered in Section 3.2.3.
According to both interpretations discussed above, spread (for the definition of spread see Definition 8, page 34) tends to zero when maturity becomes very small. It contradicts observations of significant spreads even for bonds with short maturity. It leads to the hypothesis of unobservable firm value. It can be equivalently formulated in Section 3.2.2 in terms of the firm value modeled as a geometric Brownian motion and random (unknown) boundary. This model suggested in [8] fits economical data better.

3.3 Default Intensity

Regard a stochastic basis $(\Omega, \mathcal{F}, Q, (\mathcal{F}(t))_{t\geq 0})$. Let $\{\tau_i : i \in I \subseteq \mathbb{N}\}$ be a sequence of stopping times with respect to the filtration $(\mathcal{F}(t))_{t\geq 0}$ such that

$$\lim_{i\to\infty} \tau_i = +\infty.$$

Regard the counting process N such that

$$N(t) = \sum_{i=1}^{\infty} \mathbb{1}_{[\tau_i,\infty)}(t),\ t \in [0, T].$$

Let $\mathcal{F}^N(t)$ denote the minimal filtration generated by the process N.
The following theorem can be applied in the current situation as well as for more general processes:

Theorem 10 ([21], p. 33 *Doob-Meyer Decomposition*) *Let X be an adapted process of locally integrable variation. There exists a predictable process Λ of locally integrable variation such that $\Lambda(0) = 0$, which is unique up to an evanescent set such that $X - \Lambda$ is a local martingale.*

The variation of the process N is locally bounded. In particular, the process N has locally integrable variation. Thus, by the theorem 10 there is a decomposition

$$N(t) = \Lambda(t) + M(t),\ t \in [0, T], \tag{3.24}$$

where M is a local martingale for $t \in [0, T]$.

Definition 6 The process $(\Lambda(t), \mathcal{F}(t))$ described in Theorem 10 is called the *compensator* of the process $(X(t), \mathcal{F}(t))$.

Definition 7 If there exists a predictable process $\lambda = (\lambda(t), \mathcal{F}(t)) \geq 0$ such that the compensator $(\Lambda(t), \mathcal{F}(t))$ from Theorem 10 has the representation

$$\Lambda(t) = \int_0^t \lambda(s)ds,$$

λ is called the *intensity* of the process X.

In the context of credit products, the compensator has an important interpretation. The interpretation follows from the fact that

$$Q(\tau > t) = e^{-\Lambda(t)}.$$

Consider a zero-coupon bond with recovery $R = 0$, i.e. its payment measure is

$$\mu = \delta_T$$

and the bond price $p_0^{(z)}$ is

$$p_0^{(z)} = e^{-rT} Q(\tau > T) = e^{-rT-\Lambda(T)} = e^{-(r+\frac{\Lambda(T)}{T})T}.$$

Definition 8 The difference between the yield on the defaultable bond in question $\frac{\Lambda(T)}{T} + r$ to the yield on riskless bond r is called *spread*. Spread reflects the risks of the defaultable bond. It might be often viewed as a *risk premium.*

The compensator and intensity of N can be determined with the help of the following theorem:

Theorem 11 ([32], p.245) *Let* $F_1(t) := Q(\tau_1 \leq t)$*, and let*

$$F_i(t) := Q(\tau_i \leq t | \tau_{i-1}, \dots, \tau_1), \; i \in \mathbb{N}$$

be regular conditional distribution functions. Then the compensator

$$\Lambda = (\Lambda(t), \mathcal{F}^N(t)), \; t \in [0, T]$$

of the point process $(N(t), \mathcal{F}^N(t)), \; t \in [0, T]$ *is given by the following formula:*

$$\Lambda(t) = \sum_{i \geq 1} \Lambda^{(i)}(t),$$

where

$$\Lambda^{(i)}(t) = \int_0^{t \wedge \tau_i} \frac{dF_i(s)}{1 - F_i(s-)} \quad \textit{for } i \geq 1.$$

Two examples for the calculation of a default intensity are given below. Example 10 arises from the intensity approach. Example 11 provides a connection to the firm value approach.
In both examples $d = 1$ and the process N has a simple form

$$N(t) = \mathbb{1}_{[\tau, \infty)}(t).$$

Example 10 Let τ be exponentially distributed with parameter $\lambda_0 > 0$:

$$Q(\tau \leq t) = 1 - e^{-\lambda_0 t} \text{ for } t > 0.$$

By Theorem 11, the compensator Λ is of the form

$$\Lambda(t) = \int_0^{\tau\wedge t} \frac{\lambda_0 e^{-\lambda_0 s}}{e^{-\lambda_0 s}} ds = \lambda_0 \cdot (\tau \wedge t).$$

The intensity then is just $\lambda(t) = \lambda_0 1\!\!1_{[0,\tau]}(t)$. By definition 5, the zero-coupon bond with default time τ (defined as in the current example) which pays no recovery ($R = 0$), costs

$$p_0^{(z)} = e^{-(\lambda_0 + r)T}. \tag{3.25}$$

Thus, in the case of constant intensity spread equals λ_0. The intensity of default is exactly the *risk premium.*

Example 11 Assume that the riskless interest rate $r > 0$ is given. Define the stopping time now as in Lemma 6:

$$\tau := \inf\{t : \nu t + \sigma W(t) \geq \ln \frac{x_1}{x_2}\}. \tag{3.26}$$

Here the parameters are $\sigma > 0,\ \nu \in \mathbb{R},\ x_1 \geq x_2$. According to the firm value paradigm the stopping time τ is interpreted as the first time when the firm value falls below some predefined level (see page 22).
Combining Corollary 7 and Theorem 11 we obtain the intensity of the firm's default.

Corollary 12 *The default intensity related to the process Y with τ defined by (3.26) equals*

$$\lambda(t) = \tilde{\lambda}(t) 1\!\!1_{[0,\tau]}(t), \tag{3.27}$$

where

$$\tilde{\lambda}(t) = \frac{\ln \frac{V_0}{M} e^{-(h_1(t))^2/2}}{\sigma\sqrt{2\pi t^3}\left(1 - N(h_1(t)) - (\frac{M}{V_0})^{2r\sigma^{-2}-1} N(h_2(t))\right)} \tag{3.28}$$

for $h_1(s) = \frac{\ln \frac{V_0}{M} - (r - \frac{1}{2}\sigma^2)s}{\sigma\sqrt{s}}$ *and* $h_2(s) = \frac{\ln \frac{V_0}{M} + (r - \frac{1}{2}\sigma^2)s}{\sigma\sqrt{s}}$.

Figure 3.2 shows some default intensities under the settings of the current example.
Remark: Formula (3.28) of default intensity implies that credit spread $\frac{\int_0^T \lambda(t)dt}{T}$ goes to zero as maturity goes to zero, regardless of the credit quality of issuer. Nevertheless, some empirical studies show the presence of credit spreads even for bonds with short maturity.

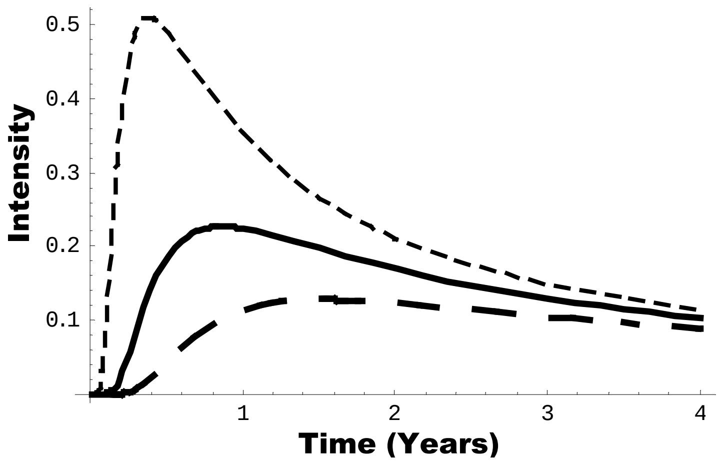

Figure 3.2: Default intensity depending on time (equation (3.28)) plotted for three different distances to default $a = \ln \frac{V}{M} = 0.2,\ 0.3,\ 0.4$. The upper dashed line corresponds to $a = 0.2$, the solid middle line is related to $a = 0.3$, the lower dashed line has parameter $a = 0.4$. The riskless interest rate $r = 2\%$ p.a., firm value volatility $\sigma = 0.2$.

As it is shown in [8], if the information about the firm value is noisy or delayed, credit spreads have a positive limit when maturity goes to zero.

An assumption that intensity is bounded will be often used later on, starting from Chapter 5. As Proposition 13 states, under some light restrictions on parameters, this is the case for intensity given by equation (3.28) when default time is defined as the first time when some transformed Brownian motion crosses a certain constant bound.
Namely, let $A \subseteq \mathbb{R}_0$ denote the set of possible distances to default and $\Sigma \subseteq \mathbb{R}_0$ denote the set of possibe volatilities. The restrictions refer to sets A and Σ of parameters and do not bring additional difficulties for practical applications. The estimations below are rather rough and can be significantly improved. But any improvement would require lengthy calculations and would be technically complicated.

Proposition 13 *Let $r > 0$ be fixed. Assume that there are positive constants $a_*, \sigma^* \in \mathbb{R}_+$ such that*

$$A \subseteq [a_*, \infty) \quad \textit{and} \quad \Sigma \subseteq (0, \sigma^*].$$

Then there is $\lambda^* \in \mathbb{R}_+$ such that

$$\tilde{\lambda}(t) \leq \lambda^*, \ \textit{for all}\ t \in [0, T],$$

where $\tilde{\lambda}$ is given by formula (3.28) with the parameters $a \in A$, $\sigma \in \Sigma$.

Proof: By Proposition 47, the distribution function of the default time F given by

$$F_{a,\sigma}(t) = Q(\tau \leq t), \quad t \in \mathbb{R}_0, \ a \in A, \ \sigma \in \Sigma$$

is continuous and, moreover, has a continuous derivative $f_{a,\sigma}$. Hence, $F_{a,\sigma}(t) = F_{a,\sigma}(t-)$ for arbitrary $t > 0$ and by theorem 11, the intensity of default $\tilde{\lambda}_{a,\sigma}$ is given by

$$\tilde{\lambda}_{a,\sigma}(t) = \frac{f_{a,\sigma}(t)}{1 - F_{a,\sigma}(t)}.$$

First, let us find an upper bound for the expression $\frac{1}{1-F_{a,\sigma}(t)}$. According to Proposition 48, $F_{a,\sigma}(t) \leq F_{a_*,\sigma}(t)$ for all $t \geq 0$.
Since $[0, \sigma^*]$ is compact, there is $\hat{\sigma} \in [0, \sigma^*]$ such that $F_{a_*,\sigma}(T) \leq F_{a_*,\hat{\sigma}}(T)$ for all $\sigma \in [0, \sigma^*]$. Notice that it is enough to consider a compact $[\sqrt{r}, \sigma^*]$ since by Remark after Proposition 49, the distribution of default time for $\sigma \in [0, \sqrt{2r})$ is degenerate.
Thus, $F_{a,\sigma}(t) \leq F_{a_*,\hat{\sigma}}(T) < 1$ and the upper bound announced above is found:

$$\frac{1}{1 - F_{a,\sigma}(t)} \leq \frac{1}{1 - F_{a_*,\hat{\sigma}}(t)}.$$

Second, let us find an upper bound for the density function $f_{a,\sigma}$.
Notice first that the function $g : \mathbb{R}_0 \to \infty$ given by

$$g(s) = s \exp(-s^{2/3})$$

for $s \in \mathbb{R}_0$ takes its maximum at $s = \left(\frac{3}{2}\right)^{\frac{3}{2}}$. Hence, for arbitrary $s \in \mathbb{R}_0$, it holds

$$g(s) = s \exp(-s^{2/3}) \leq \left(\frac{3}{2}\right)^{3/2} \exp(-3/2) \leq g\left(\left(\frac{3}{2}\right)^{\frac{3}{2}}\right).$$

Consider first $a \in A' := A \cap [0, \sqrt{2}(r + \frac{1}{2}(\sigma^*)2)T]$. Then for all $a \in A'$, $\sigma \in \Sigma$ it holds

$$-\frac{a(r - \frac{1}{2}\sigma^2)}{\sigma^2} \leq a\left(\frac{1}{2} - \frac{r}{2(\sigma^*)^2}\right) \leq \sqrt{2}\left(r + \frac{1}{2}(\sigma^*)^2\right) T \left(\frac{1}{2} - \frac{r}{2(\sigma^*)^2}\right) =: C_1.$$

It implies that for all $a \in A'$, $\sigma \in \Sigma$ it is valid

$$\begin{aligned} f_{a,\sigma}(t) &= \frac{a}{\sigma\sqrt{2\pi t^3}} \exp\left(-\frac{(a+(r-\frac{1}{2}\sigma^2)t)^2}{2\sigma^2 t}\right) \\ &= \frac{\sigma^2\sqrt{2^3}}{a^2\sqrt{2\pi}}\left(\frac{a}{\sigma\sqrt{2t}}\right)^3 \exp\left(-\left(\frac{a}{\sigma\sqrt{2t}}\right)^2\right) \\ &\quad \cdot \exp\left(-\frac{a(r-\frac{1}{2}\sigma^2)}{\sigma^2} - \frac{(r-\frac{1}{2}\sigma^2)^2 t}{2\sigma^2}\right) \\ &\leq \frac{2(\sigma^*)^2}{a_*^2\sqrt{\pi}}\left(\frac{3}{2}\right)^{3/2} \exp(-3/2)e^{C_1} =: C_2. \end{aligned}$$

Analogously, for $a \in A \setminus A'$ and $\sigma \in \Sigma$ it holds

$$\begin{aligned} f_{a,\sigma}(t) &= \frac{a}{\sigma\sqrt{2\pi t^3}} \exp\left(-\frac{(a+(r-\frac{1}{2}\sigma^2)t)^2}{2\sigma^2 t}\right) \\ &\leq \frac{\sigma^2 2^3}{a^2\sqrt{2\pi}}\left(\frac{a}{2\sigma\sqrt{t}}\right)^3 \exp\left(-\left(\frac{a}{2\sigma\sqrt{t}}\right)^2\right) \\ &\leq \frac{2^3(\sigma^*)^2}{a_*^2\sqrt{2\pi}}\left(\frac{3}{2}\right)^{3/2} \exp(-3/2) =: C_3. \end{aligned}$$

Put

$$\lambda^* := \max(C_2, C_3)(1 - F_{a_*,\hat{\sigma}}(T))^{-1}.$$

Now λ^* is the required boundary. □

Chapter 4

The Bond Price

Throughout this Chapter a bond with a constant coupon payment c is regarded. Thus,

$$\mu = c\boldsymbol{\lambda}|_{[0,T]} + \delta_T \tag{4.1}$$

is taken as a principal measure.

The structure of the current chapter is the following: it is first considered prices of defaultable bonds when the default distribution has a density (part 4.1.1). The basic example of this kind of distributions is an exponential distribution considered in part 4.1.2. It plays an important role since the default intensity in the case of exponential default distribution is constant.

Other examples of default distributions are listed in Chapter 3. Three of them can be combined in one class since their mathematical properties do not differ much. These are the distributions related to the cases of a constant, an exponentially increasing and a random bound. The bond price for this first class is given in part 4.1.3 of the current chapter.

The bond price for the bound of the forth type of default distribution listed in Chapter 3 is determined in the second part of the chapter. It is the case of a discontinuous bound with a jump at maturity. In general, it implies that default distribution is discontinuous and density does not exists. Note that 21 in part 4.2 which calculates the bond price in the case of a discontinuous bound, can be applied to arbitrary $\mu \in \mathcal{M}([0,T])$. In particular, if μ is given by (4.1) the price obtained in 4.2 might be viewed as a generalization of the price obtained in part 4.1.3.

When an interest rate is not deterministic, another generalization of the bond price given in the beginning of the chapter is given.

4.1 Absolutely Continuous Default Distribution Function

4.1.1 General Settings

The main assumption of the current chapter is the existence of the density function of the default time τ. For example, if τ is the hitting time (i.e. the first time when a Brownian motion with constant drift hits a constant bound), there is a density function given by equation (3.16), Corollary 7.
Let $A \subseteq \mathbb{R}$, $A \neq \emptyset$ be the set of parameters related to the distribution function of default time for some $d \in \mathbb{N}$. The distribution and the density function of the default can be represented in the following way:

$$F : A \times \mathbb{R}_0 \longrightarrow [0,1], \qquad f : A \times \mathbb{R}_0 \rightarrow \mathbb{R}_0$$

According to Definition 5, the price of a bond with a payment measure (4.1) (if it is evaluated at the time $t = 0$) equals

$$p_0 = E_Q\left(e^{-r\tau} R \mathbb{1}_{\{\tau \leq T\}} + c\int_0^{v\wedge T} e^{-rv}dv + e^{-rT}\mathbb{1}_{\{\tau > T\}} \middle| \mathcal{F}_W(0)\right). \tag{4.2}$$

Hence, if there is a default density parametrized by $a \in A$, the bond price is given by

$$\begin{aligned} p_0 &= (1-F(a,T))(e^{-rT} + \textstyle\int_0^T ce^{-rv}dv) \\ &\quad + \textstyle\int_0^T (Re^{-rs} + \int_0^s ce^{-rv}dv) f(a,s)ds \\ &= (1-F(a,T))(e^{-rT} + c\frac{1-e^{-rT}}{r}) \\ &\quad + \textstyle\int_0^T (Re^{-rs} + c\frac{1-e^{-rs}}{r}) f(a,s)ds \\ &= \frac{c}{r} + (1-F(a,T))e^{-rT}(1-\frac{c}{r}) \\ &\quad + (R-\frac{c}{r})\textstyle\int_0^T e^{-rs}f(a,s)ds \end{aligned} \tag{4.3}$$

Let $v : A \longrightarrow \mathbb{R}_0$ denote the function given by

$$v(a) = \frac{c}{r} + (1-F(a,T))e^{-rT}\left(1-\frac{c}{r}\right) + \left(R-\frac{c}{r}\right)\int_0^T e^{-rs}f(a,s)ds \tag{4.4}$$

which reflects the dependence of the bond price given by equation (4.2) on the parameter $a \in A$.

The next theorem is an important tool for understanding the impact of the parameter $a \in A$ on the dynamics of the bond price:

Theorem 14 *Suppose that the relation $c > rR > 0$ between the coupon, the riskless interest rate and recovery holds and that for arbitrary $t \in [0,T]$ $F(t,\cdot)$ is a strictly monotonic decreasing function:*

$$F(t,a_2) < F(t,a_1) \text{ iff } a_1 < a_2 \text{ for all } a_1, a_2 \in A. \tag{4.5}$$

Then the bond price (4.3) is a strictly monotonic increasing function of the parameter $a \in A$: $v(a_1) < v(a_2)$ for $a_1 < a_2$.

Proof: Applying expression (4.3) the difference of prices $v(a_2) - v(a_1)$ for $a_1, a_2 \in A$ such that $a_1 < a_2$ can be written as

$$\begin{aligned} v(a_2) - v(a_1) &= e^{-rT}\left(1 - \frac{c}{r}\right)(1 - F(a_1,T) - (1 - F(a_2,T)) \\ &\quad + \left(R - \frac{c}{r}\right)\int_0^T e^{-rt}(f(a_2,t) - f(a_1,t))dt \\ &= e^{-rT}\left(\frac{c}{r} - 1\right)(F(a_1,T) - F(a_2,T)) \\ &\quad + \left(R - \frac{c}{r}\right)(e^{-rT}(F(a_2,T) - F(a_1,T)) \\ &\quad + \int_0^T \frac{e^{-rt}}{r}(F(a_2,T) - F(a_1,T))dt \\ &= e^{-rT}(R-1)(F(a_2,T) - F(a_1,T)) \\ &\quad + \left(R - \frac{c}{r}\right)\int_0^T \frac{e^{-rt}}{r}(F(a_2,T) - F(a_1,T))dt. \end{aligned}$$

By the assumption of the current theorem, $c > rR$. Recall that by its definition, $R \in [0,1]$. Additionally applying assumption (4.5) of the lemma one obtains

$$e^{-rT}(R-1)(F(a_2,T) - F(a_1,T)) \geq 0$$

and

$$\left(R - \frac{c}{r}\right)\int_0^T \frac{e^{-rt}}{r}(F(a_2,T) - F(a_1,T))dt > 0.$$

Hence,

$$v(a_2) - v(a_1) > 0.$$

In other words, the bond price is a strictly monotone increasing function of the parameter $a \in A$ and the current theorem is proved. □

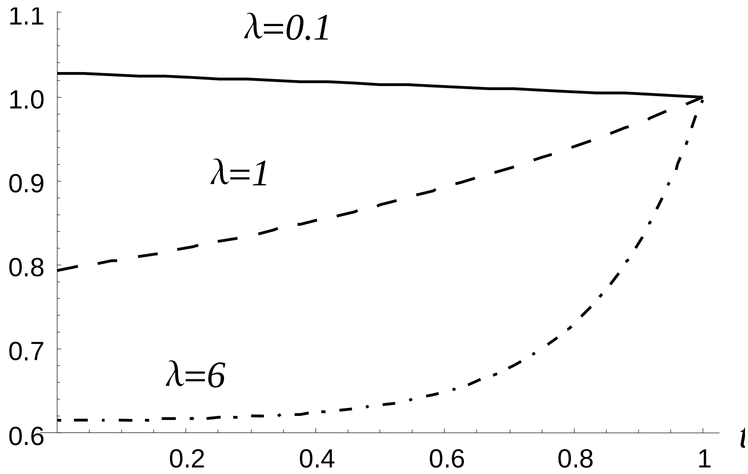

Figure 4.1: Bond prices (equation (4.7)) depending on time $t \in [0, T]$ for three constant intensities λ. The bond has maturity $T = 1$ year and pays constantly coupon $c = 9\%$ p.a. Bond recovery rate is set to $R = 0,6$. The riskless interest rate $r = 2\%$ p.a. The bond prices are calculated for the following constant default intensities: $\lambda(t) = 0,1;\ 1;\ 6$.

Remark: If in Theorem 14 instead of strictly decreasing it is taken a strictly increasing with respect to parameter a distribution function then the bond price is a strictly monotonic decreasing function, i.e. if one reverts signs in condition (4.5) and for all $a_1, a_2 \in A$ such that $a_1 < a_2$

$$F(t, a_2) < F(t, a_1)$$

then one has to do so for bond prices as well:

$$v(a_2) < v(a_1).$$

4.1.2 Constant Intensity Model

Assume that default intensity is constant, that is: $\lambda(t) = \lambda_0 \in \mathbb{R}_0$, $t \in [0, T]$. In this case the distribution of default time τ is

$$Q(\tau \leq t) = (1 - e^{-\lambda_0 t})\mathbb{1}_{\mathbb{R}_+}(t), \quad t \in \mathbb{R}$$

The corresponding density function is

$$f(t) = \lambda_0 e^{-\lambda_0 t}, \quad t > 0$$

Thus, the natural parametrization in the case of constant intensity is as follows:
the set $A = (0, +\infty)$, the parametrized distribution function F is given by

$$F(a, t) = (1 - e^{-at})\mathbb{1}_{\mathbb{R}_+}(t), \quad a \in A. \tag{4.6}$$

Theorem 15 *The arbitrage price of a defaultable coupon bond with the constant coupon rate c, maturity T, recovery R when evaluated at the time t is given by*

$$\begin{aligned} p_t &= e^{-(r+a)(T-t)} + \tfrac{c}{r}(1-e^{-r(T-t)})e^{-a(T-t)} + \tfrac{c}{r}(1-e^{-a(T-t)}) \\ &\quad + \tfrac{Ra}{r+a}(1-e^{-(r+a)(T-t)}) + \tfrac{ca}{r(a+r)}(e^{-(a+r)(T-t)}-1). \end{aligned} \tag{4.7}$$

Proof: applying formula (4.3) to the density function (4.6) we obtain formula (4.7). □

Picture 4.1 shows the dependence of bond price on time $s \in [0,T]$ represented in equation (4.7) when intensity λ is a constant. For small parameters $a \ll 1$ and, in particular, for $a = 0$ the bond price on Figure 4.1 is a decreasing function of time because in this case the probability of default is small and the bond price mostly depends on the total discounted coupon payment up to maturity.

Corollary 16 *If $c > rR$ the bond price is a strictly decreasing function of the intensity parameter $a = \lambda(t),\ a \in \mathbb{R}_+,\ t \in [0,T]$.*

Proof: According to Theorem 14 and the remark right after it, the bond price given by (4.7) is a strictly decreasing function of the parameter a since default intensity in this case is given by $F(a,t) = 1 - e^{-at}$, $t > 0$ which is a strictly increasing function of the parameter $a = \lambda(t)$ for $t \in [0,T]$. □

4.1.3 Firm Value Model

Consider the case of the constant bounding process $X_2(t) = M$ with $t \in [0,T]$, described on the page 24. Recall that in this case the firm's value is given by the equation (3.1) and the default happens when the process Y crosses the bound $a = \ln \frac{V_0}{M}$ for the first time:

$$\tau = \inf\{t : Y(t) \geq \ln \frac{V_0}{M}\} \tag{4.8}$$

The parameters of the process Y are listed in (3.17). Some prices are plotted on Figure 4.2. Similarly to Figure 4.1, due to the continuity of the default distribution function, $p_T = 1$ for all parameters $a \in A$. The specific shape of intensity (see e.e. Figure 3.2) in the firm value settings makes the price process less regular than in the case of a constant intensity as on Figure 4.1. In this case the price of a defaultable bond is given in [1] under very general settings. For the current purposes we give a simplified version of the theorem and its proof. The following theorem is valid:

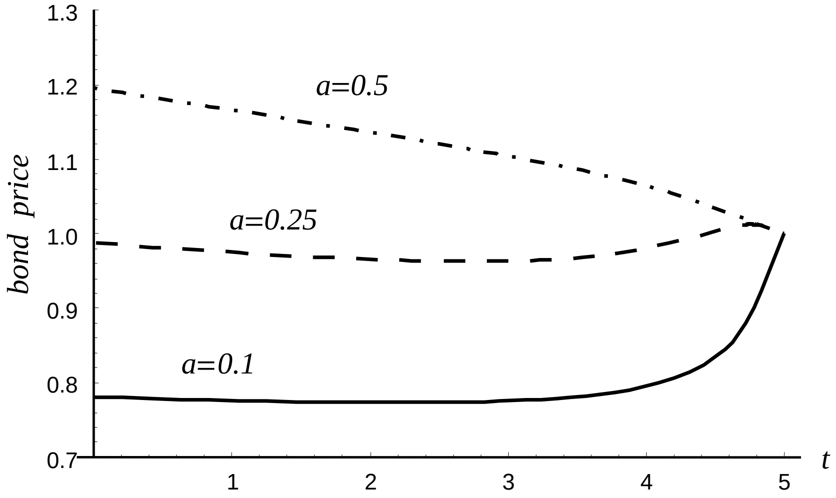

Figure 4.2: Bond prices (equation (4.9)) depending on time $t \in [0, T]$ for three constant intensities λ. The bond has maturity $T = 5$ years and pays constantly coupon $c = 9\%$ p.a. Bond recovery rate is set to $R = 0,6$. The riskless interest rate $r = 2\%$ p.a. The plotted prices are related to bonds issued by firms with volatility $\sigma_V = 0.2$ of their firm value and distance to default $a = \ln \frac{V(t)}{M} = 0,1;\ 0,25;\ 0,5$ correspondingly.

Theorem 17 ([1]) *The arbitrage price $v(a)$ at the time $t = 0$ of the defaultable coupon bond with the constant coupon rate c, time to maturity T is given by*

$$\begin{aligned} v(a) \;&=\; \tfrac{c}{r} + e^{-rT}(1 - \tfrac{c}{r})(N(l_1(T)) - \left(\tfrac{M}{V}\right)^{2\sigma^{-2}r-1} N(l_2(T))) \\ &\quad +(R - \tfrac{c}{r})(\left(\tfrac{M}{V}\right)^{2\sigma^{-2}r} N(g_1(T)) + \tfrac{V}{M} N(g_2(T))) \end{aligned} \tag{4.9}$$

Here $l_1(T) = \frac{a+(r-\frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}}$, $l_2(T) = \frac{-a+(r-\frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}}$ *and* $g_1(T) = \frac{(r+\frac{1}{2}\sigma_V^2)T-a}{\sigma_V\sqrt{T}}$, $g_2(T) = \frac{-(r+\frac{1}{2}\sigma_V^2)T-a}{\sigma_V\sqrt{T}}$.

Proof: Let us apply formula (4.3) of the bond price when the payment measure is given by

$$\mu = c\lambda|_{[0,T]} + \delta_T$$

to the current situation when default is modeled as the first hitting time. According to (4.3) the bond price is equals

$$v(a) = \frac{c}{r} + (1 - F(a,T))e^{-rT}\left(1 - \frac{c}{r}\right) + \left(R - \frac{c}{r}\right)\int_0^T e^{-rs} f(a,s)ds$$

Equation (4.9) is a partial case of equation (4.3) with corresponding density and distribution functions. The distribution and density functions of the first hitting time are already calculated in Chapter 3. Using formula (3.15) for the distribution function from Corollary 7 we obtain the probability of no default up to maturity:

$$Q(\tau > T) = 1 - Q(\tau \le T) = N(l_1(T)) - \left(\frac{M}{V}\right)^{2\sigma^{-2}r-1} N(l_2(T)),$$

where $l_1(T) = \frac{a+(r-\frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}}$, $l_2(T) = \frac{-a+(r-\frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}}$.
Substitution of the density function given by formula (3.16) from Corollary 7 under the integral sign yields (see [1], page 79):

$$\int_0^T e^{-rt} f_{\tau_{Y-a,a}}(t)dt = \left(\frac{M}{V}\right)^{2\sigma^{-2}r} N(g_1(T)) + \frac{V}{M} N(g_2(T)),$$

with $g_1(T) = \frac{(r+\frac{1}{2}\sigma_V^2)T-a}{\sigma_V\sqrt{T}}$, $g_2(T) = \frac{-(r+\frac{1}{2}\sigma_V^2)T-a}{\sigma_V\sqrt{T}}$.
This completes the proof. □

An important feature of the bond price in the currently regarded case is that it is a strictly monotonic function of the parameter $a = \ln\frac{V_0}{M}$. This fact is proved in Corollary 18:

Corollary 18 *Suppose that $c > rR$. Then the bond price from Theorem 17 is a strictly monotonic increasing function of the distance to default parameter $a = \ln\frac{V_0}{M}$.*

Proof: Proposition 48 states that F is a decreasing function of a. Applying Theorem 14, we finish the proof. □

Lemma 1 from Chapter 2 states that the bond price p belongs a.s. to the interval $[p_*, p^*]$, where the bounds are given by $p_* = Re^{-r^*T}$ and $p^* = R + \mu([0,T])]$. Specifying the model, we can sharpen the bounds. In particular, lemma 19 shows that the lower bound can be increased if default time is determined according to (4.8) and the payment measure is given by (4.1).

Lemma 19 *Assume that $c > rR$. The bond price p_0 belongs a.s. to the interval $(R, e^{-rT} + \frac{c}{r}(1-e^{-rT}))$.*
Moreover, the mapping $v : (0,\infty) \to (R, e^{-rT} + \frac{c}{r}(1-e^{-rT}))$ given by relation (4.4) is a bijection.

Proof: Notice first that for the functions $l_{1,2}$ and $g_{1,2}$ given by $l_{1,2}(T) = \frac{\pm a-(r-\frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}}$ and $g_{1,2}(T) = \frac{\pm(r+\frac{1}{2}\sigma_V^2)T-a}{\sigma_V\sqrt{T}}$ it is valid:

$$\lim_{a\to 0+}(N(l_1(T)) - e^{a(2\sigma^{-2}r-1)}N(l_2(T))) = 0, \tag{4.10}$$

$$\lim_{a\to +\infty}(N(l_1(T)) - e^{a(2\sigma^{-2}r-1)}N(l_2(T))) = 1, \tag{4.11}$$

$$\lim_{a\to 0+}(e^{-a2\sigma^{-2}r}N(g_1(T)) + e^aN(g_2(T))) = N(\kappa) - N(-\kappa) = 1, \tag{4.12}$$

where $\kappa = \frac{(r+\frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}}$ and

$$\lim_{a\to +\infty}(e^{-a2\sigma^{-2}r}N(g_1(T)) + e^aN(g_2(T))) = 0 \tag{4.13}$$

As it can be seen from formula (4.9) of the bond price, v is a continuous function. By Corollary 18, the function v decreases strict monotone with the growth of the parameter $a = \ln\frac{V_0}{M} \in (0, +\infty)$. Hence, the lower bound is given by R what is implied by (4.10), (4.12) and the following consideration:

$$\begin{aligned}
\lim_{a\to 0+} v(a) &= \lim_{a\to 0+}\Big[\frac{c}{r} + e^{-rT}\left(1-\frac{c}{r}\right)(N(l_1(T)) - e^{a(2\sigma^{-2}r-1)}N(l_2(T))) \\
&\quad + \left(R-\frac{c}{r}\right)(e^{-a2\sigma^{-2}r}N(g_1(T)) + e^aN(g_2(T))\Big] \\
&= \frac{c}{r} + 0\cdot e^{-rT}\left(1-\frac{c}{r}\right) + \left(R-\frac{c}{r}\right) = R.
\end{aligned}$$

The upper bound results from (4.10), (4.12) and equalities:

$$\begin{aligned}
\lim_{a\to +\infty} v(a) &= \lim_{a\to +\infty}\Big[\frac{c}{r} + e^{-rT}\left(1-\frac{c}{r}\right)(N(l_1(T)) - e^{a(2\sigma^{-2}r-1)}N(l_2(T))) \\
&\quad + \left(R-\frac{c}{r}\right)(e^{-a2\sigma^{-2}r}N(g_1(T)) + e^aN(g_2(T))\Big] \\
&= \frac{c}{r} + e^{-rT}\left(1-\frac{c}{r}\right) + 0\cdot\left(R-\frac{c}{r}\right) = e^{-rT} + \frac{c}{r}(1-e^{-rT}).
\end{aligned}$$

Monotonicity implies the bounds for the price. Bijectivity of the function v follows from its strict monotonicity and continuity. □

Corollary 20 *If $c > rR$ for the bond price p_t given by (4.9) it is valid: $p_t \geq R$ for all $t \in [0, T]$.*

Proof: Apply the lower bound R from Lemma 19 to the function v which corresponds to a bond with maturity $T - t$ for arbitrary $t \in [0, T]$. □

Remark: All results of the current part 4.1.3 are formulated for the firm value model described by a geometric Brownian motion and the constant bound M as in (3.17). But they can be immediately applied to the exponentially increasing (given by (3.18)) and stochastic (see (3.14)) boundary listed in Chapter 3. The distance to default parameter $a = \ln \frac{V_0}{M}$ can be left the same. The Corollary 18 remains valid.
The only thing which needs to be slightly changed is the bond price from Theorem 17. More precisely, the bond price is given as before by equation (4.9) but with other functions $l_{1,2}$ and $g_{1,2}$, namely: in the case of exponentially increasing bound

$$l_1(T) = \frac{a - (r - \mu_2 - \frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}}, \quad l_2(T) = \frac{-a - (r - \mu_2 - \frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}},$$

$$g_1(T) = \frac{(r - \mu_2 + \frac{1}{2}\sigma_V^2)T - a}{\sigma_V\sqrt{T}}, \quad g_2(T) = \frac{-(r - \mu_2 + \frac{1}{2}\sigma_V^2)T - a}{\sigma_V\sqrt{T}}$$

and in the case of a random bound

$$l_1(T) = \frac{a - (r - \mu_2 - \frac{1}{2}(\sigma_V^2 - \sigma_2^2))T}{\sqrt{(\sigma_V^2 + \sigma_2^2 - 2\rho\sigma_V\sigma_2)T}}, \; l_2(T) = \frac{-a - (r - \mu_2 - \frac{1}{2}(\sigma_V^2 - \sigma_2^2))T}{\sqrt{(\sigma_V^2 + \sigma_2^2 - 2\rho\sigma_V\sigma_2)T}},$$

$$g_1(T) = \frac{(r - \mu_2 + \frac{1}{2}(\sigma_V^2 - \sigma_2^2))T - a}{\sqrt{(\sigma_V^2 + \sigma_2^2 - 2\rho\sigma_V\sigma_2)T}}, \; g_2(T) = \frac{-(r - \mu_2 + \frac{1}{2}(\sigma_V^2 - \sigma_2^2))T - a}{\sqrt{(\sigma_V^2 + \sigma_2^2 - 2\rho\sigma_V\sigma_2)T}}.$$

4.2 Boundary with a Jump at Maturity

This case was introduced on page 26. It differs from the previous one by the bounding process which is given by (3.21) now.
The measure $\mu \in \mathcal{M}([0,T])$ is arbitrary. Expression (4.1) gives an example which can be often used in calculations. The firm's value is described by the equation (3.1). Default in this case can be equivalently interpreted as the first time when the process Y with parameters as in (3.22) crosses the time dependent bound a (3.22)

$$\tau_j := \inf\{t : Y(t) \geq a(t)\} = \inf S_j \tag{4.14}$$

where the set S_j is determined as

$$S_j = \left\{ t \in [0,T) : Y(t) \geq \ln \frac{V_0}{M} \} \cup \{ t \geq T : V(t) \geq \ln \frac{V_0}{M'} \right\}$$

Let p_t^j denote the bond price under this settings. According to definition 5,

$$p_0^j = E_Q(Re^{-r\tau_j}\mathbb{1}_{\{\tau_j\le T\}} + \mu_d(A_{\tau_j})).$$

The distribution of the default time $Q(\tau_j \le T)$ was determined in Chapter 3 (Lemma 8 and Corollary 9). The probability

$$\begin{aligned} Q(\tau_j = T) &= Q(\tau_j \le T) - Q(\tau_j < T) = Q(\tau_j \le T) - Q(\tau \le T) \\ &= N\left(\frac{\ln\frac{M'}{V_0} - \nu T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln\frac{M}{V_0} - \nu T}{\sigma\sqrt{T}}\right) \\ &+ \left(\frac{M}{V_0}\right)^{2r\sigma^{-2}-1}\left(N\left(\frac{\ln\frac{M^2}{V_0 M'} + \nu T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln\frac{M}{V_0} + \nu T}{\sigma\sqrt{T}}\right)\right) \end{aligned}$$

is used in the theorem below in order to find the price p^j of the bond if the bond's price p related to the bound with no jump at maturity T is known. More precisely, if the stopping time τ is defined as

$$\tau := \inf\{t : Y(t) \ge \ln\frac{V_0}{M}\}, \tag{4.15}$$

the price p_0 is given by

$$p_0 = E_Q\left(e^{-r\tau}R\mathbb{1}_{\{\tau\le T\}} + \mu_d(A_\tau)\right) \tag{4.16}$$

Recall that previously in the part 4.1 default was related to the stopping time τ and the measure μ was of a special form (4.1).

Theorem 21 *The price of the defaultable bond p_0^j at the time $t = 0$ in the current settings equals*

$$p_t^j = p_t - \Delta p \tag{4.17}$$

where the bond price p_t is given by (4.16) and the correction term Δp is

$$\Delta p = (1-R)e^{-rT}Q(\tau_j = T)$$

Proof: From the definitions of τ (4.15) and τ_j (4.14) it follows that $\tau_j \le \tau$. Moreover, on the set $\{\tau \le T\}$ the inequality above turns into equality $\tau = \tau_j$. The random sets $A_\tau, A_{\tau_j} \subset [0,T]$ coincide if $\tau \le T$ (in this case $\tau = \tau_j < T$) or if $\tau_j > T$ (in this case $A_\tau = A_{\tau_j} = [0,T]$). Thus,

$$A_{\tau_j} = \begin{cases} A_\tau \setminus \{T\}, & \tau_j = T,\ \tau \ne T \\ A_\tau, & \text{otherwise} \end{cases}$$

Note that from the continuity of the probability distribution of the stopping time τ (given explicitly in Corollary 7) it follows that

$$Q(\tau = T) = 0$$

and thus

$$Q(\tau_j = T, \tau \neq T) = Q(\tau_j = T).$$

By the construction of the stopping times τ and τ_j it is valid:

$$Q(\tau < T) = Q(\tau_j < T).$$

Summarizing the facts listed above, we conclude that

$$\begin{aligned} p^j &= E_Q(Re^{-r\tau_j}\mathbb{1}_{\{\tau_j<T\}} + Re^{-rT}\mathbb{1}_{\{\tau_j=T\}} + \mu(A_\tau) - \mu(\{T\})\mathbb{1}_{\{\tau_j=T,\ \tau\neq T\}}) \\ &= E_Q(Re^{-r\tau}\mathbb{1}_{\{\tau<T\}} + \mu(A_\tau)) + E_Q(Re^{-rT}\mathbb{1}_{\{\tau_j=T\}} - \mu(\{T\})\mathbb{1}_{\{\tau_j=T,\ \tau\neq T\}}) \\ &= E_Q(Re^{-r\tau}\mathbb{1}_{\{\tau\leq T\}} + \mu(A_\tau)) + (R-1)e^{-rT}Q(\tau_j = T) = p - \Delta p. \end{aligned}$$

In what follows the proof. □

Regard the difference of prices $\Delta p = p - p^j$ which represents the impact of the default at maturity risk. It is meant under the default at maturity risk the danger that the bond defaults at maturity due to the jump of the bounding process X_2 at T from M to M'.
If the parameters T and R are fixed, Δp can be interpreted as a constant times the probability $Q(\tau = T)$ of default at maturity. Figure 3.1 which shows the dependence of the probability $Q(\tau = T)$ on the quotient $\frac{M'}{M}$ gives an intuition about the dependence of Δp on $\frac{M'}{M}$. Note that from Corollary 9 which gives the explicit formula of the default probability in this case and Theorem 21 above it follows that

$$\lim_{\frac{M'}{M}\to 1+} p_j - p = (1-R)e^{-rT} \lim_{\frac{M'}{M}\to 1+} Q(\tau_j = T) = 0$$

and

$$\lim_{\frac{M'}{M}\to +\infty} p_j - p = (1-R)e^{-rT}(1 - Q(\tau < T))$$

Thus, the impact of the 'jump-of-border risk' on the bond price decreases when the time to maturity grows. Moreover, it decreases even faster than the probability

$$1 - Q(\tau < T) = Q(\tau \geq T)$$

of no jump before the maturity T.

4.3 Bond Price under Stochastic Interest Rate

Consider the probability space $(\Omega, \mathcal{F}, Q)$. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration on this space. We assume that $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions and that it is rich enough for two Brownian Motions W_r and W, which are not necessarily independent.
With the help of the Brownian Motion W_r the process of a spot interest rate r is modeled. Namely, it is a process described by the stochastic differential equation

$$dr(t) = \alpha(t, r)dt - \beta(t, r)dW_r(t). \tag{4.18}$$

The Brownian Motion W models the firm value process, which satisfies as usually the equation (3.1):

$$dV(t) = V(t)(r(t)dt + \sigma_V dW(t)), \quad V(0) = V_0.$$

Here $\sigma_V > 0$ is constant. The processes α, β are progressively measurable with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and such that the solution of the equation (4.18) exists.
Choose the bounding process X_2 to be stochastic now. Let it be given for some $m \in \mathbb{R}$ by

$$M(t) = M_0 e^{\int_0^t r(s)ds - mt}.$$

It is assumed that the savings account process $B(t) = e^{\int_0^t r(s)ds}$ is integrable on $[0, T]$.
In order to study the distribution of the first time when the processes X and M cross, let us apply an approach similar to the one from the Chapter 3. Regard the process Y defined by

$$Y(t) = \ln\left(\frac{V(t)}{M(t)}\right)$$

for $t \in [0, T]$. Define default time as a stopping time τ which indicates when the firm value V reaches the bound M for the first time:

$$\tau := \inf\{t : V(t) \leq M(t)\} = \inf\{t : Y(t) \leq 0\}.$$

Lemma 22 *The Process Y is given by*

$$Y(t) = \ln\left(\frac{V_0}{M_0}\right) + \left(m - \frac{1}{2}\sigma_V^2\right)t + \sigma_V W(t),$$

where the Brownian motion W is the same as in equation (3.1).

Proof: Since $\int_0^T r(t)dt$ is a process of bounded variation, we have:

$$dM(t) = M_0(r(t) - m)e^{\int_0^T r(t)dt - mt}dt = (r(t) - m)M(t)dt.$$

Applying the Ito formula we find that

$$dY(t) = \frac{dV(t)}{V(t)} - \frac{dM(t)}{M(t)} - \frac{dV(t)dV(t)}{2V^2(t)}.$$

The above calculated derivative of M and the stochastic differential equation (3.1) of the firm's value give us the following:

$$\begin{aligned} dY(t) &= \frac{V(t)(r(t)dt + \sigma_V dW(t))}{V(t)} \\ &\quad - \frac{M(t)(r(t) - m)dt}{M(t)} \\ &\quad - \frac{V^2(t)\sigma_V^2 dt}{2V^2(t)} \\ &= \left(m - \frac{1}{2}\sigma_V^2\right) dt + \sigma_V dW(t). \end{aligned}$$

Since the initial value of the process Y is $Y(0) = \ln\left(\frac{V(0)}{M(0)}\right)$ we get the statement of the lemma. □

Remark: As it is mentioned on page 50, the Brownian motion W from the lemma above is not necessarily independent of W_r and the parameters of the spot interest rate r from equation (4.18).

Lemma 22 implies that the default time also in the case of stochastic interest rate can be interpreted as the first passage time of the Brownian motion with a specific drift through a constant bound. This effect is due to the special choice of the boundary M in the case of stochastic interest rate.
Hence, Lemma 46 can be applied in the case of stochastic interest rate as well in order to find the distribution and the density function of default.

Corollary 23 *The density f_τ and the distribution function F_τ of the first time to passage through 0 of the process Y are given correspondingly by*

$$f_\tau(t) = \frac{\ln\frac{V_0}{M_0}}{\sigma\sqrt{2\pi t^3}} \exp\left(\frac{-\left(\ln\frac{V_0}{M_0} - \left(\alpha - \frac{1}{2}\sigma_V^2\right)t\right)^2}{2\sigma_V^2 t}\right),$$

$$F_\tau(t) = N\left(\frac{\ln\frac{V_0}{M_0} - (\alpha - \frac{1}{2}\sigma_V^2)t}{\sigma_V\sqrt{t}}\right) + \left(\frac{M_0}{V_0}\right)^{2\alpha\sigma_V^{-2}-1} N\left(\frac{\ln\frac{V_0}{M_0} + (\alpha - \frac{1}{2}\sigma_V^2)t}{\sigma_V\sqrt{t}}\right).$$

Proof: Apply Lemma 46 to the process $\tilde{W} = -Y + \ln\frac{V_0}{M_0}$ and the boundary $a = \ln\frac{V_0}{M_0}$ which gives the distribution of the default time
$\tau = \inf\{t : Y(t) \le 0\} = \inf\{t : \tilde{W}(t) \ge a\}$. □

Vasicek Interest Rate Dynamics Suppose that the interest rate has the Vasicek dynamics. It is given by the stochastic differential equation

$$dr(t) = (\theta - ar(t))dt + \sigma_r dW_r(t), \tag{4.19}$$

where $a, \theta, \sigma_r \in \mathbb{R}_+$. Integration of the equation (4.19) results in

$$r(t) = r(0)e^{-at} + \frac{\theta}{a}(1 - e^{-at}) + \sigma_r \int_0^t e^{-a(t-s)} dW_r(s).$$

Denote by $p_0^r(T) = Ee^{-\int_0^T r(s)ds}$ the price of the pure-discount bond at the time $t = 0$ with stochastic interest rate corresponding to the equation (4.19). It can be found explicitly and equals

$$p_0^r = \exp\left(\left(\frac{\theta}{a} - \frac{\sigma_r^2}{2a^2}\right)\left(\frac{1-e^{-aT}}{a} - T\right) - \frac{\sigma_r^2(1-e^{-aT})}{4a^3} - \frac{r(0)}{a}(1 - e^{-aT})\right). \tag{4.20}$$

Analogously to the Theorem 17 the price of the defaultable bond with the payment measure given by 4.1 under the assumption of stochastic interest rate with Vasicek dynamics (see equation (4.19)) equals

$$p_0 = Q(\tau > T)\left(p_0^r(T) + c\int_0^T p_0^r(t)dt\right) + \int_0^T \left(p_0^r(t) + c\int_0^t p_0^r(s)ds\right) f_\tau(t)dt, \tag{4.21}$$

where the probabilities and prices on the right hand side of (4.21) are given by (4.20) and Corollary 23.

Of course, the calculation of expression 4.21 is rather to be made using numerical procedures. Notice also that the method described here can be similarly used for other types of stochastic interest rate process.

4.4 Estimation under Lack of Information

Typically in practice, the information delivered to the investor even at default times is not complete, i.e.

$$\mathcal{F}(\tau_i) \subset \mathcal{F}_W(\tau), \quad \mathcal{F}(\tau_i) \neq \mathcal{F}_W(\tau)$$

Here it is considered the situation when the investor knows the prices of new bonds but does not know the distribution of default which determines the bond price. Recall that for the determination of this distribution the following parameters are relevant:

1. the riskless interest rate r;
2. the distance to default a;
3. the volatility of the firm value σ.

Usually, the riskless interest is an observable value. Thus, for practical purposes it is important to estimate the parameters $a \in A \subset \mathbb{R}_+$ and $\sigma \in \Sigma \subset \mathbb{R}_+$ in order to find the distribution of default. The suggested approach is the following:
Guess the volatility parameter. Let it be denoted by $\sigma_2 \in \Sigma$ (the 'real' parameter is denoted by σ_1). Then find a parameter $a_2 \in A$ such that the bond price calculated using the parameters r, σ_2 and a_2 equals to the known one which is based on the real parameters r, σ_1 and a_1. By Lemma 19, if σ_2 is fixed, such parameter a_2 is unique. It is shown in Theorem 27 that the estimated parameters determine the closest distribution of default time (in some sense specified later) to the real one. The results of Lemma 24, 26 and Corollary 25 will be used in order to prove Theorem 27.

Lemma 24 *Let $\nu = r - \frac{1}{2}\sigma^2 > 0$. Then $F : A \times \mathbb{R}_0 \to [0,1]$, where A is an open subset of $\mathbb{R}_+$ is a strictly convex function of the parameter $a \in A$.*

Proof: It is enough to prove that every summand of the formula

$$F(a,t) = N\left(\frac{-a-\nu t}{\sigma\sqrt{t}}\right) + e^{-a\sigma^{-2}\nu} N\left(\frac{-a+\nu t}{\sigma\sqrt{t}}\right), \tag{4.22}$$

which defines the distribution function F is a strictly convex function of the parameter a.
Let $N_1 : A \to \mathbb{R}$ denote the derivative of the first summand $N(\frac{-a-\nu t}{\sigma\sqrt{t}})$ with

respect to the variable a. Since $-a < 0$, it implies that $-a < \nu t$ for $t > 0, \nu > 0$. Thus, the function N_1 is an increasing function of the variable a:

$$N_1(a) = \frac{\partial N(\frac{-a-\nu t}{\sigma\sqrt{t}})}{\partial a} = \frac{-1}{\sigma^2 t\sqrt{2\pi}} \exp\left(-\frac{(a+\nu t)^2}{2\sigma^2 t}\right).$$

Consequently, the first summand is convex in a.
Regard now the second summand $e^{-a\sigma^{-2}\nu}N(\frac{-a+\nu t}{\sigma\sqrt{t}})$ of the formula (4.22). Denote by $N_2 : A \to \mathbb{R}$ its derivative with respect to the variable a. It equals

$$\begin{aligned} N_2(a) &= -2\nu\sigma^{-2}e^{-2a\nu\sigma^{-2}}\int_{-\infty}^{\frac{-a+\nu t}{\sigma\sqrt{t}}} \frac{e^{-u^2/2}du}{\sqrt{2\pi}} + \frac{\exp\left(-\frac{(-a+\nu t)^2}{2\sigma^2 t} - 2a\nu\sigma^{-2}\right)}{\sigma\sqrt{2\pi t}} \\ &= -2\nu\sigma^{-2}e^{-2a\nu\sigma^{-2}}N\left(\frac{-a+\nu t}{\sigma\sqrt{t}}\right) + N_1(a). \end{aligned}$$

Hence, N_2 is an increasing function of the variable a since $-e^{-2a\nu\sigma^{-2}}N\left(\frac{-a+\nu t}{\sigma\sqrt{t}}\right)$ and N_1 are increasing functions of a. □

Note that from the proof it follows that

$$\frac{\partial F(a,t)}{\partial t} = 2\left(\nu\sigma^{-2}e^{2a\nu\sigma^{-2}}N\left(\frac{-a+\nu t}{\sigma\sqrt{t}}\right) + N_1(a)\right).$$

Let us introduce the function $\tilde{F} : A \times \mathbb{R}_+ \times \Sigma \to [0,1]$ which represents the distribution function of the default time depending on parameters a and σ. It is given by

$$\tilde{F}(a,t,\sigma) = N\left(\frac{a-\nu(\sigma)t}{\sigma\sqrt{t}}\right) + e^{-a\sigma^{-2}\nu(\sigma)}N\left(\frac{a+\nu(\sigma)t}{\sigma\sqrt{t}}\right), \tag{4.23}$$

where $\nu(\sigma) = r - \frac{1}{2}\sigma^2$.
For $\boldsymbol{\nu} \in \mathcal{M}([0,T])$ denote by $\boldsymbol{\nu}^\perp := \{f \in C([0,T]) : \int_0^T f d\boldsymbol{\nu} = 0\}$ the class of orthogonal to the measure $\boldsymbol{\nu}$ continuous functions. Denote also by $||\cdot||_{\boldsymbol{\nu}}$ the seminorm given by

$$\| f \|_{\boldsymbol{\nu}} := \left(\int_0^T f^2 d\boldsymbol{\nu}\right)^{\frac{1}{2}}$$

for $f \in C([0,T])$.
Consider the price at the time 0 as the function $p : A \times \Sigma \to \mathbb{R}$ of parameters $a \in A$ and $\sigma \in \Sigma$.

Corollary 25 *Assume that $a_1, a_2 \in A$ and $\sigma_1, \sigma_2 \in \Sigma$ are such that*

$$p_0(a_1, \sigma_1) = p_0(a_2, \sigma_2).$$

Then the function $t \mapsto \tilde{F}(a_1, t, \sigma_1) - \tilde{F}(a_2, t, \sigma_2)$ is in $\boldsymbol{\nu}_d^{\perp}$, where $\boldsymbol{\nu} \in \mathcal{M}([0, T])$ is given by

$$\boldsymbol{\nu} = \frac{c - rR}{1 - R} \boldsymbol{\lambda}|_{[0,T]} + \delta_T.$$

Proof: Apply Theorem 2. □

Let $\boldsymbol{v_1}, \boldsymbol{v_2}$ and $\boldsymbol{v}$ be measures on $\mathcal{B}([0, T])$ such that $\boldsymbol{v_1} + \boldsymbol{v_2} \ll \boldsymbol{v}$. The $\mathcal{L}^1$-distance between measures $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ is defined (see, e.g. [50], page 136) by

$$d(\boldsymbol{v_1}, \boldsymbol{v_2}) := \int_0^T \left| \frac{\partial \boldsymbol{v_1}}{\partial \boldsymbol{v}} - \frac{\partial \boldsymbol{v_2}}{\partial \boldsymbol{v}} \right| d\boldsymbol{v}.$$

Lemma 26 *Let $g(t) := F(a_1, t, \sigma_1) - F(a_2, t, \sigma_2)$ be as in the previous corollary. Let the measure $\boldsymbol{\nu}$ be defined as*

$$\boldsymbol{\nu} = a\boldsymbol{\lambda}|_{[0,T]} + \delta_T$$

for some $a \in \mathbb{R}_+$. Assume that $g \in \boldsymbol{\nu}_d^{\perp}$.
Then for arbitrary $\eta > 0$ it exists $\epsilon > 0$ and a measure $\boldsymbol{\nu}^{\epsilon} \ll \boldsymbol{\nu}$ which satisfies the following conditions:

1. $\boldsymbol{\nu}^{\epsilon}([0, \epsilon]) = 0$
2. $d(\boldsymbol{\nu}, \boldsymbol{\nu}^{\epsilon}) < \eta$
3. $g \in (\boldsymbol{\nu}_d^{\epsilon})^{\perp}$

Proof: Consider the following cases:
Case 1: Assume $g(t) = 0$ for $\boldsymbol{\nu}$-almost all $t \in [0, T]$. The statement is obvious: take $\epsilon = \min(\frac{\eta}{2a}, T/2)$ and set $\boldsymbol{\nu}^{\epsilon} := \boldsymbol{\nu}|_{[\epsilon,T]}$. In this case clearly, $\boldsymbol{\nu}^{\epsilon} \ll \boldsymbol{\nu}$ and conditions 1-3 are satisfied, indeed:
by construction, condition 1 holds: $\boldsymbol{\nu}^{\epsilon}([0, \epsilon]) = 0$;
$d(\boldsymbol{\nu}, \boldsymbol{\nu}^{\epsilon}) = \int_0^{\epsilon} d\boldsymbol{\nu} = a\boldsymbol{\lambda}([0, \epsilon]) = a\epsilon \leq \frac{\eta}{2} < \eta$ which means that condition 2 is valid;
finally, since $g(t) = 0$ for $\boldsymbol{\nu}$-all $t \in [0, T]$, we have: $\int_0^T g d\boldsymbol{\nu}_d^{\epsilon} = \int_{\epsilon}^T g d\boldsymbol{\nu}_d = 0$. Hence, $g \in (\boldsymbol{\nu}_d^{\epsilon})^{\perp}$.
Case 2: There is $B \in \mathcal{B}([0, T])$ with $\mu(A) > 0$ such that $g(t) \neq 0$ for all $t \in A$.
1) Assume first that $T \in B$.
Fix some positive $\epsilon < \min(T, c_1, c_2)$, where

$$c_1 := (g(T))^{-1}, \quad c_2 := \frac{\eta}{a(1 + |c_1|e^{rT})},$$

For the chosen ϵ define
$$C_\epsilon := \frac{\int_0^\epsilon g d\boldsymbol{\nu}_d}{e^{-rT} g(T)}$$
and set $\boldsymbol{\nu}^\epsilon := \boldsymbol{\nu}|_{[0,T]} + C_\epsilon \delta_T$. $\boldsymbol{\nu}^\epsilon$ is a measure ($\boldsymbol{\nu}^\epsilon(B') \geq 0$ for all $B' \in \mathcal{B}([0,T])$), which is due to the fact that $\epsilon < c_1$ and consequently, $|C_\epsilon| < 1$. Then:
$\boldsymbol{\nu}([0,\epsilon]) = 0$ which means that condition 1 holds;
Since $|g(t)| \leq 1$ for all $t \in [0,T]$, we have: $|\int_0^\epsilon g d\lambda_d| \leq |\int_0^\epsilon g d\lambda| \leq \epsilon$. Using this and $\epsilon < c_2$, we obtain the following sequence of inequalities, which validates condition 2:

$$\begin{aligned}
d(\boldsymbol{\nu}, \boldsymbol{\nu}^\epsilon) &= \delta_T(T)|1 - (1 - C_\epsilon)| + \int_0^\epsilon d\boldsymbol{\nu} = a\boldsymbol{\lambda}([0,\epsilon]) + |C_\epsilon| \\
&\leq a\epsilon + \frac{a|\int_0^\epsilon g(t) d\boldsymbol{\lambda}_d|}{e^{-rT}|g(T)|} \\
&\leq a\epsilon(1 + e^{rT}|g(T)|^{-1}) < \eta.
\end{aligned}$$

Definition of C_ϵ implies that condition 3 is valid as well:

$$\begin{aligned}
\int_0^T g d\boldsymbol{\nu}_d^\epsilon &= \int_\epsilon^T g d(\boldsymbol{\nu}^\epsilon + C_\epsilon \delta_T)_d \\
&= \int_0^T g d\boldsymbol{\nu}_d - \int_0^\epsilon g d\boldsymbol{\nu}_d + C_\epsilon e^{-rT} g(T) \\
&= 0 - \int_0^\epsilon g d\boldsymbol{\nu}_d + \frac{\int_0^\epsilon g d\boldsymbol{\nu}_d}{e^{-rT} g(T)} e^{-rT} g(T) \\
&= 0.
\end{aligned}$$

2) The situation if $T \notin B$ can be treated similarly but it needs more technical details. Notice that there is $t_1 \in (0,T)$ such that $\boldsymbol{\nu}([t_1,T) \cap B) > 0$. Set $B_1 := [t_1,T) \cap B$. Obviously, $\boldsymbol{\lambda}(B_1) < T$.
Choose again some $\epsilon < \min(t_1, c_1', c_2')$, where

$$c_1' := \left(\int_{B_1} g d\boldsymbol{\lambda}_d\right)^{-1}, \quad c_2' := \eta a^{-1}\left(1 + \frac{T}{\int_{B_1} g d\boldsymbol{\lambda}_d}\right)^{-1}.$$

Set
$$C_\epsilon := \frac{\int_0^\epsilon g d\boldsymbol{\nu}_d}{\int_{A_1} g d\boldsymbol{\lambda}_d}.$$
Define $\boldsymbol{\nu}^\epsilon := \boldsymbol{\nu}|_{[\epsilon,T]} - C_\epsilon \boldsymbol{\lambda}|_{B_1}$. By construction, it was defined that $\epsilon < c_1'$ what ensures that $a - C_\epsilon > 0$. As a consequence, $\boldsymbol{\nu}^\epsilon$ is a measure absolutely continuous with respect to the measure $\boldsymbol{\nu}$. Verify conditions 1-3:

condition 1 is obviously satisfied;
condition 2 is implied by inequalities:

$$\begin{aligned} d(\boldsymbol{\nu}, \boldsymbol{\nu}^\epsilon) &= \int_0^\epsilon d\boldsymbol{\nu} + \int_{B_1} |a - (a - C_\epsilon)| d\boldsymbol{\lambda} \\ &= a\epsilon + |C_\epsilon| \boldsymbol{\lambda}(B_1) \\ &\leq a\epsilon + \frac{T \int_0^\epsilon g d\boldsymbol{\nu}_d}{\int_{B_1} g d\boldsymbol{\lambda}_d} \\ &\leq a\epsilon \left(1 + \frac{T}{\int_{B_1} g d\boldsymbol{\lambda}_d}\right) \leq \eta; \end{aligned}$$

the orthogonality required by condition 3 holds as well:

$$\begin{aligned} \int_0^T g d\boldsymbol{\nu}_d^\epsilon &= \int_\epsilon^T g d(\boldsymbol{\nu} - C_\epsilon \boldsymbol{\lambda}|_{B_1})_d \\ &= \int_0^T g d\boldsymbol{\nu}_d - \int_0^\epsilon g d\boldsymbol{\nu}_d + \int_{B_1} C_\epsilon g d\boldsymbol{\lambda}_d \\ &= 0 - \int_0^\epsilon g d\boldsymbol{\nu}_d + \frac{\int_0^\epsilon g d\boldsymbol{\nu}_d}{\int_{B_1} g d\boldsymbol{\lambda}_d} \int_{B_1} g d\boldsymbol{\lambda}_d = 0. \end{aligned}$$

□

Set now the measure $\bar{\boldsymbol{\nu}}^\epsilon$ to be defined by

$$\frac{d\bar{\boldsymbol{\nu}}^\epsilon(t)}{d\boldsymbol{\nu}^\epsilon(t)} := -\left(\frac{\partial \tilde{F}(a_2, t, \sigma_2)}{\partial a_2}\right)^{-1}. \tag{4.24}$$

Theorem 27 *If $r - \sigma_2^2/2 > 0$ for given $a_1 \in A,\ \sigma_1 \in \Sigma$ there is a unique value $a_2 \in A$ such that*

$$p(a_1, \sigma_1) = p(a_2, \sigma_2).$$

In this case the distribution function $t \mapsto \tilde{F}(a_2, t, \sigma_2)$ has the minimal $\| \cdot \|_{\bar{\boldsymbol{\nu}}^\epsilon}$-distance to the distribution function $t \mapsto \tilde{F}(a_1, t, \sigma_1)$:

$$\| \tilde{F}(a_1, t, \sigma_1) - \tilde{F}(a_2, t, \sigma_2) \|_{\bar{\boldsymbol{\nu}}^\epsilon} = \min_{a \in A} \| \tilde{F}(a_1, t, \sigma_1) - \tilde{F}(a, t, \sigma_2) \|_{\bar{\boldsymbol{\nu}}^\epsilon}.$$

Proof: The uniqueness of the parameter a_2 was stated in Lemma 19. If the function $\tilde{F}(a_2, \cdot, \sigma_2)$ with $a \in A$ has the minimal $\| \cdot \|_{\bar{\boldsymbol{\nu}}}^\epsilon$-distance to the distribution function $t \mapsto \tilde{F}(a_1, t, \sigma_1)$, the following equation is valid:

$$\frac{\partial \| \tilde{F}(a_1, t, \sigma_1) - \tilde{F}(a_2, t, \sigma_2) \|_{\bar{\boldsymbol{\nu}}^\epsilon}}{\partial a_2} = 0$$

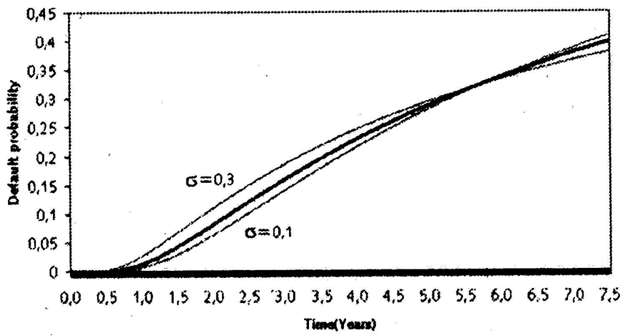

Figure 4.3: Default distributions depending on time $t \in [0, T]$ which correspond to the same bond price. For the thick line $\sigma = 0.2$, the upper line corresponds to $\sigma = 0.3$, the lower line corresponds to $\sigma = 0.1$.

or, equivalently,

$$\int_0^T (\tilde{F}(a_1, t, \sigma_1) - \tilde{F}(a_2, t, \sigma_2)) \frac{\partial \tilde{F}(a_2, t, \sigma_2)}{\partial a_2} d\bar{\boldsymbol{\nu}}^\epsilon = 0. \tag{4.25}$$

By virtue of (4.24) which defines $\bar{\boldsymbol{\nu}}^\epsilon$ it can be written as

$$\int_0^T (\tilde{F}(a_1, t, \sigma_1) - \tilde{F}(a_2, t, \sigma_2)) d\boldsymbol{\nu}^\epsilon = 0.$$

The last equality is valid by Lemma 26: the measure $\boldsymbol{\nu}^\epsilon$ is orthogonal to the difference between the distribution functions $t \mapsto (\tilde{F}(a_1, t, \sigma_1) - \tilde{F}(a_2, t, \sigma_2))$. Let us analyze the behavior of the expression under the integral sign in (4.25). For all $t \in [0, T]$, according to Proposition 48 and Lemma 24, $a \mapsto \tilde{F}(a, t, \sigma_2)$ is a strictly decreasing and convex function of the parameter a. It implies that for $a' < a'' \in A$ it is valid

$$\tilde{F}(a_1, t, \sigma_1) - \tilde{F}(a', t, \sigma_2) < \tilde{F}(a_1, t, \sigma_1) - \tilde{F}(a'', t, \sigma_2).$$

and convexity means that

$$\frac{\partial \tilde{F}(a', t, \sigma_2)}{\partial a'} < \frac{\partial \tilde{F}(a'', t, \sigma_2)}{\partial a''} < 0.$$

Inserting both inequalities above into (4.25) we obtain that a_2 corresponds indeed to the distribution with the minimal distance to the distribution $t \mapsto \tilde{F}(a_1, t, \sigma_1)$. Indeed, for $a' < a_2 < a''$ it holds

$$\frac{\partial \parallel \tilde{F}(a_1, t, \sigma_1) - \tilde{F}(a', t, \sigma_2) \parallel_{\bar{\boldsymbol{\nu}}^\epsilon}}{\partial a'} < 0 < \frac{\partial \parallel \tilde{F}(a_1, t, \sigma_1) - \tilde{F}(a'', t, \sigma_2) \parallel_{\bar{\boldsymbol{\nu}}^\epsilon}}{\partial a''}.$$

This proves the theorem. □

Picture 4.3 illustrates that distribution functions related to bonds which have the same prices are indeed close to each other.

Chapter 5

Portfolio of Bonds

In this chapter we consider a portfolio which consists of defaultable bonds. The face value of the portfolio is modeled as a point process. This process has jumps when one of the bonds in the portfolio defaults. Defaults correspond to stopping times related to a Brownian motion. Depending on the type of a stopping time, there are different methods to model the behavior of the bond's portfolio.
Throughout the current chapter the following situation is modeled:
A subcontracted organization (it can be thought of as a high-yield bond fund) manages its portfolio of defaultable bonds and delivers from time to time reports to investors. Thus, we may assume that the fund managers operate with a finer information flow in comparison to the small investors. Information available to the small investors updates at stopping times described below. The principal value of bonds in the fund portfolio is modeled as a point process X.

5.1 Principal Value Process

Regard a probability space $(\Omega, \mathcal{F}, P)$ with N-dimensional Brownian motion $W(\cdot) = \{W(t), \mathcal{F}_W(t) : 0 \le t < \infty\}$ on it with its components $W_1, \dots, W_N$:

$$W(\cdot) = \begin{pmatrix} W_1(t) \\ \vdots \\ W_N(t) \end{pmatrix}.$$

The components $W_1, \dots, W_N$ are assumed to be independent Brownian Motions on $(\Omega, \mathcal{F}, P)$. They are given by

$$W_i(\cdot) = \{W_i(t), \mathcal{F}_i(t) : 0 \le t < +\infty\} \text{ for } i = 1, \dots, N.$$

Let $(\tau_n)_{n\in\mathbb{N}_0}$ be a nondecreasing sequence of stopping times with respect to the filtration $(\mathcal{F}_W(t))_{t>0}$:

$$0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots$$

such that

$$\lim_{n\to\infty} \tau_n = \infty. \tag{5.1}$$

The corresponding counting process N is defined by

$$N(t) := \sum_{i=1}^{\infty} \mathbb{1}_{[\tau_i,+\infty]}(t) \tag{5.2}$$

for $t \geq 0$.

Fix the maturity time $T > 0$. Regard the marked point process X on $[0, T]$ given by:

$$X(t) = X_0 + \sum_{n=1}^{\infty} \mathbb{1}_{[\tau_n,T]}(t)\Delta X_{\tau_n},\ t \in [0, T], \tag{5.3}$$

where $\Delta X_{\tau_n} < \infty$ for $n \in \mathbb{N}$ are real valued random variables.

The process X has jumps exactly at stopping times $\{\tau_n\}$ of the Brownian motion W as defined above.

Economically, the process X represents the face value of the portfolio (see Definition 1).

A sample path of the process X is shown on Figure 5.1.

Let $\mathcal{F}^X(t)$ denote the minimal filtration generated by the process $X(t)$. Assume that the information about the Brownian motion W is not known completely to the investor and it is updated only at random stopping times $\{\tau_n\}$. Namely, let $\mathcal{F}'_W(t)$ be some (sub)-filtration of the filtration $\mathcal{F}_W(t)$ such that

1. $\mathcal{F}'_W(\tau_n) \subseteq \mathcal{F}_W(\tau_n),\ n \in \mathbb{N}_0$;
2. $\mathcal{F}'_W(t) = \mathcal{F}'_W(\tau_n),\ \tau_n \leq t < \tau_{n+1}$.

Define now the available information to the investor at the time $t > 0$ as

$$\mathcal{F}(t) := \sigma(\mathcal{F}^X(t) \cup \mathcal{F}'_W(t)). \tag{5.4}$$

Note that N is $(\mathcal{F}(t))_{t\geq 0}$-adapted. Moreover, condition (5.1) implies that N is locally bounded and has a locally bounded variation. Indeed, if $\{\tau_n : n \in \mathbb{N}_0\}$ is taken as a localizing sequence, then

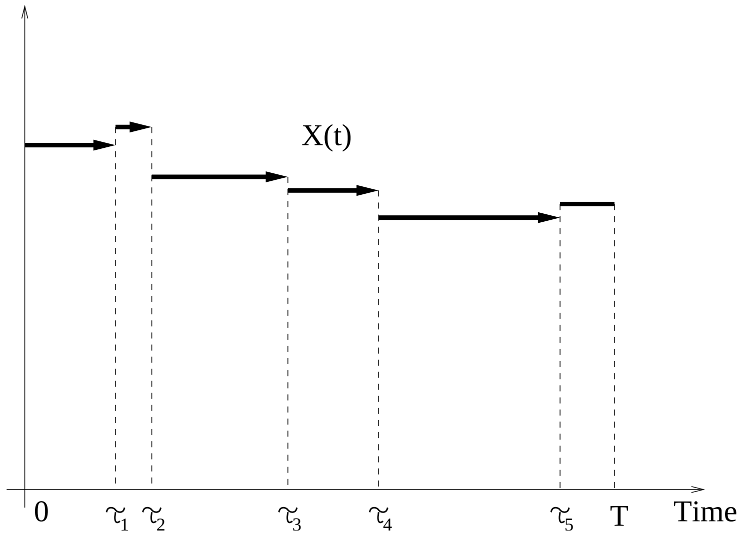

Figure 5.1: Sample path of the face value process X.

1. for arbitrary $n \in \mathbb{N}_0$ the counting process N is bounded on $[0, \tau_n]$ by the constant n:

$$n = N(\tau_n) \geq N(t) \text{ for all } t \leq \tau_n,$$

   which shows that N is **locally bounded**, and

2. the process N has **locally bounded variation** since N is a process with nonnegative increments starting in 0, its variation up to $t \in \mathbb{R}_0$ equals $N(t)$ itself.

Therefore, according to Theorem 10, there is a Doob-Meyer decomposition

$$N(t) = A(t) + M(t) \tag{5.5}$$

of the process N, where $(A, \mathcal{F})$ is predictable process and $(M, \mathcal{F})$ is a local martingale.

Motivation Economically the process X is interpreted as the face value process. The Brownian motion W acts in this model as a generator of uncertainties on the market. Information $\mathcal{F}$ about the market situation updates at default times $\{\tau_i : i \in N_0\}$. The following situation illustrates on one of the possible applications of the portfolio model described above:

It is wellknown that small investors (such as private investors, small investment or insuranse companies and banks) usually do not have a direct access to the market of high-yield bonds. It is due to many factors. One of the reasons is that this kind of investment turns to be very risky if the portfolio is not diversified. Second factor is the high volume of issued bonds. Issued bonds usually have high principal value, they are sold in packages which a small investor cannot buy. One more factor is small investor cannot easily access bond market.

As the result, small investor into high yield bonds needs a subcontracted organization (global bank or fund) which buys bonds directly from the issuer or on the bond market. This large organisation splits bonds, forms portfolios of high yield bonds and creates derivative products for small investors which become their clients. Funds deliver information about the current state of investment to their clients but of course they can not do it continuously in time.

Here it is modeled the situation when information updates at default times of bonds. Small investors receive the information about bond defaults and partially the information about actual situation on the bond market at this stopping times. Thus, every time a bond in portfolio defaults, its investor receives an updated information about the total principal value of the bonds in the portfolio. This corresponds to the filtration $(\mathcal{F}_X(t))_{t\in[0,T]}$ which is assumed to be the minimal known filtration. In addition, the partial information about the currently known market situation can be also delivered. Hence, the filtration $(\mathcal{F}(t))_{t\in[0,T]}$ available to the investor optionally includes sets from the filtration $(\mathcal{F}_W(t))_{t\in[0,T]}$.

Compensator Let us consider in more detailes Doob-Meyer decomposition of the counting process N given by equation (5.5).

Analogously to Theorem 11 we can find the compensator of the counting process N with respect to filtration $(\mathcal{F}(t))_{t\in[0,T]}$. From the construction of the filtration $(\mathcal{F}(t))_{t\in[0,T]}$ it follows that $(\mathcal{F}_N(t))_{t\in[0,T]}$ which is the minimal filtration generated by the counting process N is the subfiltration $(\mathcal{F}(t))_{t\in[0,T]}$. Recall that theorem 11 gives the compensator of N with respect to the filtration $(\mathcal{F}_N(t))_{t\in[0,T]}$:

Theorem 28 *Let N be a counting process defined by (5.2) with the filtration $(\mathcal{F}(t))_{t\in[0,T]}$ given by (5.4). The Q-compensator A of the process N with respect to the filtration $(\mathcal{F}(t))_{t\in[0,T]}$ is given by*

$$A(t) = \sum_{i\geq 1} A_i(t), \tag{5.6}$$

where

$$A_i(t) = \int_0^{t\wedge\tau_i} \frac{dQ(\tau_i \le s|\mathcal{F}(\tau_{i-1}))}{1 - Q(\tau_i < s|\mathcal{F}(\tau_{i-1}))}.$$

Proof: By the Doob-Meyer decomposition, it exists the compensator of the process N with respect to the filtration $(\mathcal{F}(t))_{t\in[0,T]}$. Let us show that formula (5.6) defines indeed the compensator of N.

Note that the process A given by (5.6) is predictable. For arbitrary $s, t \in [0, T]$ with $s < t$ it can be represented as

$$A(t) = \sum_{i:\tau_i \le s} A_i(t) + \sum_{i:\tau_i > s} A_i(t).$$

Analogously, the counting process N can be written in the form

$$N(t) = \sum_{i:\tau_i \le s} \mathbb{1}_{[\tau_i,+\infty]}(t) + \sum_{i:\tau_i > s} \mathbb{1}_{[\tau_i,+\infty]}(t).$$

Define the process M as the difference of the processes N and A:

$$M := N - A.$$

In order to proof the theorem it is enough to show that for arbitrary $s, t \in [0, T],\ s < t$ it holds

$$E_Q(M(t)|\mathcal{F}(s)) = M(s).$$

Denote by $F_i(u) := Q(\tau_i \le u|\mathcal{F}(\tau_{i-1})) = E_Q(\mathbb{1}_{\{\tau_i \le u\}}|\mathcal{F}(\tau_{i-1}))$.

On one hand we have

$$\begin{aligned}
E_Q(A(t)|\mathcal{F}(s)) &= E_Q\left(\sum_{i=1}^{\infty}\int_0^{t\wedge\tau_i} \frac{dF_i(u)}{1-F_i(u-)}\right|\left.\mathcal{F}(s)\right) \\
&= E_Q\left(\sum_{i=1}^{\infty}\int_0^{s\wedge\tau_i} \frac{dF_i(u)}{1-F_i(u-)}\right|\left.\mathcal{F}(s)\right) \\
&\quad + E_Q\left(\sum_{i=1}^{\infty}\int_{s\wedge\tau_i}^{t\wedge\tau_i} \frac{dF_i(u)}{1-F_i(u-)}\right|\left.\mathcal{F}(s)\right) \\
&= A(s) + E_Q\left(\sum_{i:\tau_i>s} E_Q\Big(\int_s^{t\wedge\tau_i} \frac{dF_i(u)}{1-F_i(u-)}|\mathcal{F}(\tau_i)\Big)\right|\left.\mathcal{F}(s)\right).
\end{aligned}$$

For i such that $\tau_i > s$ it holds

$$\begin{aligned}
E_Q(\textstyle\int_s^{t\wedge\tau_i} \frac{dF_i(u)}{1-F_i(u-)}|\mathcal{F}(\tau_i)) &= E_Q\left(\mathbb{1}_{\{\tau_i\le t\}}\textstyle\int_s^t\int_s^v \frac{dF_i(u)}{1-F_i(u-)}dF_i(v)\right|\left.\mathcal{F}(\tau_i)\right) \\
&\quad + E_Q\left(\mathbb{1}_{\{\tau_i>t\}}\textstyle\int_s^t \frac{dF_i(u)}{1-F_i(u-)}\right|\left.\mathcal{F}(\tau_i)\right).
\end{aligned} \tag{5.7}$$

By virtue of the formula (see B.2)

$$A(t)B(t) - A(s)B(s) = \int_s^t A(u-)dB(u) + \int_s^t B(u)dA(u),$$

it is valid

$$\begin{aligned}
\int_s^t \int_s^v \frac{dF_i(u)}{1 - F_i(u-)} dF_i(v) &= F_i(t) \int_s^t \frac{dF_i(u)}{1 - F_i(u-)} - \int_s^t \frac{F_i(u-)dF_i(u)}{1 - F_i(u-)} \\
&= (F_i(t) - 1) \int_s^t \frac{dF_i(u)}{1 - F_i(u-)} + F_i(t) - F_i(s).
\end{aligned}$$

Hence, equality (5.7) can be written as

$$\begin{aligned}
E_Q\left(\int_s^{t\wedge\tau_i} \frac{dF_i(u)}{1 - F_i(u-)} \middle| \mathcal{F}(\tau_i) \right) &= (F_i(t) - 1) \int_s^t \frac{dF_i(u)}{1 - F_i(u-)} + F_i(t) - F_i(s) \\
&\quad + Q(\tau_i > u|\mathcal{F}(\tau_{i-1})) \int_s^t \frac{dF_i(u)}{1 - F_i(u-)} \\
&= F_i(t) - F_i(s).
\end{aligned}$$

Finally for the process A we obtain

$$\begin{aligned}
E_Q(A(t)|\mathcal{F}(s)) &= A(s) + E_Q\left(\sum_{i:\tau_i > s} E_Q(\mathbb{1}_{\{s<\tau_i\le t\}}|\mathcal{F}(\tau_{i-1})) \middle| \mathcal{F}(s) \right) \\
&= A(s) + E_Q\left(\sum_{i:\tau_i > s} \mathbb{1}_{\{\tau_i \le t\}} \middle| \mathcal{F}(s) \right).
\end{aligned}$$

On the other hand, for the process N it is valid:

$$\begin{aligned}
E_Q(N(t)|\mathcal{F}(s)) &= E_Q\left(\sum_{i:\tau_i \le s} \mathbb{1}_{\{\tau_i,+\infty\}}(t) \middle| \mathcal{F}(s) \right) + E_Q\left(\sum_{i:\tau_i > s} \mathbb{1}_{\{\tau_i,+\infty\}}(t) \middle| \mathcal{F}(s) \right) \\
&= N(s) + E_Q\left(\sum_{i:\tau_i > s} \mathbb{1}_{\{\tau_i \le t\}} \middle| \mathcal{F}(s) \right).
\end{aligned}$$

Note that summands in both expressions are equal. Thus, for the difference of the processes N and A it holds

$$\begin{aligned}
E_Q(M(t)|\mathcal{F}(s)) &= E_Q(N(t)|\mathcal{F}(s)) - E_Q(A(t)|\mathcal{F}(s)) \\
&= N(s) - A(s) = M(s).
\end{aligned}$$

It imples that M is a local martingale. Since $A(0) = 0$ according to the Doob-Meyer decomposition, formula (5.6) defines the unique compensator process. This proves the statement of the theorem. □

Remark: Obviously, if F_i has density f_i for all $i \in \mathbb{N}$, then F_i is continuous: $F_i(s) = F_i(s-)$, $s \in [0, T]$ and there is an intensity λ given by

$$\lambda(t) = \sum_{i \geq 1} \lambda_i(t), \tag{5.8}$$

where

$$\lambda_i(s) = \mathbb{1}_{[0,t\wedge\tau_i]}(s) \frac{f_i(s)}{1 - F_i(s)}.$$

Notice that if intensity λ exists, the compensator A is continuous and hence stopping times $\{\tau_i :\in \mathbb{N}\}$ are totally inaccessible.

5.2 Portfolio 'Chain of Bonds'

5.2.1 Definition and Construction

The portfolio described in this section represents in some sense the most important in the present context example of a bond portfolio. It is the so called 'chain of bonds' portfolio. Its main property is that during the lifetime of the portfolio, there is constantly only one bond in it. Bonds which can be in the portfolio are assumed to be defaultable. Every time a bond which is currently in the portfolio defaults, some compensation will be paid by its issuer. After the compensation was paid, a new bond will be bought on that money. In other words, the constant amount of bonds in the portfolio is kept due to their consequent substitution but not due to their reliability.
It is assumed that in general the issuers of the new bond and the previous one are different. Default of the new bond depends on the development of the firm value process of its issuer. Firm value is modeled as a stochastic process (5.9) generated by a Brownian motion defined on some random interval.
We look at the situation from the point of view of the portfolio owner. Since his portfolio consists of fixed rate products for which the value of coupon payment directly depends on the face value, it is important to model the process of the face value of bonds in the portfolio rather than their market value etc. The process of the face value is constant between defaults and jumps when default happens. Notice that the portfolio holder has an information about the face value process but usually can not observe the process (5.9). This

explains the choice of filtration (5.4) we work with.
Natural questions an investor can ask himself are: what is the distribution of the face value of the portfolio at maturity? what is an expected coupon payment from this portfolio? etc. These questions are treated in the current section. The construction works in the following way:
Let $(\Omega, \mathcal{F}, P)$ be a probability space, it is assumed to be big enough to carry Brownian motion W, which generates filtration $(\mathcal{F}_W(t))_{t\geq 0}$.
Regard the sequence of stopping times with respect to the filtration $(\mathcal{F}_W(t))_{t\geq 0}$:

$$\tau_0 := 0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq \ldots .$$

Define stopping times as $\tau_i = \inf\{t \geq \tau_{i-1} | (t, V(t)) \in B_i\}$, where the firm value process V is the solution of SDE

$$dV(t) = f(V(t), t)dt + g(V(t), t)dW(t), \tag{5.9}$$

$$V(0) = v_0$$

and $B_i \subseteq \mathbb{R} \times \mathbb{R}_+$ are random sets, $i \in \mathbb{N}$.
Let $Q \ll P$ denote a measure such that the discounted firm value processes $e^{-rt}V(t)$ is a martingale.
Some relevant for praxis examples of default distributions are given in Chapter 3. Recall that in the first example the firm value processes are modeled as geometric Brownian motions and the random sets are the lower half-planes $B_i = [M_i, -\infty) \times \mathbb{R}_+$. It describes the situation when the firm value approach can be applied directly and there is no accounting noise. But in the presence of accounting noise ([8]), B_i have to be modeled as random sets.
In the current work it is studied the partial case of the general firm value process given by equation (5.9). It is based on the strong Markov property of the Brownian Motion formulated in Theorem 50 from Appendix A.2. Namely, it is assumed that on every random time interval $[\tau_{i-1}, \tau_i]$ the process V coincides with the process V_i which is given by SDE

$$dV_i(t) = f_i(V_i(t), t)dt + g_i(V_i(t), t)dW_i(t), \quad t \geq 0$$

where f_i, g_i for $i \in \mathbb{N}$ are deterministic functions and the Brownian Motion W_i is defined by

$$W_i(s - \tau_{i-1}) := W(s) - W(\tau_{i-1}) \text{ for } s \geq \tau_{i-1}.$$

More precisely, $V(t) = V_i(t + \tau_{i-1})$ for $t \in [\tau_{i-1}, \tau_i]$ with $i \in \mathbb{N}$ and the boundary condition $V_1(0) = V(0) = v_0$. Note that according to this condition, $V_{i+1}(0) = V_i(\tau_{i+1} - \tau_i)$ which can be viewed as a boundary condition for the

processes V_k, $k \geq 2$ and which makes the process V continuous on the other hand.
The process V_i represents here the face value of the firm, which issued the i-th bond in the portfolio.

Remark: Notice that according to this model any distribution $Q(\tau \leq t)$ of the stopping time τ can be modeled as a default time. Indeed, by setting tautologically

$$B_i := \left\{ (v,t) \in \mathbb{R} \times \mathbb{R}_+ : v \leq V_i(t) - \frac{1}{2} + \mathbb{1}_{[\tau,+\infty)}(t) \right\}$$

the default time τ_{i+1} equals

$$\begin{aligned} \tau_{i+1} &= \inf \left\{ t : V_i(t) \leq V_i(t) - \frac{1}{2} + \mathbb{1}_{[\tau,+\infty)}(t) \right\} \\ &= \inf \left\{ t : \frac{1}{2} \leq \mathbb{1}_{[\tau,+\infty)}(t) \right\} = \tau. \end{aligned}$$

Thus, we obtain that the distribution of the default time τ_{i+1} coincides with the distribution of the stopping time τ.
As it can be seen, the representation of default by the means of random sets B_i and firm value process is not unique. Even if the firm value process is fixed, there is a family of random sets $\mathcal{B} := \{B_i\}$ such that the distribution of the first passage time when the firm value process enters the random region B_i is the same for all $B_i \in \mathcal{B}$. Among all random sets which lead to the same default distribution usually we choose a random set which gives the most clear economic interpretation or the one which is easier to work with.

Equation (5.2) defines the corresponding counting process N.
As it is mentioned in [17], if firm value is not observable starting from the moment its bond have been purchased, then the counting process N has an intensity λ. The intensity λ is an $(\mathcal{F}(t))_{t\geq 0}$-predictable process defined on the time interval $[0,T]$. We set the technical condition that there is $\lambda^* > 0$ such that $\lambda(t) \leq \lambda^*$ for all $t \in [0,T]$.
We concentrate ourselves on the studying of the phenomena related to the marked point process X given by equation (5.3), where $X_0 \in \mathbb{R}_+$ and ΔX_{τ_i} takes values in some measurable subspace of $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.
Let the initial investment be equal to x_0. During the lifetime of the portfolio there is exactly one bond in it. The first bond was bought at the time $t = 0$ for the price p_0. Thus, the initial principal value of the portfolio is $X(0) = \frac{x_0}{p_0}$.
Let τ_1 denote the time of its default. If $\tau_1 \leq T$, recovery with the rate R will

be paid instead. On the money obtained from the recovery the next bond will be bought. Let p_1 denote its price, τ_2 denote the random time of its default. Every time the i-th bond in the chain portfolio defaults up to time T, a next one ($(i+1)$-th) will be bought for the price p_i on the money obtained from the payment of the recovery with the rate R. At the maturity time T the face value of the portfolio will be paid back to the bondholder.
The construction above implies that the jumps $\{\tau_i : i \geq 1\}$ of the process X correspond to the jumps of the counting process N. The size of the jumps is

$$\Delta X_i = X(\tau_i-) \left(\frac{R}{p_i} - 1 \right).$$

We conclude that the process X defined by (5.3) can be represented now as $X(t) = X_0 + \sum_{i=1}^{N(t)} X(\tau_i-) \left(\frac{R}{p_{\tau_i}} - 1 \right)$. Thus, it satisfies SDE

$$dX(t) = X(t-) \left(\frac{R}{p_t} - 1 \right) dN(t), \tag{5.10}$$

$$X(0) = X_0.$$

The solution of SDE (5.10) is a stochastic exponential

$$X(t) = X_0 \prod_{i=1}^{N(t)} \left(\frac{R}{p_{\tau_i}} \right). \tag{5.11}$$

5.2.2 Properties

From the construction of the process X given by equation (5.11) it follows that the process X is positive a. s. since for every $t \in [0, T]$ the value of the process X at t is a product of positive multipliers $\left(\frac{R}{p_i} \right) > 0$. But, it is not correct to think that the multipliers are less than 1.
Generally speaking, the process X is not necessarily a decreasing process. Moreover, in some cases the process X is not even a process bounded from above by some constant. But L^p-norm of the process X is bounded for arbitrary $p > 0$ (see Proposition 29).
Consider first an example of a portfolio such that for arbitrary predefined $C > 0$ the probability $Q(X(T) > C) > 0$ is positive. In order to show this, it is ensured first that there is $a > 1$ such that all multipliers from expression (5.11) exceed a:

$$\frac{R}{p_i} \geq a \text{ for } \tau_i \in [0, T/2] \text{ for all } i \in \mathbb{N},$$

and second that for arbitrary predefined $n \in \mathbb{N}$ the probability of not less than n jumps up to the time $\frac{T}{2}$ and no jumps afterwards on the time interval $[\frac{T}{2}, T]$ is positive:

$$Q(N(T/2) \geq n, N(T) = N(T/2)) > 0.$$

As a consequence, $X(T) \geq a^{N(T)}$ and $Q(a^{N(T)} > C) > 0$ what gives the required construction.

Example 12 We fix $r > 0$ and maturity $T \geq 4$.
For the sake of simplicity we consider a zero coupon bond, i.e. the payment measure is set to be $\mu := \delta_T$.
Let the counting process N constructed here be a renewal process: the successive times between defaults are independent and have the same distribution. For some $t_1 \in (0, \frac{T}{2} - 1)$ and $\lambda^* > 1$ which will be specified below set

$$\lambda_0 := \mathbb{1}_{[0,t_1]} + \lambda^* \mathbb{1}_{[\frac{T}{2}-1,T]}$$

to be the intensity which corresponds to that common distribution. The set of available intensities at $t \in [0, \frac{T}{2}]$ consist of one element, namely

$$\Lambda(t) := \{\mathbb{1}_{[t,t+t_1]} + \lambda^* \mathbb{1}_{[t+\frac{T}{2}-1,t+T]}\}.$$

In other words, the intensity λ of N is stochastic and it is given by

$$\lambda(t) = \lambda_0(t - \alpha(t-)) \text{ for } t \in [0, T],$$

where $\alpha(t-) = \sup\{\tau_i : i \in \mathbb{N}_0, \tau_i < t\}$.
It can be seen that the default probability corresponding to λ_0 in this case is given by

$$Q(\tau \leq t) = \begin{cases} 1 - e^{-t}, & \text{if } t \in [0, t_1]; \\ 1 - e^{-t_1}, & \text{if } t \in (t_1, \frac{T}{2} - 1); \\ 1 - e^{-t_1 - \lambda^*(t - \frac{T}{2} + 1)}, & \text{if } t \in (\frac{T}{2} - 1, T]; \\ 1 - e^{-t_1 - \lambda^*(\frac{T}{2} + 1)}, & \text{if } t > T. \end{cases}$$

The current goal is to demonstrate that there are pairs (t_1, λ^*) such that the prices of bond which can be bought on the time interval $[0, \frac{T}{2}]$ are less than recovery R (if the bonds are not yet defaulted). In this case the process X increases after every jump. The number n of jumps on the time interval $[0, \frac{T}{2}]$ must be big enough to achieve $X(\frac{T}{2}) = x_0 \prod_{i=1}^{n} \left(\frac{R}{p_\lambda(\tau_i)} \right) > C$.
Show that under the current settings, the price of a bond p_i purchased at

the random time $\tau_i \in [0, \frac{T}{2}],\ i \in \mathbb{N}_0$ can be bounded from above by a convex combination of $\{R, Re^{-(\frac{T}{2}-1)r}, e^{-r\frac{T}{2}}\}$.
We find the price first. It equals:

$$\begin{aligned}
p_{\tau_i} &= E_Q(Re^{-r(\tau_{i+1}-\tau_i)} + \mathbb{1}_{\{\tau_{i+1}>T\}}e^{-r(T-\tau_i)}|\mathcal{F}_W(\tau_i)) \\
&= R\int_{\tau_i}^{T} e^{-r(t-\tau_i)}dP(\tau \le t-\tau_i) + e^{-r(T-\tau_i)}Q(\tau > T-\tau_i) \\
&= R\int_{\tau_i}^{\tau_i+t_1} e^{-(r+1)(t-\tau_i)}dt \\
&\quad +Re^{-t_1}\int_{\frac{T}{2}-1+\tau_i}^{T} \lambda^* e^{-(r+\lambda^*)(t-\tau_i)}dt \\
&\quad +e^{-r(T-\tau_i)}e^{-t_1-\lambda^*(\frac{T}{2}-\tau_i+1)}.
\end{aligned}$$

Since $e^{-(1+r)s} \le e^{-s}$ for $r, t > 0$, it is valid

$$\int_0^t e^{-(1+r)s}ds \le \int_0^t e^{-s}ds = 1 - e^{-t}, \quad t \in \mathbb{R}_+.$$

Now the summands can be estimated by

$$\begin{aligned}
R\int_{\tau_i}^{\tau_i+t_1} e^{-(r+1)(t-\tau_i)}dt &\le R(1-e^{-t_1}), \\
Re^{-t_1}\int_{\frac{T}{2}-1+\tau_i}^{T} \lambda^* e^{-(r+\lambda^*)(t-\tau_i)}dt &\le e^{-r(T-\tau_i)}e^{-t_1-\lambda^*(\frac{T}{2}-\tau_i+1)}, \\
e^{-r(T-\tau_i)}e^{-t_1-\lambda^*(\frac{T}{2}-\tau_i+1)} &\le e^{-r(T-\tau_i)}e^{-t_1-\lambda^*(\frac{T}{2}-\tau_i+1)}.
\end{aligned}$$

Denote by $\alpha_1 := 1 - e^{-t_1}$ and by $\alpha_2 := e^{-t_1-\lambda^*\frac{T}{2}}$. Then $1 - \alpha_1 - \alpha_2 = e^{-t_1}(1 - e^{\lambda^*\frac{T}{2}})$. Notice that $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 \le 1$.
Hence, the price of a bond purchased at $\tau_i \in [0, \frac{T}{2}]$ is bounded indeed:

$$p_i \le K(\alpha_1, \alpha_2).$$

Here $K(\alpha_1, \alpha_2)$ is a convex combination

$$K(\alpha_1, \alpha_2) = R\alpha_1 + Re^{-(\frac{T}{2}-1)r}(1-\alpha_1-\alpha_2) + e^{-r\frac{T}{2}}\alpha_2$$

of numbers $R,\ Re^{-(\frac{T}{2}-1)r},\ e^{-r\frac{T}{2}}$ such that

$$\lim_{\alpha_1,\alpha_2\to 0} K(\alpha_1, \alpha_2) = Re^{-(\frac{T}{2}-1)r}.$$

It means that small enough α_1^*, α_2^* can be chosen, such that $Re^{-(\frac{T}{2}-1)r} \leq K(\alpha_1^*, \alpha_2^*) \leq b < R$ for some fixed b. Let α_1^* and α_2^* be such that $b := K(\alpha_1^*, \alpha_2^*) = Re^{-r\frac{T-3}{2}}$ and fix t_1 and λ^* to be such that the following equalities hold:

$$\alpha_1^* = (1 - e^{-t_1})$$

and

$$\alpha_2^* = e^{-t_1 - \lambda^* \frac{T}{2}}.$$

For the parameters t_1, λ^* fixed above, $p_{\tau_i} \leq Re^{-r\frac{T}{2}}$ and consequently, for all $t \in [0, \frac{T}{2}]$

$$a := \frac{R}{p_t} \geq e^{r\frac{T-3}{2}}.$$

For $n \in \mathbb{N}$ such that $n > \frac{T}{2t_1}$ the probability of n or more defaults up to time $\frac{T}{2}$ is positive:

$$Q(N(T/2) \geq n) > 0.$$

Hence, it is valid:

$$\begin{aligned} Q(X(T/2) > C) &\geq Q(a^{N(T/2)} > C) \\ &\geq Q\left(N\left(\frac{T}{2}\right) \geq \max\left(\frac{T}{2t_1}, \log_a C\right)\right) > 0. \end{aligned}$$

Since $Q(X(\frac{T}{2}) = X(T)) \geq Q(N(\frac{T}{2}) = N(T)) > 0$ we conclude that $Q(X(T) > C) > 0$.

In order to illustrate the example above numerically, consider $T = 6$; $R = 0,6$; $r = 0,03$ and the parameters $t_1 = 0,2$ and $\lambda^* = 20$. It provides that $p_t \leq Re^{-r(T/4-1/2)} = 0.582267$, $t \in [0, T/2]$. In what follows

$$X(t) = \frac{x_0}{p_0} \prod_{i=1}^{N(t)} \frac{R}{p_{\tau_i}} \geq 1,03045^{N(t)+1}.$$

Thus, the process X from equation (5.11) is not necessarily bounded from above.

Nevertheless, Proposition 29 states that this process is L^p bounded.

Proposition 29 *Assume that $r \geq 0$ and that default intensity λ is bounded from above, i.e. there is $\lambda^* > 0$ such that $\lambda^* \geq \lambda(t)$ for all $t \in [0, T]$. Then for $p \in (0, \infty)$ it is valid:*

i) $X(t) \in \mathcal{L}^p(\Omega, \mathcal{F}, Q)$.

ii) $X^{-1}(t) \in \mathcal{L}^p(\Omega, \mathcal{F}, Q)$.

More precisely,

$$||X(t)||_p \leq \frac{x_0}{R} \exp\left(rT + \frac{1}{p}\lambda^* T e^{prT}\right)$$

and

$$||X^{-1}(t)||_p \leq \frac{R + \mu([0,T])}{x_0} \exp\left(\frac{\lambda^* T}{p}\left(1 + \frac{\mu([0,T])}{R}\right)^p\right).$$

Proof: By Lemma 1, if a bond was bought at $s = 0$, then the following two inequalities can be established:

$$p_0 \geq Re^{-rT}$$

and

$$p_0 \leq R + \mu_d([0,T]) \leq R + \mu([0,T]).$$

Thus, for arbitrary $s \in [0,T]$ it is valid:

$$p_s \geq Re^{-r(T-s)} \geq Re^{-rT}$$

and

$$p_s \leq R + \mu([0,T]).$$

From (5.11) we conclude that the value of the process X at arbitrary time $t \in [0,T]$ is bounded from above by the random variable $x_o e^{rT(N(t)+1)}$ (where N is the counting process (5.2) corresponding to X):

$$X(t) \leq \frac{x_0}{p_0}\left(\frac{R}{Re^{-rT}}\right)^{N(t)} \leq x_0 \frac{e^{rTN(t)}}{Re^{-rT}} \leq \frac{x_0}{R} e^{rT(N(t)+1)}.$$

It implies that

$$\begin{aligned} E(X^p(t)) &\leq \sum_{k=0}^{\infty} \left(\frac{x_0}{R} e^{(k+1)rT}\right)^p Q(N(t) = k) \\ &\leq \left(\frac{x_0}{R} e^{rT}\right)^p \sum_{k=0}^{\infty} e^{kprT} e^0 \frac{(\lambda^* T)^k}{k!} = \left(\frac{x_0}{R}\right)^p \exp(rTp + \lambda^* T e^{prT}), \end{aligned}$$

which proves immediately part i).
Analogously, it can be seen that for arbitrary $t \in [0,T]$ the value $\frac{1}{X(t)}$ is

bounded from above by the random variable $x_0^{-1}R\left(1+\frac{\mu([0,T])}{R}\right)^{N(t)+1}$. In what follows that

$$\begin{aligned} X^{-1}(t) &\leq \frac{p_0}{x_0}\left(\frac{R+\mu([0,T])}{R}\right)^{N(t)} \\ &\leq \frac{R+\mu([0,T])}{x_0}\left(1+\frac{\mu([0,T])}{R}\right)^{N(t)} \\ &= \frac{R}{x_0}\left(1+\frac{\mu([0,T])}{R}\right)^{N(t)+1}. \end{aligned}$$

The proof of part ii) follows from the inequalities below

$$\begin{aligned} E\left(X^{-p}(t)\right) &\leq \sum_{k=0}^{\infty}\left(\frac{R}{x_0}\right)^p\left(1+\frac{\mu([0,T])}{R}\right)^{p(k+1)}Q(N(t)=k) \\ &\leq \left(1+\frac{\mu([0,T])}{R}\right)^p\left(\frac{R}{x_0}\right)^p\sum_{k=0}^{\infty}\left(1+\frac{\mu([0,T])}{R}\right)^{pk}\frac{(\lambda^*T)^k}{k!} \\ &= \left(\frac{R+\mu([0,T])}{x_0}\right)^p\exp\left(\left(1+\frac{\mu([0,T])}{R}\right)^p\lambda^*T\right). \end{aligned}$$

□

Remark: In addition to showing that the process X is not bounded the construction used in example 12 also implies that the process X is, generally speaking, not necessarily a supermartingale.
Notice first that the study of sub- (super-)martingale property of the process X has sense due to Proposition 29. It states in particular that

$$||X(t)||_1 = E|X(t)| < \infty, \text{ for all } t \in [0,T].$$

It is intuitively clear that the submartingale property

$$E(X(t)|\mathcal{F}(s)) \geq X(s)$$

is not satisfied for arbitrary $s < t,\ s,t \in [0,T]$.
Example 12 demonstrates a construction of the process X which is not a supermartingale.
Indeed, according to the construction,

$$E[X(t_1)|\mathcal{F}(0)] \geq x_0 Q(\tau_1 > t_1) + x_0 e^{r\frac{T}{2}} Q(\tau_1 \leq t_1) > x_0$$

since $Q(\tau_1 \le t_1) > 0$.
This implies that X fails to be a supermartingale. It means that under certain conditions and in particular in Example 12 default leads to increase of the firm value.
Nevertheless, in the most cases considered in this work, the process X is a supermartingale. Corollary 20 provides us with an example of the face value process X which is a supermartingale. The portfolio is constructed in the following way:
All the bonds have the payment measure $\mu = c\boldsymbol{\lambda}|_{[0,T]} + \delta_T$ such that $c > rR$ for the recovery rate $R \in [0,1]$. The default times $\{\tau_i : i \in \mathbb{N}\}$ arise from the firm value model. They are defined as $\tau_i := \inf\{t \ge \tau_{i-1} : V(t) \ge M_i\}$. Corollary 20 states that for the bond price at arbitrary $t \in [0,T]$ we have $p_t > R$. Hence, $\frac{R}{p_{\tau_i}} < 1$ and the face value process X given by $X(t) = \frac{x_0}{p_0} \prod_{\tau_i \le t} \left(\frac{R}{p_{\tau_i}}\right)$ is a submartingale:

$$\begin{aligned} E_Q(X(t)|\mathcal{F}(s)) &= E_Q\left(X(s) \prod_{s<\tau_i\le t} \left(\frac{R}{p_{\tau_i}}\right) \middle| F(s)\right) \\ &= X(s) E_Q\left(\prod_{s<\tau_i\le t} \left(\frac{R}{p_{\tau_i}}\right)\right) \le X(s) \end{aligned}$$

We restrict further ourselves to the studying of a certain kind of chain portfolios.
Let the measure μ describe the payment received by holders of a bond. Assume that all the bonds in our chain portfolio correspond to the same measure μ. In other words, the i-th bond inherits the measure μ from the $(i-1)$-th bond. Economically it means that the manager of the portfolio chooses bonds in a way that they have the same coupon payments at the same fixed points of time on the interval $[0,T]$. For example, he might concentrate himself only on bonds which pay some fixed in advance at the time $t = 0$ coupon constantly or in the beginning of every month, every quarter or every year etc. Thus, the total discounted payment obtained from the portfolio in the case we regard can be written in a relatively simple form

$$\int_0^T e^{-rt} X(t) d\mu(t) \tag{5.12}$$

A natural question an investor can ask himself is what is the expectation of the discounted payment (5.12). In order to answer this question it should be specified:
-the measure P which describes investor's believes

-distribution of X with respect to P.
If measure P is chosen to be 'risk-neutral' i.e. the same as is used in calculation of bond prices, the expectation 5.12 can be found.
Regard first a special case of the chain portfolio (5.11), a chain portfolio with restricted amount of crashes. It can be interpreted as a highest possible amount of defaults a portfolio can suffer, determined by a financial institution in order to protect investors against a credit risk.
The only way to bound the maximal number of defaults by k is to buy a default free bond not later than right after the k-th default. It means that the default free must be bought at one of the stopping times from the set $\{\tau_i \wedge T : i = 0, 1, \dots, k\}$.
The following lemma finds an expected discounted payoff of the portfolio with maximal k defaults under the condition that all the bonds in this chain portfolio were bought for their fair prices. Note that the lemma regards a more general case when the first bond was bought at some random time from the time interval $[0, T]$.

Lemma 30 *Let $\tau : \Omega \to [0, T]$ be an $(\mathcal{F}_W)_t$-stopping time on $[0, T]$. Fix $k \geq 0$.*
The discounted expected payoff with respect to the risk-neutral measure Q evaluated at τ is given by

$$E_Q\left(\int_\tau^T X(t) d\mu_d(t) \middle| \{N(T) - N(\tau) \leq k\}, \mathcal{F}_W(\tau)\right) = X(\tau) p_\tau e^{-\tau r}, \quad (5.13)$$

where

$$p_\tau = E_Q\left(\int_\tau^{T \wedge \tau'} e^{-r(t-\tau)} d\mu(t) + Re^{-r\tau} \mathbb{1}_{\tau \leq T} \middle| \mathcal{F}_W(\tau)\right)$$

denotes the fair price evaluated at τ of the bond which at the time τ belongs to the portfolio and defaults at stopping time τ'.

Proof: Let $I_l := E_Q(\int_\tau^T X(t) d\mu_d(t) | \{N(T) - N(\tau) \leq l\}, \mathcal{F}_W(\tau))$ denote throughout the current proof a discounted expected payoff in the case of not more than l jumps on $[\tau, T]$.
If the bond in the portfolio is riskless, the principal value stays unchanged on the time interval $[\tau, T]$:

$$X^0(\tau) = X^0(s) \text{ for all } s \in [0, T].$$

This consideration infact implies the statement of the lemma for the case $k = 0$. Indeed, since the price of the riskless bond equals

$$p_\tau = E_Q\left(\int_\tau^T e^{-r(t-\tau)} d\mu(t) \middle| \{N(T) - N(\tau) = 0\}, \mathcal{F}_W(\tau)\right)$$

one obtains

$$
\begin{aligned}
I_0 &= E_Q\left(X(\tau)\int_\tau^T d\mu_d(t)\middle|\{N(T)-N(\tau)=0\},\mathcal{F}_W(\tau)\right)\\
&= X(\tau)e^{-r\tau}E_Q\left(e^{r\tau}\int_\tau^T d\mu_d(t)\middle|\mathcal{F}_W(\tau)\right)\\
&= X(\tau)p_\tau e^{-r\tau}.
\end{aligned}
$$

By induction the statement of the lemma extends to all natural numbers. Notice first that by the construction of the process X it holds

$$X(\tau)=X(s) \text{ for all } s\in[0,\tau')$$

and if $\tau'<T$, then

$$X(\tau')=X(\tau)\frac{R}{p_{\tau'}}. \tag{5.14}$$

Assume that equation (5.13) is valid for $k\geq 0$. Now, the expected discounted payback of the process X at τ equals.

$$
\begin{aligned}
I_{k+1} &= E_Q\left(\int_\tau^{\tau'\wedge T} X(t)d\mu_d(t)\middle|\ \{N(T)-N(\tau)\leq k+1\},\mathcal{F}_W(\tau)\right)\\
&\quad+E_Q\left(\int_{\tau'\wedge T}^{T} X(t)d\mu_d(t)\middle|\ \{N(T)-N(\tau')\leq k\},\mathcal{F}_W(\tau)\right)\\
&= X(\tau)E_Q\left(\int_\tau^{\tau'\wedge T} d\mu_d(t)\middle|\ \mathcal{F}_W(\tau)\right)\\
&\quad+E_Q\left(E_Q\left(\int_{\tau'\wedge T}^{T} X(t)d\mu_d(t)\middle|\{N(T)-N(\tau')\leq k\},\mathcal{F}_W(\tau'\wedge T)\right)\middle|\mathcal{F}_W(\tau)\right).
\end{aligned}
$$

Denote by $\tau'':=\tau'\wedge T$. Using the assumption of induction, one obtains that for the $\mathcal{F}_W(\tau'')$-measurable random variable from the transformations above it holds:

$$E_Q\left(\int_{\tau''}^{T} X(t)d\mu_d(t)\middle|\{N(T)-N(\tau')\leq k\},\mathcal{F}_W(\tau'')\right)=X(\tau'')p_{\tau''}e^{-r(\tau'')}.$$

In its turn equality (5.14) yields

$$E_Q\left(\int_{\tau''}^{T} X(t)d\mu_d(t)\middle|\{N(T)-N(\tau')\leq k\},\mathcal{F}_W(\tau'')\right)=X(\tau)Re^{-r(\tau'')}.$$

Hence,

$$\begin{aligned} I_{k+1} &= X(\tau) E_Q \left(\int_{\tau}^{\tau' \wedge T} e^{-rt} d\mu(t) + R e^{-r(\tau' \wedge T)} \mathbb{1}_{\{\tau' \leq T\}} \middle| \mathcal{F}_W(\tau) \right) \\ &= X(\tau) p_\tau e^{-r\tau}. \end{aligned}$$

and the statement is valid for $N = k+1$ as well. Hence, lemma is proved for arbitrary finite N. $\square$

Corollary 31 *For the process X as in lemma 30 we have for all $k \in \mathbb{N}_0$*

$$E_Q \left(\int_0^T e^{-rt} X(t) d\mu(t) \middle| N(T) \leq k \right) = x_0.$$

Proof: By the construction of the process X with $x_0 = X(0)p_0$. Thus, the statement follows from lemma 30 if we set $\tau = 0$. $\square$

The claim of Lemma 30 and the statement of Corollary 31 can be extended to the general face value process X (5.11) which has no restrictions on the maximal amount of defaults.

Theorem 32 *Let $\tau : \Omega \to [0,T]$ be a $(\mathcal{F}_W(t))_{t>0}$-stopping time on $[0,T]$. Assume that $r > 0$ and that the intensity λ of the process X (5.11) is bounded by some deterministic constant $\lambda^* \in \mathbb{R}_+$. The discounted expected payoff is given by:*

$$E_Q \left(\int_\tau^T X(t) d\mu_d(t) \middle| \mathcal{F}_W(\tau) \right) = X(\tau) p_\tau e^{-\tau r}. \tag{5.15}$$

In particular,

$$E_Q \left(\int_0^T X(t) d\mu_d(t) \right) = x_0. \tag{5.16}$$

Proof: Denote correspondingly by I in the case of no restrictions:

$$I := E_Q \left(\int_\tau^T X(t) d\mu_d(t) \middle| \mathcal{F}_W(\tau) \right),$$

by I_n as in the Proof of Lemma 30 the expected payoff in the case of maximal n jumps:

$$I_n := E_Q \left(\int_\tau^T X^n(t) d\mu_d(t) \middle| \mathcal{F}_W(\tau) \right)$$

and by $I_{>n}$ the expected payoff in the case of more than n jumps:

$$I_{>n} := E_Q\left(\int_\tau^T X(t)d\mu_d(t)\middle| \{N(T)-N(\tau)>n\}, \mathcal{F}_W(\tau)\right).$$

According to Lemma 30,

$$I_n = E_Q\left(\int_\tau^T X(t)d\mu_d(t)\middle| \{N(T)-N(\tau)\leq n\}, \mathcal{F}_W(\tau)\right) = X(\tau)p_\tau e^{-\tau r}.$$

For $n \in \mathbb{N}$ denote by $q_n := Q(N(T)-N(\tau) \leq n)$. Then the following estimations are valid:

$$\begin{aligned}
|I - I_n| &= |I_n q_n + I_{>n}(1-q_n) - I_n| \\
&= |I_{>n} - I_n|(1-q_n) \\
&\leq \left(\mu_d[\tau,T] E_Q(\max_{t\in[\tau,T]} X(t)|\mathcal{F}_W(\tau), N(T)-N(\tau)>n) + I_n\right)(1-q_n) \\
&\leq \mu_d[\tau,T]X(\tau)\sum_{i=n+1}^{\infty}\left(\frac{R}{p_*}\right)^i \frac{(\lambda^* T)^i}{i!} + I_1 \sum_{i=n+1}^{\infty}\frac{(\lambda^* T)^i}{i!},
\end{aligned}$$

where $p_* = Re^{-rT}$ is the lower bound of the bond's price from Lemma 1. Notice that

$$\sum_{i=n+1}^{\infty}\left(\frac{R}{p_*}\right)^i \frac{(\lambda^* T)^i}{i!} = \sum_{i=n+1}^{\infty}\frac{(e^{rT}\lambda^* T)^i}{i!} \to 0, \quad n \to +\infty$$

and

$$\sum_{i=n+1}^{\infty}\frac{(\lambda^* T)^i}{i!} \to 0, \quad n \to +\infty.$$

The limit behavior of the summands implies that $|I - I_n| = 0$.
Hence,

$$E_Q\left(\int_\tau^T X(t)d\mu_d(t)\middle| \mathcal{F}_W(\tau)\right) = I_n$$

which proves equality (5.15). Analogously to the proof of Corollary 31, in the particular case when $\tau = 0$ we obtain (5.16).
This completes the proof. □

Remark: The statement of Theorem 32 is an important motivation to consider the model introduced in Section 5.1 and in particular the face value process (5.11). In fact, Theorem 32 can be interpreted as a 'no arbitrage'

condition. The measure Q can be interpreted as the risk neutral measure in the current model. Note that the determination of the expected payoff under the real world measure P is beyond the scope of the current research.
Equation (5.16) demonstrates the absence of arbitrage in the beginning of investment process at $t = 0$. The more general equation (5.15) reflects the fact that there is no arbitrage during the whole time period from $t = 0$ up to maturity T in the sense that there is no arbitrage opportunity at arbitrary $\mathcal{F}_W$-stopping time on $[0, T]$. This shows that the model is worth to work with and it has a right to be considered from the point of view of financial mathematics.
The second message which delivers Theorem 32 is the rationality of portfolio construction (5.11). It shows that an investor does not loose his money in the investment process described in Section 5.1.
Nevertheless, it needs to be studied in more details which profit brings an investment into the portfolio of high-yield bonds and its derivatives under the real-world measure. It is a complicated question which can not be immediately answered.

5.2.3 Markov Process Related to the 'Chain of Bonds' Portfolio.

The portfolio process described currently is, generally, not Markov. It happens since the default intensity is not to be determined just basing on the value of the process X. Indeed,

$$\begin{aligned} Q(X(t) = X(s)|\mathcal{F}(s)) &= Q(N(t) - N(s) = 0|\mathcal{F}(s)) \\ &= e^{-\int_s^t \lambda(u)du} \end{aligned} \tag{5.17}$$

Thus, the information about the current non-deterministic intensity which is contained in $\mathcal{F}(s)$ must be used in order to predict the future behavior of the process X. Since this information can not be extracted from the information about the value of the process X and the default intensity on the time interval $[s, t]$, if it is not deterministic, can not be evaluated in general, we conclude that

$$Q(X(t) = X(s)|\mathcal{F}(s)) \neq Q(X(t) = X(s)|X(s))$$

If λ is deterministic, the process X is Markov. If it is not the case and the default intensity λ changes randomly, there is a standard procedure which can be used. According to the procedure, the dimension of the process will be increased in a way that the resulting process is Markov. New supplementary

variables relevant to the prediction of the future behavior of the process will be included. These additional variables transmit the information which is missing by the process X in order to be Markov.
It follows from the expression (5.17) that in order to predict the process X the following supplementary variables can be used:

- A variable $\lambda \in \Lambda$ which indicates default intensity after the last jump of X.
- A time variable $t \in R_+$ which in its turn indicates the time passed since the last jump and as a consequence it shows the further development of the default intensity λ.

Following the procedure as in ([7], p. 62), construct a piecewise deterministic process $\tilde{X}$ which corresponds to the process X.
The process X is one-dimensional and it takes values on E_1. Denote by $E_2 := \Lambda$, $E_3 = E_4 := \mathbb{R}_+$. It is assumed here that the space Λ is compact and that it has finite dimension, which is the case if Λ is parameterized by finitely many one-dimensional parameters. Moreover, let P_Λ denote a probability measure on Λ. We can think of (Λ, P_Λ) as of a compact Borel subset of $\mathbb{R}^n$, $n \in \mathbb{N}$ which is equipped with a probability measure induced by Lebesgue measure $\boldsymbol{\lambda}$ restricted on this subset: Λ
$slshapein\mathbb{R}^n$, $P_\Lambda = \frac{1}{\boldsymbol{\lambda}(\Lambda)}\boldsymbol{\lambda}|_\Lambda$. Let $\mathcal{B}(\Lambda)$ denote the σ-algebra on the space Λ.
Regard now the space $\tilde{E} := \times_{i=1}^4 E_i = E_1 \times \Lambda \times \mathbb{R}_+ \times \mathbb{R}_+$ and its projections $\Psi_i : E \to E_i$, $i = 1, \dots, 4$. Sigma-algebra $\mathcal{E}$ on E is defined as a product sigma-algebra. Let $\mathcal{P}(\mathcal{E})$ denote the class of probability measures on $(E, \mathcal{E})$.
Denote by $\tilde{E}_0 := E_1 \times \Lambda \times \{0\} \times \mathbb{R}_+$.
The Markov piecewise deterministic process $\tilde{X} : \Omega \times [0, T] \to E$

$$\tilde{X}(t) := (X(t), \lambda(t), t - \alpha(t), t)$$

is defined by the means of:

1. The intensity $\lambda : \tilde{E} \to \mathbb{R}_+$.
2. The family $(Q_{\tilde{x}})_{\tilde{x} \in E} \subseteq \mathcal{P}(\mathcal{E})$ of transition probabilities on $\mathcal{E}$ which satisfies the following conditions:
   (a) $Q_{\tilde{x}}(\{\tilde{x}\}) = 0$ for every $\tilde{x} \in \tilde{E}$ and $t \in [0, T]$.
   Due to this condition it is possible to recognize jumps of the process $\tilde{X}$.

(b) For $\tilde{x} = (x, \lambda, s, t) \in \tilde{E}$ the measure $Q_{\tilde{x}}$ has a support $A_{\tilde{x}}$. It is given by

$$A_{\tilde{x}} \subseteq \left\{ \tilde{x}_1 = (x_1, \lambda_1, 0, t_1) : x_1 = x \frac{R}{p_{t_1}^{\lambda_1}} \right\} \subseteq \tilde{E}_0$$

This definition of the content $A_{\tilde{x}}$ of transition probability is based on the explicit formula (6.3) of the process X. Here $p_{t_1}^{\lambda_1}$ denotes the price of a defaultable bond (calculated according to Definition 5) which was bought at the time t_1 and has default intensity λ_1.

(c) $Q^{\tilde{x}_1} = Q^{\tilde{x}_2}$ if $\Psi_1(\tilde{x}_1) = \Psi_1(\tilde{x}_2)$ and $\Psi_4(\tilde{x}_1) = \Psi_4(\tilde{x}_2)$. According to this condition, the choice of the next bond is affected only by the value of initial process X at the time t.

3. The trajectories are determined by the vector-field $\mathcal{U} = \Psi_3 \times \Psi_4$ with the flow

$$\phi(v, (x, \lambda, s, t)) = (x, \lambda, s + v, t + v) \tag{5.18}$$

The processes X and λ are constant between the jumps and the time after the last jump grows linearly with coefficient 1.

Let $Q_{\tilde{x}}^{\Lambda} := Q_{\tilde{x}}|_{\Lambda}$ denote the image measure on the subspace Λ. From the condition 2(b) it follows that for a set $A \in \mathcal{E}$

$$Q_{\tilde{x}}(A) = Q_{\tilde{x}}(A \cap A_{\tilde{x}}) = Q_{\tilde{x}}^{\Lambda}(\Psi_2(A \cap A_{\tilde{x}}))$$

Note that $\Psi_2(A \cap A_{\tilde{x}}) \in \mathcal{B}(\Lambda)$. The projections on Λ are measurable due to the construction of the set $A_{\tilde{x}}$.
If $Q_{\tilde{x}}^{\Lambda} \ll P_{\Lambda}$ denote by

$$q_{\tilde{x}} := \frac{\partial Q_{\tilde{x}}^{\Lambda}}{\partial P_{\Lambda}}$$

the corresponding Radon-Nikodym density.
For $\tilde{y} = (y, \lambda_{\tilde{y}}, s_{\tilde{y}}, t_{\tilde{y}}) \in E_1 \times \Lambda \times [v, +\infty)^2 \subseteq \tilde{E}$ denote by

$$\phi(-v, \tilde{y}) := (y, \lambda_{\tilde{y}}, s_{\tilde{y}} - v, t_{\tilde{y}} - v)$$

This symbolic writing is based on the equation (5.18) which defines the flow. For $A \subseteq \tilde{E}$ it is natural to introduce the set

$$\phi(-v, A) := \{\phi(-v, \tilde{y}) : \tilde{y} \in A \cap E_1 \times \Lambda \times [v, +\infty)^2\}$$

Obviously,

$$\phi(-s_{\tilde{y}}, \tilde{y}) = (y, \lambda_{\tilde{y}}, 0) \in \tilde{E}_0$$

An important feature of the process constructed above is the following theorem.

Theorem 33 ([7], p.64) *The process $\tilde{X}$ is a homogeneous strong Markov process, i.e. for any $\tilde{x} \in \tilde{E}$, $\mathcal{F}(t)$-stopping time τ and bounded measurable function f,*

$$E_{\tilde{x}}(f(\tilde{x}_{\tau+s}\mathbb{1}_{\{\tau<\infty\}})|\mathcal{F}(\tau)) = P_s f(\tilde{x}_\tau)\mathbb{1}_{\{\tau<\infty\}}$$

Proof: see [7] p. 64. □

Due to 3. which establishes linear flow, the process $\tilde{X}$ defined on E is a piecewise linear Markov process.
From now on we write $\tilde{X}(t) = (X(t), \lambda(t), t - \alpha(t))$ instead of $\tilde{X}(t) = (X(t), \lambda(t), t-\alpha(t), t)$. It saves space and obviously leaves Markov property of the process $\tilde{X}$ untouched. But the process $\tilde{X}$ is not homogeneous any more. We write now the transition measure as $(Q_{\tilde{x},t})_{\tilde{x}\in E}$ and denote $\tilde{E} = \times_{i=1}^{3} E_i$.
For the characterization of the Markov process $\tilde{X}$ let us use its transition kernels:

Definition 9 A mapping $K_{s,t} : \mathcal{E} \times \mathcal{E} \to \mathbb{R}_0$ for $s, t \in [0, T]$ with $s \leq t$ such that

1. the function $K_{s,t}^A : E \to \mathbb{R}_0$ defined as

   $$K_{s,t}^A(\tilde{x}) := K_{s,t}(\tilde{x}, A)$$

   is $\mathcal{E}$-measurable in $\tilde{x} \in E$ for fixed $A \in \mathcal{E}$

2. $K_{s,t}^{\tilde{x}} : \mathcal{E} \to \mathbb{R}_0$ such that $K_{s,t}^{\tilde{x}}(A) = K_{s,t}(\tilde{x}, A)$ where $A \in \mathcal{E}$ is a probability measure on E for fixed $\tilde{x} \in E$

3. $K_{s,t}(\tilde{X}(s), A) = Q(\tilde{X}(t) \in A|\tilde{X}(s)) = Q(\tilde{X}(t) \in A|\mathcal{F}(s))$ a.s.,$A \in \mathcal{E}$ if the regular conditional distributions exist

is called a *transition kernel*.

The product of kernels $K_{s_1,s_2}K_{t_1,t_2}$ is defined by

$$K_{s_1,s_2}K_{t_1,t_2}(\tilde{x}, A) := \int K_{s_1,s_2}(\tilde{x}, dy)K_{t_1,t_2}(y, A)$$

for every $\tilde{x} \in \tilde{E}$, $A \in \mathcal{E}$.

Let us formulate a Chapman-Kolmogorov relation in terms of transition kernels. This relation is a basic consistency requirement in the theory of Markov processes.

Theorem 34 ([24] p. 142) *(Chapman, Smoluchovsky) For any Markov process $\tilde{X}$ in a Borel space S with transition kernel $(K_{s,t})_{0\geq s\geq t}$, we have*

$$K_{s,t} = K_{s,v}K_{v,t} \text{ a.s. } \mathcal{L}(\tilde{X}_s), \quad s \leq v \leq t.$$

5.2.4 Distribution of the 'Chain of Bonds' Process.

Apply now the construction above to calculation of the distribution of the principal value process X and in particular the final principal value $X(T)$. From the construction of the Markov process $\tilde{X}$ related to X it follows that the distribution of X is completely defined by the distribution of $\tilde{X}$ by the mean of the following relation:

$$Q(X(t) \in A_1) = Q(\tilde{X}(t) \in A_1 \times \Lambda \times [0,t])$$

for $t \in [0,T]$ and a set $A_1 \in \mathcal{E}_1$.
For $t, s > 0$ and a fixed $\tilde{x} = (x, \lambda_{\tilde{x}}, s_{\tilde{x}}, t_{\tilde{x}}) \in E$ denote by

$$\pi^{\tilde{x}}(i) := Q(N(t+s) - N(t) = i | \tilde{X}(t) = \tilde{x}),\ i \in \mathbb{N}_0$$

the probability of exactly i jumps on the time interval $[t, t+s]$.
It will be used later on that $\Psi_2(\tilde{X}(t)) = \Psi_2(\tilde{x}) = \lambda_{\tilde{x}}$ and that $\Psi_3(\tilde{X}(t)) = s_{\tilde{x}}$ if $\tilde{X}(t) = \tilde{x}$.
In order to find the distribution of the process $\tilde{X}$ regard first its transition kernel $K_{t,t+s}$. It satisfies the equality

$$K^{\tilde{x}}_{t,t+s}(A) = \sum_{i=0}^{\infty} K^{\tilde{x}}_{t,t+s}(A | N(t+s) - N(t) = i) \pi^{\tilde{x}}(i). \tag{5.19}$$

Lemma below characterizes probabilities of 0,1 and ≥ 2 jumps which appear in expression (5.19):

Lemma 35 *Let $t, t+s \in [0,T]$, $s > 0$. Assume that at the time t the intensity function $\lambda_{\tilde{x}}$ of the process $\tilde{X}$ is continuous on $[s_{\tilde{x}} + t, s_{\tilde{x}} + t + s]$, and that the intensity process (5.8) of $\tilde{X}$ is bounded by a constant $\lambda^* > 0$. Then we have:*

$$\pi^{\tilde{x}}(0) = \exp\left(-\int_0^s \lambda_{\tilde{x}}(s_{\tilde{x}} + v) dv\right), \tag{5.20}$$

$$\lim_{s \to 0+} (s^{-1}(\pi^{\tilde{x}}(1) - \lambda_{\tilde{x}}(s_{\tilde{x}}))) = 0, \tag{5.21}$$

$$\lim_{s \to 0+} \left(s^{-1} \sum_{i=2}^{\infty} \pi^{\tilde{x}}(i)\right) = 0. \tag{5.22}$$

Proof: i) the equality (5.20) is the usual probability of no default on $[t, t+s]$ if the default intensity is $\lambda : [t, t+s] \to \mathbb{R}_0$ given by

$$\lambda(v) = \lambda_{\tilde{x}}(s_{\tilde{x}} + v - t),\ v \in [t, t+s].$$

This probability equals

$$\pi^{\tilde{x}}(0) = \exp\left(-\int_t^{t+s} \lambda_{\tilde{x}}(s_{\tilde{x}} + v - t)dv\right) = \exp\left(-\int_0^{s} \lambda_{\tilde{x}}(s_{\tilde{x}} + v)dv\right).$$

ii) Let $\tau := \inf_{v\in[t,t+s]}\{v : \tilde{X}(t) \neq \tilde{X}(v)\}$ denote the random time of the first default. Denote by $F^{\tilde{x}}(v) := Q(\tau \leq v|\tilde{X}(t) = \tilde{x})$. Note that $f^{\tilde{x}}(v) := dF^{\tilde{x}}_{\tau}(v) = \lambda_{\tilde{x}}(s_{\tilde{x}} + v - t)\exp(-\int_t^v \lambda_{\tilde{x}}(s_{\tilde{x}} + w - t)dw)$. The probability of one default on $[t, t+s]$ equals

$$\begin{aligned}
\pi^{\tilde{x}}(1) &= \int_t^{t+s} Q(N(t+s) - N(v) = 1|\tilde{X}(t) = \tilde{x})dF^{\tilde{x}}(v) \\
&= \int_t^{t+s} \exp\left(-\int_v^{t+s} \lambda(w)dw\right) f^{\tilde{x}}(v)dv \\
&= \int_0^{s} \lambda_{\tilde{x}}(s_{\tilde{x}} + u)\exp\left(-\int_t^{t+s} \lambda(w)dw\right) du.
\end{aligned}$$

Thus, for $\pi^{\tilde{x}}(1)$ it is valid:

$$se^{-\lambda^* s} \min_{v\in[0,s]} \lambda_{\tilde{x}}(s_{\tilde{x}} + v) \leq \pi^{\tilde{x}}(1) \leq s \max_{v\in[0,s]} \lambda_{\tilde{x}}(s_{\tilde{x}} + v) \tag{5.23}$$

The inequalities above are obtained by substitution of λ by λ^* and 0 under the integral sign. The first inequality with the lower bound is due to the condition $\lambda^* \geq \lambda(t)$ and the upper bound results from the fact that $\lambda(t) \geq 0$ for $t \in [0, T]$. Limit (5.21) follows immediately from inequalities (5.23).
iii) Limit (5.22) is implied by the bound

$$\sum_{i=2}^{\infty} \pi^{\tilde{x}}(i) \leq \sum_{i=2}^{\infty} \frac{(\lambda^* s)^i}{i!} = (\lambda^* s)^2 \sum_{i=0}^{\infty} \frac{(\lambda^* s)^i}{(i+2)!}$$

and the fact that series $\sum_{i=0}^{\infty} \frac{(\lambda^* s)^i}{(i+2)!}$ converge uniformly for $s \in [0, T]$. □

Return now to expression (5.19). Note first that $K^{\tilde{x}}_{t,t+s}(A|N(t+s) - N(t) = 0)$ is completely defined by the flow (5.18) and equals

$$K^{\tilde{x}}_{t,t+s}(A|N(t+s) - N(t) = 0) = \delta_{\phi(s,\tilde{x})}(A)\exp\left(-\int_0^{s} \lambda_{\tilde{x}}(s_{\tilde{x}} + v)dv\right),$$

second that $K^{\tilde{x}}_{t,t+s}(A|N(t+s) - N(t) = 1)$ is determined by the mean of a subfamily of transition probabilities $(Q_{\tilde{x},v})_{v\in[t,t+s]}$ and third that

$K^{\tilde{x}}_{t,t+s}(A|N(t+s)-N(t)\geq 2)$ is a convolution of two or more transition probabilities from the subfamily $(Q_{\tilde{y},v})_{\tilde{y}\in\tilde{E},v\in[t,t+s]}$.
Using Lemma 35 the equality (5.19) transforms into

$$\begin{aligned} K^{\tilde{x}}_{t,t+s}(A) &= \delta_{\phi(s,\tilde{x})}(A)\exp\left(-\int_0^s \lambda_{\tilde{x}}(s_{\tilde{x}}+v)dv\right) \\ &\quad +\lambda_{\tilde{x}}(s_{\tilde{x}})s\int_t^{t+s} Q_{v,\tilde{x}}(\phi(u-t-s,A))dF^{\tau,1}(v) \\ &\quad +o(s)\nu_{t,s,\tilde{x}}(A), \end{aligned}$$

where $F^{\tilde{x},1}(v) := Q(\tau\leq v|\tilde{X}(t)=\tilde{x},\ N(t+s)-N(t)=1)$ denotes the conditional probability and $\nu_{t,s,\tilde{x}}$ is a probability measure such that for any set $B\in\mathcal{E}$, where $B\cap E_1\times\Lambda\times[0,s]=\emptyset$ it follows that $\nu_{t,s,\tilde{x}}(B)=0$.

Theorem 36 *Assume that transition measure $(Q_{v,\tilde{y}})$ satisfies the following regularity condition:*
There are constants $L>0$ and $q>0$ such that for every set $A\in\mathcal{E}$

$$|Q_{t_1,\tilde{x}_1}(A)-Q_{t_2,\tilde{x}_1}(A)|\leq L|t_1-t_2|^{1+q} \text{ for } t_1,t_2\in[0,T],\ \tilde{x}_1\in\tilde{E}. \quad (5.24)$$

Then the distribution of the process $\tilde{X}$ is given by

$$Q(\tilde{X}(t)\in A)=\int_A e^{-\int_0^{s_{\tilde{y}}}\lambda_{\tilde{y}}(v)dv}\mu_K(\phi(-s_{\tilde{y}},d\tilde{y}),t-s_{\tilde{y}}), \quad (5.25)$$

where μ_K satisfies the equation

$$\frac{\partial\mu_K(A_0,t)}{\partial t}=\int_{\tilde{E}_0}\int_0^t \lambda_{\tilde{x}}(w)e^{-\int_0^w\lambda_{\tilde{x}}(v)dv}Q_{t,\phi(w,\tilde{x})}(A_0)\mu_K(d\tilde{x},t-dw). \quad (5.26)$$

Proof: Under condition (5.24),

$$\int_t^{t+s}|Q_{v,\tilde{x}}(\phi(u-t-s,A))-Q_{t,\tilde{x}}(\phi(u-t-s,A))|dF^{\tilde{x},1}(v)\leq Ls^{1+q}$$

it means that the expression (5.19) can be finally written as

$$\begin{aligned} K^{\tilde{x}}_{t,t+s}(A) &= \delta_{\phi(s,\tilde{x})}(A)\exp\left(-\int_0^s \lambda_{\tilde{x}}(s_{\tilde{x}}+v)dv\right) \\ &\quad +\lambda_{\tilde{x}}(s_{\tilde{x}})s\int_t^{t+s} Q_{t,\tilde{x}}(\phi(u-t-s,A))dF^{\tilde{x},1}(v) \\ &\quad +o(s)\nu'_{t,s,\tilde{x}}(A), \end{aligned}$$

where similarly to the measure $\nu_{t,s,\tilde{x}}$, the probability measure $\nu'_{t,s,\tilde{x}} = 0$ for any set $B \in \mathcal{E}$, such that $B \cap E_1 \times \Lambda \times [0, s] = \emptyset$.
By Chapman-Kolmogorov relation given by Theorem 34 the following equality holds for the transition kernel $(K_{t_1,t_2})_{0\le t_1\le t_2}$ of the Markov process $\tilde{X}$

$$K_{0,t+s} = K_{0,t}K_{t,t+s}. \tag{5.27}$$

Thus, by equation (5.27) the distribution of the process $\tilde{X}$ at $t+s$ is given by

$$Q(X(t+s) \in A) = K_{0,t+s}(\tilde{x}_0, A) = \int K_{t,t+s}(\tilde{x}, A)K_{0,t}(\tilde{x}_0, d\tilde{x})$$

for arbitrary set $A \in \mathcal{E}$. Combining it with expression (5.19) we obtain that

$$\begin{aligned} Q(X(t+s) \in A) = & \int \delta_{\phi(s,\tilde{x})}(A) \exp(-\textstyle\int_t^{t+s} \lambda_{\tilde{x}}(s_{\tilde{x}}+v)dv)K_{0,t}(\tilde{x}_0, d\tilde{x}) \\ & + \int \lambda_{\tilde{x}}(s_{\tilde{x}})s \int\limits_t^{t+s} Q_{t,\tilde{x}}(\phi(v-t-s, A))dF^{\tilde{x},1}(v)K_{0,t}(\tilde{x}_0, d\tilde{x}) \\ & + o(\lambda^* s) \int \nu'_{t,s,\tilde{x}}(A)K_{0,t}(\tilde{x}_0, d\tilde{x}) \\ = & \, I_1 + I_2 + \; o(\lambda^* s) \int \kappa_t(\tilde{x})\nu_{t,s,\tilde{x}}(A)d\tilde{x}. \end{aligned} \tag{5.28}$$

1. Regard first a compact set $A \in \mathcal{E}$ such that $A \cap \tilde{E}_0 = \emptyset$. Then there is $s_1 > 0$ such that $A \subset E_1 \times \Lambda \times [s_1, \infty)$. Let $s \in [0, s_1]$ be arbitrary from the interval. It is now valid:

$$\nu'_{t,s,\tilde{y}}(A) = 0 \text{ for } \tilde{y} \in \tilde{E},$$

$$\phi(v-t-s, A) \cap \tilde{E}_0 = \emptyset \text{ for } v \in [t, t+s].$$

It implies that the last two summands in expression (5.28) turn into zero and it can be written in this case

$$Q(X(t+s) \in A) = \int_{\phi(-s,A)} \exp\left(-\int_t^{t+s} \lambda_{\tilde{x}}(s_{\tilde{x}}+v)dv\right) K_{0,t}(\tilde{x}_0, d\tilde{x})$$

in particular, for $\tilde{y} \notin \tilde{E}_0$ it holds

$$K_{0,t+s}(\tilde{x}_0, d\tilde{y}) = \exp\left(-\int_0^s \lambda_{\tilde{y}}(s_{\tilde{y}}+v-s)dv\right) K_{0,t}(\tilde{x}_0, d\phi(-s,\tilde{y})) \tag{5.29}$$

$$= \exp\left(-\int_0^{s_{\tilde{y}}} \lambda_{\tilde{y}}(v)dv\right) K_{0,t+s-s_{\tilde{y}}}(\tilde{x}_0, d\phi(-s_{\tilde{y}},\tilde{y})) \tag{5.30}$$

It means that for the compact set A such that $A \cap \tilde{E}_0 = \emptyset$ it is valid:

$$Q(\tilde{X}(t) \in A) = K_{0,t}(\tilde{x}_0, A) = \int_A e^{-\int_0^{s_{\tilde{y}}} \lambda_{\tilde{y}}(v)dv} K_{0,t-s_{\tilde{y}}}(\tilde{x}_0, d\phi(-s_{\tilde{y}}, \tilde{y})) \quad (5.31)$$

Equality (5.31) is the motivation to introduce a measure μ_K on the measurable space $(\tilde{E}_0^T, \mathcal{E}_0^T)$, where $\tilde{E}_0^T := \tilde{E}_0 \times [0,T] = E_1 \times \Lambda \times [0,T]$ and $\mathcal{E}_0^T$ is a product σ-algebra generated by $\mathcal{E}_0 = \mathcal{E}|_{\tilde{E}_0}$ and $\mathcal{B}([0,T])$ by the mean of transformation formula

$$\mu_K(B) = \int_B e^{\int_0^{t-s} \lambda_{\tilde{x}}(v)dv} K_{0,t}(\tilde{x}_0, \phi(t-s, d\tilde{x}))ds \quad (5.32)$$

here the set $B \subseteq \mathcal{E}_0^T$, time $t \in [0,T]$ is chosen so that $t \geq \sup\{s : (\tilde{x}, s) \in B\}$. In particular, if $t = T$ then $\mu_K(B) = \int_B e^{\int_0^{T-s} \lambda_{\tilde{x}}(v)dv} K_{0,T}(\tilde{x}_0, \phi(T-s, d\tilde{x}))ds$. From (5.31) we conclude that the distribution of the process $\tilde{X}$ on the space $\tilde{E}$ is completely determined by μ_K via reverse to (5.32) transformation

$$Q(\tilde{X}(t) \in A) = \int_A e^{-\int_0^{s_{\tilde{y}}} \lambda_{\tilde{y}}(v)dv} \mu_K(\phi(-s_{\tilde{y}}, d\tilde{y}), t - s_{\tilde{y}}) \quad (5.33)$$

2. From the previous subparagraph 1 (formula (5.31)) it follows that it is crucial to find the measure μ_K on $\tilde{E}_0^T$. In order to do this, consider now the case when $A \subset E_1 \times \Lambda \times [0,s)$ for some small s. Under this assumption, $\phi(-s, A) = \emptyset$ and the first summand in the formula (5.28) disappears.
Fix a set $A_0 \subset \tilde{E}_0$. Let $A = \{\phi(s_1, A_0) : s_1 \in [0,s)\}$ be a set of a special form. Then

$$\int_t^{t+s} Q_{t,\tilde{x}}(\phi(v-t-s), A) dF^{\tilde{x},1}(v) = Q_{t,\tilde{x}}(A_0)$$

since $F^{\tilde{x},1}(v)$ is a distribution function related to the conditional probability measure on $[t, t+s]$ and $\phi(v-t-s, A) = A_0$ for arbitrary $v \in [t, t+s]$.
The expression (5.28) can be written now as follows:

$$\begin{aligned} Q(\tilde{X}(t+s) \in A) &= s \int \lambda_{\tilde{x}}(s_{\tilde{x}}) Q_{t,\tilde{x}}(A_0) K_{0,t}(\tilde{x}_0, d\tilde{x}) \\ &\quad + o(\lambda^* s) \int \nu'_{t,s,\tilde{x}}(A) K_{0,t}(\tilde{x}_0, d\tilde{x}) \end{aligned} \quad (5.34)$$

From (5.25) it follows that

$$Q(\tilde{X}(t+s) \in A) = \int_0^s \int_{A_0} e^{-\int_0^{s_1} \lambda_{\tilde{x}}(v)dv} \mu_K(d\tilde{x}, ds_1)$$

and

$$K_{0,t}(\tilde{x}_0, d\tilde{x}) = e^{-\int_0^{s_{\tilde{x}}} \lambda_{\tilde{x}}(v)dv} \mu_K(d\phi(-s_{\tilde{x}}, \tilde{x}), d(t - s_{\tilde{x}})) \tag{5.35}$$

Finally, using (5.35), equation (5.34) is equivalent to

$$\begin{aligned}
\int_0^s \int_{A_0} e^{-\int_0^{s_1} \lambda_{\tilde{x}}(v)dv} \mu_K(d\tilde{x}, ds_1) &= s \int \lambda_{\tilde{x}}(s_{\tilde{x}}) Q_{t,\tilde{x}}(A_0) K_{0,t}(\tilde{x}_0, d\tilde{x}) \\
&\quad + o(\lambda^* s) \int \nu'_{t,s,\tilde{x}}(A) K_{0,t}(\tilde{x}_0, d\tilde{x}) \\
&= s \int_{\tilde{E}_0} \int_0^t \lambda_{\tilde{x}}(w) Q_{t,\phi(w,\tilde{x})}(A_0) e^{-\int_0^w \lambda_{\tilde{x}}(v)dv} \mu_K(d\tilde{x}, t - dw) \\
&\quad + o(\lambda^* s) \int \nu'_{t,s,\tilde{x}}(A) K_{0,t}(\tilde{x}_0, d\tilde{x})
\end{aligned}$$

Divide both parts by s and let now $s \to 0+$. It follows that if $\mu_K(A_0, t)$ is partially differentiable w.r.t. time then the following equation holds true:

$$\frac{\partial \mu_K(A_0, t)}{\partial t} = \int_{\tilde{E}_0} \int_0^t \lambda_{\tilde{x}}(w) e^{-\int_0^w \lambda_{\tilde{x}}(v)dv} Q_{t,\phi(w,\tilde{x})}(A_0) \mu_K(d\tilde{x}, t - dw) \tag{5.36}$$

and the theorem is proven. □

We obtained an analog of the Kolmogorov forward differential equation for purely discontinuous processes in the case of certain kind of piecewise linear Markov processes.

5.3 Portfolio with N Bonds

Consider a bond portfolio which is built according to the following rules: Every moment the portfolio consists of a considerable amount of defaultable bonds $N \gg 1$. Every single bond was issued by some public firm. It is assumed that the values of firms which issued different bonds are not correlated. In particular, one firm can not issue bonds of two different types which would be at the same time in the portfolio.
A bond defaults if the value of the corresponding firm hits some predefined bound. If some bond in the portfolio defaults, its recovery will be immediately paid. At the same moment, a new defaultable bond will be bought with the money which was obtained from the recovery. From now on, the new

bond will be a part of the portfolio.
It is assumed that a portfolio is completely determined by the fond's managers. They decide which bond to buy. The next important assumption is that the fond's managers have the complete information about the values of firms whose bonds are in the portfolio. They observe not only the assets of the firms but the default bounds as well.
To the contrary, investors do not decide by themselves which bond will be the component of the portfolio. Moreover, they do not have constant information about the assets of the firms. This information is updated only when the portfolio's structure changes, i.e. at the stopping times $\tau_i \wedge T,\ i = 1, \ldots, N$. Thus, the process (5.10) of the portfolio's principal value has jumps at stopping times $\{\tau_i \cap [0, T]\}_{i \in \mathbb{N}}$. The approximative behavior of the process X is shown on the figure 5.1. Let us now find the jumps ΔX_{τ_i} of the process X.
Introduce two processes $(X_i(t))_{t \in [0,T]}$ and $(x_i(t)_{t \in [0,T]}$. The process $(X_i(t))_{t \in [0,T]}$ denotes the principal value of the bond i in the portfolio, $i = 1, \ldots, N$ and has its values in $\mathbb{R}_0$. Note that the principal value of the portfolio is the sum of the principal values of its components:

$$X(t) = \sum_{k=1}^{N} X_k(t),\ t \in [0, T].$$

The process $(x_i(t))_{t \in [0,T]}$ with its values in $[0, 1]$ represents the relative part of the bond i in the whole portfolio. It is defined as

$$x_i(t) := \frac{X_i(t)}{X(t)}.$$

Obviously, $\sum_{i=1}^{N} x_i(t) = 1$. It is assumed that there is no borrowing:

$$x_i(t) > 0,\ t \in [0, T].$$

Let $R_i : \Omega \to [0, 1)$ denote the recovery rate of the i-th bond. If the bond i defaulted at the time $\tau_i \in [0, T]$, the total payment from the recovery was

$$R_i X_i(\tau_i-) = x_i(\tau_i-) X(\tau_i-) R_i.$$

Assume that the bond $j \in \{1, \ldots, N\}$ was bought instead and exactly the amount of money $x_i(\tau_i-)X(\tau_i-)R_i$ was invested into it. Let p_j be the price of the bond j which is determined according to (5). It follows that the principal value of the bond j is given by

$$X_j(t) = \begin{cases} \frac{x_i(\tau_i-)X(\tau_i-)R_i}{p_j(\tau_i)}, & t \in [\tau_i, \tau_j) \cap [0, T]; \\ 0, & t \in [0, T]/[\tau_i, \tau_j). \end{cases}$$

Finally, jumps of the process X(t) are given by

$$\begin{aligned}\Delta X_{\tau_i} &= X_j(\tau_i) - X_i(\tau_i-)\\ &= x_i(\tau_i-)X(\tau_i-)\left(\frac{R_i}{p_j(\tau_i)} - 1\right).\end{aligned}$$

It can be seen now that the process X satisfies under these settings the 'exponential' stochastic differential equation

$$dX(t) = x_i(t-)\left(\frac{R_i}{p_j(\tau_i)} - 1\right) X(t-)dN(t),\ t \in [0, T], \tag{5.37}$$

$$X(0) = x_0,$$

where the counting process N is given by

$$N(t) = \sum_{i=1}^{N} \mathbb{1}_{\{\tau \geq t\}}(t),\ t \in [0, T]$$

and $x_i(t-)\left(\frac{R_i}{p_j(\tau_i)} - 1\right)$ is predictable. The process (5.10) as the solution of the equation (5.37) can be represented as

$$X(t) = x_0 \prod_{i=1}^{N(t)} \left(1 + x_i(\tau_i-)\left(\frac{R_i}{p_j(\tau_i)} - 1\right)\right).$$

5.4 Examples

5.4.1 Deterministic Intensity

Regard now a simplified situation when default intensity $\lambda : [0, T] \to \mathbb{R}_0$ given by (5.8) is a deterministic function. In this case the corresponding counting process N given by (6.1) is a Poisson process (in general, non-homogeneous). Recall that the compensator A of the process N is calculated according to the formula

$$\Lambda(t) = \int_0^t \lambda(s)ds$$

Poisson measure is a natural extention of the idea of compensator. Denote by $\Lambda : \mathcal{B}(\mathbb{R}) \to \mathbb{R}_0$ the Poisson measure defined by $\Lambda(B) = \int_B \lambda(s)ds$. Here $\Lambda(t) = \Lambda([0, t])$.

For $t \in [0,T]$ consider the principal value process X defined in (5.10). Its logarithm under the current settings has the following representation:

$$\begin{aligned} \ln X(t) &= \ln \tfrac{x_0}{p_0} + \textstyle\sum_{i=1}^{N(t)} \ln \left(\tfrac{R}{p_i}\right) \\ &= \ln \tfrac{x_0}{p_0} + \textstyle\int_0^t \ln \tfrac{R}{p_s} dN(s), \end{aligned} \tag{5.38}$$

where p_s denotes the price of a bond purchased at the time $s \in [0,T]$. According to Definition 5, it equals

$$p_s = e^{rs}\mu_d(A_s)e^{-\int_s^T \lambda(u)du} + e^{rs}\int_s^T \lambda(t)e^{-\int_s^v \lambda(u)du}(Re^{-rv} + \mu_d(A_s))dv \tag{5.39}$$

Picture 4.1 shows the dependence of bond price on time $s \in [0,T]$ represented in equation (5.39) when deterministic λ is a constant.

Let us show that the random variable defined as $Z^t := \ln X(t) - \ln \frac{x_0}{p_0}$ can be represented in the form

$$Z^t := \sum_{i=1}^{N(t)} Z_i^t, \tag{5.40}$$

where $Z_i^t,\ i \in \mathbb{N}$ are i.i.d. random variables. It means then that $\ln X(t) - \ln \frac{x_0}{p_0}$ is an infinitely divisible random variable. These statements constitute Theorem 38. We need Lemma 37 in order to prove it. From Lemma 37 follows representation 5.40 of the random variable $\ln X(t)$.

Lemma 37 *Let N be a Poisson process on $(\Omega, \mathcal{F}, P, \mathcal{F}(s)_{s\in[0,t]})$ with finite compensator $\Lambda(t) < \infty$ and intensity $\lambda : [0,t] \to \mathbb{R}_0$. Let $W : [0,t] \to \mathbb{R}$ be a measurable function and Z_i be iid random variables independent of the process N and distributed according to*

$$P(Z_1 \in B) = (\Lambda(t))^{-1}\int_0^t \mathbb{1}_{W^{-1}(B)}(s)\lambda(s)ds = \frac{\Lambda(W^{-1}(B))}{\Lambda(t)}, \tag{5.41}$$

for $B \in \mathcal{B}([0,t])$.
Then the random variables $\tilde{W} := \int_0^t W(s)dN(s)$ and $Z := \sum_{i=1}^{N(t)} Z_i$ are identically distributed:

$$\tilde{W} \overset{d}{=} Z$$

Proof: The random variable Z is a random sum. For $z \in \mathbb{R}$,

$$\begin{aligned} P(Z \le z) &= \sum_{i=1}^{\infty} P\left(\sum_{k=1}^{i} Z_k \le z \,\middle|\, N(t) = k\right) P(N(t) = i) \\ &= \sum_{i=1}^{\infty} P\left(\sum_{k=1}^{i} Z_k \le z\right) P(N(t) = i), \end{aligned} \tag{5.42}$$

for $S \in \mathcal{B}(\mathbb{R})$.
In its turn, the random variable $\tilde{W}$ equals

$$\tilde{W} = \sum_{k=1}^{N(t)} W(\tau_k),$$

where $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_{N(t)}$ is the increasing sequence of jumps of the Poisson process N on the time interval $[0, t]$. For the random variable $\tilde{W}$ and $z \in \mathbb{R}$ it is valid:

$$P(\tilde{W} \leq z) = \sum_{i=1}^{\infty} P\left(\sum_{k=1}^{i} W(\tau_k) \leq z \middle| N(t) = k \right) P(N(t) = i) \tag{5.43}$$

We prove Lemma by showing that the corresponding summands from expressions (5.42) and (5.43) are pairwise equal. From (5.41) which defines the distribution of the random variable Z_1 it follows that

$$P_{Z_1}(z) := P(Z_1 \leq z) = \frac{\Lambda(W^{-1}((-\infty, z]))}{\Lambda(t)}, \; z \in \mathbb{R}$$

Compare it with the distribution of the random variable $\tilde{W}$ given that there is exactly one jump before t. In the calculations below by $\int_B dN(s) = k$ we mean the event that there are exactly k jumps on the set $B \in \mathcal{B}(\mathbb{R})$:

$$\begin{aligned} P(\tilde{W} \leq z | N(t) = 1) &= \frac{P(\tilde{W} \leq z, N(t) = 1)}{P(N(t) = 1)} \\ &= \frac{P(\int_{W^{-1}(-\infty, z]} dN(s) = 1, \int_{W^{-1}(z, \infty)} dN(s) = 0)}{e^{-\Lambda(t)} \Lambda(t)} \\ &= \frac{e^{-\Lambda(W^{-1}((-\infty, z]))} \Lambda(W^{-1}((-\infty, z])) e^{-\Lambda(W^{-1}((z, \infty)))}}{e^{-\Lambda(t)} \Lambda(t)} \\ &= P(Z_1 \leq z). \end{aligned}$$

It shows the equality of the first summands from expressions (5.42) and (5.43). Now using induction we show that all other components are equal as well.
Since the random variables Z_j, $j \in \mathbb{N}$ are independent, the distribution function $F_{\sum_{k=1}^{i} Z_k}$ of finite sums from (5.42) is determined via convolution

$$F_{\sum_{k=1}^{i} Z_k}(z) = F_{Z_1} * \cdots * F_{Z_i}(z)$$

of distribution functions $F_{Z_1}, \ldots, F_{Z_i}$ related to the random variables Z_l with $l \in \{1, \ldots, i\}$. Assume now that for $z \in \mathbb{R}$, $k \in \mathbb{N}$ it is valid:

$$P(\tilde{W} \leq z, N(t) = k) = \frac{(\Lambda(t))^k}{k!} F_{Z_1} * \cdots * F_{Z_k}(z). \tag{5.44}$$

Note that this relation holds for $k = 1$. Indeed, as it was shown above,

$$P(\tilde{W} \leq z, N(t) = 1) = P(Z_1 \leq z)P(N(t) = 1) = F_{Z_1}(z)e^{-\Lambda(t)}\Lambda(t).$$

Using the assumption, show that the relation holds for $k + 1$:

$$\begin{aligned}
P(\tilde{W} \leq z, N(t) = k+1) &= \frac{1}{k+1}\int_0^t \lambda(s)P(\tilde{W} \leq z - W(s), N(t) = k)ds \\
&= \frac{1}{k+1}\int_0^t \lambda(s)\frac{(\Lambda(t))^k}{k!}F_{\sum_{k=1}^i Z_k}(z - W(s))ds \\
&= \frac{(\Lambda(t))^k}{(k+1)!}\int_{\mathbb{R}} \lambda(W^{-1}(w))F_{\sum_{k=1}^i Z_k}(z - w)dw \\
&= \frac{(\Lambda(t))^k}{(k+1)!}\int_{\mathbb{R}} \Lambda(t)F_{\sum_{k=1}^i Z_k}(z - w)dF_{Z_{k+1}}(w) \\
&= \frac{(\Lambda(t))^{k+1}}{(k+1)!}F_{Z_1} * \cdots * F_{Z_{k+1}}(z).
\end{aligned}$$

Thus, by induction, it was shown that expression (5.44) holds $\forall k \in \mathbb{N}$. Since

$$\begin{aligned}
P(\tilde{W} \leq z | N(t) = k) &= \frac{P(\tilde{W} \leq z, N(t) = k)}{P(N(t) = k)} \\
&= \frac{\frac{(\Lambda(t))^k}{k!}F_{Z_1} * \cdots * F_{Z_k}(z)}{e^{-\Lambda(t)}(\Lambda(t))^k(k!)^{-1}} \\
&= F_{Z_1} * \cdots * F_{Z_k}(z),
\end{aligned}$$

it is finally derived the equality of summands from expressions (5.42) and (5.43). We conclude now that for all $S \in \mathcal{B}(\mathbb{R})$ we have

$$P(\tilde{W} \in S) = P(Z \in S)$$

which proves the statement of Lemma 37. □

Applying Lemma 37 to the current situation, we obtain that the random variable

$$\ln X(t) - \ln \frac{x_0}{p_0} = \sum_{s \leq t} Z(s)\Delta N(s),$$

where $Z(s) = \ln \frac{R}{p_s}$ has the representation (5.40), i.e. as it was claimed before, it can be written as a compound Poisson random variable:

$$\ln X(t) - \ln \frac{x_0}{p_0} = \sum_{i=1}^{N(t)} Z_i^t.$$

This statement is proven in Corollary 38.

Corollary 38 *Assume that the counting process N corresponding to the principal value process X in (5.10) has deterministic intensity λ. Then for every $t \in [0,T]$ the random variable $\ln X(t) - \ln \frac{x_0}{p_0}$ has compound Poisson distribution:*

$$\ln X(t) - \ln \frac{x_0}{p_0} \stackrel{\mathcal{D}}{=} \sum_{i=1}^{N(t)} Z_i^t,$$

where the random variables Z_i^t are i.i.d. with distribution function given by

$$F_{Z_1^t}(z) = (\Lambda(t))^{-1}\Lambda([0,t] \cap p^{-1}([Re^{-z}, +\infty))). \tag{5.45}$$

Proof: the statement follows from Lemma 37 if we set

$$W := \ln \frac{R}{p_s},$$

where p_s is calculated according to (5.39). Function W defined this way is measurable since p is continuous and according to Lemma 1, $p_s \geq Re^{-rT} > 0$ for all $s \in [0,T]$.
In addition, since the following sets coincide:

$$W^{-1}((-\infty, z]) = \{s : W(s) \leq z\} = \{s : \ln \frac{R}{p_s} \leq z\} = \{s : p_s \geq Re^{-z}\}$$

it implies the definition of the distribution of the random variables Z_i^t, $i \in \mathbb{N}$ as in Lemma 37. □

In Corollary 38 the distribution of the random variables Z_i^t plays an essential role. Along with the counting process N it determines the distribution function of the face value process X at t (see Figure 5.3). Figure 5.2 shows an example of the distribution of Z_i^T. As formula (5.45) shows, bond price (5.39) on $[0,t]$ generates the distribution of Z_i^t, $i \in \mathbb{N}$. In our case the distribution shown on Figure 5.2 is generated by one of the prices from Figure 4.1. The parameter λ in both cases equals $0,1$. The graphic of the corresponding bond price is the middle graphic of Figure 4.1. The bond parameters are taken the same as before for Figure 4.1: The maturity $T = 1$ year, constant coupon $c = 9\%$ p.a., recovery rate is $R = 0,6$. The riskless interest rate $r = 2\%$ p.a. The distribution of the final face value $X(T)$ itself, $X(T) = \frac{x_0}{p_0} \prod_{i=1}^{N(t)} \frac{R}{p_{\tau_i}} = \exp\left(\ln \frac{x_0}{p_0} \sum_{i=1}^{N(t)} Z_i^T\right)$ for constant intensity $\lambda = 0,5$ of the counting process N, random variables Z_i^T for $i \in \mathbb{N}$ distributed as on Figure 5.2, is shown on Figure 5.3.
Now we can give the characterization of the process X and its distribution

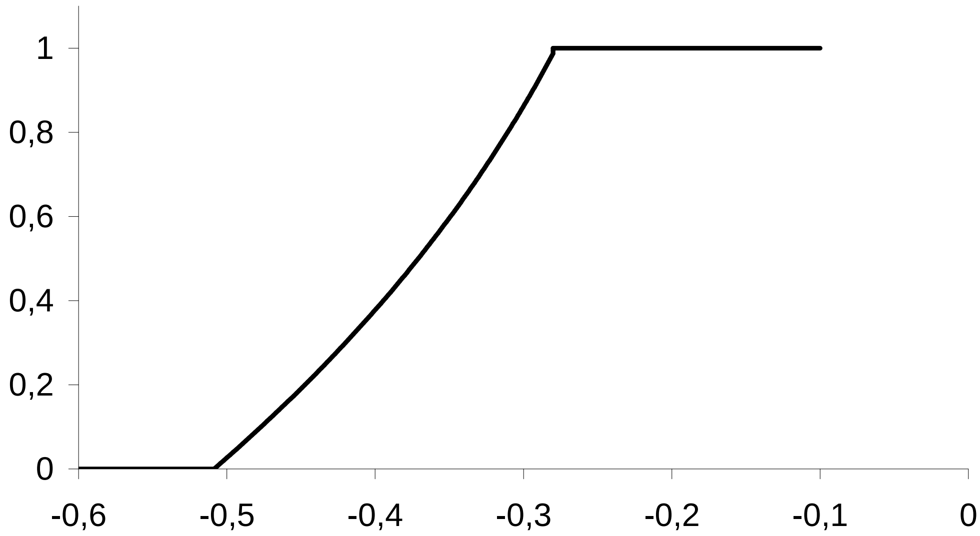

Figure 5.2: Distribution of the random variable Z_1.

in terms of the characteristic function of the random variables $\ln X(t)$ with $t \in [0,T]$. The characteristic function carries the whole information about the distribution of $\ln X$. But it seems like the explicit form of the distribution of the random variable $\ln X(t)$ for every $t \in [0,T]$ can be in general case obtained only using numerical procedures.

Theorem 39 *Under assumptions of Corollary 38,*

i) the characteristic function φ_t of the random variable $\ln X(t)$ admits the representation

$$\varphi_t(u) = \exp\left(iau - \Lambda(t) + \int_{\mathbb{R}} e^{iuz} \nu_t(dz) \right),$$

where $a = \ln \frac{x_0}{p_0}$ and $\nu_t((-\infty, z]) = \Lambda(p_t^{-1}([Re^{-z}, +\infty)))$ for $t \in [0,T]$,

ii) if $\Lambda(p_t^{-1}(\{R\})) = 0$, the random variable $\ln X(t)$ is infinitely divisible for all $t \in [0,T]$.

Proof: i) from Corollary 38 it follows that for every $t \in [0,T]$ it is valid:

$$\ln X(t) \stackrel{d}{=} \ln \frac{x_0}{p_0} + \sum_{i=1}^{N(t)} Z_i^t.$$

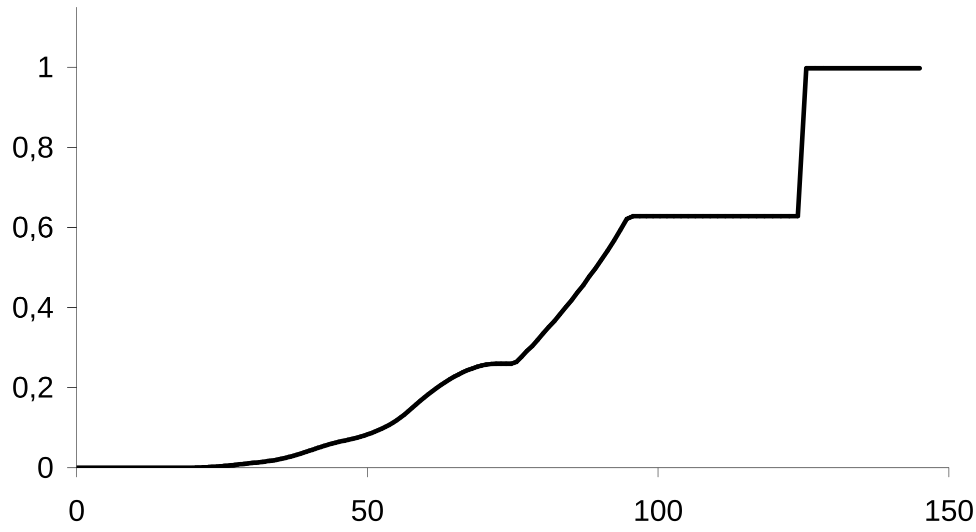

Figure 5.3: Distribution of the final face value $X(T)$.

Here $N(t)$ is the Poisson distributed random variable and the distribution function of Z_1 is given by (5.45).
More formal, denote by φ_i given by

$$\varphi_i(u) = Ee^{iuZ_i}$$

the characteristic function of the random variable Z_i, $i \in \mathbb{N}$. The characteristic function φ of the random variable $Z := \ln \frac{x_0}{p_0} + \sum_{i=1}^{N(t)} Z_i^t$ is calculated according to

$$\begin{aligned}
\varphi(u) &= E\exp(iuZ) \\
&= \exp\left(iu\ln\frac{x_0}{p_0}\right) E\exp\left(iu\sum_{j=1}^{N(t)} Z_j^t\right) \\
&= \left(\ln\frac{x_0}{p_0}\right)^{iu} \left(\sum_{n=1}^{\infty}\prod_{j=1}^{n} E(\exp\left(iuZ_j^t\right)|N(t)=n)P(N(t)=n)\right) \\
&= \left(\ln\frac{x_0}{p_0}\right)^{iu} \left(\sum_{i=1}^{\infty}(\varphi_1(u))^n e^{-\Lambda(t)}\frac{(\Lambda(t))^n}{n!}\right) \\
&= \exp\left(iu\ln\frac{x_0}{p_0} + \Lambda(t)(\varphi_1(u)-1)\right).
\end{aligned}$$

Using explicit formula (5.45) one obtains that

$$\begin{aligned}\Lambda(t)(\varphi_1(u)-1) &= \Lambda(t)\left(\int_{\mathbb{R}} e^{iuz}(-1+\Lambda(t))^{-1}d\Lambda(p_t^{-1}([Re^{-z},+\infty)))\right)\\ &= \int_{\mathbb{R}}(e^{iuz}-1)d\Lambda(p^{-1}([Re^{-z},+\infty))).\end{aligned}$$

ii) In the proof of the part ii) it was shown that

$$\varphi(u)=\exp\left(iu\ln\frac{x_0}{p_0}+\int_{\mathbb{R}}(e^{iuz}-1)\nu(dz)\right).$$

The measure ν has a compact support and it is bounded by $\Lambda(t)<\infty$ in $\mathcal{L}^1$ since $(\Lambda(t))^{-1}\nu$ defines a probability measure. It implies that $\int_{|z|\leq 1}|z|\nu(dz)<\infty$. For measures μ such that

$$\int_{|z|\leq 1}|z|\mu(dz)<\infty,\ \int_{\mathbb{R}}(z^2\wedge 1)\mu(dz)<\infty \text{ and } \mu(\{0\})=0$$

the Lévy-Khinchin representation of the characteristic function ϕ of an infinitely divisible random variable (see [43], p. 39) has the form

$$\phi(u)=\exp\left(-\frac{cu^2}{2}+i\gamma_0 u+\int_{\mathbb{R}}(e^{iuz}-1)\mu(dz)\right)$$

with $\gamma_0\in\mathbb{R}$ and $c\geq 0$.
In our case, the components of the triplet $(c,\mu,\gamma_0)_0$ which specifies the infinitely divisible distribution are as follows:

$$c=0,\ \mu=\nu,\ \gamma_0=\ln\frac{x_0}{p_0}.$$

□

Remark: Note that the distribution of the random variables Z_t depends on t. This shows that in general the process $\ln X$ is not a stationary process. In particular, it is not a Lévy process.

If the previous theorem gave the characterization of the process X with the help of characteristic functions of the random variables $\ln X(t)$ for $t\in[0,T]$, the next theorem gives the explicit form of the predictable projection of the process $\ln X$:

Theorem 40 *i) the processes X and $\ln X$ are Markov processes,* *ii) the compensator* $\Lambda_{\ln X} : [0,T] \to \mathbb{R}_0$ *of the process* $\ln X$ *is a deterministic function given by*

$$\Lambda_{\ln X}(t) = \int_0^t \ln \frac{R}{p_t} \lambda(t) dt \tag{5.46}$$

Proof: Notice that in the case of deterministic intensity the increments of the counting process N on arbitrary disjunktiv sets B_1, B_2 are independent random variables. Moreover, the distribution of $N(v) - N(s)$ is independent on $\mathcal{F}(s)$ for all $v \in (s,T]$.
Note also that the bond price is given by formula (5.39) in what follows that it does not depend on the filtration $(\mathcal{F}(t))_{t\geq 0}$.
i) Hence, for the process X it is valid:

$$\begin{aligned}
E_Q(X(t)|\mathcal{F}(s)) &= E_Q\left(X(s) \prod_{i=N(s)}^{N(t)} \left(\frac{R}{p_{\tau_i}}\right) \middle| \mathcal{F}(s)\right) \\
&= X(s) E_Q\left(\prod_{i=N(s)}^{N(t)} \left(\frac{R}{p_{\tau_i}}\right)\right) \\
&= E_Q(X(t)|X(s)),
\end{aligned}$$

since the product $\prod_{i=N(s)}^{N(t)} \left(\frac{R}{p_{\tau_i}}\right)$ depends neither on $\mathcal{F}(s)$ nor on $X(s)$. This implies that the process X is Markov.
Analogously, the sum $\sum_{i=N(s)}^{N(t)} \ln \left(\frac{R}{p_{\tau_i}}\right)$ is independent from $\mathcal{F}(s)$ and $\ln X(s)$
Thus, we have

$$\begin{aligned}
E_Q(\ln X(t)|\mathcal{F}(s)) &= E_Q\left(\ln X(s) + \sum_{i=N(s)}^{N(t)} \ln \frac{R}{p_{\tau_i}} \middle| \mathcal{F}(s)\right) \\
&= \ln X(s) + E_Q\left(\sum_{i=N(s)}^{N(t)} \ln \frac{R}{p_{\tau_i}}\right) \\
&= E_Q(\ln X(t)| \ln X(s))
\end{aligned}$$

which implies that $\ln X$ is Markov as well.
ii) Denote in this proof by $M := \ln X - \Lambda_{\ln X}$, where $\Lambda_{\ln X}$ is defined by formula (5.46). We need to show first that M is a local martingale with

respect to the filtration $(\mathcal{F}(t))_{t\geq 0}$.
Using the argumentation of the part *i)*,

$$E_Q(\ln X(t) - \ln X(s)|\mathcal{F}(s)) = E_Q(\ln X(t) - \ln X(s)).$$

Let us find first $E_Q(\ln X(t) - \ln X(s))$. Notice that according to the representation of the process $\ln X$ in equation (5.38) it is valid

$$\ln X(t) - \ln X(s) = \sum_{i=N(s)+1}^{N(t)} \ln \frac{R}{p_{\tau_i}}.$$

Corollary 38 applied to the $\mathcal{F}(s)$-measurable process $\ln X - \ln X(s)$ implies the following representation:

$$\ln X(t) - \ln X(s) = \sum_{i=N(s)+1}^{N(t)} Z_i^{t,s},$$

where the independent random variables $Z_i^{t,s}$ are identically distributed according to

$$Q(Z_i^{t,s} \leq z) = (\Lambda(t) - \Lambda(s))^{-1} A\left(\left\{v \in (s,t] : \ln \frac{R}{p_v} \leq z\right\}\right).$$

Hence, the expectation of the compound Poisson random variable $\sum\limits_{i=N(s)+1}^{N(t)} Z_i^{t,s}$ equals

$$\begin{aligned}
E_Q(\ln X(t) - \ln X(s)) &= E_Q\left(\sum_{i=N(s)+1}^{N(t)} Z_i^{t,s}\right) \\
&= E_Q(N(t) - N(s)) E_Q(Z_1^{t,s}) \\
&= (\Lambda(t) - \Lambda(s)) \int z \mathbb{1}_{\{\ln \frac{R}{p_v}:\ s<v\leq t\}}(z) dQ(Z_1^{t,s} \leq z) \\
&= \int z \mathbb{1}_{\{\ln \frac{R}{p_v}:\ s<v\leq t\}}(z) \Lambda\left(\left\{v \in (s,t] : \ln \frac{R}{p_v} \in dz\right\}\right).
\end{aligned}$$

For $v \in (s,t]$ denote by $z' := \frac{R}{p_v}$. Inserting z' under the integral sign, we obtain:

$$\int z \mathbb{1}_{\{\ln \frac{R}{p_v}:\ s<v\leq t\}}(z) A\left(\left\{v \in (s,t] : \ln \frac{R}{p_v} \in dz\right\}\right) = \int_s^t \ln\left(\frac{R}{p_v}\right) \lambda(v) dv$$

Finally , for the process M it is valid:

$$
\begin{aligned}
E_Q(M(t)|\mathcal{F}(s)) &= E_Q(\ln X(t) - \Lambda_{\ln X}(t)|F(s)) \\
&= \ln X(s) + E_Q(\ln X(t) - \ln X(s)|\mathcal{F}(s)) - \int_0^t \ln\left(\frac{R}{p_v}\right)\lambda(v)dv \\
&= \ln X(s) - \int_s^t \ln\frac{R}{p_v} + \left(\int_s^t \lambda(v)\ln\frac{R}{p_v}dv - \int_s^t \lambda(v)\ln\frac{R}{p_v}dv\right) \\
&= M(s),
\end{aligned}
$$

where $0 \le s \le t \le T$.
The process $\ln X$ has locally bounded and integrable variation. In order to see it, take $\{\tau_i : i \in \mathbb{N}_0\}$ as a localizing sequence of stopping times and notice that the jump size is bounded:

$$
\ln\frac{R}{R+\mu[0,T]} \le \ln\frac{R}{p_v} \le rT, \quad v \in [0,T]
$$

what follows from Lemma 1 which states that $Re^{-rT} \le p_v \le R + \mu[0,T]$ for all $v \in [0,T]$.
Hence, the Doob-Meyer decomposition (Theorem 10) can be applied to the process $\ln X$. It implies that the process given by formula (5.46) is indeed a unique compensator s.th. $\Lambda_{\ln X}(0) = 0$. It proves the part *ii)*. □

Figures 5.4 and 5.5 depict the surface formed by the distribution functions of the final face value $X(T)$ for $T = 1$ year which are set into correspondence to the defining default intensities. Recall that the example of a single distribution function (when intensity $\lambda = 0,5$ and maturity $T = 7$ years) is shown on Figure 5.3.
For both figures 5.4 and 5.5, the initial investment $x_0 = 80$, it is invested in bond which pays constant coupon $c = 0,09$ and recovery $R = 0,6$; the riskless interest rate is set to be $r = 0,02$. Default intensity λ is constant for all distributions and the set of admissible intensities Λ has constantly one element from the start up to maturity T, i.e. $\Lambda(t) = \{\lambda_0\}$ for all $t \in [0,T]$. The parameter λ_0 is taken from the set $[0.1, 6]$.
Figure 5.4 gives the general picture, it shows the surface for a wide range of parameters $\lambda \in [0.1, 6]$. Figure 5.5 pays an attention to the extreme values of λ. The surface on Figure 5.4 is split into two parts: the left picture corresponds to small parameters $\lambda \in [0.1, 2]$; the right one gives a more detailed picture in the case of relatively large $\lambda \in [2, 6]$.

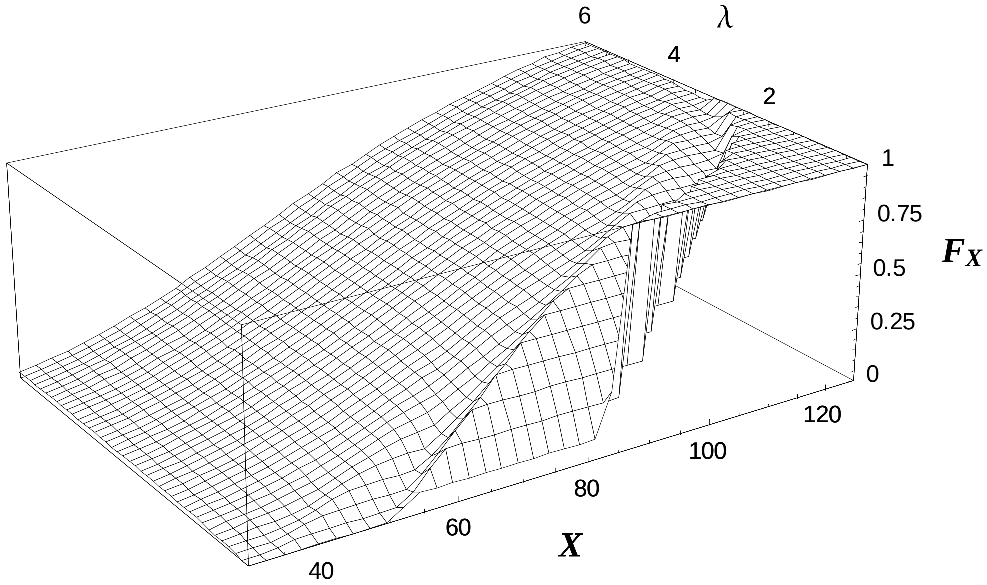

Figure 5.4: The family of cumulative distribution functions F_X for different values of constant default intensity. The initial investment $x_0 = 100$, it is invested in bond which pays constant coupon $c = 9\%$ and recovery $R = 0,6$; the riskless interest rate is set to be $r = 2\%$. Default intensity $\lambda(t) = \lambda_0$ for all $t \in [0, T]$, where $\lambda_0 \in [0.1, 6]$.

5.4.2 Value at Risk

Corollary 38 can be used in some important for practical purposes cases for calculating the risk of investment into the 'Chain of Bonds' portfolio via calculating its VaR. The cases when it is possible to find VaR of the face value $X(s)$ at arbitrary $s \in [0, T]$ are listed below. Let s be fixed. For the calculation of value at risk we use the representation validated by Corollary 38

$$\ln X(s) \stackrel{D}{=} \ln \frac{x_0}{p_0} + \sum_{i=1}^{N(s)} Z_i^s,$$

where Z_i^s are iid random variable.

Definition 10 Let $Y : \Omega \to \mathbb{R}$ be a random variable. Its *Value-at-Risk at level* $\alpha \in (0, 1)$ is determined by

$$VaR_\alpha(Y) := \inf\{y \in \mathbb{R} : Q(Y \leq y) > \alpha\}$$

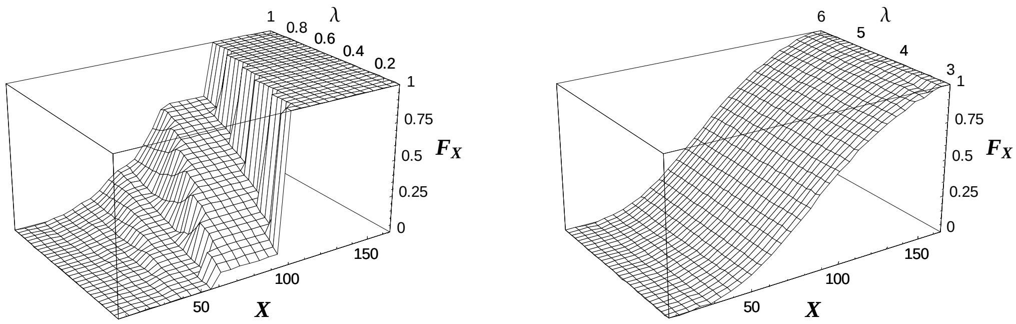

Figure 5.5: The family of cumulative distribution functions F_X for different values of constant default intensity λ in detail. Left panel corresponds to the region $0.1 < \lambda_0 \leq 2$. Right panel corresponds to the region $2 \leq \lambda_0 \leq 6$. Here $\lambda(t) = \lambda_0$ for $t \in [0, T]$. The initial investment $x_0 = 80$, it is invested in bond which pays constant coupon $c = 9\%$ and recovery $R = 0,6$; the riskless interest rate is set to be $r = 2\%$.

Let the level α be given. In order to find $VaR_\alpha(X(s))$, introduce first the upper and the lower bound of $\ln \frac{R}{p_t}$ for $t \in [0, s]$:

$$Z_* := \inf\{\ln \frac{R}{p_t} : t \in [0, s]\}$$

and

$$Z^* := \sup\{\ln \frac{R}{p_t} : t \in [0, s]\}.$$

Consider the common situation on the market which is characterized by the following two assumptions:

1. $\Lambda(s) = \int_0^s \lambda(t)dt$ is small: it is likely that default does not happened at all.

2. $Z^* < 0$, i.e. every default diminishes the face value.

Denote by $n_z := \lfloor \frac{-Z_*}{Z^* - Z_*} \rfloor$. By virtue of assumption 2, $n_z \in \mathbb{N}$. Notice, that n_z can be equivalently defined as

$$n_z = \max \left\{ n : P\left(\ln \frac{x_0}{p_0} + \sum_{i=1}^{n} Z_i^s > \ln \frac{x_0}{p_0} + \sum_{i=1}^{n-1} Z_i^s \right) = 0 \right\}$$

We calculate $VaR_\alpha(X(s))$ only in the case when

$$P(N(s) \geq n_z) = e^{-\Lambda(s)} \sum_{i=n_z}^{\infty} \frac{(\Lambda(s))^k}{k!} \leq \alpha.$$

Thus, assumption 1 means that we restrict ourselves to the case when $n_z > k_\alpha$, where

$$k_\alpha := \max\left\{ n : e^{-\Lambda(s)} \sum_{i=n}^{\infty} \frac{(\Lambda(s))^i}{i!} > \alpha \right\}.$$

Using the definition of k_α and n_z we conclude that for

$$\beta := \alpha - e^{-\Lambda(s)} \sum_{i=k_\alpha+1}^{\infty} \frac{(\Lambda(s))^k}{k!} \tag{5.47}$$

$VaR_\alpha(\ln X(s))$ is the β-quantile of the random variable $\ln \frac{x_0}{p_0} + \sum_{i=1}^{k_\alpha} Z_i^s$ and that it belongs to the interval

$$VaR_\alpha(\ln X(s)) \in \left[\ln \frac{x_0}{p_0} + k_\alpha Z_* \ , \quad \ln \frac{x_0}{p_0} + k_\alpha Z^* \right].$$

Finally, using Corollary 38 we conclude that

$$VaR_\alpha(\ln X(s)) = \ln \frac{x_0}{p_0} + \inf\{y : F_{Z_1^s}^{*k_\alpha}(y) > \beta\},$$

where $F_{Z_1^s}^{*k_\alpha}$ is the k_α-th convolution of the distribution function $F_{Z_1^s}$ given by formula (5.45). In particular, for $k_\alpha = 0$ obviously,

$$VaR_\alpha(\ln X(s)) = \ln \frac{x_0}{p_0}$$

and for $k_\alpha = 1$,

$$\begin{aligned} VaR_\alpha(\ln X(s)) &= \ln \frac{x_0}{p_0} + \inf\{y : F_{Z_1^s}(y) > \beta\} \\ &= \ln \frac{x_0}{p_0} + \inf\left\{ y : \frac{\int_{B_y} \lambda(t)dt}{\Lambda(s)} > \beta \right\}, \end{aligned}$$

where $B_y = \{t \in [0, s] : p_t \geq Re^y\}$.

Conclusion: Thus, we obtain that if $k_\alpha > n_z$, it can be found VaR_α. Denote by $u := \inf\{y : F_{Z_1^s}^{*k_\alpha}(y) > \beta\}$ with β given by expression (5.47). Then

$$VaR_\alpha(X(s)) = \frac{x_0}{p_0} e^u.$$

5.4.3 Firm Value Model

Consider an example where the firm value process is given by equation (5.9) with

$$f(V_i, t) = V_i(t)r \text{ and } g_i(V_i, t) = V_i(t)\sigma_i.$$

In other words, the firm value is described by a geometric Brownian motion:

$$dV_i(t) = V_i(t)(rdt + \sigma dW(t))$$

with initial condition $V_i(0) = v_o^{(i)}$.
Stopping time τ_i is defined as the first time when V_i crosses a constant boundary:

$$\tau_i := \inf\{t > \tau_{i-1} : V_{i-1}(t) \leq M_{i-1}\}.$$

Recall that this model and the distribution of default time τ_i for $i \in \mathbb{N}$ was considered above in Chapter 3. The distribution and the density function are calculated in Corollary 7. In this case it exists an intensity of default time as well. It is given in Corollary 3.28.
Notice that as it is shown in Proposition 13 there is $\lambda^* > 0$ such that the intensity λ of the corresponding counting process N satisfies the condition

$$\lambda(t) \leq \lambda^*, \quad t \in [0, T]$$

if there are predefined constants $\sigma^* > 0$ and $a_* > 0$ such that for the parameters σ_i and $a_i = \ln \frac{V_i(0)}{M_i}$ for $i \in \mathbb{N}$ it is valid:

$$\sigma_i \leq \sigma^*, \quad a_i \geq a_*.$$

The price of a bond which pays constant coupon c and in the case of default recovery with rate R evaluated at $t \in [0, T]$ see [1]) is given by

$$\begin{aligned} p_t &= \frac{c}{r} + e^{-r(T-t)}\left(1 - \frac{c}{r}\right)\left(N(l_1(T-t)) - \left(\frac{M_i}{V_i}\right)^{2\sigma^{-2}r-1} N(l_2(T-t))\right) \\ &\quad + \left(R - \frac{c}{r}\right)\left(\left(\frac{M_i}{V_i}\right)^{2\sigma^{-2}r} N(g_1(T-t)) - \left(\frac{V_i}{M_i}\right)^{2\sigma^{-2}r-1} N(g_2(T-t))\right) \end{aligned}$$

where N is a standard normal distribution function, the functions

$$l_1(s) = \frac{\ln \frac{V_i}{M_i} - (r - \frac{1}{2}\sigma_i^2)s}{\sigma_i\sqrt{s}}, \quad l_2(s) = \frac{-\ln \frac{V_i}{M_i} - (r - \frac{1}{2}\sigma_i^2)s}{\sigma_i\sqrt{s}}$$

and

$$g_1(s) = \frac{\ln \frac{M_i}{V_i} \pm (r + \frac{1}{2}\sigma_i^2)s}{\sigma_i\sqrt{s}}, \quad g_2(s) = \frac{\ln \frac{M_i}{V_i} - (r + \frac{1}{2}\sigma_i^2)s}{\sigma_i\sqrt{s}}$$

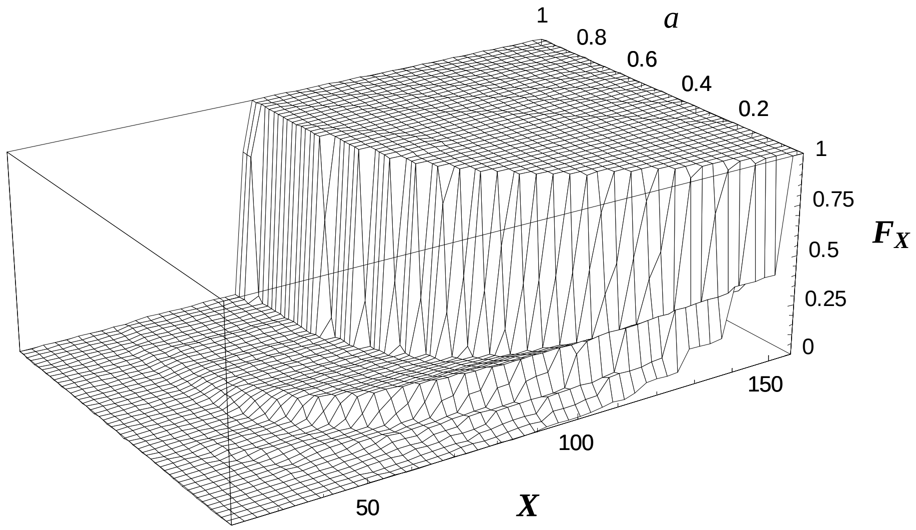

Figure 5.6: The family of cumulative distribution functions F_X for the range of parameters $a = \ln \frac{V}{M} \in [0.03, 1]$, $r = 2\%$, $c = 9\%$, $T = 5.0$, $R = 0.6$. The initial investment $x_0 = 100$. Firm value volatility $\sigma_V = 0.2$

with $s \in [0, T]$. Some prices are plotted on Figure 4.2.

If we restrict ourselves on portfolios consisting of homogeneous bonds. Bonds are homogeneous if they were issued by firms which have similar volatility and at the time they were bought they had equal distance from default $\frac{V_i}{M_i}$. The counting process N which corresponds to portfolio of homogeneous bonds is a renewal process. The face value at maturity $X(T)$ is shown on Figure 5.6.

Chapter 6

Optimal Control of the Principal Value Process

The current section deals with the problem of finding an optimal investment strategy for credit portfolios. While choosing a defaultable bond to buy, an investor decides between two contradictive possible actions: a purchase of a relatively cheap (read: risky) bond or an investment into a secure bond with low probability of default. These two factors make it hard for an investor to take a decision. Therefore, it is crucial to determine an optimal bond to invest in or, equivalently, a bond with an optimal intensity of default. The expectation of some certain kind of payment received by the bondholder is chosen to be a criterion of optimality.

6.1 Introduction and Setup

Regard the stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}(t))_{t\geq 0})$ with the process X as in (5.10) on it. The filtration $(\mathcal{F}(t))_{t\geq 0}$ is given by (5.4).
Recall that the process X represents the principal value of a bond which belongs to the portfolio at time t. When the bond defaults, the purely jump process X changes its value. The same happens to the intensity λ of the jump counting process N. Here

$$\lambda(t) = \sum_{i=0}^{\infty} \lambda_i(t) \mathbb{1}_{(\tau_i, \tau_{i+1}]}(t),$$

$$N(t) = \sum_{i=0}^{\infty} \mathbb{1}_{[\tau_i, T]}(t), \quad t \in [0, T]. \tag{6.1}$$

For $t \in [0,T]$ denote by

$$\alpha(t) := \max\{\{0\} \cup \{\tau_i : \tau_i \leq t\}\}$$

the last default time before t and by

$$\beta(t) := \min\{\{T\} \cup \{\tau_i : \tau_i > t\}\}$$

the next following after $\alpha(t)$ default time.
According to this definition, $t \in [\alpha(t), \beta(t))$ for $t \in [0,T]$ and $\alpha(\tau_i) = \tau_i$, $i \geq 0$.
Let p_i denote the price of the new bond which was purchased at τ_i as in the definition 5. Note that the price p_i of a new bond depends, among other factors, on the intensity of default. Thus, the distribution of the process X is completely determined by default intensity of purchased bonds. In other words, the process X is controlled by the mean of default intensities λ_i. Every time a bond in portfolio defaults, a new control λ must be determined. Equivalently, the process X is controlled at random stopping times $(\tau_i)_{i\geq 0}$ of default.
In order to give a definition of the controlled process X_u define first the class of admissible controls:

Definition 11 $\Lambda(t)$ is a family of admissible available intensities bounded in $L_\infty([t,T])$ by some $\lambda^* \in \mathbb{R}_+$:

$$\sup_{s\in[t,T]} |\lambda(s)| \leq \lambda^*, \lambda \in \Lambda.$$

Denote also by $\Lambda := \bigcup_{t\in[0,T]} \Lambda(t)$ *the collection of all admissible available intensities on* $[0,T]$.

Definition 12 An adapted process

$$u(t) := \lambda(t+) = \sum_{i=0}^{\infty} \lambda_i(t) \mathbb{1}_{[\tau_i,\tau_{i+l})}(t), \quad t \in [0,T], \tag{6.2}$$

with $\lambda_i(t) \in \Lambda(\tau_i),\ i \geq 0$, is called *a control process* in the current settings.

Definition 13 The process X associated with the control $u(\cdot)$ is called *the controlled process* X_u.

As before $R \in [0,1)$ denotes the recovery rate which is assumed to be equal for all bonds. The process X defined by equation (5.10) can be represented now as follows

$$X_u(t) = x_0 R^{N(t)} \left(\prod_{i=0}^{N(t)} p_i^u(\tau_i) \right)^{-1}, \tag{6.3}$$

where N is a counting process (6.1) with compensator A_u given by

$$A_u(t) = \int_0^t \lambda(s)ds = \int_0^t u(s)ds.$$

Regard the monotonously increasing function

$$g : \mathbb{R}_+ \to \mathbb{R}.$$

The function g reflects the investor's preferences or an income which investor gets from his investment.

Consider the following examples arising form the practical applications:

Example 1. A fund which invests into junk bonds offers the following type of product to its clients. A client receives an amount of money which depends on the total discounted payment from the beginning of the investment up to maturity T:

$$g\left(\int_0^T e^{-rs} X_{u^*}(s) d\mu(s) \right) = g\left(\int_0^T X_{u^*}(s) d\mu_d \right). \tag{6.4}$$

The payment $\int_0^T e^{-rs} X_{u^*}(s) d\mu$ results from the total fund's investment into high-yield bonds. It includes coupon payment during the lifetime of bond and principal payment at maturity. The function g represents beliefs of clients and fund manager about the future development of the investment.

Typically the function g is defined as

1. $g(x) = (x - K)^+$. According to this contract a client gets only that part of the total payment which exceeds some predefined level $K \in \mathbb{R}_+$.

2. $g(x) = \min(K_1, (K_2 - x)^+)$ with $K_1, K_2 \in \mathbb{R}_0$. Similarly to 1), a client receives some part of the total payment only in the case it belongs to some range.

3. The fund manager himself can take any utility function g (for example, $g(x) = \ln x$) and maximize the expected utility of the investment.

Example 2. The client receives a certain part of the coupon payment during the whole investment period and some part of the face value at maturity. Comparing to the Example 1 above it is more typical for funds oriented on long term investments.
In this case one intends to maximize the expectation of the expression

$$\int_0^T e^{-rs} g(X_{u^*}(s)) d\mu(s) = \int_0^T g(X_{u^*}(s)) d\mu_d(s). \tag{6.5}$$

The function g can be taken as in Example 1.
The optimization problem in this case is to find an *optimal control* u^* in the sense that

$$E \int_0^T g(X_{u^*}(s)) d\mu_d(s) = \max_{u \in \Lambda} E \int_0^T g(X_u(s)) d\mu_d(s) \tag{6.6}$$

Nevertheless, the process X is not Markov. But there is a Markov process $\tilde{X}$ of higher dimension than the process X itself, which contains X as a component. For Markov process $\tilde{X}$ Dynkin-type and HJB equations for the problem (6.6) can be found and a control at any time t is determined only by the information which is contained in the process $\tilde{X}$ itself.
Note that the solution of the optimization problem corresponding to (6.4) in some cases relevant to the practice can be immediately found. The statement below is a straightforward application of Theorem 32 for the case when g is a concave function (for example when it is a utility function).

Corollary 41 *If $g : \mathbb{R}_+ \to \mathbb{R}$ is a concave function, then it is optimal to invest into a purely riskless bond.*

Proof: By Jensen's inequality,

$$E_Q \left(g \left(\int_0^T X(t) d\tilde{\mu}_d(t) \right) \right) \leq g \left(E_Q \left(\int_0^T X(t) d\tilde{\mu}_d(t) \right) \right).$$

As it was shown in Theorem 32 for the process X,

$$E_Q \left(\int_0^T X(t) d\tilde{\mu}_d(t) \right) = x_0.$$

Thus,

$$E_Q \left(g \left(\int_0^T X(t) d\tilde{\mu}_d(t) \right) \right) \leq g(x_0).$$

Since for a purely riskless bond it is valid

$$E_Q \left(g \left(\int_0^T X_0(t) d\tilde{\mu}_d(t) \right) \right) = g \left(\int_0^T X_0(t) d\tilde{\mu}_d(t) \right) = g(x_0),$$

we conclude that for every process X

$$E_Q\left(g\left(\int_0^T X(t)d\tilde{\mu}_d(t)\right)\right) \leq E_Q\left(g\left(\int_0^T X_0(t)d\tilde{\mu}_d(t)\right)\right).$$

The last inequality exactly means that an investment into a purely riskless bond is optimal. □

6.2 Controlled Markov Processes Related to the 'Chain of Bonds' Portfolio.

In order to define a controlled Markov piecewise deterministic process $\tilde{X}_u$ related to the current problem, let us follow the procedure as in Section 5.2.3. Analogously, we set

$$\tilde{X}_u(t) = (X(t), u(X(t), t), t - \alpha(t), t).$$

The four components of the process $\tilde{X}_u$ take values in the following sets: X is one-dimensional and it takes values on E_1, u is $E_2 := \Lambda$-valued, $t - \alpha(t)$ and t are correspondingly in $E_3 = E_4 := \mathbb{R}_+$. Hence, $\tilde{X}_u$ is a $E = \times_{i=1}^4 E_i$ - valued process.
As before $\Psi_i : E \to E_i$, denote the projections on the corresponding subspace. The controlled Markov piecewise deterministic process $\tilde{X}_u$ is defined by the mean of:

1. The $(\mathcal{F}_t)$-adapted process u is the control of the process $\tilde{X}_u$. If t, s are such that $s, t \in (\alpha(t), \beta(t))$, then $u(\tilde{X}(s)) = u(\tilde{X}(t))$. Remind that according to the construction of $\tilde{X}_u$, $\Psi_i(\tilde{X}(s)) = \Psi_i(\tilde{X}(t))$, $i = 1, 2$. The control u changes when the first component X_u of the process $\tilde{X}_u$ has a jump.

2. The intensity measure λ_u is defined completely by the control u according to the equality (6.2) which implies that $\lambda_u(\tilde{X}_u(t)) = u(\tilde{X}_u(t-), t-)$, for $t \in [0, T]$.

3. The transition probability Q_u satisfies the following conditions:

   (a) $Q_u^{\{x\}}(x) = 0$, for every control u, and every $\tilde{x} \in E$.

   (b) $Q_u^A(x) = \delta_{xf(u)}(A)$. This condition means that the size of a jump of the process X_u at τ is completely determined by the value of the process at this time $\tilde{X}_u(\tau)$ and the control u applied at

τ. This dependence is established by the formula (6.3) and by definition 5 of the bond's price p_u as a function of default intensity (read:control u) and reward measure. According to this, $f(u) = \frac{R}{p_u}$.

4. The trajectories are determined by the vector-field $\mathcal{U} = \Psi_3 \times \Psi_4$ with the flow $\phi(t, (x, \lambda, s, v)) = (x, \lambda, t + s, t + v)$. The processes X and λ are constant between the jumps and the time after the last jump grows linearly with coefficient 1.

An important feature of the process constructed above is the following theorem.

Theorem 42 ([7], p.64) *The process $\tilde{X}_u$ is a homogeneous strong Markov process, i.e. for any $x \in E$, $\mathcal{F}(t)$-stopping time τ and bounded measurable function f,*

$$E_x(f(\tilde{X}_{\tau+s}\mathbb{1}_{\{\tau<\infty\}})|\mathcal{F}(\tau)) = P_s f(\tilde{X}_\tau)\mathbb{1}_{\{\tau<\infty\}}$$

Proof: see [7] p. 64. □

Due to 4., the process $\tilde{X}$ defined on E is a piecewise linear process. Thus, the controlled Markov process $\tilde{X}_u$ which corresponds to the initial process X was constructed. Now the which corresponds to $\tilde{X}$ can be properly defined and an equation which defines an optimal control for the initial problem will be found (see 6.3).

6.3 Value Function

Let the functions $G_u : [0, T] \times E \to \mathbb{R}_+$ be such that

$$G_u(t, \tilde{x}) := E\left(\int_t^T g(\Psi_1(\tilde{X}_u(s)))d\mu_d(s) \middle| \tilde{X}_u(t) = \tilde{x}\right). \tag{6.7}$$

The value function corresponding to the current problem is defined as a maximum over all admissible functions G_u.

Definition 14 The *value function* $G^* : [0, T] \times E \to \mathbb{R}_+$ for the optimization problem (6.6) is defined as

$$G^*(t, \tilde{X}(t)) := G_{u^*}(t, \tilde{X}(t)) = \max_u G_u(t, \tilde{X}(t)).$$

By definition 14, the value function at time 0 is the optimal expected value of the functional (6.5) which solves the optimization problem (6.6), indeed: $G^*(0,\tilde{x}_0) = \max_u E(\int_0^T g(s,\tilde{X}_u(s))ds)$. For the function G_u the following relations hold:

Lemma 43 *Let $t \in [0,T]$ and $\tilde{x} = (x,\lambda,s,v) \in E$. Define the set*

$$A_{t,\tau} := [t,T]/[\tau,\infty)$$

(here $\tau := \beta(t)$). The function G_u can be found the following way:

1. *if t is not a default time ($s > 0$) then*

$$G_u(t,\tilde{x}) = g(x)E(\mu_d(A_{t,\tau})) + \int_t^T G_u(s,\tilde{x}')P(\tau \in ds|\mathcal{F}(t))$$

   where $\tilde{x}' = (x,u,0) \in E$.

2. *if t is a time of default ($s = 0$) then*

$$G_u(t,\tilde{x}) = g(x_u)E(\mu_d(A_{t,\tau})) + \int_t^T G_u(s,x_u)P(\tau \in ds|\mathcal{F}(t)), \quad (6.8)$$

   where $x_u := x\frac{R}{p_u(t)}$, $\tilde{x}'_u = (x_u,u,0,t) \in E$.

Proof: Fix a control u and a time $t \in [0,T]$.
Denote by

$$x_+ := \begin{cases} x, & s > 0, \\ x\frac{R}{p_u(t)}, & s = 0. \end{cases}$$

If $s > 0$ then X_u is continuous in the time point t (it is constant on $\alpha(t) < t < \beta(t)$ in this case). The process X_u can be represented then as

$$X_u(s') = x \prod_{\tau_i \in (t,s]} \frac{R}{p_u(\tau_i)} = x_+ \prod_{\tau_i \in (t,s]} \frac{R}{p_u(\tau_i)}, \quad s' \in [t,T] \quad (6.9)$$

If $s = 0$, then $t = \alpha(t)$ and t is the jump time. The representation of the process X_u in this case is similar to the representation (6.9):

$$X_u(s') = x\frac{R}{p_u(t)} \prod_{\tau_i \in (t,s]} \frac{R}{p_u(\tau_i)} = x_+ \prod_{\tau_i \in (t,s]} \frac{R}{p_u(\tau_i)}, \quad s' \in (t,T] \quad (6.10)$$

The process X_u stays constant on $(t,\beta(t))$. Intensity which is chosen at the time $\alpha(t)$ also stays the same on $(t,\beta(t))$. In this sense the control u and the information about the value of the Markov process $\tilde{X}_u$ at the time t define the values of the process $\tilde{X}_u$ just before the following after t default time $\beta(t)$. This value is

$$\tilde{X}_u(\beta(t)) = \begin{cases} (x_+, u, 0, \beta(t)), & \beta(t) \in [t, T), \\ (x_+, u, s+T-t, \beta(t)), & \beta(t) = T. \end{cases}$$

Moreover, by the construction of the Markov process $\tilde{X}_u$ the distribution of $\beta(t)$ given the information up to time t is completely determined by the two last components $\Psi_2(\tilde{X}_u)$ and $\Psi_3(\tilde{X}_u)$ of $\tilde{X}_u$. It equals to

$$P(\beta(t) \in dw|\mathcal{F}(t)) = \lambda(w)e^{-\int_s^w \lambda(v)dv}, \; w \in (t, T].$$

By Markov property,

$$\begin{aligned} & E\left(\int_t^{\beta(t)} g(\Psi_1 \circ \tilde{X}_u(s))d\mu_d(s)\middle| \tilde{X}_u(t) = \tilde{x}\right) \\ =& E\left(\int_t^{\beta(t)} g(x_+)d\mu_d(s)|\mathcal{F}(t)\right) \\ =& g(x_+)\int_t^T \int_t^w d\mu_d(s)P(\beta(t) \in dw|\mathcal{F}(t)) \\ =& g(x_+)\int_t^T \mu_d([t,w))P(\beta(t) \in dw|\mathcal{F}(t)). \end{aligned}$$

By the linearity of expectation and taking (6.9) and (6.10) into account, the following holds:

$$\begin{aligned} & E\left(\int_t^T g(\Psi_1 \circ \tilde{X}_u(s))d\mu_d(s)\middle| \tilde{X}_u(t) = \tilde{x}\right) \\ = & \int_t^T \int_t^w g(x_+)d\mu_d(s)P(\beta(t) \in dw|\mathcal{F}(t)) \\ & + E\left(\int_{\beta(t)}^T g\left(x_+ \prod_{\tau_i \in (t,s]} \frac{R}{p_u(\tau_i)}\right) d\mu_d(s)\middle| \mathcal{F}(t)\right). \end{aligned} \tag{6.11}$$

Denote by $I := E\left(\int_{\beta(t)}^T g\left(x_+ \prod_{\tau_i \in (t,s]} \frac{R}{p_u(\tau_i)}\right) d\mu_d(s)|\mathcal{F}(t)\right)$ and note that

$$I = E\int_t^T \left(\int_w^T g\left(x_+ \prod_{\tau_i \in (t,s]} \frac{R}{p_u(\tau_i)}\right) d\mu_d(s)\middle| \mathcal{F}(t)\right) P_\lambda(\beta(t) \in dw|\mathcal{F}(t)).$$

Applying Fubini's theorem, it results in

$$I = \int_t^T E\left(\int_w^T g(P_1\tilde{X}_u(s))d\mu_d|\tilde{X}_u(w) = (x_+, u, 0, w)\right) P_\lambda(\beta(t) \in dw|\mathcal{F}(t)).$$

By the definition of the function G_u,

$$I = \int_t^T G_u(w, (x_+, u, 0, w))P_\lambda(\beta(t) \in dw|\mathcal{F}(t)).$$

Thus, $G_u(t, x)$ can be represented as follows

$$G_u(t, x) = \int_t^T \left(\int_t^w g(x)d\mu_d(s) + G_u(w, (x_+, u, 0, w))\right) P_\lambda(\beta(t) \in dw|\mathcal{F}(t)).$$

It implies the statements 1., 2. of the lemma. □

Regard now a restricted version X^n, $n \in \mathbb{N}_0$ of the process X which has at most n jumps on $[0, T]$. Denote by G^{n*} the value function as in the definition 14 which corresponds to the process X^n.
Note that G^{0*} can be easily found:

$$G^{0*}(0, x_0) = g(x_0(\mu_d([0, T]))^{-1})\mu_d([0, T]).$$

The following lemma interprets the fact that the process X^n has less restrictions than X^m if $m < n \in \mathbb{N}$.

Lemma 44 *Assume that* $0 \in \Lambda(t)$ *for* $t \in [0, T]$. *Then* $\{G^{n*}(t, x) : n \in \mathbb{N}_0\}$ *is a monotone increasing sequence for fixed* $t \in [0, T]$ *and* $x \in E$.

Proof: If $0 \in \Lambda(t)$, $t \in [0, T]$, then every control $u \in \Lambda$ of the process X^{n-1} is also a control which can be applied to the process X^n. This is due to the fact that $u = 0$ can be chosen after the $(n-1)$-th default. It implies the monotonicity

$$G^{(n-1)*}(t, x) = \max_u E\left(\int_t^T g(s, \tilde{X}_u^{n-1}(s))ds\,\middle|\, \tilde{X}_u(t-) = x\right)$$

$$\leq \max_u E\left(\int_t^T g(s, \tilde{X}_u^{n}(s))ds\,\middle|\, \tilde{X}_u(t-) = x\right) = G^{n*}(t, x) \text{ for } n \in \mathbb{N}.$$

□

Note that the process X is controlled only at default times $\{\tau_i\}$. Thus, in order to solve the optimization problem it is enough to regard value functions of a special type, namely $\mathbb{G}^* = G^*|_{E_0}$, where $E_0 := \{\tilde{x} \in E : \Psi_3\tilde{x} = 0\}$.
If $\mathbb{G}^{n*}$ is known, the value function $\mathbb{G}^{(n+1)*}$ can be calculated. Lemma 43 yields this recursive formula:

Theorem 45 *Set* $\tilde{x} = (x, \lambda, 0, t) \in E_0$ *and* $\tilde{x}_u = (\frac{xR}{p_u(t)}, \lambda_u, 0, t) \in E_0$*. Denote by* $A_t := [t, T]/[\tau, \infty)$*. Then*

$$\mathbb{G}^{n*}(t, \tilde{x}) = \max_u \left(g\left(\frac{xR}{p_u(t)}\right) \int_t^T \mu_d(A_\tau) dQ(s) + \int_t^T \mathbb{G}^{(n-1)*}(s, \tilde{x}_u) dQ(s) \right). \tag{6.12}$$

Proof: follows from the definition of $\mathbb{G}^{n*}$ and Lemma 43. □

By the construction, the value function is bounded from above for every $(t, x) \in [0, T] \times E$. Indeed, since the function g is monotone increasing,

$$G^*(t, x) \leq g(x_0)\mu_d([0, T]).$$

In particular, for all $n \in \mathbb{N}$ and $G^{n*}(t, x) \leq g(x_0)\mu_d([0, T])$. By Lemma 44, $\{G^{n*}(t, x)\}$ increases monotonously. Thus, for every $(t, x) \in [0, T] \times E$ there is a limit

$$\mathbb{G}^*(t, x) := \lim_{n\to\infty} \mathbb{G}^{n*}(t, x)$$

Theorem 45 implies that

$$\mathbb{G}^*(t, \tilde{x}) = \max_u \left(g\left(\frac{xR}{p_u(t)}\right) \int_t^T \mu_d(A_t) dQ(s) + \int_t^T \mathbb{G}^*(s, \tilde{x}_u) dQ(s) \right)$$

Appendix A

Facts from the Theory of Brownian Motion

A.1 Distribution of the First Time to Passage

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis. The filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions and supports the standard Brownian motion W. For constants $\mu \in \mathbb{R}$, $\tilde{\sigma} > 0$ regard its transformation $\tilde{W}$ defined as

$$\tilde{W}(t) = \mu t + \tilde{\sigma} W(t).$$

This is a Brownian motion with drift parameter $\mu \in \mathbb{R}$ and standard deviation $\tilde{\sigma} > 0$. Let $\mathbf{N} : \mathbb{R} \to (0,1)$ denote the standard normal distribution function. Fix some constant $a > 0$.

Definition 15 The random variable

$$\tau_{\tilde{W},a} = \inf_{t>0}\{t : \tilde{W}(t) \geq a\}$$

is called the *first passage time* of the process $\tilde{W}$ through the border a.

If the border a and the parameters of the process $\tilde{W}$ are specified, $\tau_{\tilde{W},a}$ will be often denoted by τ.
A well-known fact of the theory of Brownian Motion is that τ has an inverse-Gaussian distribution:

Lemma 46 *The distribution function of the random variable $\tau_{\tilde{W},a}$ is given by the formula*

$$P(\tau_{\tilde{W},a} \leq t) = N\left(\frac{\mu t - a}{\tilde{\sigma}\sqrt{t}}\right) + e^{2a\mu\tilde{\sigma}^{-2}} N\left(-\frac{\mu t + a}{\tilde{\sigma}\sqrt{t}}\right), \tag{A.1}$$

and its density function is

$$f_{\tau_{\tilde{W},a}}(t) = \frac{a}{\tilde{\sigma}\sqrt{2\pi t^3}} e^{-\frac{(a-\mu t)^2}{2\tilde{\sigma}^2 t}}. \tag{A.2}$$

Proof: See e.g. [45]. □

Consider the function

$$P^t_{\tilde{W}} : (0, +\infty) \to [0, 1],$$

which is defined by the relation

$$P^t_{\tilde{W}}(a) = P(\tau_{\tilde{W},a} \leq t).$$

The function $P^t_{\tilde{W}}$ shows the default probability depending only on the parameter a when the parameters $\tilde{\sigma}$ and μ of the process $\tilde{W}$ and the time t are fixed.
We list some properties of the function $P^t_{\tilde{W}}$.

Proposition 47 $P^t_{\tilde{W}} \in C^\infty(\mathbb{R}_+)$.

Proof: The statement follows directly from Lemma 46, equation (A.1). □

Proposition 48 $P^t_{\tilde{W}}$ *is a strictly decreasing function.*

Proof: choose some positive real numbers $a_1 < a_2$. Then there is the following inclusion of the events:

$$\{t : \tilde{W}(t) \geq a_1\} \subset \{t : \tilde{W}(t) \geq a_2\}.$$

Thus, the probability of the first set is not greater than the probability of the second one. By the definition of the first passage time, the inclusion above shows that the function $P^t_{\tilde{W}}$ decreases. Moreover, $P^t_{\tilde{W}}$ is a strictly decreasing function:

$$\begin{aligned} P^t_{\tilde{W}}(a_2) &= P(\tau_{\tilde{W},a_2} \leq t, \tau_{\tilde{W},a_1} \leq t) \\ &= P(\tau_{\tilde{W},a_2} \leq t | \tau_{\tilde{W},a_1} \leq t) P(\tau_{\tilde{W},a_1} \leq t) \\ &= P(\tau_{\tilde{W},a_2} \leq t | \tau_{\tilde{W},a_1} \leq t) P^t_{\tilde{W}}(a_1) \end{aligned}$$

since $P(\tau_{\tilde{W},a_2} \leq t | \tau_{\tilde{W},a_1} \leq t) < 1$ for $a_1 < a_2$ it follows that $P^t_{\tilde{W}}(a_2) < P^t_{\tilde{W}}(a_1)$. □

Proposition 49 *For arbitrary $t > 0$, $\tilde{\sigma}^2 > 0$, and $\mu \in \mathbb{R}$, the mapping*

$$P^t_{\tilde{W}} : (0, +\infty) \to (0, 1)$$

is a bijection.

Proof: It follows from Properties 47, 48 and the limits that if a tends to infinity,

$$\begin{aligned}
\lim_{a\to+\infty} P^t_{\tilde{W}}(a) &= \lim_{a\to+\infty} \left(N\left(\frac{\mu s - a}{\tilde{\sigma}\sqrt{s}}\right) + e^{2a\mu\tilde{\sigma}^{-2}} N\left(-\frac{\mu s + a}{\tilde{\sigma}\sqrt{s}}\right)\right) \\
&= 0 + \lim_{a\to+\infty} e^{2a\mu\tilde{\sigma}^{-2}} N\left(-\frac{\mu s + a}{\tilde{\sigma}\sqrt{s}}\right) \\
&= \lim_{a\to+\infty} \frac{\frac{\partial}{\partial a} N\left(-\frac{\mu s + a}{\tilde{\sigma}\sqrt{s}}\right)}{\frac{\partial}{\partial a} e^{-2a\mu\tilde{\sigma}^{-2}}} = \lim_{a\to+\infty} \frac{-\frac{1}{\tilde{\sigma}\sqrt{2\pi s}} e^{-\frac{(\mu s + a)^2}{2\tilde{\sigma}^2 s}}}{-2\mu\tilde{\sigma}^{-2a\mu\tilde{\sigma}^{-2}}} \\
&= \frac{\tilde{\sigma} e^{-\frac{\mu^2 s}{2\tilde{\sigma}^2}}}{2\mu\sqrt{2\pi s}} \lim_{a\to+\infty} e^{-\frac{a^2}{2\tilde{\sigma}^2 s}} = 0.
\end{aligned}$$

The third equality from expression above follows from de L'Hopital's rule. If a tends to 0 from the right, we have

$$\begin{aligned}
\lim_{a\to 0+} P^t_{\tilde{W}}(a) &= \lim_{a\to 0+} \left(N\left(\frac{\mu s - a}{\tilde{\sigma}\sqrt{s}}\right) + e^{2a\mu\tilde{\sigma}^{-2}} N\left(-\frac{\mu s + a}{\tilde{\sigma}\sqrt{s}}\right)\right) \\
&= N\left(\frac{\mu s}{\tilde{\sigma}\sqrt{s}}\right) + N\left(-\frac{\mu s}{\tilde{\sigma}\sqrt{s}}\right) = 1.
\end{aligned}$$

□

Remark: The statement of Lemma 49 does not contradict of course the fact that $P(\tau < \infty) = e^{2a\mu\tilde{\sigma}^{-2}} < 1$ if $\mu < 0$. Indeed, if $\tilde{\sigma} > 0$, $\mu < 0$ then

$$\lim_{a\to 0+} P(\tau < \infty) = \lim_{a\to 0+} e^{2a\mu\tilde{\sigma}^{-2}} = 0, \quad \lim_{a\to+\infty} P(\tau < \infty) \lim_{a\to+\infty} e^{2a\mu\tilde{\sigma}^{-2}} = 1$$

and the same limit values as for the probability $P(\tau < \infty)$ will be reached by the probability $P(\tau \leq t)$.

A.2 Stopping Times and Strong Markov Property of Brownian Motion

Consider a measurable space $(\Omega, \mathcal{F})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$.

Definition 16 A random time $\tau : \Omega \to \mathbb{R}_0$ is a *stopping time* of the filtration, if the event $\{\tau \leq t\}$ belongs to the sigma-field $\mathcal{F}_t$, for every $t \in \mathbb{R}_0$.

Theorem 50 [25], p. 86 *Let τ be an a.s. finite stopping time of the filtration $(\mathcal{F}_t)_{t>0}$ for the d-dimensional Brownian motion $W = \{W_t, F_t | t \geq 0\}$. Then with $Y_t := W_{\tau+t} - W_\tau$, the process $Y = \{Y_t, F_t^Y | t \geq 0\}$ is a d-dimensional Brownian motion, independent of $\mathcal{F}_{\tau_+}$.*

Proof: see [25] Theorem 6.16, p. 86. □

Appendix B

Semimartingales

B.1 The Exponential of a Semimartingale

Theorem 51 ([37], p. 203) *Let Z be a real semimartingale defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}), P)$.*

1. *For every $\mathcal{F}_0$-measurable random variable U there are exists a unique (up to P-equality) real regular right-continuous process such that*

$$dX(t) = X(t-)dZ(t), \quad X(0) = U$$

2. *One has*

$$X(t) = U \exp\left(Z(t) - \frac{1}{2}[Z(t)]^c \right) \prod_{0<s\leq t} (1 + \Delta Z(s)) \exp(-\Delta Z_s), \tag{B.1}$$

   where the infinite product is a.s. absolutely convergent.

B.2 Integration by Parts

A real-valued process V will be said *regular* when V is adapted and its paths have left and right limits for every $t \in \mathbb{R}_0$.
A process V is called *right-continuous* when its paths are right-continuous functions.

Lemma 52 ([37], p. 192) *If Z is a semimartingale and V is a regular right-continuous process of locally bounded variation, then*

$$Z(t)V(t) = Z(0)V(0) + \int_{(0,t]} V(s-)dZ(s) + \int_{(0,t]} Z(s)dV(s).$$

Bibliography

[1] T. R. Bielecki, M. Rutkowski (2004): *Credit risk: Modeling, valuation and hedging.* Springer Finance, Berlin.

[2] F. Black, J. Cox (1976): *Valuating corporate securities: Some effects of bond indenture provisions.* Journal of Finance, **31**, 351-367.

[3] F. Black, M. Scholes (1973): *The pricing of options and corporate liabilities.* Journal of Political Economy, **81**, 637-654.

[4] A. N. Borodin, P. Salminen (2002): *Handbook of Brownian motion: Facts and formulae.* Birkhäuser Verlag, Basel.

[5] P. Brémaud (1981): *Point Processes and queues. Martingale dynamics.* Springer-Verlag, New York - Heidelberg - Berlin.

[6] D. Brigo, F. Mercurio (2001): *Interest rate models. Theory and practice.* Springer, Berlin.

[7] M. H. A. Davis (1993): *Markov models and optimization.* Chapman & Hall, London.

[8] D. Duffie, D. Lando (2001): *Term structures of credit spreads with incomplete accounting information.* Econometrica, **69(3)**, 633-664.

[9] D. Duffie, K. J. Singleton (1999): *Modeling term structures of default bonds.* The Review of Financial Studies, **12(4)**, 687-720.

[10] E. B. Dynkin, A. A. Yushkevich (1979): *Controlled Markov processes.* Springer, Berlin.

[11] R. J. Elliott, P. E. Kopp (2001): *Mathematics of Financial Markets.* Springer, New York.

[12] S. Emmer, C. Klüppelberg, R. Korn (2001): *Optimal portfolios with bounded capital at risk.* Mathematical Finance, **11(4)**, 365-384.

[13] W. Feller (1966): *An introduction to probability theory and its applications. Volume 1-2.* John Wiley & Sons, New York.

[14] W. H. Fleming, R. W. Rishel (1975): *Deterministic and stochastic optimal control.* Springer, Berlin.

[15] H. Föllmer, A. Schied (2002): *Stochastic Finances.* De Gruyter, Berlin.

[16] K. Giesecke (2002): *Default compensator, incomplete information, and the term structure of credit spreads.* www.defaultrisk.com.

[17] K. Giesecke (2003): *Default and Information.* www.defaultrisk.com.

[18] I. I. Gihman, A. V. Skorohod (1979): *Controlled stochastic processes.* Springer, New York.

[19] B. V. Gnedenko, V. Yu. Korolev (1996): *Random summation. Limit theorems and applications.* CRC Press, Boca Raton.

[20] J. Michael Harrison (1985): *Brownian motion and stochastic flow systems.* Wiley, New York.

[21] J. Jacod, A. N. Shiryaev (2003): *Limit theorems for stochastic processes.* Springer, Berlin.

[22] R. A. Jarrow, D. Lando, S. M. Turnbull (1997): *A Markov model for the term structure of credit risk spreads.* The Review of Financial Studies, **10(2),** 481-523.

[23] R. A. Jarrow, S. M. Turnbull (1995): *Pricing derivatives on financial securities subject to credit risk.* Journal of Finance, **50**, 53-85.

[24] O. Kallenberg (2002): *Foundations of modern probability.* Springer, New York.

[25] I. Karatzas, S. E. Shreve (1991): *Brownian motion and stochastic calculus.* Springer-Verlag, New York.

[26] I. Karatzas (1989): *Optimization problems in the theory of continuous trading.* SIAM Journal on Control and Optimization **27**, 1221-1259.

[27] R. Korn (1997): *Optimal portfolios.* World Scientific, Singapore.

[28] R. Korn, H. Kraft (2002): *Optimal portfolios with defaultable securities: A firm value approach.* International Journal of Applied and Theoretical Finance, **6**, 793-819.

[29] R. Korn, O. Menkens (2003): *Worst-case scenario optimization: A new stochastic control approach.* Working Paper, University of Kaiserslautern, Kaiserslautern.

[30] A. L. Lewis (1998): *Application of eigenfunction expansions in continuous-time finance.* Mathematical Finance, **8(4)**, 349-383.

[31] R.S. Liptser, A.N. Shiryaev (1989): *Theory of martingales.* Kluwer Academic Publishers, Dordrecht.

[32] R.S. Liptser, A.N. Shiryaev (2001): *Statistics of random processes. Books 1-2.* Springer, New York.

[33] D. Madan, H. Unal (1998): *Pricing of risks of default.* Review of Derivative Research, **2(2/3)**, 121-160.

[34] R. C. Merton (1971): *Optimal consumption and portfolio rules in a continuous-time model.* Journal of Economic Theory **3**, 373-413.

[35] R. C. Merton (1974): *On the pricing of corporate debt: The risk structure of interest rates.* Journal of Finance, **29**, 449-479.

[36] R. C. Merton (1999): *Continuous-time finance.* Blackwell, Cambridge.

[37] M. Métivier (1982): *Semimartingales, a course on stochastic processes.* De Gruyter, Berlin.

[38] B. Øksendal (2000): *Stochastic differential equations. An introduction with applications.* Springer-Verlag, New York.

[39] A. Pelsser (2000): *Efficient methods for valuing interest rate derivatives.* Springer, London.

[40] V. V. Petrov (1995): *Limit theorems of probability theory. Sequences of independent random variables.* Clarendon Press, Oxford.

[41] V. V. Petrov (1975): *Sums of independent random variables.* Springer, Berlin.

[42] G. Ch. Pflug (2000): *Some remarks on the value-at-risk and the conditional value-at-risk.* Nonconvex Optim. Appl. **49**, 272-281.

[43] K.-I. Sato (1999): *Lévy processes and infinitely divisible distributions.* Cambridge University Press, Cambridge.

[44] Ph. J. Schönbucher (2000): *Credit risk modelling and credit derivatives.* PhD Thesis, Bonn.

[45] V. Seshadri (1999): *The inverse Gaussian distribution. Statistical theory and applications.* Springer, New York.

[46] A. N. Shiryaev (1999): *Essentials of stochastic finance. Facts, models, theory.* World Scientific, Singapore.

[47] A. V. Skorohod (1990): *Lectures on the theory of random processes. (in Ukrainian)* Lybid, Kiyv.

[48] S. Uryasev, R.T. Rockafellar (2000): *Optimization of conditional value-at-risk.* The Journal of Risk, **2(3)**, 21-41.

[49] O. Vasicek (1977): *An equilibrium Characterization of the term structure.* Journal of Financial Economics, **5**, 513-531.

[50] O. Witting (1985): *Mathematische Statistik 1. Parametrische Verfahren bei festem Stichprobenumfang.* B. G. Teubner, Stuttgart.

Lebenslauf

Persönliche Daten

Name	Helen Kovilyanskaya
Anschrift	Max-Born Str. 38, 40 591 Düsseldorf
Geburtsdaum/-ort	24.11.1977 in Nikolaev (Ukraine)
Staatsangehörigkeit	ukrainisch
Familienstand	ledig

Schulausbildung

09/1984-11/1990	Mittelschule 11, Nikolaev, Ukraine
11/1990-05/1994	Mathematisch-Informatisches Lytzeum 38, Nikolaev, Ukraine. Abschluß : Ausgezeichnet (Goldene Medallie)
1992	1^{ste} Platz an der Ukrainianische Mathematik Olympiade
1994	2^{te} Platz an der Ukrainianische Mathematik Olympiade

Studium

09/1994-06/1999	Studium der Mathematik an der Taras Shevtchenko Universität Kiew (Ukraine). Abschluß : Magister. Note: mit Auszeichnung.
10/1998-06/1999	paralleles Studium an der Universität Kaiserslautern, Deutschland. Teilnahme an der Austauschprogramm 'Mathematics International'
06/1999-09/2000	Studium an der Universität Kaiserslautern (Deutschland). Abschluß : Diplom-Wirtschaftsmathematikerin. Note: mit Auszeichnung.

Berufserfahrung

10/1999-03/2000	Wissenschafliche Hilfskraft an der Universität Kaiserslautern.
11/2000-03/2003	Wissenschaftliche Angestellte an der Universität Kaiserslautern. Arbeit an dem gemeinsamen Forschungsprojekt mit Versicherungsfirma 'DBV-Winterthur': 'Riskmaße in Finanz- und Versicherungsmathematik'.
Seit April 2003	Wissenschaftliche Angestellte an der Heinrich-Heine Universität Düsseldorf.