A homotopy between the solutions of the elastic and the elastoplastic boundary value problem

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Abstract

In this article, we give an explicit homotopy between the solutions (i.e. stress, strain, displacement) of the quasistatic linear elastic and the quasistatic nonlinear elastoplastic boundary value problems, where we assume a linear kinematic hardening material law. We give error estimates with respect to the homotopy parameter.

Keywords. Elastic BVP, elastoplastic BVP, variational inequalities, rate-independency, hysteresis, linear kinematic hardening, stop- and play-operator, homotopy.

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1 Introduction

This paper is a continuation of the previous articles of Lang et al. [6], in which the difference of the solutions of the quasistatic elastic and elastoplastic boundary value problem (with hysteretic phenomena) has been analysed. We assume again linear kinematic hardening material, where we have the linear coupling

$$\alpha = B\varepsilon^{pl}$$

between the backstress α and the plastic strain ε^{pl}. Additionally in [6], the error between a postprocessing correction method and the elastoplastic solution has been studied. And it is exactly this correction method that gives us now an explicit homotopy between both solutions. We give as well error estimates in dependence of the homotopy parameter $\lambda \in [0,1]$.

Throughout the paper, we will make frequently use of the following abbreviations.

(CS)	Cauchy-Schwarz inequality	a.e.	almost everywhere
$\xrightarrow{n}$	limit $n \to \infty$, strong convergence	e.	everywhere

2 The two boundary value problems

Let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega = \partial_1\Omega \,\dot\cup\, \partial_2\Omega$. We summarize the gouverning equations. The reader finds more details in [6].

The elastic model (E). The *Newton balance equations* are in its weak form

$$^{e}\varepsilon(t) = \mathcal{D}^{e}u(t), \qquad \mathcal{D}^{\star\,e}\sigma(t) = F(t). \tag{1}$$

The constitutive material law is *linear Hooke's law*

$$^{e}\sigma(t) = \mathcal{C}^{e}\varepsilon(t). \tag{2}$$

For each t equations (1) and (2) build up a linear elliptic static problem.

The elastoplastic model (EP). The *Newton balance equations* are in its weak form

$$\varepsilon(t) = \mathcal{D}u(t), \qquad \mathcal{D}^\star\sigma(t) = F(t). \tag{3}$$

Linear Hooke's law and the *linear kinematic hardening law*

$$\sigma(t) = \mathcal{C}\varepsilon(t), \qquad \alpha(t) = \mathcal{B}\varepsilon^{pl}(t), \tag{4}$$

together with the additive decomposition of *strain* and *stress*

$$\varepsilon(t) = \varepsilon^{el}(t) + \varepsilon^{pl}(t), \qquad \sigma(t) = \alpha(t) + \beta(t) \tag{5}$$

and the *normality rule*

$$\dot{\varepsilon}^{pl}(t) \in \partial\chi_Z\big(\beta(t)\big) \tag{6}$$

build up the constitutive material law.

All the relations (1), ..., (6) are assumed to be valid for almost every $t \in [0, T]$, where $T > 0$. In both models, quasi-staticity and linearised geometry is assumed. With the scalar products/norms

$$\begin{aligned}
\langle u, v\rangle_U &= \int_\Omega u(x)\cdot v(x) + \nabla u(x) : \nabla v(x)\,\mathrm{d}V(x), & \|u\|_U^2 &= \langle u, u\rangle_U,\\
\langle \varepsilon, \eta\rangle_\Sigma &= \int_\Omega \varepsilon(x) : \eta(x)\,\mathrm{d}V(x), & \|\varepsilon\|_\Sigma^2 &= \langle \varepsilon, \varepsilon\rangle_\Sigma,
\end{aligned}$$

the Sobolev space $U = W_0^{1,2}(\Omega, \partial_1\Omega, \mathbb{R}^3)$ and the Lebesgue space $\Sigma = L^2(\Omega, \mathbb{R}_s^{3\times 3})$ are separable Hilbert spaces. We have

$$\begin{aligned}
U &= \text{space of displacements } {}^{(e)}u, & \Sigma &= \text{space of strains } {}^{(e)}\varepsilon,\, {}^{(e)}\varepsilon^{el},\, {}^{(e)}\varepsilon^{pl},\\
U^\star &= \text{space of outer forces } F, & \Sigma^\star &= \text{space of stresses } {}^{(e)}\sigma,\, {}^{(e)}\alpha,\, {}^{(e)}\beta.
\end{aligned}$$

By the Riesz-Fréchet representation theorem, we identify $U \simeq U^\star$, $\Sigma \simeq \Sigma^\star$ in the usual sense. Hooke's tensor and the linear kinematic hardening tensor

$$\mathcal{B},\, \mathcal{C} \in \mathcal{L}\big(\Sigma,\, \Sigma^\star\big)$$

are assumed to linear, continuous, symmetric and strongly positive

$$\|\mathcal{B}\varepsilon\|_\Sigma \le \|\mathcal{B}\|\,\|\varepsilon\|_\Sigma, \qquad \big\langle \mathcal{B}\varepsilon, \eta\big\rangle = \big\langle \varepsilon, \mathcal{B}\eta\big\rangle, \qquad \big\langle \mathcal{B}\varepsilon, \varepsilon\big\rangle_\Sigma \ge \kappa_\mathcal{B}\|\varepsilon\|_\Sigma^2, \tag{7}$$

with $\kappa_\mathcal{B} > 0$. Analogeously for $\mathcal{C}$. The differential operator $\mathcal{D} \in \mathcal{L}(U, \Sigma)$, and its adjoint $\mathcal{D}^\star \in \mathcal{L}(\Sigma^\star, U^\star)$ are given by

$$\mathcal{D}u = \frac{1}{2}\big(\nabla u + \nabla u^t\big), \qquad \big\langle \mathcal{D}^\star\sigma, v\big\rangle = \int_\Omega \sigma : \mathcal{D}v\,\mathrm{d}V. \qquad \big(\sigma \in \Sigma^\star,\, u, v \in U\big)$$

We apply the same outer force F

$$\big\langle F(t), v\big\rangle = \int_\Omega f(t)\cdot v\,\mathrm{d}V + \int_{\partial_2\Omega} g(t)\cdot v\,\mathrm{d}V \qquad \big(v \in U, t \in [0,\, T]\big) \tag{8}$$

with given volume and boundary terms

$$f \in W^{1,2}\big([0,T], L^2(\Omega)\big), \qquad g \in W^{1,2}\big([0,T], L^2(\partial_2\Omega)\big) \tag{9}$$

to the body Ω in both models. We set further for abbreviation

$$S = \mathcal{C}\mathcal{D}(\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1} \in \mathcal{L}(U^\star, \Sigma^\star), \qquad \mathcal{R} = P\mathcal{C} + \mathcal{B} \in \mathcal{L}(\Sigma, \Sigma^\star),$$

where

$$Q = S\mathcal{D}^\star \in \mathcal{L}(\Sigma^\star, \Sigma^\star), \qquad P = I - Q \in \mathcal{L}(\Sigma^\star, \Sigma^\star).$$

The operator $\mathcal{R}$ is strongly positive and symmetric with

$$\big\langle \mathcal{R}\varepsilon, \varepsilon \big\rangle \geq \kappa_{\mathcal{R}} \|\varepsilon\|^2, \qquad \kappa_{\mathcal{R}} > 0. \tag{10}$$

(See again [6].) Consequently, the inverse $\mathcal{R}^{-1}$ exists and is strongly positive and symmetric. The same holds for $\mathcal{B}$ and its existing inverse $\mathcal{B}^{-1}$. We define equivalent scalar products on $\Sigma^\star$ by

$$\big\langle \sigma, \tau \big\rangle_{\mathcal{R}} = \big\langle \mathcal{R}^{-1}\sigma, \tau \big\rangle, \qquad \big\langle \sigma, \tau \big\rangle_{\mathcal{B}} = \big\langle \mathcal{B}^{-1}\sigma, \tau \big\rangle$$

with equivalence constants, given by

$$c_{\mathcal{B}} \| \cdot \| \leq \| \cdot \|_{\mathcal{B}} \leq C_{\mathcal{B}} \| \cdot \|, \qquad c_{\mathcal{R}} \| \cdot \| \leq \| \cdot \|_{\mathcal{R}} \leq C_{\mathcal{R}} \| \cdot \|. \tag{11}$$

The elastic domain is the convex, closed set

$$Z = \big\{ \beta \in \Sigma^\star : \| \operatorname{dev} \beta(x) \| \leq \rho \text{ for almost every } x \text{ in } \Omega \big\}$$

for a fixed radius $\rho > 0$. The subdifferential of its indicator function is denoted by

$$\partial\chi_Z(\beta) = \big\{ \tau \in \Sigma^\star : \chi(*) \geq \chi(\beta) + \big\langle \tau, * - \beta \big\rangle \text{ for all } * \in \Sigma^\star \big\}.$$

We have the following result.

2.1 Theorem (Existence and uniqueness of solution) *(1) For each given outer force*

$$F \in W^{1,q}\big([0,T], U^\star\big), \qquad (1 \leq q < \infty) \tag{12}$$

there exists a uniquely determined triple

$${}^{e}u \in W^{1,q}\big([0,T], U\big), \qquad {}^{e}\varepsilon \in W^{1,q}\big([0,T], \Sigma\big), \qquad {}^{e}\sigma \in W^{1,q}\big([0,T], \Sigma^\star\big),$$

satisfying the elastic relations (E).

(2) For each given outer force and initial memory

$$F \in W^{1,2}\big([0,T], U^\star\big) \qquad \text{and} \qquad \beta_0 \in Z,$$

there exists a uniquely determined septuple

$$u \in W^{1,2}\big([0,T], U^\star\big), \quad \varepsilon,\, \varepsilon^{el}, \varepsilon^{pl} \in W^{1,2}\big([0,T], \Sigma\big), \quad \sigma,\, \alpha,\, \beta \in W^{1,2}\big([0,T], \Sigma^\star\big),$$

satisfying the elastoplastic relations (EP) with initial condition $\beta(0) = \beta_0$.

Proof: Cf. [6], theorems 3.1 and 3.2. A proof of (2) given in [7], theorem 66.A. ■

There holds for the elastic stress at each $t \in [0,T]$

$${}^{e}\sigma(t) \;=\; \mathcal{C}\mathcal{D}(\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}F(t) \;=\; SF(t) \tag{13}$$

All unknowns of the elastoplastic model can be expressed with the aid of

$$\beta(t), \qquad \gamma(t) = \big(SF - \beta\big)(t) \qquad \text{and} \qquad \beta_0 \in Z. \tag{14}$$

It is possible to express α, β and γ as stop and play operators with respect to different sclar products, cf. [6].

(A) **W.r.t. the input σ and the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{B}}$:**

$$\begin{cases} \left\langle \mathcal{B}^{-1}\dot{\alpha},\, \beta - *\right\rangle = \left\langle \dot{\alpha},\, \beta - *\right\rangle_{\mathcal{B}} \geq 0 & \text{for all } * \in Z \quad \text{a.e. in } [0,T] \\ \alpha + \beta = \sigma & \text{e. in } [0,T] \\ \beta(0) = \beta_0 & \end{cases}$$

or equivalently

$$\alpha = \mathcal{P}_{\mathcal{B}}(\sigma, \beta_0), \qquad \beta = \mathcal{S}_{\mathcal{B}}(\sigma, \beta_0), \tag{15}$$

(B) **W.r.t. the input ${}^e\sigma$ and the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{R}}$:**

$$\begin{cases} \left\langle \mathcal{R}^{-1}\dot{\gamma},\, \beta - *\right\rangle = \left\langle \dot{\gamma},\, \beta - *\right\rangle_{\mathcal{R}} \geq 0 & \text{for all } * \in Z \quad \text{a.e. in } [0,T] \\ \gamma + \beta = {}^e\sigma & \text{e. in } [0,T] \\ \beta(0) = \beta_0 & \end{cases}$$

or equivalently

$$\gamma = \mathcal{P}_{\mathcal{R}}({}^e\sigma, \beta_0), \qquad \beta = \mathcal{S}_{\mathcal{R}}({}^e\sigma, \beta_0), \tag{16}$$

where $\mathcal{P}_\cdot$, $\mathcal{S}_\cdot$ denote the stop and play operator with the respective scalar products. For definition of the stop and play operators, we refer to the articles of Brokate and Krejci [1, 2, 4] or Lang et al. [5] or the monograph [3].

3 Homotopy between (E) and (EP)

In this section, we assume for the sake of simplicity that we have virgin material, i.e.

$$\varepsilon^{pl}(0) = 0. \tag{17}$$

For $\lambda \in [0,1]$, the operator ${}^e\mathcal{B}_\lambda : \Sigma \to \Sigma^\star$ defined by

$${}^e\mathcal{B}_\lambda = \lambda\mathcal{R} + (1-\lambda)\mathcal{B} = \mathcal{B} + \lambda P\mathcal{C}, \tag{18}$$

is linear, continuous, symmetric and strongly positive, such that

$$\left\langle \sigma, \tau\right\rangle_{{}^e\mathcal{B}_\lambda} = \left\langle {}^e\mathcal{B}_\lambda{}^{-1}\sigma, \tau\right\rangle, \qquad c_{{}^e\mathcal{B}_\lambda}\|\cdot\| \leq \|\cdot\|_{{}^e\mathcal{B}_\lambda} \leq C_{{}^e\mathcal{B}}\|\cdot\|. \tag{19}$$

We introduce the *elastic decomposition* of

(${}^e\mathcal{B}_\lambda$) **the input ${}^e\sigma$ with respect to the product $\langle\cdot,\cdot\rangle_{{}^e\mathcal{B}_\lambda}$**, namely

$$\begin{cases} \left\langle {}^e\mathcal{B}_\lambda{}^{-1}\,{}^e\dot{\alpha}_\lambda,\, {}^e\beta_\lambda - *\right\rangle \geq 0 & \text{for all } * \in Z \quad \text{a.e. in } [0,T] \\ {}^e\alpha_\lambda + {}^e\beta_\lambda = SF & \text{e. in } [0,T] \\ {}^e\beta_\lambda(0) = \beta_0 & \end{cases},$$

i.e. with the definition of stop and play

$${}^e\alpha_\lambda = \mathcal{P}_{{}^e\mathcal{B}_\lambda}(SF,\, \beta_0), \qquad {}^e\beta_\lambda = \mathcal{S}_{{}^e\mathcal{B}_\lambda}(SF,\, \beta_0). \tag{20}$$

We define the λ-*stresses* by

$$\hat{\alpha}_\lambda = \mathcal{B}\,{}^e\mathcal{B}_\lambda{}^{-1}\,{}^e\alpha_\lambda, \qquad \hat{\beta}_\lambda = {}^e\beta_\lambda, \qquad \hat{\sigma}_\lambda = \hat{\alpha}_\lambda + \hat{\beta}_\lambda, \tag{21}$$

the λ-*strains* by

$$\hat{\varepsilon}^{pl}_\lambda = \lambda\,{}^e\mathcal{B}_\lambda{}^{-1}\,{}^e\alpha_\lambda, \qquad \hat{\varepsilon}^{el}_\lambda = \mathcal{C}^{-1}\hat{\sigma}_\lambda, \qquad \hat{\varepsilon}_\lambda = \hat{\varepsilon}^{el}_\lambda + \hat{\varepsilon}^{pl}_\lambda, \tag{22}$$

and the λ-*displacements* by

$$\hat{u}_\lambda = (\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}\mathcal{D}^\star\mathcal{C}\,\hat{\varepsilon}_\lambda. \tag{23}$$

We arrive at the following error expressions w.r.t. the elastoplastic solution

$$\Delta\alpha = \hat{\alpha}_\lambda - \alpha = \mathcal{B}{}^e\mathcal{B}_\lambda{}^{-1}\mathcal{P}_{{}^e\mathcal{B}_\lambda}(SF,\beta_0) - \mathcal{B}\mathcal{R}^{-1}\mathcal{P}_{\mathcal{R}}(SF,\beta_0), \tag{24}$$

$$\Delta\beta = \hat{\beta}_\lambda - \beta = \mathcal{S}_{{}^e\mathcal{B}_\lambda}(SF,\beta_0) - \mathcal{S}_{\mathcal{R}}(SF,\beta_0). \tag{25}$$

Then we receive

$${}^e\mathcal{B}_\lambda - \mathcal{B} = \lambda\, P\mathcal{C}, \qquad \mathcal{R} - {}^e\mathcal{B}_\lambda = (1-\lambda)P\mathcal{C}, \qquad \mathcal{R} - \mathcal{B} = P\mathcal{C},$$

for $\lambda = 0$, i.e. ${}^e\mathcal{B}_0 = \mathcal{B}$, the **elastic solution**

$$\hat{\sigma}_0 = {}^e\sigma, \quad \hat{\varepsilon}_0 = {}^e\varepsilon, \quad \hat{u}_0 = {}^eu,$$

and for $\lambda = 1$, i.e. ${}^e\mathcal{B}_1 = \mathcal{R}$, the **elastoplastic solution**

$$\hat{\alpha}_1 = \alpha, \quad \hat{\beta}_1 = \beta, \quad \hat{\sigma}_1 = \sigma, \quad \hat{\varepsilon}_1 = \varepsilon, \quad \hat{\varepsilon}_1^{pl} = \varepsilon^{pl}, \quad \hat{\varepsilon}_1^{el} = \varepsilon^{el}, \quad \hat{u}_1 = u.$$

The choice $\lambda = 0$ gives rise to define the remaining *elastic quantities*, which are originally not present in model (E), namely

$${}^e\alpha := \hat{\alpha}_0, \qquad {}^e\beta := \hat{\beta}_0, \qquad {}^e\varepsilon^{el} := \hat{\varepsilon}_0 = \hat{\varepsilon}_0^{el}, \qquad {}^e\varepsilon^{pl} := \hat{\varepsilon}_0^{pl} = 0.$$

3.1 Remark. For ${}^eu$ resp. u note that – cf. proof of theorem 4.1 in [6] –

$${}^eu = (\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}F = (\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}\mathcal{D}^\star\,{}^e\sigma = (\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}\mathcal{D}^\star\mathcal{C}\,{}^e\varepsilon$$

and

$$u = (\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}\mathcal{D}^\star(\sigma + \mathcal{C}\varepsilon^{pl}) = (\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}\mathcal{D}^\star(\mathcal{C}\varepsilon^{el} + \mathcal{C}\varepsilon^{pl}) = (\mathcal{D}^\star\mathcal{C}\mathcal{D})^{-1}\mathcal{D}^\star\mathcal{C}\,\varepsilon.$$

3.2 Theorem (Homotopy) *(a) The map*

$$\mathfrak{H} : [0,1] \times W^{1,2}\big([0,T],U^\star\big) \to C\big([0,T],(\Sigma^\star)^3 \times \Sigma^3 \times U\big)$$
$$(\lambda,F) \mapsto \big(\hat{\alpha}_\lambda,\, \hat{\beta}_\lambda,\, \hat{\sigma}_\lambda,\, \hat{\varepsilon}_\lambda^{el},\, \hat{\varepsilon}_\lambda^{pl},\, \hat{\varepsilon}_\lambda,\, \hat{u}_\lambda\big)(F)$$

is a homotopy

from the elastic solution $\mathfrak{H}(0,F) = \big({}^e\alpha, {}^e\beta, {}^e\sigma, {}^e\varepsilon^{el}, {}^e\varepsilon^{pl}, {}^e\varepsilon, {}^eu\big)(F)$

to the elastoplastic solution $\mathfrak{H}(1,F) = \big(\alpha,\, \beta,\, \sigma,\, \varepsilon^{el},\, \varepsilon^{pl},\, \varepsilon,\, u\big)(F).$

(b) For each $\lambda \in [0,1]$*, the map*

$$\mathfrak{H}(\lambda,\cdot) : W^{1,2}\big([0,T],U^\star\big) \to W^{1,2}\big([0,T],(\Sigma^\star)^3 \times \Sigma^3 \times U\big)$$

is continuous.

Proof: First, some preliminaries.

(1) It is sufficient to prove the homotopy property for the maps $(\lambda,F) \mapsto \hat{\alpha}_\lambda(F)$, $(\lambda,F) \mapsto \hat{\beta}_\lambda(F)$ because of (21), ..., (23) and the definition of the product topology.

(2) We estimate downwards

$$\begin{aligned}\big\langle {}^e\mathcal{B}_\lambda\varepsilon, \varepsilon\big\rangle &\overset{(18)}{=} \lambda\big\langle\mathcal{R}\varepsilon,\varepsilon\big\rangle + (1-\lambda)\big\langle\mathcal{B}\varepsilon,\varepsilon\big\rangle \\ &\overset{(7),(10)}{\geq} \lambda\kappa_{\mathcal{R}}\|\varepsilon\|^2 + (1-\lambda)\kappa_{\mathcal{B}}\|\varepsilon\|^2 \\ &= \big(\lambda\kappa_{\mathcal{R}} + (1-\lambda)\kappa_{\mathcal{B}}\big)\|\varepsilon\|^2 \\ &\geq \min\{\kappa_{\mathcal{B}},\kappa_{\mathcal{R}}\}\|\varepsilon\|^2,\end{aligned} \tag{26}$$

i.e. ${}^e\mathcal{B}_\lambda$ is strongly positive. From this

$$\|{}^e\mathcal{B}_\lambda\varepsilon\| \overset{(CS)}{\geq} \min\{\kappa_\mathcal{B},\kappa_\mathcal{R}\}\|\varepsilon\|,$$

so that the inverse ${}^e\mathcal{B}_\lambda^{-1}$ exists. Further – due to the symmetry of ${}^e\mathcal{B}_\lambda$ and the subsitution $\sigma = {}^e\mathcal{B}_\lambda\varepsilon$ –

$$\begin{aligned} \|{}^e\mathcal{B}_\lambda^{-1}\sigma\|\,\|\sigma\| &\overset{(CS)}{\geq} \left\langle {}^e\mathcal{B}_\lambda^{-1}\sigma,\sigma\right\rangle \\ &= \left\langle \varepsilon, {}^e\mathcal{B}_\lambda\varepsilon\right\rangle \\ &\overset{(26)}{\geq} \big(\lambda\kappa_\mathcal{R} + (1-\lambda)\kappa_\mathcal{B}\big)\big)\|{}^e\mathcal{B}_\lambda^{-1}\sigma\|^2, \end{aligned} \tag{27}$$

thus

$$\|{}^e\mathcal{B}_\lambda^{-1}\sigma\| \leq \frac{1}{\lambda\kappa_\mathcal{R}+(1-\lambda)\kappa_\mathcal{B}}\|\sigma\| \leq \frac{1}{\min\{\kappa_\mathcal{B},\kappa_\mathcal{R}\}}\|\sigma\|$$

and

$$\|{}^e\mathcal{B}_\lambda^{-1}\| \leq \frac{1}{\lambda\kappa_\mathcal{R}+(1-\lambda)\kappa_\mathcal{B}} \leq \frac{1}{\min\{\kappa_\mathcal{B},\kappa_\mathcal{R}\}}. \tag{28}$$

Further

$$\|\sigma\|^2_{{}^e\mathcal{B}_\lambda} = \left\langle {}^e\mathcal{B}_\lambda^{-1}\sigma,\sigma\right\rangle \overset{(CS)}{\leq} \frac{1}{\min\{\kappa_\mathcal{R},\kappa_\mathcal{B}\}}\|\sigma\|^2 =: C^2_{\mathcal{R}\mathcal{B}}\|\sigma\|^2.$$

All rightmost estimates are independent of λ.

(3) We estimate upwards

$$\|{}^e\mathcal{B}_\lambda\varepsilon\| \overset{(18)}{\leq} \big(\lambda\|\mathcal{R}\| + (1-\lambda)\|\mathcal{B}\|\big)\,\|\varepsilon\|,$$

thus – with $\sigma = {}^e\mathcal{B}_\lambda\varepsilon$ –

$$\|{}^e\mathcal{B}_\lambda^{-1}\sigma\| \geq \frac{1}{\lambda\|\mathcal{R}\|+(1-\lambda)\|\mathcal{B}\|}\|\sigma\| \geq \frac{1}{\max\{\|\mathcal{R}\|,\|\mathcal{B}\|\}}\|\sigma\|$$

and

$$\begin{aligned} \|\sigma\|^2_{{}^e\mathcal{B}_\lambda} = \left\langle {}^e\mathcal{B}_\lambda^{-1}\sigma,\sigma\right\rangle &\overset{(27)}{\geq} \frac{\lambda\kappa_\mathcal{R}+(1-\lambda)\kappa_\mathcal{B}}{(\lambda\|\mathcal{R}\|+(1-\lambda)\|\mathcal{B}\|)^2}\|\sigma\|^2 \\ &\geq \frac{\min\{\kappa_\mathcal{R},\kappa_\mathcal{B}\}}{\max^2\{\|\mathcal{R}\|,\|\mathcal{B}\|\}}\|\sigma\|^2 =: c^2_{\mathcal{R}\mathcal{B}}\|\sigma\|^2. \end{aligned}$$

Again, all the rightmost estimates are independent of λ.

(4) From the steps (2) and (3) above, we have a norm equivalence, which is independent of λ

$$c_{\mathcal{R}\mathcal{B}}\|\cdot\| \leq \|\cdot\|_{{}^e\mathcal{B}_\lambda} \leq C_{\mathcal{R}\mathcal{B}}\|\cdot\|. \tag{29}$$

(5) Let a sequence (λ_n) be given such that $[0,1] \ni \lambda_n \to \lambda \in [0,1]$. Then there holds

$$\begin{aligned} \|{}^e\mathcal{B}_{\lambda_n} - {}^e\mathcal{B}_\lambda\| &= \|(\lambda_n\mathcal{R} + (1-\lambda_n)\mathcal{B}) - (\lambda\mathcal{R} + (1-\lambda)\mathcal{B})\| \\ &= |\lambda_n - \lambda|\,\|\mathcal{R}-\mathcal{B}\| \xrightarrow{n} 0, \end{aligned}$$

i.e.

$$ {}^e\mathcal{B}_{\lambda_n} \stackrel{n}{\longrightarrow} {}^e\mathcal{B}_\lambda \qquad \text{in } \mathcal{L}\big(\Sigma, \Sigma^\star\big). $$

Also for the inverse, we have

$$ \begin{aligned} \|{}^e\mathcal{B}_{\lambda_n}^{-1} - {}^e\mathcal{B}_\lambda^{-1}\| &= \|{}^e\mathcal{B}_{\lambda_n}^{-1}({}^e\mathcal{B}_\lambda - {}^e\mathcal{B}_{\lambda_n}){}^e\mathcal{B}_\lambda^{-1}\| \\ &\leq \|{}^e\mathcal{B}_{\lambda_n}^{-1}\|\,\|{}^e\mathcal{B}_\lambda^{-1}\|\,\|{}^e\mathcal{B}_{\lambda_n} - {}^e\mathcal{B}_\lambda\| \\ &\stackrel{(28)}{\leq} \frac{1}{\min^2\{\kappa_\mathcal{B}, \kappa_\mathcal{R}\}}\,\|{}^e\mathcal{B}_{\lambda_n} - {}^e\mathcal{B}_\lambda\| \stackrel{n}{\longrightarrow} 0, \end{aligned} $$

i.e.

$$ {}^e\mathcal{B}_{\lambda_n}^{-1} \stackrel{n}{\longrightarrow} {}^e\mathcal{B}_\lambda^{-1} \qquad \text{in } \mathcal{L}\big(\Sigma, \Sigma^\star\big), \tag{30} $$

thus

$$ {}^e\mathcal{B}_{\lambda_n}^{-1} \stackrel{n}{\longrightarrow} {}^e\mathcal{B}_\lambda^{-1} \qquad \text{in } \mathcal{L}\big(W^{1,2}([0,T],\Sigma^\star), W^{1,2}([0,T],\Sigma)\big) $$

and

$$ {}^e\mathcal{B}_{\lambda_n}^{-1} \stackrel{n}{\longrightarrow} {}^e\mathcal{B}_\lambda^{-1} \qquad \text{in } \mathcal{L}\big(C([0,T],\Sigma^\star), C([0,T],\Sigma)\big). \tag{31} $$

(a) Let us put all the pieces together. Let sequences

$$ \lambda_n \stackrel{n}{\longrightarrow} \lambda \text{ in } [0,1], \qquad F_n \stackrel{n}{\longrightarrow} F \text{ in } W^{1,2}\big([0,T],U^\star\big) $$

be given. According to (21) and (20), we have to show

$$ \mathcal{B}\,{}^e\mathcal{B}_{\lambda_n}^{-1}\mathcal{P}_{{}^e\mathcal{B}_{\lambda_n}}(SF_n,\beta(0)) = \hat\alpha_{\lambda_n}(F_n) \stackrel{n}{\longrightarrow} \hat\alpha_\lambda(F) = \mathcal{B}\,{}^e\mathcal{B}_\lambda^{-1}\mathcal{P}_{{}^e\mathcal{B}_\lambda}(SF,\beta(0)), \tag{32} $$

$$ \mathcal{S}_{{}^e\mathcal{B}_{\lambda_n}}(SF_n,\beta(0)) = \hat\beta_{\lambda_n}(F_n) \stackrel{n}{\longrightarrow} \hat\beta_\lambda(F) = \mathcal{S}_{{}^e\mathcal{B}_\lambda}(SF,\beta(0)) \tag{33} $$

in $C\big([0,T],\Sigma^\star\big)$. To this end we show amongst others

$$ \mathcal{P}_{{}^e\mathcal{B}_{\lambda_n}}(SF_n,\beta(0)) = {}^e\alpha_{\lambda_n}(F_n) \stackrel{n}{\longrightarrow} {}^e\alpha_\lambda(F) = \mathcal{P}_{{}^e\mathcal{B}_\lambda}(SF,\beta(0)) \tag{34} $$

in $C([0,T],\Sigma^\star)$. First, clearly due to the continuity of S,

$$ {}^e\sigma_n = SF_n \stackrel{n}{\longrightarrow} SF = \sigma \qquad \text{in } W^{1,q}([0,T],\Sigma^\star) \qquad \text{for each } 1 \leq q \leq 2 \tag{35} $$

thus

$$ {}^e\sigma_n = SF_n \stackrel{n}{\longrightarrow} SF = \sigma \qquad \text{in } C([0,T],\Sigma^\star) \tag{36} $$

We apply the generalised stop estimate theorem 3.1 in [5] with (29) and identical initial memories in order to receive

$$ \begin{aligned} \|\hat\beta_{\lambda_n}(F_n)(t) - \hat\beta_\lambda(F)(t)\| &= \big\|\mathcal{S}_{{}^e\mathcal{B}_{\lambda_n}}(SF_n,\beta(0))(t) - \mathcal{S}_{{}^e\mathcal{B}_\lambda}(SF,\beta(0))(t)\big\| \\ &\leq \frac{C_{\mathcal{RB}}}{c_{\mathcal{RB}}}\int_0^t \|S(\dot F_n - \dot F)\|\mathrm{d}\tau \\ &\quad + \frac{1}{c_{\mathcal{RB}}^2}\|{}^e\mathcal{B}_{\lambda_n}^{-1} - {}^e\mathcal{B}_\lambda^{-1}\| \int_0^t \|\mathrm{d}_t\mathcal{P}_{{}^e\mathcal{B}_\lambda}(SF,\beta(0))\|\mathrm{d}\tau. \end{aligned} $$

Thus – with of n independent constants, because of (30), (35) –

$$ \begin{aligned} \|\hat\beta_{\lambda_n}(F_n) - \hat\beta_\lambda(F)\|_\infty &\leq \text{const}\int_0^T \|S(\dot F_n - \dot F)\|\mathrm{d}\tau \\ &\quad + \text{const}\,\|{}^e\mathcal{B}_{\lambda_n}^{-1} - {}^e\mathcal{B}_\lambda^{-1}\| \stackrel{n}{\longrightarrow} 0, \end{aligned} $$

giving (33). We next show (34) in the same way. Applying the generalised play estimate, corollary 3.4 in [5], to obtain

$$\begin{aligned} \|{}^e\alpha_{\lambda_n}(F_n)(t) - {}^e\alpha_\lambda(F)(t)\| &= \big\|\mathcal{P}_{{}^e\mathcal{B}_{\lambda_n}}(SF_n,\beta(0))(t) - \mathcal{P}_{{}^e\mathcal{B}_\lambda}(SF,\beta(0))(t)\big\| \\ &\le \|S(F-F_n)(t)\| + \frac{C_{\mathcal{RB}}}{c_{\mathcal{RB}}}\int_0^t \|S(\dot F_n - \dot F)\|\mathrm{d}\tau \\ &\quad + \frac{1}{c_{\mathcal{RB}}^2}\|{}^e\mathcal{B}_{\lambda_n}^{-1} - {}^e\mathcal{B}_\lambda^{-1}\| \int_0^t \|\mathrm{d}_t\mathcal{P}_{{}^e\mathcal{B}_\lambda}(SF,\beta(0))\|\mathrm{d}\tau. \end{aligned}$$

Thus – with of n independent constants, because of (30), (35), (36) –

$$\begin{aligned} \|{}^e\alpha_{\lambda_n}(F_n) - {}^e\alpha_\lambda(F)\|_\infty &\le \mathrm{const}\Big(\|S(F_n - F)\|_\infty + \int_0^T \|S(\dot F_n - \dot F)\|\mathrm{d}\tau\Big) \\ &\quad + \mathrm{const}\,\|{}^e\mathcal{B}_{\lambda_n}^{-1} - {}^e\mathcal{B}_\lambda^{-1}\| \overset{n}{\longrightarrow} 0, \end{aligned}$$

giving (34). Finally clearly – because of (34) and (31) –

$$\mathcal{B}{}^e\mathcal{B}_{\lambda_n}^{-1}{}^e\alpha_{\lambda_n}(F_n) \overset{n}{\longrightarrow} \mathcal{B}{}^e\mathcal{B}_\lambda^{-1}{}^e\alpha_\lambda(F) \qquad \text{in } C([0,T],\Sigma^\star),$$

which is (32).
(b) This assertion is clear because of the continuity of stop and play $W^{1,2} \to W^{1,2}$, see [3, 4]. ■

Recall from theorem 4.6 in [6]

$$\varphi_F(t) = \int_0^t \|S\dot F\|\mathrm{d}\tau, \quad \varphi_F^{{}^e\alpha}(t) = \|{}^e\alpha_\lambda(0)\| + \varphi_F(t), \quad \varphi_F^{\gamma}(t) = \|\gamma(0)\| + \varphi_F(t)$$

and

$$h = \big\|{}^e\mathcal{B}_\lambda^{-1} - \mathcal{R}^{-1}\big\|_{\Sigma,\Sigma^\star}, \qquad h_{\mathcal{B}} = \big\|\mathcal{B}({}^e\mathcal{B}_\lambda^{-1} - \mathcal{R}^{-1})\big\|_{\Sigma^\star,\Sigma^\star}.$$

3.3 Theorem (Error Homotopy) *Let $\Delta^\lambda\cdot = \hat\cdot_\lambda - \cdot$. For $\Delta^\lambda\alpha$, $\Delta^\lambda\beta$, $\Delta^\lambda\sigma$, $\Delta^\lambda\varepsilon^{el}$ and ${}^e\alpha_\lambda - \gamma$, there hold the same estimates as in [6], theorem 4.6.*

For $\Delta^\lambda\varepsilon^{pl}$, there holds

$$\begin{aligned} \|\Delta^\lambda\varepsilon^{pl}(t)\|_\Sigma &\le r_1\|\Delta^\lambda\beta(t)\|_{\Sigma^\star} + q_{{}^e\mathcal{B}}h^{(\lambda)}\,\varphi_F^{{}^e\alpha}(t), \\ \|\Delta^\lambda\varepsilon^{pl}(t)\|_\Sigma &\le \lambda{}^e b_1\|\Delta^\lambda\beta(t)\|_{\Sigma^\star} + q_{\mathcal{R}}h^{(\lambda)}\,\varphi_F^{\gamma}(t). \end{aligned}$$

For $\Delta^\lambda\varepsilon$, there holds

$$\begin{aligned} \|\Delta^\lambda\varepsilon(t)\|_\Sigma &\le \big(r_1 + c_1(r_1^b+1)\big)\|\Delta^\lambda\beta(t)\|_{\Sigma^\star} + q_{{}^e\mathcal{B}}\big(h^{(\lambda)} + c_1h_{\mathcal{B}}\big)\,\varphi_F^{{}^e\alpha}(t), \\ \|\Delta^\lambda\varepsilon(t)\|_\Sigma &\le \big(\lambda{}^e b_1 + c_1({}^e b_1^b+1)\big)\|\Delta^\lambda\beta(t)\|_{\Sigma^\star} + q_{\mathcal{R}}\big(h^{(\lambda)} + c_1h_{\mathcal{B}}\big)\,\varphi_F^{\gamma}(t). \end{aligned}$$

For $\Delta^\lambda u$, there holds

$$\begin{aligned} \|\Delta^\lambda u(t)\|_U &\le d\big(r_1 + c_1(r_1^b+1)\big)\|\Delta^\lambda\beta(t)\|_{\Sigma^\star} + q_{{}^e\mathcal{B}}d\big(h^{(\lambda)} + c_1h_{\mathcal{B}}\big)\,\varphi_F^{{}^e\alpha}(t), \\ \|\Delta^\lambda u(t)\|_U &\le d\big(\lambda{}^e b_1 + c_1({}^e b_1^b+1)\big)\|\Delta^\lambda\beta(t)\|_{\Sigma^\star} + q_{\mathcal{R}}d\big(h^{(\lambda)} + c_1h_{\mathcal{B}}\big)\,\varphi_F^{\gamma}(t). \end{aligned}$$

Here

$$h^{(\lambda)} = \big\|\lambda{}^e\mathcal{B}_\lambda^{-1} - \mathcal{R}^{-1}\big\|_{\Sigma,\Sigma^\star}. \tag{37}$$

All estimates are valid e. in $[0,T]$. For $\lambda = 1$, all rightmost sides are vanishing.

Proof: It is similar to the proof of the cited theorem. The estimates for $\Delta^\lambda\alpha$, $\Delta^\lambda\beta$, $\Delta^\lambda\sigma$ and $\Delta^\lambda\varepsilon^{el}$ are clearly identical. But we have now

$$\Delta^\lambda\varepsilon^{pl} = \lambda{}^e\!\mathcal{B}_\lambda^{-1}{}^e\alpha_\lambda - \mathcal{R}^{-1}\gamma. \tag{38}$$

We receive

$$\begin{aligned} \|\Delta^\lambda\varepsilon^{pl}(t)\|_{\Sigma^\star} &\leq \|\mathcal{R}^{-1}\Delta^\lambda\beta(t)\|_{\Sigma^\star} + \|(\lambda{}^e\!\mathcal{B}_\lambda^{-1} - \mathcal{R}^{-1}){}^e\alpha(t)\|_{\Sigma^\star}, && (39)\\ \|\Delta^\lambda\varepsilon^{pl}(t)\|_{\Sigma^\star} &\leq \lambda\|{}^e\!\mathcal{B}_\lambda^{-1}\Delta^\lambda\beta(t)\|_{\Sigma^\star} + \|(\lambda{}^e\!\mathcal{B}_\lambda^{-1} - \mathcal{R}^{-1})\gamma(t)\|_{\Sigma^\star}. && (40) \end{aligned}$$

The second summand in (39) resp. (40) is estimated with (11), (19) and the aid of [5](16). The rest is again straightforward. ■

3.4 Corollary (Error homotopy II) *Assume (17). Then there exist positive constants – dependencies in brackets – such that the following estimates are valid e. in the interval* $[0,T]$.

1. *Estimates for the* ***stresses.*** *There holds*

$$\begin{aligned} \|\Delta^\lambda\alpha(t)\|_{\Sigma^\star} &\leq c_\alpha(\mathcal{R},\mathcal{B})h^\lambda\varphi_F(t),\\ \|\Delta^\lambda\beta(t)\|_{\Sigma^\star} &\leq c_\beta(\mathcal{R},\mathcal{B})h^\lambda\varphi_F(t),\\ \|\Delta^\lambda\sigma(t)\|_{\Sigma^\star} &\leq c_\sigma(\mathcal{R},\mathcal{B})h^\lambda\varphi_F(t). \end{aligned}$$

2. *Estimates for the* ***strains.*** *There holds*

$$\begin{aligned} \|\Delta^\lambda\varepsilon^{pl}(t)\|_{\Sigma} &\leq \big(c_{\varepsilon^{pl}}(\mathcal{R},\mathcal{B})h^\lambda + k_{\varepsilon^{pl}}(\mathcal{R},\mathcal{B})h^{(\lambda)}\big)\varphi_F(t),\\ \|\Delta^\lambda\varepsilon^{el}(t)\|_{\Sigma} &\leq c_{\varepsilon^{el}}(\mathcal{R},\mathcal{B},\mathcal{C})h^\lambda\varphi_F(t),\\ \|\Delta^\lambda\varepsilon(t)\|_{\Sigma} &\leq \big(c_\varepsilon(\mathcal{R},\mathcal{B},\mathcal{C})h^\lambda + k_\varepsilon(\mathcal{R},\mathcal{B},\mathcal{C})h^{(\lambda)}\big)\varphi_F(t). \end{aligned}$$

3. *Estimates for the* ***displacement.*** *There holds*

$$\|\Delta^\lambda u(t)\|_U \leq \big(c_u(\mathcal{R},\mathcal{B},\mathcal{C},\mathcal{D})h^\lambda + k_u(\mathcal{R},\mathcal{B},\mathcal{C},\mathcal{D})h^{(\lambda)}\big)\varphi_F(t).$$

4. *Estimates for* ${}^e\alpha - \gamma$. *There holds*

$$\|({}^e\alpha_\lambda - \gamma)(t)\|_{\Sigma^\star} \leq c_\beta(\mathcal{R},\mathcal{B})h^\lambda\varphi_F(t).$$

Here $\Delta^\lambda\cdot = \tilde{\cdot}_\lambda - \cdot$ *again. The function*

$$[0,1]\ni\lambda\mapsto h^\lambda = (1-\lambda)\frac{\|\mathcal{R}^{-1}\|\,\|P\mathcal{C}\|}{\min\{\kappa_\mathcal{B},\kappa_\mathcal{R}\}}$$

is strictly decreasing. The functions h^λ *and* $h^{(\lambda)}$ *satisfy*

$$h^1 = h^{(1)} = 0, \qquad h^0 = \frac{h^{(0)}\,\|P\mathcal{C}\|_{\Sigma^\star,\Sigma^\star}}{\min\{\kappa_\mathcal{B},\kappa_\mathcal{R}\}}, \qquad h^{(0)} = \|R^{-1}\|_{\Sigma,\Sigma^\star}. \tag{41}$$

Proof: Note (28) and (29), thus

$$\lambda{}^e\!b_1 \leq {}^e\!b_1, \qquad {}^e\!b_1 \leq \frac{1}{\min\{\kappa_\mathcal{R},\kappa_\mathcal{B}\}}, \qquad {}^e\!b_1^b \leq \frac{\|\mathcal{B}\|_{\Sigma^\star,\Sigma}}{\min\{\kappa_\mathcal{R},\kappa_\mathcal{B}\}}, \qquad q_{{}^e\mathcal{B}} \leq \frac{C_{\mathcal{R}\mathcal{B}}}{c_{\mathcal{R}\mathcal{B}}}$$

in theorem 3.3 is possible. We have $h \leq h^\lambda$ and $h_\mathcal{B} \leq \|\mathcal{B}\|\,h^\lambda$. For $h^{(\lambda)}$ such an estimate is not possible. ■

3.5 Corollary (Model differences) *Assume (17). There exist positive constants – dependencies in brackets – such that*

$$\begin{aligned}\|({}^e\sigma - \sigma)(t)\|_{\Sigma^\star} &\leq C_\sigma(\mathcal{R}, \mathcal{B})\varphi_F(t),\\ \|({}^e\varepsilon - \varepsilon)(t)\|_{\Sigma} &\leq C_\varepsilon(\mathcal{R}, \mathcal{B}, \mathcal{C})\varphi_F(t),\\ \|({}^e u - u)(t)\|_{U} &\leq C_u(\mathcal{R}, \mathcal{B}, \mathcal{C}, \mathcal{D})\varphi_F(t).\end{aligned}$$

All estimates are valid e. in $[0, T]$.

Proof: Let $\lambda = 0$ in theorem 3.3 and corollary 3.4, giving h^0 and $h^{(0)}$ as stated in (41). ■

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