Lipschitz estimates for the stop and the play operator

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Abstract

In this article, we give some generalisations of existing Lipschitz estimates for the stop and play operator with respect to an arbitrary convex and closed characteristic in a separable Hilbert space. We are especially concerned with the dependence of their outputs with respect to different scalar products.

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Abbreviations. Throughout this paper, we will make frequently use of the following shortcuts, in order to help the reader.

(CS)	Cauchy-Schwarz inequality		
(Ho)	Hölder inequality	a.e.	almost everywhere
(Mi)	Minkowski inequality	e.	everywhere
(FT)	Fundamental theorem		

1 Definition of the stop and play operator

According to KREJCI [9, 10, 11] and BROKATE [1, 2, 3, 4], for a given scalar product $\langle\cdot,\cdot\rangle$ on a separable Hilbert space X, a convex and closed *caracteristic* $Z \subseteq X$ and an *initial memory* $s_0 \in Z$, there exist a unique decomposition of the unity

$$\mathcal{I} = \mathcal{P}_{Z,\langle\cdot,\cdot\rangle} + \mathcal{S}_{Z,\langle\cdot,\cdot\rangle} \tag{1}$$

into operators

$$\begin{aligned} \mathcal{P}_{Z,\langle\cdot,\cdot\rangle} &: W^{1,q}([0,T],X) \times Z \to W^{1,q}([0,T],X) \\ \mathcal{S}_{Z,\langle\cdot,\cdot\rangle} &: W^{1,q}([0,T],X) \times Z \to W^{1,q}([0,T],X) \end{aligned}$$

named

$$\textbf{play } p(t) = \mathcal{P}_{Z,\langle\cdot,\cdot\rangle}(f,s_0)(t) \qquad \text{and} \qquad \textbf{stop } s(t) = \mathcal{S}_{Z,\langle\cdot,\cdot\rangle}(f,s_0)(t)$$

(for all $t \in [0,T]$), defined by the uniquely determined solutions of the *differential evolution variational inequality*

$$\left\{ \begin{array}{rcll} \langle \dot{p}(t),\, s(t) - * \rangle & \geq & 0 \quad \text{for all } * \in Z & \text{a.e. in } [0,T] \\ s(t) + p(t) & = & f(t) & \text{e. in } [0,T] \\ s(0) & = & s_0 & \end{array} \right. \tag{2}$$

For definition of $W^{1,q}([0,T],X)$ see the appendix in [10]. They play an essential role in plasticity theory, see [1, 4, 9], and contact mechanics, wherever rate-independent hysteresis phenomena occur.

1.1 Remark. (a) In fact domain and range of stop and play can be chosen larger, see Krejci [10], theorem 4.1 or [9] theorem 3.1, functions of class CBV, i.e. continuous functions of bounded variation. Note that the inclusions

$$W^{1,q}([0,T],\cdot) \subset W^{1,1}([0,T],\cdot) = AC([0,T],\cdot) \subset CBV([0,T],\cdot)$$

($1 \le q < \infty$) hold.
(b) $\mathcal{S}$ and $\mathcal{P}$ are continuous with respect to the norm $\|\cdot\|_{W^{1,q,*}}$, see [10] theorem 4.2, and – by the following lemma 1.2 – to $\|\cdot\|_{W^{1,q}}$. □

Before starting, we introduce

$$q' = \begin{cases} q/(q-1) & \text{if } q > 1 \\ 1 & \text{if } q = 1 \end{cases}, \qquad \text{thus} \quad \frac{q'}{q} = q - 1 \tag{3}$$

the *(quasi) Hölder conjugate* for a given real number $1 \le q < \infty$. We set further

$$\|f\|_{\infty,t} = \max_{\tau \in [0,t]} \|f(\tau)\|, \qquad f \in \mathcal{C}\big([0,T],X\big)$$

for the maximum of $\|f\|$ over the interval $[0,t]$.

1.2 Lemma (equivalent norms) *Let X be a Banach space. On the space $Y = W^{1,q}([0,T],X)$, the norms*

$$\begin{aligned}\|f\|_{W^{1,q}} &= \left(\int_0^T \|f(\tau)\|^q d\tau\right)^{1/q} + \left(\int_0^T \|\dot f(\tau)\|^q d\tau\right)^{1/q} && \textit{(usual Sobolev norm)}\\ \|f\|_{W^{1,q,*}} &= \|f(0)\| + \left(\int_0^T \|\dot f(\tau)\|^q d\tau\right)^{1/q} && \textit{(unusual Sobolev norm)}\end{aligned}$$

(which make Y as well a Banach space) are equivalent for each $1 \le q < \infty$. Additionally, we have

$$\|f(t)\| \le \|f\|_\infty \le \|f\|_{W^{1,1,*}} \qquad \textit{for all } t \in [0,T] \tag{4}$$

Proof: We make several steps.

(1) Norm $\|\cdot\|_{W^{1,q,*}}$ is stronger than norm $\|\cdot\|_{W^{1,q}}$. The case $q = 1$. According to the fundamental theorem of Lebesgue integration theory we have for all $t \in [0,T]$

$$\begin{aligned}\|f\|_{W^{1,1}} &= \int_0^T \|f\|\mathrm{d}t + \int_0^T \|\dot f\|\mathrm{d}t\\ &\overset{(FT)}{=} \int_0^T \left\|f(0) + \int_0^t \dot f(\tau)\mathrm{d}\tau\right\|\mathrm{d}t + \int_0^T \|\dot f\|\mathrm{d}t\\ &\le \int_0^T \left(\|f(0)\| + \int_0^T \|\dot f\|\mathrm{d}\tau\right)\mathrm{d}t + \int_0^T \|\dot f\|\mathrm{d}t\\ &= T\|f(0)\| + (T+1)\int_0^T \|\dot f\|\mathrm{d}t\\ &\le (T+1)\|f\|_{W^{1,1,*}}\end{aligned}$$

The case $1 < q < \infty$. Let q' be defined by (3). Then with Hölder

$$\left\|\int_0^t \dot f(\tau)\mathrm{d}\tau\right\|^q \le \left(\int_0^t 1\cdot\|\dot f(\tau)\|\mathrm{d}\tau\right)^q \overset{(Ho)}{\le} t^{q/q'} \int_0^t \|\dot f(\tau)\|^q \mathrm{d}\tau \tag{5}$$

and

$$\int_0^T \left\| \int_0^t \dot{f}\mathrm{d}\tau \right\|^q \mathrm{d}t \overset{(5)}{\leq} \left(\int_0^T \|\dot{f}\|^q \mathrm{d}\tau \right) \left(\int_0^T t^{q-1} \mathrm{d}t \right) = \frac{T}{q} \int_0^T \|\dot{f}\|^q \mathrm{d}\tau \qquad (6)$$

We estimate with Minkowski and the fundamental theorem

$$\begin{aligned}
\|f\|_{W^{1,q}} &= \left(\int_0^T \|f(t)\|^q \mathrm{d}t \right)^{1/q} \\
&\overset{(FT)}{=} \left(\int_0^T \left\| f(0) + \int_0^t \dot{f}(\tau)\mathrm{d}\tau \right\|^q \mathrm{d}t \right)^{1/q} + \left(\int_0^T \|\dot{f}\|^q \mathrm{d}t \right)^{1/q} \\
&\overset{(Mi)}{\leq} \left(\int_0^T \|f(0)\|^q \mathrm{d}t \right)^{1/q} + \left(\int_0^T \left\| \int_0^t \dot{f}\mathrm{d}\tau \right\|^q \mathrm{d}t \right)^{1/q} \\
&\qquad + \left(\int_0^T \|\dot{f}\|^q \mathrm{d}t \right)^{1/q} \\
&\overset{(6)}{\leq} T^{1/q}\|f(0)\| + \left(\left(\frac{T}{q} \right)^{1/q} + 1 \right) \left(\int_0^T \|\dot{f}(\tau)\|^q \mathrm{d}\tau \right)^{1/q} \\
&\leq \mathrm{const}(T,q)\, \|f\|_{W^{1,q,*}}
\end{aligned}$$

(2) Norm $\|\cdot\|_W^{1,q}$ is stronger than norm $\|\cdot\|_{W^{1,q,*}}$: Due to the continuous embedding (see Evans [5], chapter 5.9, theorem 2 or Zeidler [12], section 23.6, proposition 23.23)

$$W^{1,q}([0,T],\Sigma^\star) \hookrightarrow C([0,T],\Sigma^\star) \qquad \text{(for each } 1 \leq q \leq \infty)$$

with

$$\|f\|_\infty \leq \mathrm{const}(T)/2\|f\|_{1,q},$$

where the constant can be chosen in dependence only on T, we have the contrary inequality

$$\|f\|_{1,q,*} \leq \mathrm{const}(T)\|f\|_{W^{1,q}}.$$

(3) Additionally let $s \in \operatorname{argmax}\{\|f(t)\|, t \in [0,T]\}$. Then there holds

$$\|f\|_\infty = \|f(s)\| \overset{(FT)}{\leq} \|f(0)\| + \int_0^s \|\dot{f}(\tau)\|\mathrm{d}\tau \leq \|f\|_{W^{1,1,*}}.$$

Alternative proof: X is complete with respect to

- $\|\cdot\|_{W^{1,q}}$ (see e.g. Han/Reddy [6], section 5.2.3, or [7], section 1.4)
- $\|\cdot\|_{W^{1,q,*}}$ (see Krejci [10], section 9.2)

Now prove *either* (1) *or* (2) and apply the following lemma 1.3. ■

1.3 Lemma. *For a vector space X with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ such that*

$$(X,\|\cdot\|_1) \text{ and } (X,\|\cdot\|_2) \text{ are Banach}$$

Then

$$\exists c > 0 : \|\cdot\|_1 \leq c\,\|\cdot\|_2 \qquad \Longrightarrow \qquad \exists C > 0 : \|\cdot\|_2 \leq C\,\|\cdot\|_1$$

Proof: : See Heuser [8], exercise 39.6. (use the *open mapping theorem*). ■

1.4 Lemma. *Let two functions*

$$\begin{array}{lcll} f & \in & L^1([0,T],\mathbb{R}), & f \geq 0 \text{ a.e. in } [0,T] \\ g & \in & C([0,T],\mathbb{R}), & g(0) \geq 0 \end{array}$$

be given. Then the inequality

$$\frac{1}{2}\, g^2(t) \leq \frac{1}{2}\, g^2(0) + \int_0^t f(\tau)g(\tau)d\tau \qquad \text{for all } t \in [0,T] \tag{7}$$

implies the inequality

$$|g(t)| \leq |g(0)| + \int_0^t f(\tau)d\tau \qquad \text{for all } t \in [0,T]. \tag{8}$$

Proof: Define for $\epsilon \geq 0$

$$h_\epsilon(t) := \frac{1}{2}(g(0)+\epsilon)^2 + \int_0^t f(\tau)g(\tau)\mathrm{d}\tau \qquad \text{for } t \in [0,T] \tag{9}$$

Thus

$$h_\epsilon(t) \geq \frac{1}{2}\, g^2(0) + \frac{1}{2}\,\epsilon^2 + \int_0^t f(\tau)g(\tau)\mathrm{d}\tau \overset{(7)}{\geq} \frac{1}{2}\, g^2(t) + \frac{1}{2}\,\epsilon^2$$

for each $\epsilon > 0$, from which

$$\text{both} \qquad \frac{1}{2}\, g^2(t) \leq h_\epsilon(t) \qquad \text{and} \qquad \frac{1}{2}\,\epsilon^2 \leq h_\epsilon(t) \qquad \text{e. in } [0,T]. \tag{10}$$

Further

$$\mathrm{d}_t h_\epsilon(t) = f(\tau)g(\tau) \overset{(10)}{\leq} f(\tau)\sqrt{2h_\epsilon(\tau)} \qquad \text{a.e. in } [0,T] \tag{11}$$

The function h_ϵ is of class $W^{1,1}([0,T],\mathbb{R})$ and

$$\mathrm{d}_t \sqrt{h_\epsilon(t)} = \frac{h'_\epsilon(t)}{2\sqrt{h_\epsilon(t)}} \overset{(11)}{\leq} \frac{1}{\sqrt{2}}\, f(t) \qquad \text{a.e. in } [0,T]$$

Thus

$$\sqrt{h_\epsilon(t)} \leq \sqrt{h_\epsilon(0)} \overset{(FT)}{\leq} \frac{1}{\sqrt{2}} \int_0^t f(\tau)\mathrm{d}\tau$$

and

$$|g(t)| \overset{(10)}{\leq} \sqrt{2h_\epsilon(t)} \leq \sqrt{2h_\epsilon(0)} + \int_0^t f(\tau)\mathrm{d}\tau \overset{(9)}{=} |g(0)| + \epsilon + \int_0^t f(\tau)\mathrm{d}\tau$$

valid for each $\epsilon > 0$. Letting $\epsilon \to 0$ gives the desired result (8). ■

2 Basic properties

From the defining variational inequality (2), the difference quotients and a passing to the limit to zero, one sees that there holds orthogonality

$$\langle \dot{p}(t), \dot{s}(t) \rangle = 0 \qquad \text{a.e. } [0,T]$$

which implies Pythagoras relation

$$\|\dot{f}(t)\|^2 = \|\dot{s}(t)\|^2 + \|\dot{p}(t)\|^2 \qquad \text{a.e. in } [0,T]$$

from which clearly

$$\|\dot p(t)\|^q \le \|\dot f(t)\|^q, \qquad \|\dot s(t)\|^q \le \|\dot f(t)\|^q \qquad \text{a.e. in } [0,T] \tag{12}$$

for each $1 \le q < \infty$. thus we find from the fundamental theorem the following smallness results

$$\|s(t)\| \overset{(FT)}{\le} \|s(0)\| + \int_0^t \|\dot f(\tau)\| \mathrm{d}\tau \overset{(Ho)}{\le} \|s(0)\| + t^{1/q'} \Big(\int_0^t \|\dot f\|^q \mathrm{d}\tau \Big)^{1/q} \tag{13}$$

$$\|p(t)\| \overset{(FT)}{\le} \|p(0)\| + \int_0^t \|\dot f(\tau)\| \mathrm{d}\tau \overset{(Ho)}{\le} \|p(0)\| + t^{1/q'} \Big(\int_0^t \|\dot f\|^q \mathrm{d}\tau \Big)^{1/q} \tag{14}$$

everywhere, where q' is defined by (3). Further, we have for two decompositions

$$p_i = \mathcal{P}(f_i, s_{i,0}) \qquad s_i = \mathcal{S}(f_i, s_{i,0}), \qquad f_i = p_i + s_i \qquad (i \in \{1,2\})$$

and $\Delta\cdot = \cdot\big|_2^1 = \cdot_1 - \cdot_2$ the basic Lipschitz estimates

- for stop

$$\|\Delta s(t)\| \le \|\Delta s(0)\| + \int_0^t \|\Delta \dot f(\tau)\| \mathrm{d}\tau \tag{15}$$

- for play

$$\begin{aligned} \|\Delta p(t)\| &\le \|\Delta s(0)\| + \|\Delta f(t)\| + \int_0^t \|\Delta \dot f(\tau)\| \mathrm{d}\tau \\ &\overset{(FT)}{\le} \|\Delta s(0)\| + \|\Delta f(0)\| + 2\int_0^t \|\Delta \dot f(\tau)\| \mathrm{d}\tau \\ &\le \|\Delta p(0)\| + 2\Big(\|\Delta f(0)\| + \int_0^t \|\Delta \dot f(\tau)\| \mathrm{d}\tau \Big) \end{aligned} \tag{16}$$

everywhere in the interval $[0,T]$. The estimate (16) is a simple consequence of (15) and (1). For proof of (15) see Brokate/Krejci [1, 4]). This implies the *Lipschitz continuity* of stop and play, considered as operators

$$\mathcal{S}, \mathcal{P} : \big(W^{1,1}([0,T],X), \|\cdot\|_{W^{1,1}} \text{ or } \|\cdot\|_{W^{1,1,*}}\big) \to \big(C([0,T],X), \|\cdot\|_\infty\big)$$

with respect to both norms. The constants w.r.t. $\|\cdot\|_{W^{1,q,*}}$ are 1 resp. 2. For the ones w.r.t. $\|\cdot\|_{W^{1,q}}$ cf. the proof of the lemma 1.2.)

2.1 Remark. (a) The proofs of the estimates (15) and (16) do *not* need the interior of Z being non-empty.

- The assumption $0 \in \mathrm{Int}(Z)$ in [4] A.1 - A.3 can be weakened to $0 \in Z$. For this, see [9], theorem 1.9, proposition 3.9 and remark 3.10.
- It is important, that with the aid of lemma 1.4, one can get rid of the factor 2 before the integral in (3.24) of remark 3.10 in [9]

(b) Note that the smallness result (13) for stop follows as well from the Lipschitz result (15) for stop by taking

$$f_1 \equiv f, \quad f_2 \equiv 0, \quad s_1(0) = s(0), \quad s_2(0) = 0$$

as inputs. □

3 Generalisations for Stop

We now generalise Brokate's and Krejci's results in the sense that we allow different inputs, different initial memories and *different scalar products*.

3.1 Theorem (stop operator) *Let $(X, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space, $A_1,\ A_2 \in \mathcal{L}(X,\ X)$ linear continuous, symmetric and strongly positive, inducing equivalent scalar products*

$$\big\langle x,\ y\big\rangle_{A_1} = \big\langle A_1^{-1}x,\ y\big\rangle, \qquad \big\langle x,\ y\big\rangle_{A_2} = \big\langle A_2^{-1}x,\ y\big\rangle \tag{17}$$

and equivalent norms

$$c_{A_1}\|\cdot\| \le \|\cdot\|_{A_1} \le C_{A_1}\|\cdot\|, \qquad c_{A_2}\|\cdot\| \le \|\cdot\|_{A_2} \le C_{A_2}\|\cdot\|. \tag{18}$$

Let further a convex, closed set $Z \subseteq X$ and two inputs

$$f_1,\ f_2 \in W^{1,1}([0,T], X), \qquad s_{0,1},\ s_{0,2} \in Z$$

be given. For the outputs

$$p_1 = \mathcal{P}_{A_1}(f_1, s_{0,1}), \quad s_1 = \mathcal{S}_{A_1}(f_1, s_{0,1}), \quad p_2 = \mathcal{P}_{A_2}(f_2, s_{0,2}), \quad s_2 = \mathcal{S}_{A_2}(f_2, s_{0,2})$$

there holds

$$\|\Delta s(t)\| \ \le\ \frac{C_{A_1}}{c_{A_1}}\|\Delta s_0\| + \int_0^t \left(\frac{C_{A_1}}{c_{A_1}}\|\Delta\dot f(\tau)\| + \frac{1}{c_{A_1}^2}\|\Delta A^{-1}\dot p_2(\tau)\|\right)d\tau, \tag{19}$$

$$\|\Delta s(t)\| \ \le\ \frac{C_{A_2}}{c_{A_2}}\|\Delta s_0\| + \int_0^t \left(\frac{C_{A_2}}{c_{A_2}}\|\Delta\dot f(\tau)\| + \frac{1}{c_{A_2}^2}\|\Delta A^{-1}\dot p_1(\tau)\|\right)d\tau, \tag{20}$$

where

$$\Delta\cdot = \cdot\Big|_2^1, \qquad \textit{especially} \quad \Delta A^{-1} = A_1^{-1} - A_2^{-1}.$$

Proof: By definition, we have for $i \in \{1,2\}$

$$\left\{\begin{array}{rcll} \big\langle \dot p_i,\ s_i - *\big\rangle_{A_i} & \ge & 0 \quad \text{for all } * \in Z & \text{a.e. in } [0,T] \\ p_i + s_i & = & f_i & \text{e. in } [0,T] \\ s_i(0) & = & s_{i,0} & \end{array}\right. . \tag{21}$$

Letting $s_1(t)$ and $s_2(t)$ chiastically into the defining variational inequalities (21), we see

$$\begin{aligned} \big\langle \dot p_1, s_1 - s_2\big\rangle_{A_1} &= \big\langle A_1^{-1}\dot p_1, s_1 - s_2\big\rangle \ \ge\ 0, \\ \big\langle \dot p_2, s_2 - s_1\big\rangle_{A_2} &= \big\langle A_2^{-1}\dot p_2, s_2 - s_1\big\rangle \ \ge\ 0, \end{aligned}$$

thus by addition

$$\big\langle A_1^{-1}\dot p_1 - A_2^{-1}\dot p_2, \Delta s\big\rangle \ \ge\ 0. \tag{22}$$

Adding a zero term

$$0 = -A_1^{-1}\dot p_2 + A_1^{-1}\dot p_2 \qquad \text{resp.} \qquad 0 = -A_2^{-1}\dot p_1 + A_2^{-1}\dot p_1$$

in the left side of (22) and using the centered relation in (21), we get – after rearranging – both

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta s\|_{A_1}^2 = \big\langle A_1^{-1}\Delta\dot s,\ \Delta s\big\rangle \ \le\ \big\langle A_1^{-1}\Delta\dot f + \Delta A^{-1}\dot p_2,\ \Delta s\big\rangle \tag{23}$$

$$\text{resp.} \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta s\|_{A_2}^2 = \big\langle A_2^{-1}\Delta\dot s,\ \Delta s\big\rangle \ \le\ \big\langle A_2^{-1}\Delta\dot f + \Delta A^{-1}\dot p_1,\ \Delta s\big\rangle. \tag{24}$$

Then the argument is similar to the proofs of the existing Lipschitz estimates in [1, 9].

• Integrating (23), we arrive at

$$\begin{aligned}
\frac{1}{2}\|\Delta s(t)\|_{A_1}^2 - \frac{1}{2}\|\Delta s(0)\|_{A_1}^2 &\overset{(FT)}{\leq} \int_0^t \left\langle \Delta\dot{f}, \Delta s\right\rangle_{A_1} + \left\langle \Delta A^{-1}\dot{p}_2, \Delta s\right\rangle \mathrm{d}\tau \\
&\overset{(CS)}{\leq} \int_0^t \Big(\|\Delta\dot{f}\|_{A_1}\,\|\Delta s\|_{A_1} + \|\Delta A^{-1}\dot{p}_2\|\,\|\Delta s\| \Big)\mathrm{d}\tau \\
&\overset{(18)}{\leq} \int_0^t \left(C_{A_1}\|\Delta\dot{f}\| + \frac{1}{c_{A_1}}\|\Delta A^{-1}\dot{p}_2\| \right)\|\Delta s\|_{A_1}\mathrm{d}\tau,
\end{aligned}$$

so that lemma 1.4 yields

$$\|\Delta s(t)\|_{A_1} \quad\leq\quad \|\Delta s(0)\|_{A_1} + \int_0^t \left(C_{A_1}\|\Delta\dot{f}\| + \frac{1}{c_{A_1}}\|\Delta A^{-1}\dot{p}_2\| \right)\mathrm{d}\tau.$$

With (18) we arrive at

$$c_{A_1}\|\Delta s(t)\| \quad\leq\quad C_{A_1}\|\Delta s(0)\| + \int_0^t \left(C_{A_1}\|\Delta\dot{f}\| + \frac{1}{c_{A_1}}\|\Delta A^{-1}\dot{p}_2\| \right)\mathrm{d}\tau,$$

which finally gives (19).

• The same procedure for p_1 instead of p_2, $\langle\cdot,\cdot\rangle_{A_2}$ instead of $\langle\cdot,\cdot\rangle_{A_1}$, $\langle\cdot,\cdot\rangle_{A_1}$ instead of $\langle\cdot,\cdot\rangle_{A_2}$ together with (24), (18) and (17) yields (20).

The rest is clear. ■

3.2 Corollary (special cases) *Let the assumptions of theorem 3.1 hold. Then especially*

(a) ***Same inputs, different scalar products.*** *If $f_1 = f_2$, then there holds*

$$\Delta s(t) = -\Delta p(t) \tag{25}$$

and

$$\|\Delta p(t)\| \;=\; \|\Delta s(t)\| \;\leq\; \frac{C_{A_1}}{c_{A_1}}\|\Delta s_0\| + \frac{1}{c_{A_1}^2}\int_0^t \|\Delta A^{-1}\dot{p}_2(\tau)\| d\tau \tag{26}$$

$$\|\Delta p(t)\| \;=\; \|\Delta s(t)\| \;\leq\; \frac{C_{A_2}}{c_{A_2}}\|\Delta s_0\| + \frac{1}{c_{A_2}^2}\int_0^t \|\Delta A^{-1}\dot{p}_1(\tau)\| d\tau. \tag{27}$$

(b) ***Same scalar products, different inputs.*** *If $A_1 = A_2 = A$, which allows $c_{A_1} = c_{A_2} = c_A$, $C_{A_1} = C_{A_2} = C_A$, then there holds*

$$\|\Delta s(t)\| \;\leq\; \frac{C_A}{c_A}\left(\|\Delta s_0\| + \int_0^t \|\Delta\dot{f}(\tau)\| d\tau \right). \tag{28}$$

(c) ***Original scalar product each, different inputs.*** *If $A_1 = A_2 = A = I$, which allows $c_{A_1} = c_{A_2} = 1 = C_{A_1} = C_{A_2}$, then*

$$\|\Delta s(t)\| \;\leq\; \|\Delta s_0\| + \int_0^t \|\Delta\dot{f}(\tau)\| d\tau,$$

which is identical to (15).

All estimates are valid e. in the interval $[0, T]$.

Proof: Clear. ■

3.3 Corollary (w.r.t input) *Let the assumptions of theorem 3.1 hold. Then in any case, we have – with respect to the inputs –*

$$\|\Delta s(t)\| \leq \frac{C_{A_1}}{c_{A_1}}\left(\|\Delta s_0\| + \int_0^t \|\Delta \dot{f}(\tau)\| d\tau\right) + \frac{C_{A_2}\|\Delta A^{-1}\|}{c_{A_1}^2 c_{A_2}} \int_0^t \|\dot{f}_2(\tau)\| d\tau \quad (29)$$

$$\|\Delta s(t)\| \leq \frac{C_{A_2}}{c_{A_2}}\left(\|\Delta s_0\| + \int_0^t \|\Delta \dot{f}(\tau)\| d\tau\right) + \frac{C_{A_1}\|\Delta A^{-1}\|}{c_{A_1} c_{A_2}^2} \int_0^t \|\dot{f}_1(\tau)\| d\tau \quad (30)$$

e. in the interval $[0, T]$.

Proof: Note

$$\begin{aligned} \|\Delta A^{-1}\dot{p}_2\| &\leq \|\Delta A^{-1}\|\,\|\dot{p}_2\| \\ &\overset{(18)}{\leq} \frac{1}{c_{A_2}}\|\Delta A^{-1}\|\,\|\dot{p}_2\|_{A_2} \\ &\overset{(12)}{\leq} \frac{1}{c_{A_2}}\|\Delta A^{-1}\|\,\|\dot{f}_2\|_{A_2} \overset{(18)}{\leq} \frac{C_{A_2}}{c_{A_2}}\|\Delta A^{-1}\|\,\|\dot{f}_2\| \end{aligned}$$

(and its counterpart) for the derivation of (29) and (30). ■

3.4 Corollary (play operator) *Let the assumptions of theorem 3.1 hold. Then in any case, we have*

$$\|\Delta p(t)\| \leq \|\Delta f(t)\| + \frac{C_{A_1}}{c_{A_1}}\left(\|\Delta s_0\| + \int_0^t \|\Delta \dot{f}\| d\tau\right) + \frac{C_{A_2}\|\Delta A^{-1}\|}{c_{A_1}^2 c_{A_2}} \int_0^t \|\dot{f}_2\| d\tau \quad (31)$$

$$\|\Delta p(t)\| \leq \|\Delta f(t)\| + \frac{C_{A_2}}{c_{A_2}}\left(\|\Delta s_0\| + \int_0^t \|\Delta \dot{f}\| d\tau\right) + \frac{C_{A_1}\|\Delta A^{-1}\|}{c_{A_1} c_{A_2}^2} \int_0^t \|\dot{f}_1\| d\tau \quad (32)$$

e. in the interval $[0, T]$.

Proof: Note $\Delta s(t) = \Delta f(t) - \Delta p(t)$, implying $\|\Delta s(t)\| \geq \big|\|\Delta p(t)\| - \|\Delta f(t)\|\big| \geq \|\Delta p(t)\| - \|\Delta f(t)\|$. The rest follows from (29), (30). ■

4 Generalisation for Play

We generalise now the result (4.4) of Krejci [10], begin of section 4, which is given by the estimate

$$\|\Delta p(t)\|^2 \leq \|\Delta p_0\|^2 + 2\|\Delta f\|_{\infty,t}\left(\int_0^t \|\dot{p}_1(\tau)\|\,\mathrm{d}\tau + \int_0^t \|\dot{p}_2(\tau)\|\,\mathrm{d}\tau\right). \quad (33)$$

4.1 Theorem (play operator) *Let the assumptions and notations of theorem 3.1 hold. Then we have*

$$\begin{aligned} \|\Delta p(t)\|^2 &\leq \frac{C_{A_1}^2}{c_{A_1}^2}\|\Delta p(0)\|^2 + \frac{2}{c_{A_1}^2}\int_0^t \big\langle \Delta(A^{-1}\dot{p}), \Delta f\big\rangle - \big\langle \Delta A^{-1}\dot{p}_2, \Delta p\big\rangle\, d\tau, \\ \|\Delta p(t)\|^2 &\leq \frac{C_{A_2}^2}{c_{A_2}^2}\|\Delta p(0)\|^2 + \frac{2}{c_{A_2}^2}\int_0^t \big\langle \Delta(A^{-1}\dot{p}), \Delta f\big\rangle - \big\langle \Delta A^{-1}\dot{p}_1, \Delta p\big\rangle\, d\tau. \end{aligned}$$

Here especially

$$\Delta A^{-1} = \Delta(A^{-1}) = A_1^{-1} - A_2^{-1}, \qquad \Delta(A^{-1}\dot{p}) = A_1^{-1}\dot{p}_1 - A_2^{-1}\dot{p}_2.$$

Proof: From (22), we find

$$\left\langle A_1^{-1}\dot p_1 - A_2^{-1}\dot p_2, \Delta p\right\rangle \quad \le \quad \left\langle A_1^{-1}\dot p_1 - A_2^{-1}\dot p_2, \Delta f\right\rangle$$

thus – by adding appropriate zeros –

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta p\|_{A_1}^2 = \left\langle \Delta\dot p, \Delta p\right\rangle_{A_1} \quad \le \quad \left\langle \Delta(A^{-1}\dot p), \Delta f\right\rangle - \left\langle \Delta A^{-1}\dot p_2, \Delta p\right\rangle, \tag{34}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta p\|_{A_2}^2 = \left\langle \Delta\dot p, \Delta p\right\rangle_{A_2} \quad \le \quad \left\langle \Delta(A^{-1}\dot p), \Delta f\right\rangle - \left\langle \Delta A^{-1}\dot p_1, \Delta p\right\rangle. \tag{35}$$

Integration with (18) yields the assertion. ■

4.2 Corollary (special cases) *Let the assumptions of theorem 4.1 hold. Then especially*

(a) ***Same inputs, different scalar products.*** *If $f_1 = f_2$, then there holds*

$$\|\Delta s(t)\| = \|\Delta p(t)\| \quad \le \quad \frac{C_{A_1}}{c_{A_1}}\|\Delta p(0)\| + \frac{1}{c_{A_1}^2}\int_0^t \|\Delta A^{-1}\dot p_2\|\, d\tau,$$

$$\|\Delta s(t)\| = \|\Delta p(t)\| \quad \le \quad \frac{C_{A_2}}{c_{A_2}}\|\Delta p(0)\| + \frac{1}{c_{A_2}^2}\int_0^t \|\Delta A^{-1}\dot p_1\|\, d\tau,$$

which is in fact the same as (26), (27).

(b) ***Same scalar products, different inputs.*** *If $A_1 = A_2 = A$, which allows $c_{A_1} = c_{A_2} = c_A$, $C_{A_1} = C_{A_2} = C_A$, then there holds*

$$\|\Delta p(t)\|^2 \quad \le \quad \frac{C_A^2}{c_A^2}\|\Delta p(0)\|^2 + \frac{2}{c_A^2}\|\Delta f\|_{\infty,t}\int_0^t \|A^{-1}\Delta\dot p(\tau)\|\, d\tau.$$

(c) ***Original scalar product each, different inputs.*** *If $A_1 = A_2 = A = I$, which allows $c_{A_1} = c_{A_2} = 1 = C_{A_1} = C_{A_2}$, then*

$$\|\Delta p(t)\|^2 \quad \le \quad \|\Delta p(0)\|^2 + 2\|\Delta f\|_{\infty,t}\int_0^t \|\Delta\dot p(\tau)\|\, d\tau,$$

which is more general than (33).

All estimates are valid e. in the interval $[0, T]$.

Proof: (b), (c) use Cauchy-Schwarz. (a) Going a step back to (34), we find after integration with $\Delta f = 0$ similarly as in the proof of 3.1

$$\frac{1}{2}\|\Delta p(t)\|_{A_1}^2 \quad \overset{(CS),(18)}{\le} \quad \frac{1}{2}\|\Delta p(0)\|_{A_1}^2 + \frac{1}{c_{A_1}}\int_0^t \|\Delta A^{-1}\dot p_2\|\,\|\Delta p\|_{A_1}\mathrm{d}\tau.$$

Application of lemma 1.4 yields

$$\|\Delta p(t)\|_{A_1} \quad \le \quad \|\Delta p(0)\|_{A_1} + \frac{1}{c_{A_1}}\int_0^t \|\Delta A^{-1}\dot p_2\|\,\mathrm{d}\tau$$

and in view of (18) the assertion. ■

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