A Topology Primer

Lecture Notes 2001/2002

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Preface

These lecture notes were written to accompany my introductory courses of topology starting in the summer term 2001. Students then were, and readers now are expected to have successfully completed their first year courses in analysis and linear algebra. The purpose of the first part of this course, comprising the first thirteen sections, is to make familiar with the basics of topology. Students will mostly encounter notions they have already seen in an analytic context, but here they are treated from a more general and abstract point of view. That this brings about a certain simplification and unification of results and proofs may have its own esthetic merits but is not the point. It is only with the introduction of quotient spaces in Section 10 that the general topological approach will prove to be indispensable and interesting, and indeed it might fairly be said that the true subject of topology begins there. Therefore the reader who wants to see something really new is asked to be patient and meanwhile study the introductory sections carefully.

Sections 14 up to and including Section 22 give a concise introduction to homotopy. A first culminating point is reached in Section 19 with the determination of the equidimensional homotopy group of a sphere. Together with more technical material presented in Sections 21 and 22 this lays the ground for the definition of homology in the third and final part of these notes. To make students familiar with (topologists' notion of) homology has indeed been the main goal of the courses. Rather than the traditional approach via singular homology I have chosen a cellular one which allows to make the geometric meaning of homology much clearer. I also wanted my students to experience homology as something inherently computable, and have included a fair number of explicit matrix calculations.

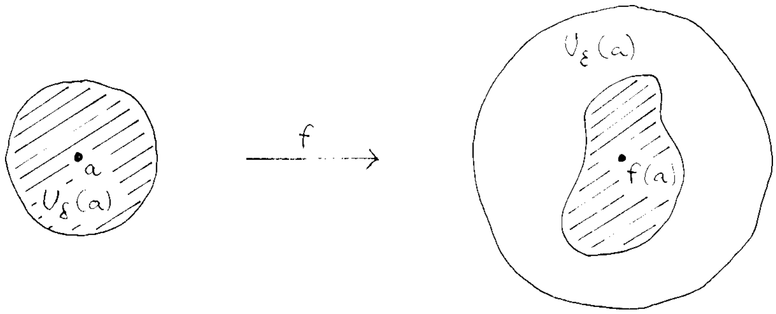
The text proper contains no exercises but in order to clarify new notions I have included a variety of simple questions which students should address right away when reading the text. More extensive exercises intended for homework are collected at the end of these notes. Those set in the original two hour course on Sections 1 to 13 are in German and called Aufgaben while the rest are in English and named problems.

Contents

1 Topological Spaces

A map $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^p$ ist continuous at $a \in \mathbb{R}^n$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \quad \text{for all } x \in \mathbb{R}^n \text{ with } |x - a| < \delta.$$



Every second year student of mathematics will be familiar with this definition. We will try to reformulate it in a more geometric manner, eliminating all that fuss about epsilons and deltas.

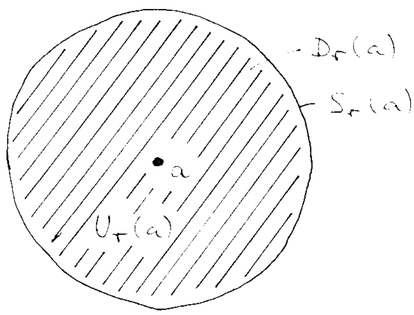
1.1 Terminology and **Notation** Let $a \in \mathbb{R}^n$ and $r \geq 0$. The open and the closed ball of radius r around a are the subsets of $\mathbb{R}^n$

$$U_r(a) := \big\{x \in \mathbb{R}^n \big| \, |x - a| < r\big\} \quad \text{and} \quad D_r(a) := \big\{x \in \mathbb{R}^n \big| \, |x - a| \leq r\big\}$$

respectively, while their difference

$$S_r(a) := D_r(a) \setminus S_r(a) = \big\{x \in \mathbb{R}^n \big| \, |x - a| = r\big\}$$

is called the sphere of radius r around a.



In the special case $a = 0$ and $r = 1$ of unit ball and sphere it is customary to drop these data from the notation and include the "dimension" instead:

$$U^n,\ D^n,\ S^{n-1} \subset \mathbb{R}^n$$

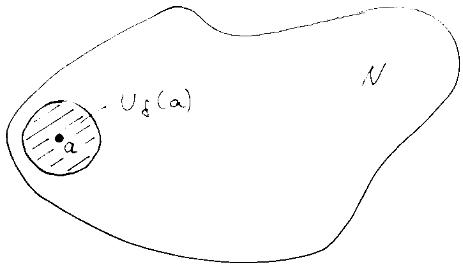
It is also common to talk of disks instead of balls throughout, and you will probably prefer to do so when thinking of the two-dimensional case.

1.2 Question Does the case $r = 0$ still make sense? What are U^1, D^1, S^0, and U^0, D^0, S^{-1} ?

Using the terminology of balls we now may say: a map $f: \mathbb{R}^n \longrightarrow \mathbb{R}^p$ ist continuous at a if for each open ball V of positive radius around $f(a)$ there exists an open ball U of positive radius around a with $f(U) \subset V$. This still is rather an awkward statement, and another notion helps to smooth it out:

1.3 Definition Let $a \in \mathbb{R}^n$ be a point. A neighbourhood of a in $\mathbb{R}^n$ is a subset $N \subset \mathbb{R}^n$ which contains a non-empty open ball around a:

$$a \in U_\delta(a) \subset N \quad \text{for some } \delta > 0$$



Thus we have another way of re-writing continuity: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^p$ is continuous at a if for each neighbourhood P of $f(a)$ in $\mathbb{R}^p$ there exists a neighbourhood N of a with $f(N) \subset P$. Or yet another one, still equivalent: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^p$ is continuous at a if for each neighbourhood $P \subset \mathbb{R}^p$ of $f(a)$ the inverse image $f^{-1}(P)$ is neighbourhood of a in $\mathbb{R}^n$.

Topology begins with the observation that the basic laws ruling continuity can be derived without even implicit reference to epsilons and deltas, from a small set of axioms for the notion of neighbourhood. While this approach to topology is perfectly viable a more convenient starting point is the global version of continuity. By definition a function $f: \mathbb{R}^n \longrightarrow \mathbb{R}^p$ is continuous if it is continuous at each point $a \in \mathbb{R}^n$. In your analysis course you may have come across the following elegant characterisation of continuous maps: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^p$ is continuous if and only if for every open set $V \subset \mathbb{R}^p$ its inverse image $f^{-1}(V)$ is an open subset of $\mathbb{R}^n$. Here, of course, the notion of openness is presumed: a set $V \subset \mathbb{R}^p$ is open in $\mathbb{R}^p$ if it is a neighbourhood of each of its points.

At this moment we will not bother to prove equivalence with the previous point-by-point definitions. The point I rather wish to make is that continuity of mappings can be phrased in terms of openness of sets, and as we will see shortly a set of very simple axioms governing this notion is all that is needed to do so. Thus we have arrived at the most basic of all definitions in topology.

1.4 Definition Let X be a set. A topology on X is a set $\mathcal{O}$ of subsets of X subject to the following axioms.

- $\emptyset \in \mathcal{O}$ and $X \in \mathcal{O}$,
- if $U \in \mathcal{O}$ and $V \in \mathcal{O}$ then $U \cap V \in \mathcal{O}$,
- if $(U_\lambda)_{\lambda\in\Lambda}$ is any family of sets with $U_\lambda \in \mathcal{O}$ for all $\lambda \in \Lambda$ then $\bigcup_{\lambda\in\Lambda} U_\lambda \in \mathcal{O}$.

A pair consisting of a set X and a topology $\mathcal{O}$ on X is called a topological space. The elements of $\mathcal{O}$ are then referred to as the open subsets of X.

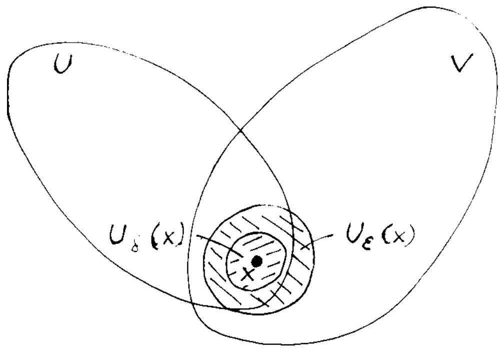
While a given set X will (in general) allow many different topologies, in practice a particular one is usually implied by the context. Then the notation X is commonly used as shorthand for $(X, \mathcal{O})$, just like a simple V is used to denote a vector space $(V, +, \cdot)$.

It goes without saying that the second axiom implies that the intersection of finitely many open sets is open again, using induction. Examples will show at once that this property does not usually extend to infinite intersections. By contrast the third axiom requires the union of any family of open sets to be open — including infinite, even uncountable families.

1.2 Question Which are the sets that carry a unique topology?

1.6 Examples (1) $\mathbb{R}^n$ is the typical example to keep in mind. We already have defined what the open subsets of $\mathbb{R}^n$ are: $U \subset \mathbb{R}^n$ is open if for each $x \in U$ there exists some $\delta > 0$ such that $U_\delta(x) \subset U$. Let us verify that these sets form a topology on $\mathbb{R}^n$.

Obviously $\emptyset$ is open (there is nothing to show) and $\mathbb{R}^n$ is open (any δ will do). Next let U and V be open and $x \in U \cap V$. Since U is open we find $\delta > 0$ such that $U_\delta(x) \subset U$, and since V is also open we can choose $\varepsilon > 0$ with $U_\varepsilon(x) \subset V$. Replacing both δ and ε by the smaller of the two we may assume $\delta = \varepsilon$ and obtain $U_\delta(x) \subset U \cap V$. Thus $U \cap V$ is open.



Finally consider a family $(U_\lambda)_{\lambda\in\Lambda}$ of open subsets $U_\lambda \subset \mathbb{R}^n$, and let x be in their union. Then Λ is non-empty, and we pick an arbitrary $\lambda \in \Lambda$ and a $\delta > 0$ with $U_\delta(x) \subset U_\lambda$. In view of

$$U_\delta(x) \subset U_\lambda \subset \bigcup_{\lambda\in\Lambda} U_\lambda$$

we have thereby shown that $\bigcup_{\lambda\in\Lambda} U_\lambda$ is an open subset of $\mathbb{R}^n$. This completes the proof.

(2) Let X be any set. Declaring all subsets open trivially satisfies the axioms for a topology on X. It is called the discrete topology, making X a discrete (topological) space.

(3) At the other extreme $\{\emptyset, X\}$ is the smallest possible topology on X. Let us call it the lump topology on X, and such an X a lump space because the open sets of this topology provide no means to separate the distinct points of X.

Examples (2) and (3) are, of course, quite uncharacteristic of topological spaces in general. Their purpose rather was to show that the notion of topology is a very wide one.

2 Continuous Mappings and Categories

At this point you should already have guessed the definition of a continuous map between topological spaces, so the following is mainly for the record:

2.1 Definition Let X and Y be topological spaces. A map $f: X \longrightarrow Y$ is continuous if for each open $V \subset Y$ the inverse image $f^{-1}(V) \subset X$ is open.

Note that it is *inverse* images of open sets that count. In the opposite direction, a map $f: X \longrightarrow Y$ may or may not send open subsets of X to open subsets of Y but this has nothing to do with continuity of f.

A few formal properties of continuity are basically obvious:

2.2 Facts about continuity

- Every constant map $f: X \longrightarrow Y$ is continuous (because $f^{-1}(V)$ is either empty or all X).
- The identity mapping $\mathrm{id}_X: X \longrightarrow X$ of any topological space is continuous.
- If $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ are continuous then so is the composition $g \circ f: X \longrightarrow Z$.

There is another way of phrasing the latter two properties by saying that topological spaces are the objects, and continuous maps the morphisms of what is called a *category*:

2.3 Definition A category $\mathbf{C}$ consists of

- a class $|\mathbf{C}|$ of objects,
- a set $\mathbf{C}(X, Y)$ for any two objects $X, Y \in |\mathbf{C}|$ whose elements are called the morphisms from X to Y, and
- a composition map $\mathbf{C}(Y, Z) \times \mathbf{C}(X, Y) \ni (g, f) \longmapsto gf \in \mathbf{C}(X, Z)$ defined for any three objects $X, Y, Z \in |\mathbf{C}|$.

For different $X, Y \in |\mathbf{C}|$ the sets $\mathbf{C}(X, Y)$ are supposed to be pairwise disjoint. Composition must be associative and for each object X there must exist a unit element $1_X \in \mathbf{C}(X, X)$ such that

$$f\,1_X = f = 1_Y f$$

holds for all $f \in \mathbf{C}(X, Y)$.

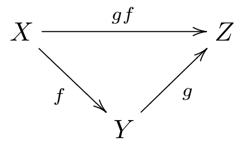
Remarks The well-known argument from group theory shows that the unit elements are automatically unique, thereby justifying the special notation for them. — It is now clear that there is a category **Top** of topological spaces and continuous maps, the composition map being ordinary composition of maps, of course. Thus

$$\mathbf{Top}(X, Y) = \{f: X \longrightarrow Y \mid f \text{ is continuous}\}$$

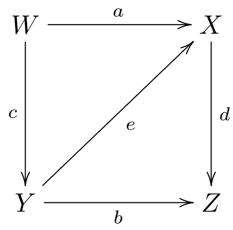
and the identity mappings $1_X = \mathrm{id}_X: X \longrightarrow X$ are the unit elements. — I have avoided to refer to $|\mathbf{C}|$ as a *set* of objects because many interesting categories have just too many objects for that. For instance, our category **Top** includes among its objects at least all sets together with their discrete topology, and you may be familiar with the logical contradiction inherent in the notion of *set of all sets* (Russell's Antinomy). Substituting the word *class* we indicate that we just wish to identify its members but have no intention to perform set theoretic operations with that class as a whole.

2.4 Question What other categories do you know?

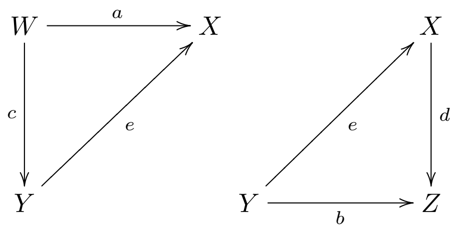
Often the objects of a category are sets with or without an additional structure, and morphisms the maps preserving that structure, like in **Top**. But this is not required by the definition, and we will meet various categories of a different kind later on. Nevertheless it is customary to write a morphism $f \in \mathbf{C}(X, Y)$ in an arbitrary category as an arrow $f: X \longrightarrow Y$ like a map from X to Y — quite an adequate notation as the laws governing morphisms are, by definition, just those of the composition of maps. In particular diagrams of objects and morphisms of a category make sense, and exactly as in the case of mappings they may or may not commute. Commutative diagrams allow to present calculations with morphisms an a graphic yet precise way. The simplest case is that of a triangular diagram



where commutativity merely restates the definition of gf. By contrast saying that, for example



commutes involves the statements $a = ec$, $b = de$, and $da = bc\,(= dec)$. Commutative diagrams with common morphisms can be combined into a bigger diagram which still commutes. So the above diagram could have been built from two commutative diagrams



and, indeed, $da = bc$ is a consequence of the identities $a = ec$ and $b = de$ represented by the triangles.

At first glance a notion as general as that of category seems of rather limited use. In fact, though, they turn out to be indispensable in more than one branch of mathematics as they allow to separate notions and results that are completely formal from those that are particular to a specific situation. A first example is provided by

2.5 Definition Let X, Y be objects of a category **C**. A morphism $f: X \longrightarrow Y$ is an <u>isomorphism</u> if there exists a morphism $g: Y \longrightarrow X$ such that $gf = 1_X$ and $fg = 1_Y$. If such an isomorphism exists then the objects X and Y are called <u>isomorphic</u>.

Again it is seen at once that the morphism g is uniquely determined by f. Naturally, it is called the <u>inverse</u> to f and written f^{-1}. In the category **Ens** of sets and mappings the isomorphisms are just the bijective maps, and in many algebraic categories like $\mathbf{Lin}_K$, the category of vector spaces and linear maps over a fixed field K isomorphism likewise turns out to be the same as bijective morphism. Not so in **Top**: let $\mathcal{O}$ and $\mathcal{P}$ be two topologies on one and the same set X and consider the identity mapping:

$$(X, \mathcal{O}) \xrightarrow{\text{id}} (X, \mathcal{P})$$

It is continuous, hence a morphism in **Top** if and only if $\mathcal{O} \supset \mathcal{P}$ — read $\mathcal{O}$ is <u>finer</u> than $\mathcal{P}$, or $\mathcal{P}$ <u>coarser</u> than $\mathcal{O}$. This morphism is certainly bijective as a map but its inverse is discontinuous unless $\mathcal{O} = \mathcal{P}$.

Note By a classical theorem on real functions of one variable the inverse of any continuous injective function defined on an interval again is continuous. This result strongly depends on the fact that such functions must be strictly increasing or decreasing, it does not generalise to several variables let alone to mappings between topological spaces in general.

As to isomorphisms in **Top**, topologists have their private

2.6 Terminology Isomorphisms in **Top** (continuous maps which have a continuous inverse) are called homeomorphisms, and isomorphic objects $X, Y \in |\mathbf{Top}|$, homeomorphic (to each other): $X \approx Y$.

Explicitly, a continuous bijective map $f: X \longrightarrow Y$ is a homeomorphism if and only if it sends open subsets of X to open subsets of Y.

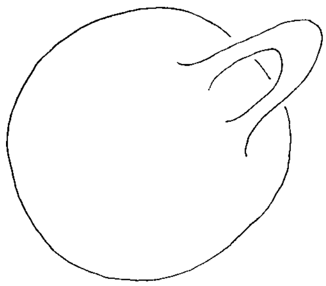
I can now roughly describe what topology is about. Let us first discuss what something you already know like linear algebra is about. Linear algebra is about vector spaces (over a field K) and K-linear maps, and it aims at understanding both — no matter whether this be considered as a goal in itself or as prerequisite to some application. And the theory does provide a near perfect understanding at least of finite dimensional vector spaces (each is isomorphic to K^n for exactly one $n \in \mathbb{N}$) and linear maps between them (think of the various classification results on matrices by rank, eigenvalues, diagonalizability, Jordan normal form).

So the first of topologists' aims would be to describe a reasonably large class of topological spaces. Let us have a closer look at how the corresponding problem for finite dimensional vector spaces is solved. It is solved by introducing the dimension as a numerical *invariant* of such spaces. The characteristic property of an invariant in linear algebra is that isomorphic vector spaces give the same value of the invariant — a property that also makes sense for categories other than $\mathbf{Lin}_K$. Thus passing to **Top** one should try to construct *topological invariants* by assigning to each topological space X one or several numbers in such a manner that homeomorphic space are assigned identical values. But topology differs from linear algebra in that it is much more difficult to construct good invariants, an invariant being good or powerful if it is able to differentiate between many topological spaces that are not homeomorphic.

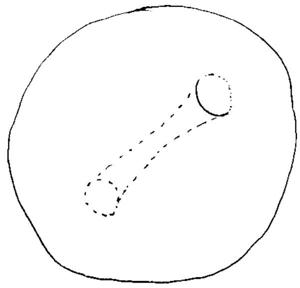
Looking at our (admittedly still very poor) list 1.6 of examples, the obvious question is whether we obtain a topological invariant by assigning the number n to the topological space $\mathbb{R}^n$. In other words: is it true that $\mathbb{R}^n$ and $\mathbb{R}^p$ are homeomorphic *only* if $n = p$? While the answer will turn out to be yes this is not easy to prove at all. Nor is this a surprising fact, once the analogy with linear algebra is pushed a bit further. Given any two finite dimensional, say real vector spaces V and W the homomorphisms (including the isomorphisms) between V and W correspond to real matrices of a fixed format and thereby form a manageable set. By comparison the set of all continuous maps from $\mathbb{R}^n$ to $\mathbb{R}^p$ is vast, no chance to describe it by a finite set of real coefficients. Can you imagine an easy direct way of proving that for $n \neq p$ there can be no homeomorphism among them?

Still, difficult does not mean impossible, and in its seventy or eighty years of existence as an independent mathematical field topology has come up with an impressive variety of clever topological invariants, some of which you will soon get to know.

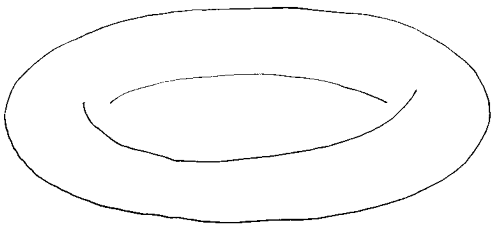
This programme would hardly be worth the effort if the question of whether $\mathbb{R}^n$ and $\mathbb{R}^p$ can be homeomorphic were the only one in topology — as at this point you have all the right to think. But this is by no means the case. In fact every mathematical object that may be called geometric in the widest sense of the word has an underlying structure as a topological space, and thus falls into the realms of topology. Topologists are geometers but they study only those properties of geometrical objects that are preserved by homeomorphisms, thereby excluding the more familiar points of view based on the notions of metric or linearity. It is only the deepest geometric qualities that are invariant under homeomorphisms. Thus, for instance, the four bodies of quite different appearance



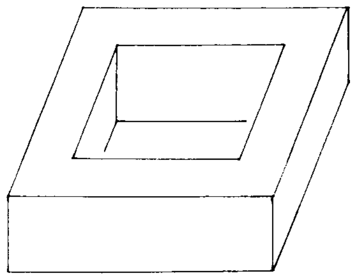
ball with handle



pierced ball



(solid) torus



pierced cube

can be shown to be all homeomorphic to each other. While it is intuitively clear what their characteristic common property is — a certain type of hole — in order to phrase this in precise mathematical terms some properly chosen topological invariant is needed (here the so-called *Euler characteristic* will do nicely). The common value of the invariant on each of these bodies will then differ from its value on an unpierced ball, say, and it follows that the former cannot be homeomorphic to the latter.

In view of our very restricted supply of rigorous examples pictures as those above appear to be rather far-fetched as a means of representing topological spaces. But it figures among the results of the next section that the notion of topological space includes everything that can be drawn, and much more.

3 New Spaces from Old: Subspaces and Embeddings

3.1 Definition Let X be a topological space. Every subset $S \subset X$ carries a natural topology given by

$$U \subset S \text{ is open} \quad :\Longleftrightarrow \quad \text{there exists an open } V \subset X \text{ such that } U = S \cap V$$

and thus is a topological space in its own right. As such it is called a (topological) subspace of X.

3.2 Question Why do these U form a topology on S ?

3.3 Question Let $X := \mathbb{R}$ be the real line. Which of the subspaces

$$S_1 := \mathbb{Z}, \quad S_2 := \left\{ \frac{1}{k} \;\middle|\; 0 \neq k \in \mathbb{Z} \right\} \quad \text{and} \quad S_3 := \{0\} \cup S_2$$

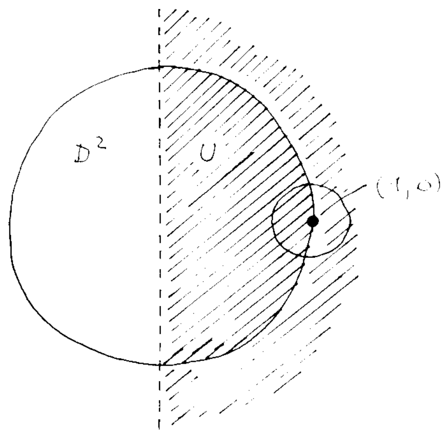
are discrete?

The notion of a subspace $S \subset X$ is a straightforward one, but one has to be careful though when talking about openness. Consider the following statements:

- U is an open subset of X, contained in S
- U is an open subset of S

While U is a subset of S in either case the former statement makes it clear that openness is with respect to the bigger topological space X. By contrast the second statement would usually be meant as referring to the subspace topology of S, so U is the intersection of S with some open subset $V \subset X$. Expressions like “U is open in S” and the more elaborate “U is relatively open in S” may be used in order to avoid ambiguity.

3.4 Example



The subset

$$U := \{(x, y) \in D^2 \mid x > 0\} \subset D^2 \subset \mathbb{R}^2$$

is not an open subset of $\mathbb{R}^2$ since no ball of positive radius around $(1, 0) \in U$ is completely contained in U. But U is relatively open in D^2 as it can be written as the intersection of D^2 with an open subset of $\mathbb{R}^2$, for instance the open half-plane $\{(x, y) \in \mathbb{R}^2 \mid x > 0\}$.

Nevertheless let us record two simple

3.5 Facts Let X be a topological space $S \subset X$ a subspace, and $U \subset S$ a subset. Then

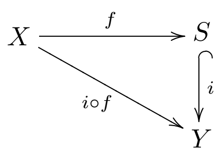
- if U is open in X then it is also open in S, and
- if S itself is open in X then the converse holds as well.

Proof The first part follows from $U = S \cap U$. As to the second, a relatively open $U \subset S$ can be written $U = S \cap V$ for some open $V \subset X$. Being the intersection of two open sets U itself is open in X.

With the introduction of the subspace topology we have passed in one small step from scarcity of examples of topological spaces to sheer abundance: all subsets of $\mathbb{R}^n$ are topological spaces in a natural way. Note that Definitions 2.1 ans 3.1 together give an immediate meaning to continuity of a function which is defined on a subset of a topological space, and of $\mathbb{R}^n$ in particular.

The following proposition states a characteristic property of the subspace topology which is familiar from elementary analysis: continuous maps into a subspace $S \subset Y$ are essentially the same as continuous maps into Y, with all values contained in S.

3.6 Proposition Let $i: S \hookrightarrow Y$ be the inclusion of a subspace and let $f: X \longrightarrow S$ be a map defined on another topological space X. Then f is continuous if and only if the composition $i \circ f$ is continuous. In particular i itself is continuous.



Proof Let f be continuous, and $V \subset Y$ an open set. Then $S \cap V$ is open in S by definition of the subspace topology, hence

$$(i \circ f)^{-1}(V) = f^{-1}\big(i^{-1}(V)\big) = f^{-1}(S \cap V) \subset X$$

is also open. Thus $i \circ f$ is continuous. Conversely make this the assumption and let $U \subset S$ be open. Again by definition $U = S \cap V$ for some open $V \subset Y$, and it follows that

$$f^{-1}(U) = f^{-1}(S \cap V) = f^{-1}\big(i^{-1}(V)\big) = (i \circ f)^{-1}(V) \subset X$$

likewise is open. Therefore f is continuous. The last clause of the proposition follows by choosing $f := \mathrm{id}_S$ which certainly is continuous.

In topology as in other branches of mathematics the concept of sub-object (like subspace) often is too narrow. Let me explain this by a familiar example, the construction of the rational numbers from the integers. A rational is defined as an equivalence class of pairs of integers $p \in \mathbb{Z}$ and $0 \neq q \in \mathbb{Z}$ with respect to the equivalence relation

$$(p, q) \sim (p', q') \iff pq' = p'q$$

and the class of (p, q) is written $\frac{p}{q}$. It may appear a problem that the set $\mathbb{Q}$ thus defined does not contain $\mathbb{Z}$ as a subset. There are two ways around it. The first is by removing from $\mathbb{Q}$ all fractions that can be written $\frac{p}{1}$ und putting in the integer $p \in \mathbb{Z}$ instead. As all the arithmetics would have to be redefined on a case by case basis this turns out to be an awkward if not ridiculous procedure. The more intelligent solution is to realize that it there is truely no need at all to insist that $\mathbb{Z}$ be a subset of $\mathbb{Q}$ since one has the canonical *embedding*

$$\mathbb{Z} \ni p \overset{e}{\longmapsto} \frac{p}{1} \in \mathbb{Q}$$

that respects the arithmetic operations and behaves in any way like the inclusion of a subring. If, as is usually done, $\mathbb{Z}$ is identified with a subring of $\mathbb{Q}$ this amounts to an implicit application of the embedding e.

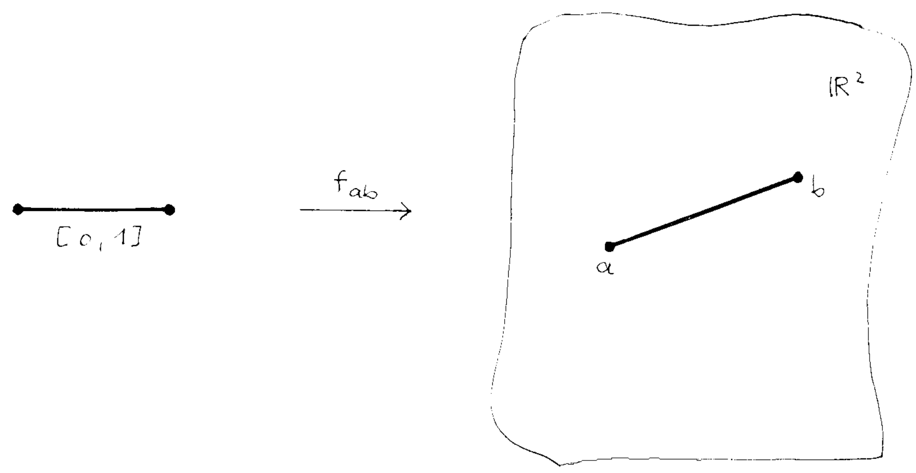
If use of the word embedding is not common in the category **Ens** the reason is that there it is synonymous with injective map. Not so in other categories like **Top**:

3.7 Definition A map between topological spaces $e: S \longrightarrow Y$ is a (topological) embedding if the map

$$S \ni s \longmapsto e(s) \in e(S)$$

(which differs from e by its smaller target set) is a homeomorphism with respect to the subspace topology on $e(S) \subset Y$.

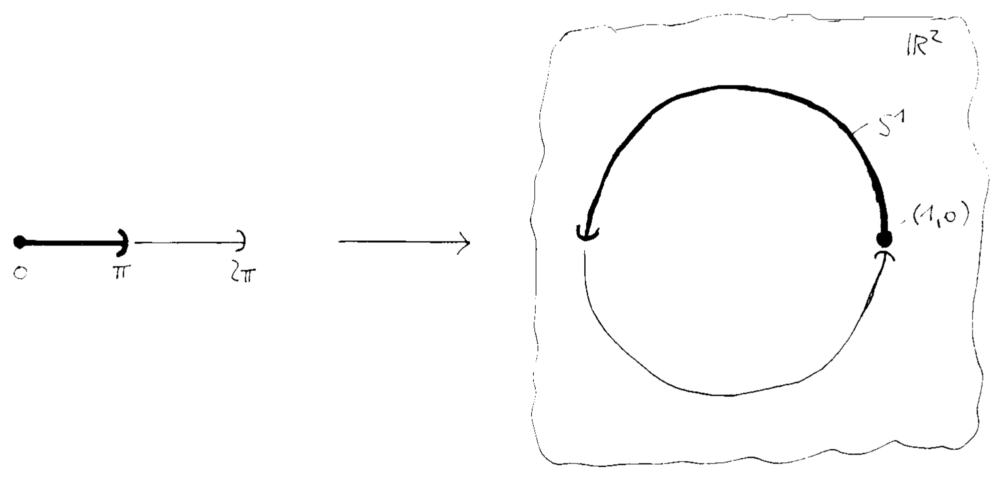
3.8 Examples (1) While the unit interval $[0,1]$ is not a subset of the plane $\mathbb{R}^2$ it can be embedded in it in manifold ways. A particularly simple example is



$$f_{ab}: t \longmapsto (1-t)\, a + t\, b,$$

depending on an arbitrary choice of two distinct points $a, b \in \mathbb{R}^2$. Clearly f_{ab} is a continuous mapping and its image is the segment S joining a to b. In view of 3.6 it still is continuous when considered as a bijective map from $[0,1]$ to S. But the inverse of this map also is continuous since it can be obtained by restriction from some linear function $\mathbb{R}^2 \longrightarrow \mathbb{R}$ which you may care to work out. It follows that f_{ab} is an embedding.

(2)



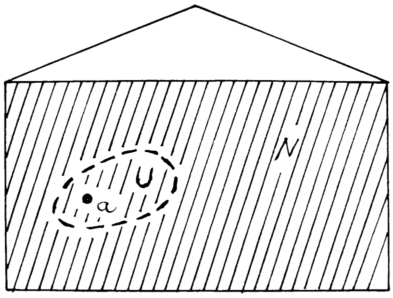
The map $[0, 2\pi) \longrightarrow \mathbb{R}^2$ sending t to $(\cos t, \sin t)$ is continuous and injective but not an embedding: while its image is the circle $S^1 \subset \mathbb{R}^2$ the bijective map

$$[0, 2\pi) \ni t \longmapsto (\cos t, \sin t) \in S^1$$

sends the open subset $[0, \pi) \subset [0, 2\pi)$ to the non-open subset $\{(1,0)\} \cup \{(x,y) \in S^1 \mid y > 0\}$ of S^1, and so cannot be a homeomorphism.

4 Neighbourhoods, Continuity, and Closed Sets

4.1 Definition Let X be a topological space, and $a \in X$ a point. A neighbourhood of a is a subset $N \subset X$ such that there exists an open set U with $a \in U \subset N$.



Remarks In case $X = \mathbb{R}^n$ one easily recovers Definition 1.3. — It is not unusual in analysis courses to restrict the term neighbourhood to open balls ("ε-neighbourhoods") but this would be meaningless in the general topological setting. You may at first be puzzled by the fact that in topology neighbourhoods may be large: by definition every set containing a neighbourhood itself is a neighbourhood. While this seems to contradict the intuitive notion of a neighbourhood of a being something close to a neighbourhoods do allow to formulate properties that are local near a as we will see in due course.

One first simple but useful

4.2 Observation A set $V \subset X$ is open if and only if it is a neighbourhood of each of its points.

Proof If V is open then the other property follows trivially. Thus assume now that V is a neighbourhood of each of its points. For each $a \in V$ let us choose an open $U_a \subset X$ with $a \in U_a \subset V$. Then in

$$V = \bigcup_{a \in V} \{a\} \subset \bigcup_{a \in V} U_a \subset V$$

equality of sets must hold throughout. Thus V is the union of the open sets U_a and therefore is open.

In general, a subset A of a topological space X will be a neighbourhood of some but not all of its points. There is another notion taking this fact into account.

4.3 Definition Let A be a subset of the topological space X. The union of those subsets of A which are open in X is called the interior of A:

$$A^\circ := \bigcup_{\substack{U \subset A \\ U \text{ open}}} U$$

Thus A° is the biggest open subset of X which is contained in A; a point $a \in X$ belongs to A° if and only if A is a neighbourhood of a.

Depending on how much you already knew about topological notions you may up to this point still wonder whether topologists' continuity as defined in 2.1 really is the same as what you have learnt in terms of epsilons and deltas. The question comes down to the equivalence of global continuity and continuity at each point, and is affirmatively settled by the following proposition. Continuity at a point has already been discussed in the case of $\mathbb{R}^n$, and the definition generalizes well:

4.4 Definition Let $X \xrightarrow{f} Y$ be a map between topological spaces. Then f is called continuous at $a \in X$ if for each neighbourhood P of $f(a)$ the inverse image $f^{-1}(P)$ is a neighbourhood of a

4.5 Proposition A map $X \xrightarrow{f} Y$ between topological spaces is continuous if and only if it is continuous at every point of X.

Proof Let f be continuous. Given $a \in X$ and a neighbourhood $P \subset Y$ of $f(a)$ we choose an open $V \subset Y$ with $f(a) \in V \subset P$. Then

$$a \in f^{-1}\{f(a)\} \subset f^{-1}(V) \subset f^{-1}(P),$$

and as $f^{-1}(V) \subset X$ is open $f^{-1}(P)$ is a neighbourhood of a. Thus f is continuous at a.

Conversely assume that f is continuous at every point of X, and let $V \subset Y$ be an arbitrary open set: we must show that $U := f^{-1}(V) \subset X$ is open. To this end consider an arbitrary $a \in U$. Since V is a neighbourhood of $f(a)$ its inverse image U is a neighbourhood of a. By 4.2 we conclude that U is open.

When the notion of neighbourhood is applied it is often not necessary to consider all neighbourhoods of a given point but only sufficiently many, with emphasis upon those which are small. An auxiliary notion is used to make this idea precise.

4.6 Definition Let a be a point in X, a topological space. A basis of neighbourhoods of a is a set $\mathcal{B}$ of neighbourhoods of a such that for each neighbourhood N of a there exists a $B \in \mathcal{B}$ with $B \subset N$.

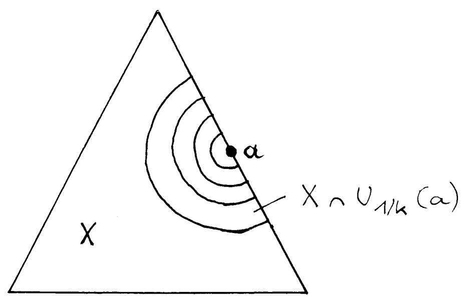
4.7 Example Let $X \subset \mathbb{R}^n$ be a subspace. Then

$$\mathcal{U} := \{X \cap U_{1/k}(a) \mid 0 < k \in \mathbb{N}\}$$

is a countable basis of neighbourhoods of $a \in X$, and

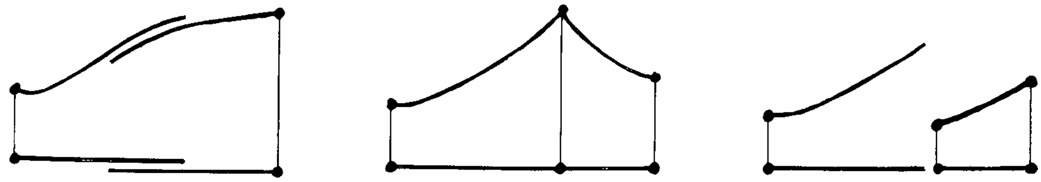
$$\mathcal{D} := \{X \cap D_{1/k}(a) \mid 0 < k \in \mathbb{N}\}$$

is another one.



Continuity of maps $f: X \longrightarrow Y$ at a point $a \in X$ is a good example that illustrates the usefulness of neighbourhood bases. If $\mathcal{B}$ is such a basis at $f(a)$ then in order to see that f is continuous at a it clearly suffices to check that $f^{-1}(B)$ is a neighbourhood of a for each $B \in \mathcal{B}$.

Real functions of one variable are often constructed by piecing together continuous functions defined on two intervals. It is a familiar fact that the resulting function is continuous if these intervals are both open, or both closed, but not in general.

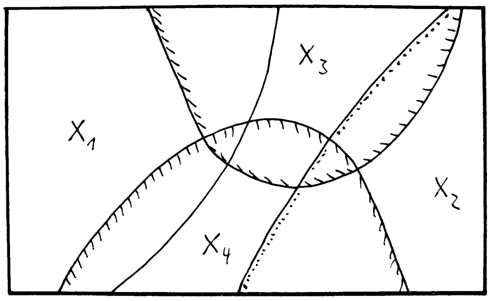


$[a, b'') \cup (b', c]$ $\qquad$ $[a, b] \cup [b, c]$ $\qquad$ $[a, b) \cup [b, c]$

We want to put this fact in the proper topological framework.

4.8 Definition Let X be a topological space. A covering of X is a family $(X_\lambda)_{\lambda\in\Lambda}$ of subsets $X_\lambda \subset X$ with

$$\bigcup_{\lambda\in\Lambda} X_\lambda = X.$$



$\Lambda = \{1,2,3,4\}$

A covering is finite or countable if Λ is finite or countable. Somewhat inconsistently we will call $(X_\lambda)_{\lambda\in\Lambda}$ an open covering if each X_λ is open (and later on we will proceed likewise with any other topological attributes the X_λ may have).

4.9 Proposition Let $(X_\lambda)_{\lambda\in\Lambda}$ be an open covering of the topological space X, and let $f\colon X \longrightarrow Y$ be a mapping into a further space Y. Then f is continuous if and only if for each $\lambda \in \Lambda$ the restriction

$$f_\lambda := f|X_\lambda\colon X_\lambda \longrightarrow Y$$

is continuous.

Proof If f is continuous then so is every restriction of f, by Proposition 3.6. Conversely assume that the f_λ are continuous, and let $V \subset Y$ be an open set. Then for each $\lambda \in \Lambda$ the set

$$X_\lambda \cap f^{-1}(V) = f_\lambda^{-1}(V)$$

is open in X_λ — and, by 3.5, also in X since X_λ itself is open in X. It follows that

$$f^{-1}(V) = \bigcup_\lambda X_\lambda \;\cap\; f^{-1}(V) = \bigcup_\lambda \big(X_\lambda \cap f^{-1}(V)\big)$$

is open in X. This proves continuity of f.

Closedness likewise is a topological notion:

4.10 Definition Let X be a topological space. A subset $F \subset X$ is closed if its complement $X \backslash F \subset X$ is an open set.

Don't let yourself even be tempted to think that openness and closedness were complementary notions: complements do play a role in the definition, but on the set theoretic, not the logical level! Thus typically "most" subsets of a given topological space X are neither open nor closed — for $X = \mathbb{R}$ think of the subsets $[0,1)$, of $\{1/k \mid 0 < k \in \mathbb{N}\}$, or $\mathbb{Q}$ to name but a few. On the other hand, we will see that usually only very special subsets of X are open and closed at the same time, with $\emptyset$ and X always among them.

It goes without saying that openness and closedness are completely equivalent in the sense that each of these notions determines the other. Thus the very axioms of topology could be re-cast in terms of closed rather than open sets, stipulating that the empty and the full set be closed, that the union of finitely many and the intersection of any number of closed sets be closed again. More important are the following facts.

- A mapping is continuous if and only if it pulls back closed sets to closed sets.

- If $S \subset X$ is a subspace, then a subset $F \subset S$ is relatively closed (that is, closed in S) if and only if it is the intersection of S with a closed subset of X.

The proofs are obvious, by taking complements.

4.11 Question Closed intervals in $\mathbb{R}$ and, more generally, closed balls in $\mathbb{R}^n$ are closed subsets indeed. Why?

While it would be rather awkward to express the notion of neighbourhood in terms of closed sets there is a useful construction dual to that of the interior introduced in 4.3.

4.12 Definition Let A be a subset of the topological space X. The intersection of all closed subsets of X that contain A is called the closure of A:

$$\overline{A} := \bigcap_{\substack{F \supset A \\ F \text{ closed}}} F$$

A is called dense (often dense in X) if $\overline{A} = X$.

Of course $\overline{A}$ is the smallest closed subset of X which contains A, and a point $a \in X$ belongs to $\overline{A}$ if and only if every neighbourhood of a intersects A. Interior and closure are related via $X \backslash A^\circ = \overline{X \backslash A}$. Do keep in mind that open and closed are relative notions, so for instance, $(D^n)^\circ = U^n$ is a true statement when referring to D^n as a subset of $\mathbb{R}^n$ but not if D^n is considered as a topological space in its own right. Likewise, $\overline{U^n} = U^n$, not $\overline{U^n} = D^n$ (of course not!) if the bar means closure in U^n.

4.13 Question Let $S \subset X$ be a subspace, and A a subset of S. Is $S \cap \overline{A}$ the same as the closure of A in S ?

We finally state the closed analogue to Proposition 4.9, which is proved in exactly same way:

4.14 Proposition Let $(X_\lambda)_{\lambda \in \Lambda}$ be a finite closed covering of the topological space X, and let $f\colon X \longrightarrow Y$ be a map. Then f is continuous if and only if

$$f_\lambda := f|X_\lambda \colon X_\lambda \longrightarrow Y$$

is continuous for each $\lambda \in \Lambda$.

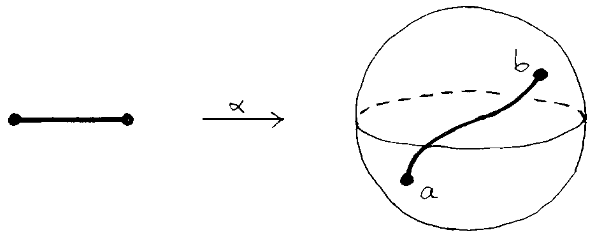
4.15 Question Why the finiteness condition?

5 Connected Spaces and Topological Sums

In a connected space any two points can be joined by a path: we have all the tools ready to make this idea precise.

5.1 Definition Let X be a topological space and let $a, b \in X$ be points. A path in X from a to b is a continuous map

$$\alpha\colon [0,1] \longrightarrow X$$

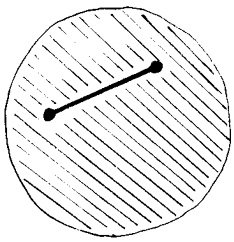


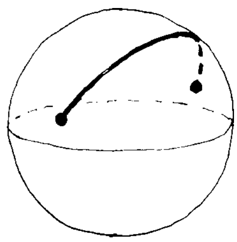
with $\alpha(0) = a$ and $\alpha(1) = b$. The space X is called connected if for any two points $a, b \in X$ there exists a path in X from a to b.

Remark In the literature this property is usually called *pathwise* connectedness in order to distinguish it from another related but slightly different version of this notion.

5.2 Examples (1) Every interval $X \subset \mathbb{R}$ clearly is connected. In fact the connected subspaces of the real line are *precisely* the intervals (including unbounded intervals and $\emptyset$): this is just a reformulation of the classical intermediate value theorem.

(2) Open and closed balls are connected (points may be joined by segments as in 3.8), and so are the spheres S^n for $n \neq 0$: join two given points along a great circle on S^n.

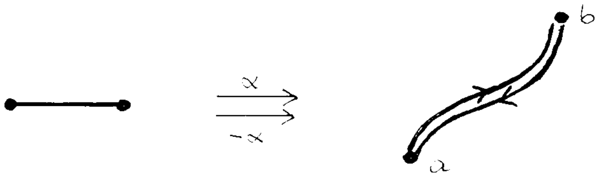




In order to study connectedness one should first know how to handle paths. The following lemma explains how paths can be reversed and composed.

5.3 Lemma and **Notation** Let X be a topological space, a, b, c points in X, and α, β two paths in X joining a to b and b to c, respectively. Then

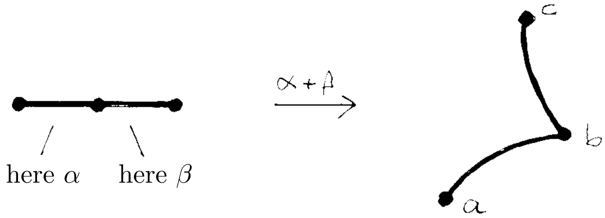
- the assignment $t \mapsto \alpha(1-t)$ defines a path $-\alpha\colon [0,1] \longrightarrow X$ from b to a, and



- the formula

$$[0,1] \ni t \longmapsto \begin{cases} \alpha\,(2t) & \text{if } t \le 1/2 \\ \beta\,(2t-1) & \text{if } t \ge 1/2 \end{cases}$$

gives a path $\alpha+\beta\colon [0,1] \longrightarrow X$ from a to c.



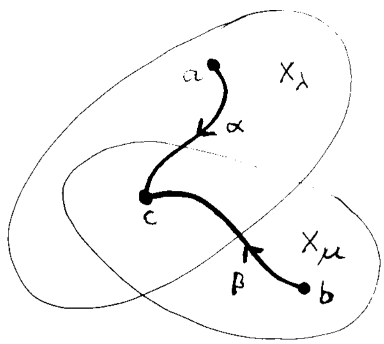
Proof The first statement is obvious while the second is a simple application of Proposition 4.14: the intervals $[0,1/2]$ and $[1/2,1]$ form a closed covering of $[0,1]$, the path $\alpha+\beta$ is well defined, and clearly continuous when restricted to one of the subintervals.

5.4 Corollary If the topological space X admits a connected covering $(X_\lambda)_{\lambda\in\Lambda}$ such that any two X_λ intersect:

$$X_\lambda \cap X_\mu \neq \emptyset \quad \text{for all } \lambda, \mu \in \Lambda$$

then X is connected.

Proof Let $a, b \in X$ be arbitrary and choose $\lambda, \mu \in X$ with $a \in X_\lambda$ and $b \in X_\mu$. By assumption we also find some $c \in X_\lambda \cap X_\mu$. Since X_λ and X_μ are connected there are paths $\alpha\colon [0,1] \longrightarrow X_\lambda$ from a to c, and $\beta\colon [0,1] \longrightarrow X_\mu$ from b to c. Reading α and β as paths in X we may form $\alpha + (-\beta)$ which is a path from a to b.



5.5 Proposition Let X and Y be topological spaces. If X is connected and if $f\colon X \longrightarrow Y$ continuous and surjective then Y is connected.

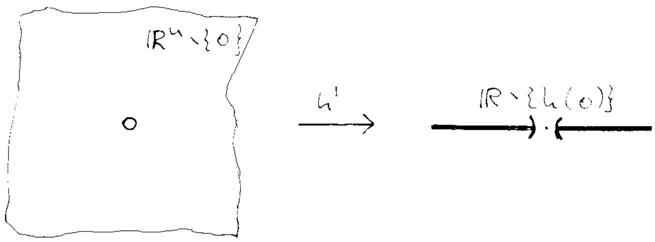
Proof Given arbitrary $a, b \in Y$ we choose $x, y \in X$ with $f(x) = a$ and $f(y) = b$. Since X is connected we find a path $\alpha\colon [0,1] \longrightarrow X$ with $\alpha(0) = x$ and $\alpha(1) = y$. The composition $f\circ\alpha$ is a path in Y which joins $c = f(x)$ to $d = f(y)$.

Thus continuous images of connected spaces are connected. In particular two homeomorphic spaces are either both connected or both disconnected, which shows that connectedness is an example of a topological invariant, albeit a very simple one. But at least it can be used to settle a small part of a question raised at the end of Section 2:

5.6 Application The real line $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for $n > 1$.

Proof Assume that there exists a homeomorphism $h:\mathbb{R}^n \longrightarrow \mathbb{R}$. Removing the origin from $\mathbb{R}^n$ we obtain another homeomorphism

$$\mathbb{R}^n\backslash\{0\} \xrightarrow{h'} \mathbb{R}\backslash\{h(0)\}$$



by restriction. Since $n > 1$ the space $\mathbb{R}^n\backslash\{0\}$ is connected: use the composition of two segments in order to avoid the origin if necessary. Thus we have arrived at a contradiction, for $\mathbb{R}\backslash\{h(0)\}$ clearly is disconnected.

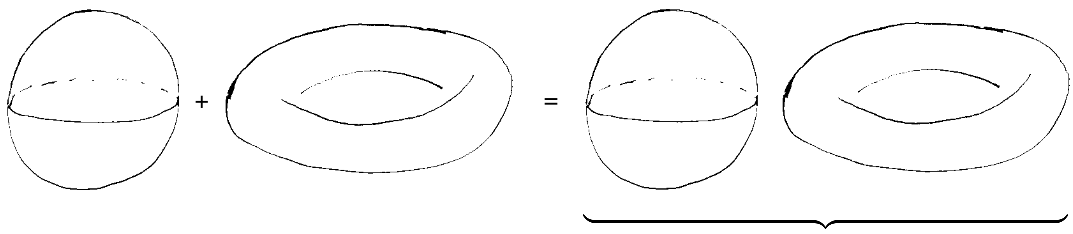
5.7 Definition Let X be an arbitary topological space. In view of 5.3 (and the existence of constant paths) the relation

$$x \sim y \quad :\Longleftrightarrow \quad \text{there exists a path in } X \text{ from } x \text{ to } y$$

is an equivalence relation on X. The equivalence classes are called the connected or path components of X.

Clearly the path components of X are the maximal non-empty connected subspaces of X, in particular a non-empty space is connected if and only if it has just one such component.

What is the simplest way of producing disconnected spaces? The obvious idea is to take two spaces X_1 and X_2 (or more) and make them the disjoint parts of a new topological space $X_1 + X_2$:



On the set theoretic level this amounts to forming what is called the disjoint union, and is defined, in greater generality, as follows:

5.8 Definition Let $(X_\lambda)_{\lambda\in\Lambda}$ be a family of sets. Their disjoint union or sum is the set

$$\sum_{\lambda\in\Lambda} X_\lambda := \bigcup_{\lambda\in\Lambda} \{\lambda\}\times X_\lambda \subset \Lambda \times \bigcup_{\lambda\in\Lambda} X_\lambda.$$

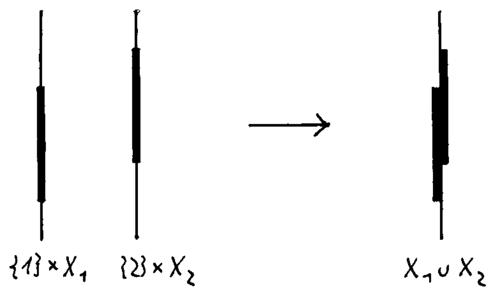
For small index sets like $\Lambda = \{1, 2\}$ one would rather write $X_1 + X_2$ of course.

The purpose of the factors $\{\lambda\}$ is to force disjointness of the summands X_λ. Note that for each λ one has a canonical injection

$$i_\lambda : X_\lambda \longrightarrow \sum_{\lambda\in\Lambda} X_\lambda$$

sending x to (λ, x). For the sake of convenience X_λ is usually considered to be a subset of the sum via i_λ. If the X_λ happen to be disjoint, does their sum give the same result as their union? Formally no but essentially, yes. For in that case the canonical surjective mapping

$$\begin{array}{ccc} \sum_{\lambda\in\Lambda} X_\lambda & \longrightarrow & \bigcup_{\lambda\in\Lambda} X_\lambda \\ \{\lambda\}\times X_\lambda \ni (\lambda, x) & \longmapsto & x \in X_\lambda \end{array}$$

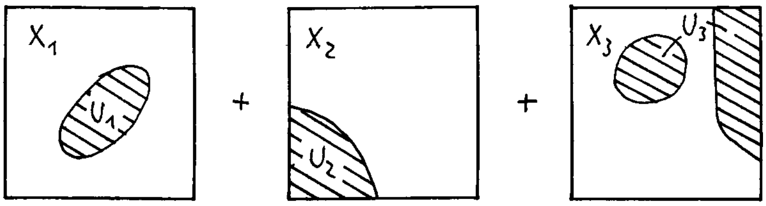


is bijective and one may use it to identify the two sets.

In any case, if each of the summands is a topological space then there is a natural way to put a topology on the disjoint union.

5.9 Definition Let $(X_\lambda)_{\lambda\in\Lambda}$ be a family of topological spaces. Then the sum topology $\mathcal{O}$ on $\sum_{\lambda\in\Lambda} X_\lambda$ is

$$\mathcal{O} = \Big\{ \sum_{\lambda\in\Lambda} U_\lambda \;\Big|\; U_\lambda \text{ open in } X_\lambda \text{ for each } \lambda\in\Lambda \Big\}.$$



Thus to build an open subset of the sum space $\sum_{\lambda\in\Lambda} X_\lambda$ one must pick one open $U_\lambda \subset X_\lambda$ for each λ and throw them together into the disjoint union. Alternatively one could say that a subset U of the sum space is open if and only if each intersection $X_\lambda \cap U$ is open in X_λ.

5.8 Question Writing $X_\lambda \cap U$ is an abuse of language. Which set does this really stand for?

5.9 Question Is there a difference between the topological spaces $[-1, 0) \cup [0, 1]$ and $[-1, 0) + [0, 1]$?

5.10 Proposition Let $(X_\lambda)_{\lambda\in\Lambda}$ be a family of topological spaces and let

$$i_\lambda : X_\lambda \longrightarrow \sum_{\lambda\in\Lambda} X_\lambda =: X$$

denote the inclusions as above. Then

- i_λ embeds X_λ as an open subspace of X, and

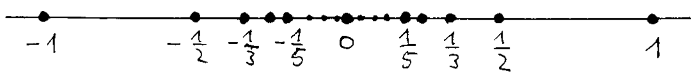
- a mapping $f: X \longrightarrow Y$ into a further topological space Y is continuous if and only if for each $\lambda \in \Lambda$ the restriction $f \circ i_\lambda: X_\lambda \longrightarrow Y$ is continuous.

Proof i_λ is continuous since it pulls back $V \subset X$ to $X_\lambda \cap V \subset X_\lambda$. On the other hand i_λ sends $U \subset X_\lambda$ to $\sum_{\kappa \in \Lambda} U_\kappa$ with $U_\lambda := U$ and $U_\kappa = \emptyset$ for $\kappa \neq \lambda$, and this set is open in X by definition. In particular X_λ itself is open in X, and i_λ a topological embedding. The second statement now is a special case of Proposition 4.9 as $(X_\lambda)_{\lambda \in \Lambda}$ is an open covering of X.

5.11 Question Why is X_λ also a closed subspace of X?

Let us return to the notions of connectedness and connected components. It should by now be clear that the connected components of the topological sum of a family of non-empty connected spaces are just the original spaces. One would naturally ask whether the converse holds: is every space X the topological sum of its path components? The answer is no in general as the example

$$X = \{0\} \cup \left\{ \frac{1}{k} \;\middle|\; 0 \neq k \in \mathbb{Z} \right\} \subset \mathbb{R}$$



shows: all components of X are one-point spaces with their unique topology, therefore the sum of them is a discrete space while X is not discrete.

From a geometric point of view the X studied in the counterexample is not a very natural space to consider, and in fact one obtains a positive answer if one is willing to exclude such spaces by imposing a mild condition on X, as follows.

5.12 Definition A topological space X is locally connected if for each point $a \in X$ the connected neighbourhoods form a basis of neighbourhoods of a.

Explicitly, the condition is that every neighbourhood of a contain a connected one. Let us note in passing that the definition may — later will — serve as a model for other local notions: imagine there were a topological property with the name *funny*, well-defined in the sense that each topological space either is or isn't funny. Then the notion of *local funniness* is defined automatically: X is locally funny if and only if for each $a \in X$ the funny neighbourhoods form a basis of neighbourhoods of a.

5.13 Proposition Let X be a topological space with connected components X_λ. If X is locally connected then the canonical bijective mapping

$$\sum_\lambda X_\lambda \xrightarrow{i} X$$

is a homeomorphism. (On the level of sets i is, essentially, the identity.)

Proof i is continuous because restricted to X_λ it is the inclusion $X_\lambda \hookrightarrow X$. In order to prove the proposition it remains to show that i sends open sets to open sets. Thus let $\sum_\lambda U_\lambda \subset \sum_\lambda X_\lambda$ be an open set in the sum topology, which means that each U_λ is open in X_λ. The local connectedness of X tells us that each component X_λ is open in X, so that U_λ is open in X too, by 3.5. Therefore $i(\sum_\lambda U_\lambda) = \bigcup_\lambda U_\lambda$ is open in X as was to be shown.

In view of the simplicity of the sum topology the previous proposition essentially reduces the study of locally connected spaces to that of spaces which are both locally and globally connected.

6 New Spaces from Old: Products

The cartesian product of a family of sets is one of the basics of set theory. It does not surprise that it carries a natural topology provided the factors are given as topological spaces. While the construction of this product topology works for arbitrary products (including those of uncountable families) for our purposes the product of just a finite number of spaces will do. As a finite product is easily reduced to that of two factors we will concentrate on this latter case.

An auxiliary notion will come in useful:

6.1 Definition Let $(X, \mathcal{O})$ be a topological space. A subset $\mathcal{B} \subset \mathcal{O}$ is a basis of $\mathcal{O}$ if each open set $U \in \mathcal{O}$ is a union of elements of $\mathcal{B}$.

For example, the standard topology 1.6(1) of $\mathbb{R}^n$ admits as a basis the set of all open balls $U_\delta(a)$, with arbitary $a \in \mathbb{R}^n$ and $\delta > 0$. Another basis of the same topology would consist of all open cubes

$$(a_1 - \delta, a_1 + \delta) \times \cdots \times (a_n - \delta, a_n + \delta) \subset \mathbb{R}^n,$$

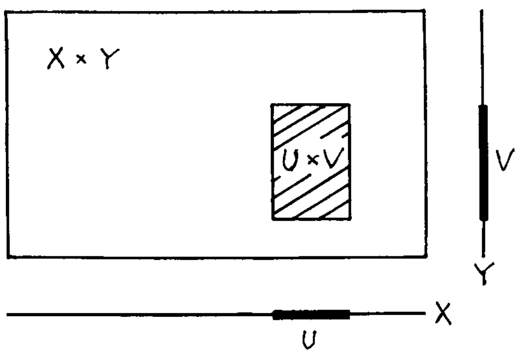
again for arbitary $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $\delta > 0$. While a given topology $\mathcal{O}$ allows many different basis conversely any of these, say $\mathcal{B}$, determines $\mathcal{O}$ as the set of all possible unions of members of $\mathcal{B}$:

$$\mathcal{O} = \Big\{ \bigcup_{U \in \mathcal{A}} U \;\Big|\; \mathcal{A} \subset \mathcal{B} \text{ any subset} \Big\}$$

6.2 Question Let $f: X \longrightarrow Y$ be a map between topological spaces, and let $\mathcal{B}$ be a basis for the topology of Y. Show that in order to prove that f is continuous it suffices to check that $f^{-1}(V)$ is open for all $V \in \mathcal{B}$.

6.3 Definition Let X and Y be topological spaces. Then

$$\mathcal{B} := \{U \times V \mid U \subset X \text{ open and } V \subset Y \text{ open}\}$$



is a basis of a topology on the cartesian product $X \times Y$. Topologized in this way, $X \times Y$ is called the product space of X and Y.

6.4 Question Verify the claim made in the definition.

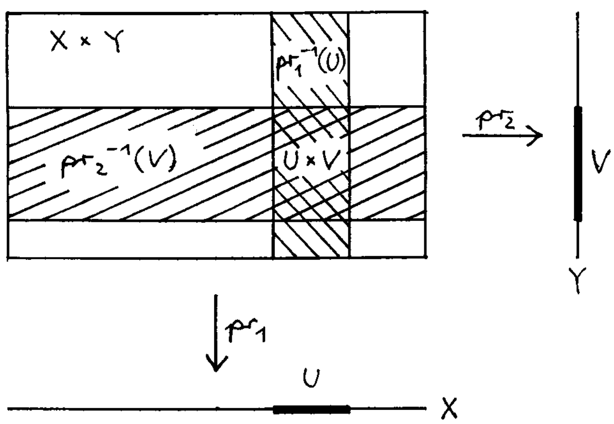
Of course, $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ may serve as a familiar example, with $\mathcal{B}$ the set of all open "rectangles". The following property of products likewise is well-known in that particular case: a vector valued function is continuous if and only if each of its components is continuous.

6.5 Proposition Let W, X, Y be topological spaces, and $f: W \longrightarrow X \times Y$ any map. Then

- the projections $\mathrm{pr}_1: X \times Y \longrightarrow X$ and $\mathrm{pr}_2: X \times Y \longrightarrow Y$ are continuous, and
- f is continuous if and only if both $\mathrm{pr}_1 \circ f: W \longrightarrow X$ and $\mathrm{pr}_2 \circ f: W \longrightarrow Y$ are continuous.

Proof If $U \subset X$ is open then $\mathrm{pr}_1^{-1}(U) = U \times Y$ is open in $X \times Y$, thus pr_1 and similarly pr_2 are continuous. Therefore in the second part, continuity of f implies that of $\mathrm{pr}_1 \circ f$ and $\mathrm{pr}_2 \circ f$. Conversely assume that these compositions are continuous, and let us prove continuity of f. As noted in 6.2 it is only open "rectangles" $U \times V$ that we must pull back by f. So let $U \subset X$ and $V \subset Y$ be open: then

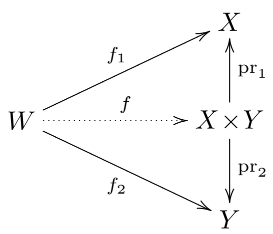
$$f^{-1}(U \times V) = (\mathrm{pr}_1 \circ f)^{-1}(U) \,\cap\, (\mathrm{pr}_2 \circ f)^{-1}(V)$$



is open in W, and we are done.

Note that the first part of the lemma, continuity of projections, is a formal consequence of the second (put $f := \mathrm{id}_{X \times Y}$). In fact the full lemma can be restated purely in terms of the category **Top**:

6.6 Proposition (**Top** version of 6.5) Let W, X, Y be topological spaces. Then for any given continuous maps $f_1: W \longrightarrow X$ and $f_2: W \longrightarrow Y$ there exists a unique continuous map $f: W \longrightarrow X \times Y$ such that $\mathrm{pr}_1 \circ f = f_1$ and $\mathrm{pr}_2 \circ f = f_2$.



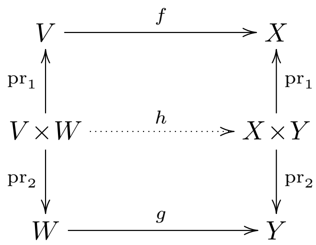
f is usually written $f = (f_1, f_2)$.

Proof The set theoretic part is obvious as f has to send $w \in W$ to the pair $\big(f_1(w), f_2(w)\big)$. The previous proposition takes care of the topological statement.

Remark The converse statement is trivial: a given morphism $f \in \mathbf{Top}(W, X \times Y)$ determines morphisms $f_1 = \mathrm{pr}_1 \circ f \in \mathbf{Top}(W, X)$ and $f_2 = \mathrm{pr}_2 \circ f \in \mathbf{Top}(W, Y)$.

In the framework of categories the so-called *universal property* described in 6.6 serves as a characterisation of products. It turns out that the familiar properties of direct products can be derived from 6.6 in a comletely formal way. The following is an example.

6.7 Construction and **Notation** Let $V \xrightarrow{f} X$ and $W \xrightarrow{g} Y$ be morphisms in **Top**. Then according to 6.6 there is a unique morphism $h = (f\circ\mathrm{pr}_1, g\circ\mathrm{pr}_2)$ that makes the diagram



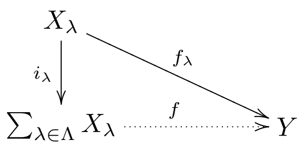
commutative. We denote this *product morphism* h by $f\times g\colon V\times W \longrightarrow X\times Y$.

In any category the notions of sum and product are dual to each other, and in the case of **Top** this is confirmed by the fact that Proposition 5.10 (at least the second part) can be stated in a way which is perfectly analogous to 6.6 but with all arrows reversed.

6.8 Proposition (**Top** version of 5.10) Let $(X_\lambda)_{\lambda\in\Lambda}$ be a family of topological spaces, and let

$$i_\lambda\colon X_\lambda \longrightarrow \sum_{\lambda\in\Lambda} X_\lambda$$

denote the inclusions as before. Then for any given family $(f_\lambda)_{\lambda\in\Lambda}$ of continuous maps $f_\lambda\colon X_\lambda \longrightarrow Y$ into a further space Y there exists a unique continuous map $f\colon \sum_{\lambda\in\Lambda} X_\lambda \longrightarrow Y$ such that the diagram



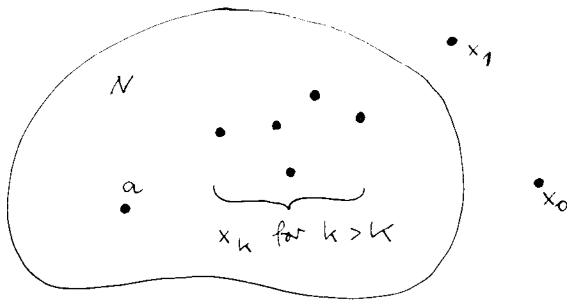
commutes for each $\lambda \in \Lambda$.

6.9 Question Explain what will be meant by a sum of morphisms $\sum_{\lambda\in\lambda} f_\lambda$.

7 Hausdorff Spaces

We now have ample evidence that whatever can be said about continuous functions makes sense in the general setting of topology. It may come as a surprise that the notion of limits — which is so closely related to continuity in the classical context — does not generalize in the same way. Take for example a sequence $(x_k)_{k=0}^{\infty}$ in a topological space X. It is straightforward enough that

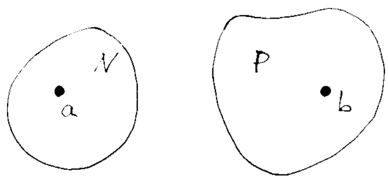
$$\lim_{k\to\infty} x_k = a$$



should mean that for every neighbourhood N of $a \in X$ there exists a $K \in \mathbb{N}$ such that $x_k \in N$ for all $k > K$. But in general this does not make sense unless carefully rephrased because (x_k) may converge to more than one limit. It certainly will do so if X is a lump space with more than one point: whatever the choice of a there is but one neighbourhood $N = X$ of a, and convergence to a involves no condition on the sequence at all!

It is, stricly speaking, a matter of taste whether to consider this kind of phenomenon a possibly interesting aspect of topology, but by the most common point of view it is a pathology. In order to exclude it one prefers to work with topological spaces that have sufficiently many open sets in order to separate distinct points.

7.1 Definition A Hausdorff space is a topological space X with the following property: if $a \neq b$ are distinct points of X then there exist neighbourhoods N of a and P of b such that $N \cap P = \emptyset$.



7.2 Question If *neighbourhoods* were replaced by *open neighbourhoods*, would that make a difference?

$\mathbb{R}^n$ is a Hausdorff space: use $U_{\delta/2}(a)$ and $U_{\delta/2}(b)$ with $\delta = |a-b|$ to separate a and b. From this observation we obtain a host of further examples since the Hausdorff property is easily seen to carry over to subspaces, products, and sums.

In a Hausdorff space X limits clearly are unique: staying with the notation of the definition, a and b cannot both be limits of the same sequence (x_k) since no x_k belongs to N and to P.

7.3 Question If a is point in a topological space X, is it always true that $\{a\} \subset X$ is a closed subset? Is it true if X is a Hausdorff space?

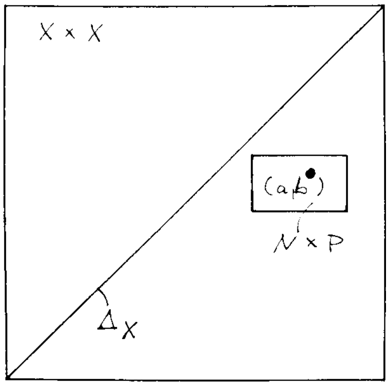
There is a another nice and useful way to state the Hausdorff property.

7.4 Proposition Let X be a topological space. X is a Hausdorff space if and only if the diagonal

$$\Delta_X := \{(x,x) \mid x \in X\} \subset X \times X$$

is a closed subset of $X \times X$.

Proof Assuming first that X has the Hausdorff property we will show that the complement $(X \times X) \setminus \Delta_X$ is open in $X \times X$. Let $(a,b) \in (X \times X) \setminus \Delta_X$ be arbitrary. Since $a \neq b$ we can pick disjoint open neighbourhoods N of a and P of b. Their product $N \times P$ is completely contained in $(X \times X) \setminus \Delta_X$ and is an open neighbourhood of (a,b) in $X \times X$. Therefore $(X \times X) \setminus \Delta_X$ is open, and Δ_X closed in $X \times X$.



Conversely assume this and let $a \neq b$ be distinct points in X. Then (a,b) belongs to the open set $(X \times X) \setminus \Delta_X$ and we find open $N, P \subset X$ with

$$(a,b) \in N \times P \subset (X \times X) \setminus \Delta_X$$

because products of this type form a basis of the topology on $X \times X$. In view of $N \cap P = \emptyset$ we have thereby established that X is a Hausdorff space.

7.5 Corollary Let Y be a Hausdorff and X an arbitrary topological space, $X \xrightarrow{f,g} Y$ two continuous mappings. Then

$$\{x \in X \mid f(x) = g(x)\}$$

is a closed subspace of X.

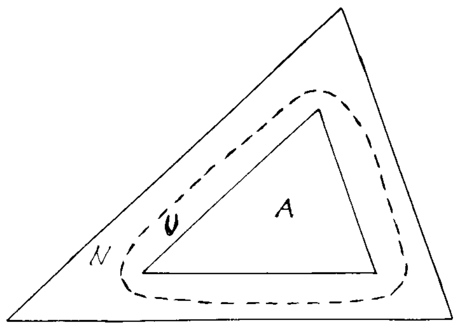
Proof It is the inverse image of the diagonal Δ_Y under the continuous mapping $(f,g)\colon X \longrightarrow Y \times Y$.

Remarks The Hausdorff property seems so natural, and its possible failure so counterintuitive that one is tempted to include it among the set of axioms for a "reasonable" topological space — as indeed Hausdorff did when he laid the foundations of topology in 1914. But this has turned out to be a liablity rather than an asset, and in modern presentations the Hausdorff axiom is relegated to the status of a topological property a space may or may not have. — Clearly, continuous mappings take convergent sequences to convergent sequences but the Hausdorff property is not by itself sufficient to ensure the converse. This continuity test by sequences, familiar from real analysis, is valid though if additionally the point in question admits a countable basis of neighbourhoods.

8 Normal Spaces

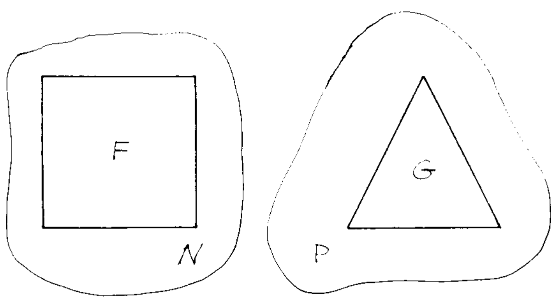
It is often necessary to separate not only distinct points but also disjoint closed sets. This is not always possible even in a Hausdorff space, and thus a narrower class of topological spaces is singled out. The definition is in terms of a straightforward generalisation of 4.1.

8.1 Definition Let A be a subset of a topological space X. A subset $N \subset X$ is a neighbourhood of A if there exists an open set U with $A \subset U \subset N$, that is if $A \subset N^\circ$.



8.2 Definition A topological space X is normal if

- it is a Hausdorff space, and
- for any two closed subsets $F, G \subset X$ with $F \cap G = \emptyset$ there exist neighbourhoods N of F and P of G such that $N \cap P = \emptyset$.



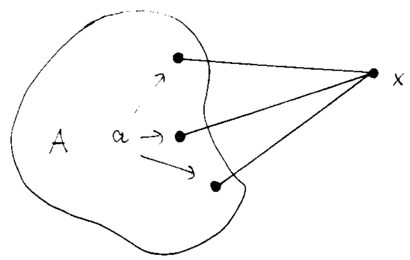
8.3 Question Why is the first condition not implied by the second?

There exist examples of Hausdorff spaces that are not normal but at least every subspace of $\mathbb{R}^n$ is normal:

8.4 Proposition Let $X \subset \mathbb{R}^n$ be a arbitrary. Then X is a normal space.

Proof Every non-empty subset $A \subset \mathbb{R}^n$ gives rise to a function

$$d_A: \mathbb{R}^n \longrightarrow [0, \infty); \quad d_A(x) := \inf \{ |x-a| \mid a \in A \}$$



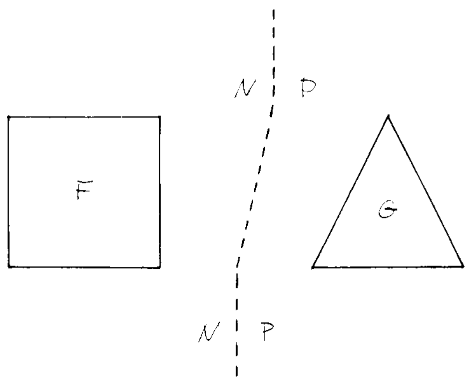
measuring distance from A. The triangle inequality shows that this function is continuous. It clearly vanishes on A and therefore on $\overline{A}$, by 7.5. In fact one has $d_A^{-1}\{0\} = \overline{A}$ precisely: in case $x \notin \overline{A}$ there exists a $\delta > 0$ with $A \cap U_\delta = \emptyset$ and thus $d_A(x) \geq \delta$.

Let now $X \subset \mathbb{R}^n$ be a subspace, and let $F, G \subset X$ be disjoint and closed in X. Then $X \cap \overline{F} = F$ where the bar means closure in $\mathbb{R}^n$, so that d_F is positive on G and vice versa. Therefore the function

$$\varphi: X \longrightarrow \mathbb{R}; \quad \varphi(x) := d_F(x) - d_G(x)$$

is negative on F, positive on G, and

$$N := \varphi^{-1}(-\infty, 0) \quad \text{and} \quad P := \varphi^{-1}(0, \infty)$$



are separating open neighbourhoods of F and G.

Rather surprisingly, the possibility of separating closed sets by a continuous function rather than by neighbourhoods is not particular to subspaces of $\mathbb{R}^n$ and similar examples, as the following important result shows.

8.5 Urysohn's Theorem Let X be a Hausdorff space. Then X is normal if and only if for any two closed subsets $F, G \subset X$ with $F \cap G = \emptyset$ there exists a continuous function $\varphi: X \longrightarrow [0, 1]$ such that $F \subset \varphi^{-1}\{0\}$ and $G \subset \varphi^{-1}\{1\}$.

Proof One direction is trivial: if φ is given then $N := \varphi^{-1}[0, \frac{1}{2})$ and $P := \varphi^{-1}(\frac{1}{2}, 1]$ are open subsets of X that separate F and G.

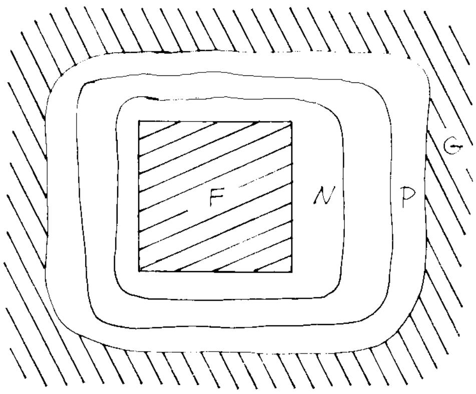
So the point of the theorem is the converse. What are the input data for the proof? A normal space X, and disjoint closed sets $F, G \subset X$. What do we look for? A continuous function $\varphi: X \longrightarrow [0, 1]$ which vanishes on F and is identically one on G. If X were a subspace of $\mathbb{R}^n$ the function

$$\varphi: X \longrightarrow \mathbb{R}; \quad \varphi(x) := \frac{1}{2}\left(1 + \frac{d_F(x) - d_G(x)}{d_F(x) + d_G(x)}\right)$$

would do nicely. But in general this approach leads nowhere, for as you now will realise it is quite unclear even how to construct *any* non-constant continuous real-valued function on X at all, let alone one with the prescribed properties. Nevertheless it can be done, and the proof is really clever and beautiful. It works by applying the defining property 8.2 not just once to F and G but likewise to many other pairs of closed subsets of X. For the purpose it is convenient to first reformulate normality as follows.

8.6 Lemma Let X be normal, and $F \subset X$ closed. Then every neighbourhood of F contains a closed neighbourhood of F.

Proof Let a neighbourhood $V \supset F$ be given: we may assume that V is open. Thus $G := X \setminus V$ is closed and disjoint from F. Since X is normal there exist open neighbourhoods N of F and P of G with $N \cap P = \emptyset$. The complement $X \setminus P$ is closed, contained in $X \setminus G = V$, and is a neighbourhood of F because it contains the open set N.

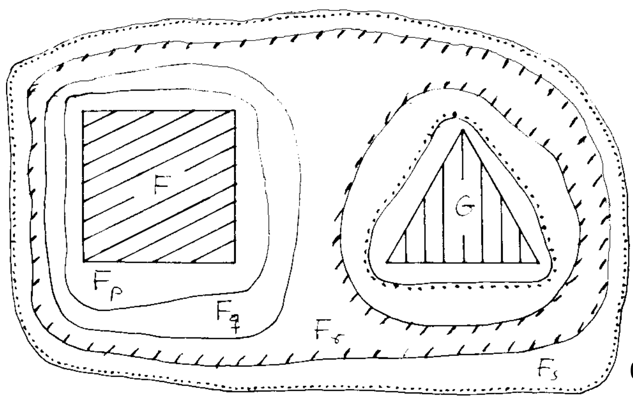


Proof of 8.5 (continuation) Let

$$Q := \{q \in \mathbb{Q} \,|\, 2^k q \in \mathbb{Z} \text{ for some } k \in \mathbb{N}\}$$

be the set of all rationals with denominator a power of two. We will construct a family $(F_q)_{q \in Q}$ closed subsets $F_q \subset X$ with the following properties.

- $F_q = \emptyset$ for $Q \ni q < 0$, and $F \subset F_0$
- $F_1 \subset X \setminus G$ and $F_q = X$ for $1 < q \in Q$
- $F_q \subset F_r^\circ$ for all $q < r$



$0 < p < q < r < s < 1$

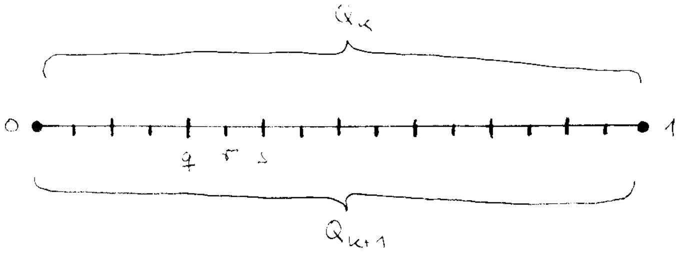
We read $F_q = \emptyset$ for $q < 0$, and $F_q = X$ for $1 < q \in Q$ as the definition of F_q for these q, and put $F_0 := F$. The open set $X \setminus G$ is a neighbourhood of F, and according to Lemma 8.6 we can choose F_1 as some closed neighbourhood of F contained in $X \setminus G$. Then the first two conditions are satisfied, and the third one too, as far as F_q and F_r have been defined.

In order to complete the definition of the family (F_q) it remains to construct F_q for $q \in [0,1] \cap Q$, and this will be done by induction. Put

$$Q_k := \big\{q \in [0,1] \,\big|\, 2^k q \in \mathbb{N}\big\}$$

so that $[0,1] \cap Q = \bigcup_{k=0}^{\infty} Q_k$. The sets F_q for $q \in \{0,1\} = Q_0$ already have been defined. Assuming inductively that closed sets F_q have been defined for all $q \in Q_k$, and satisfy the third condition above we extend the definition to Q_{k+1}. Thus let $r \in Q_{k+1} \setminus Q_k$, and define $q \in Q_k$ by

$$q < r < q + \frac{1}{2^k} =: s.$$



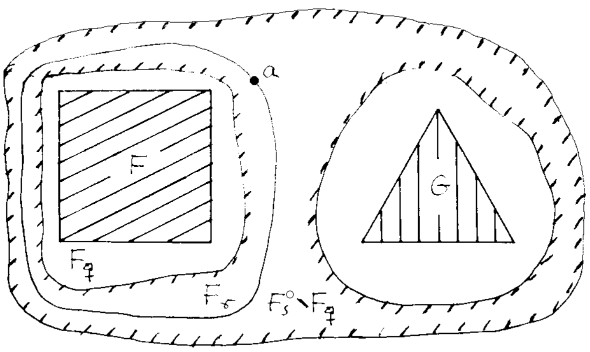
Since by inductive assumption F_s° is a neighbourhood of F_q it contains a closed neighbourhood F_r of F_q by Lemma 8.6. Then $F_q \subset F_r^\circ$ and $F_r \subset F_s^\circ$ hold, and thereby the construction of F_r is achieved.

We define $\varphi: X \longrightarrow [0,1]$ by

$$\varphi(x) := \inf \{q \in Q \mid x \in F_q\}$$

which makes sense since $F_q = X$ for $q > 1$ and $F_q = \emptyset$ for $q < 0$. In view of $F \subset F_0$ it is clear that $\varphi|F$ vanishes identically. Consider some $x \in G$. As $F_1 \subset X \setminus G$ this implies $x \notin F_1$, and therefore $\varphi(x) \geq 1$. Thus φ is identically one on G.

It remains to see why φ is continuous at each point $a \in X$. Since Q is dense in $\mathbb{R}$ we need only show that for arbitrary $q, s \in Q$ with $q < \varphi(a) < s$ the inverse image $\varphi^{-1}[q,s]$ is a neighbourhood of a. Pick some $r \in Q$ with $\varphi(a) < r < s$. From the definition of $\varphi(a)$ we know that $a \in F_r$ hence $a \in F_s^\circ$ but $a \notin F_q$. Therefore the open set $U := F_s^\circ \setminus F_q$ is a neighbourhood of a. Again from the definition of φ it follows that $\varphi \leq s$ on F_s and that $\varphi \geq q$ on $X \setminus F_q$. Therefore φ maps U into $[q,s]$ or, equivalently, $U \subset \varphi^{-1}[q,s]$. This completes the proof.



The theme underlying Urysohn's theorem, the construction of continuous real-valued functions on normal spaces, can be developped much further. Let me quote just one important result in this direction: the proof, which is pretty, can be found in any standard text on general topology.

8.7 Tietze's Extension Theorem Let X be a Hausdorff space. Then X is normal if and only if for every closed subset $F \subset X$ and every continuous function $\varphi: F \longrightarrow \mathbb{R}$ there exists a continuous function $\Phi: X \longrightarrow \mathbb{R}$ with $\Phi|F = \varphi$.

Note that Urysohn's theorem deals with a special case of this extension problem, that of extending the continuous function

$$F \cup G \longrightarrow \mathbb{R}; \quad x \mapsto \begin{cases} 0 & \text{if } x \in F \\ 1 & \text{if } x \in G \end{cases}$$

to all of X.

9 Compact Spaces

Compactness is topologists' notion of finiteness. Not literally speaking, for finite topological spaces are quite uninteresting: a finite Hausdorff space necessarily is discrete. But many properties of compact topological spaces parallel those of finite sets. The definition of this very important and powerful notion is based on that of coverings in 4.8. If $(X_\lambda)_{\lambda\in\Lambda}$ is such a covering of a space X then every subset $\Lambda' \subset \Lambda$ defines, of course, a restricted family $(X_\lambda)_{\lambda\in\Lambda'}$ which will be called a subcovering of $(X_\lambda)_{\lambda\in\Lambda}$ in case

$$\bigcup_{\lambda\in\Lambda'} X_\lambda = X$$

still holds.

9.1 Definition A topological space X is compact if every open covering of X contains a finite subcovering.

From your analysis course you will be familiar with the notion of compactness as such but not necessarily with this particular definition. In that case you will find that there are no immediate examples of topological spaces that satisfy 9.1, beyond finite and pathological ones like lump spaces. So let us prove here at least that the intervals $[a, b]$ — which, of course, are known as the compact ones — are indeed compact subspaces of the real line.

9.2 Question Why is it sufficient to prove this for the unit interval $[0, 1]$?

9.3 Proposition $[0, 1]$ is compact.

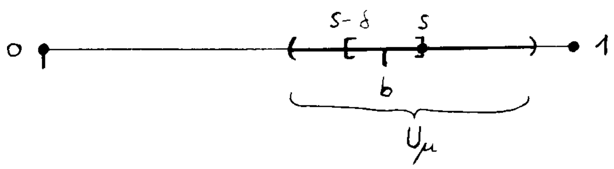
Proof Let $(U_\lambda)_{\lambda\in\Lambda}$ be an open covering of $[0, 1]$. The fact that $0 \in U_\lambda$ for at least one λ shows that

$$s := \sup\left\{b \in [0,1] \,\middle|\, \text{there exists a finite } \Lambda' \subset \Lambda \text{ with } [0,b] \subset \bigcup_{\lambda\in\Lambda'} U_\lambda\right\}$$

is not only defined but positive: $0 < s \leq 1$. We will show that the infimum is in fact a maximum.

To this end we pick a $\mu \in \Lambda$ with $s \in U_\mu$. Being a neighbourhood of s the set U_μ contains an interval $[s-\delta, s]$ with $0 < \delta < s$. By definition of s there exists some $b \in (s-\delta, s]$ such that $[0, b]$ is contained in a union of finitely many covering sets U_λ:

$$[0, b] \subset \bigcup_{\lambda\in\Lambda'} U_\lambda$$



Therefore

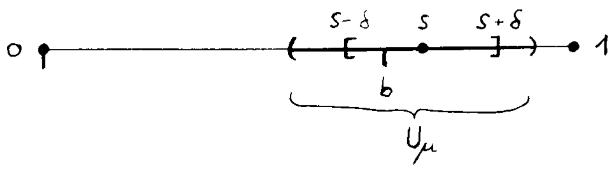
$$[0, s] = [0, b] \cup [s-\delta, s] \subset \Big(\bigcup_{\lambda\in\Lambda'} U_\lambda\Big) \cup U_\mu$$

also is contained in the union of finitely many covering sets.

This in turn implies that $s = 1$. For if s were smaller than 1 then the open set U_μ would, for sufficiently small $\delta > 0$ contain the interval $[s, s+\delta]$. But then even

$$[0, s+\delta] \subset \Big(\bigcup_{\lambda\in\Lambda'} U_\lambda\Big) \cup U_\mu$$

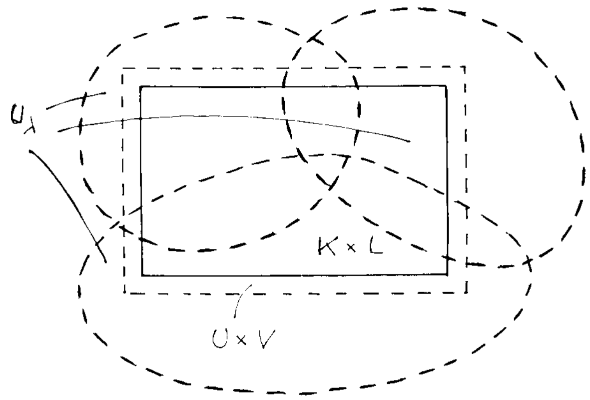
would be contained in a finite union of covering sets — a contradiction to the definition of s.



We will recognize many more examples of compact spaces once a few formal properties of compactness are established. Most of them can be conveniently derived from a single key lemma, which we have copied from [tom Dieck].

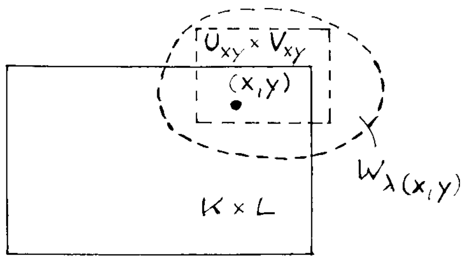
9.4 Key Lemma Let X, Y be topological spaces and let $K \subset X$ and $L \subset Y$ be compact subspaces. If $(W_\lambda)_{\lambda\in\Lambda}$ is a family of open subsets $W_\lambda \subset X\times Y$ with $K\times L \subset \bigcup_{\lambda\in\Lambda} W_\lambda$ then there exist open sets $U \subset X$ and $V \subset Y$, and a finite subset $\Lambda' \subset \Lambda$ such that

$$K \times L \subset U \times V \subset \bigcup_{\lambda\in\Lambda'} W_\lambda.$$



Proof For each point $(x, y) \in K\times L$ there exists an index $\lambda(x, y)$ with $(x, y) \in W_{\lambda(x,y)}$, and by definition of the product topology we may pick open sets $U_{xy} \subset X$ and $V_{xy} \subset Y$ with

$$(x, y) \in U_{xy} \times V_{xy} \subset W_{\lambda(x,y)}.$$



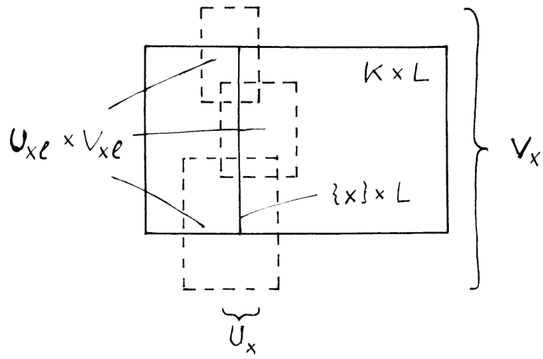
Now fix an arbitrary $x \in K$. The sets $L \cap V_{xl}$ for $l \in L$ form an open covering of the compact space L, and we choose a finite subcovering

$$(L \cap V_{xl})_{l\in L_x} \quad \text{with a finite index set } L_x \subset L \text{ depending on } x.$$

The corresponding set

$$U_x := \bigcap_{l\in L_x} U_{xl} \subset X$$

is open and contains x. Therefore the family $(K\cap U_k)_{k\in K}$ is an open covering of the compact space X: let $K'\subset K$ be a finite set so that $(K\cap U_k)_{k\in K'}$ is a subcovering.



We now put $V_x := \bigcup_{l\in L_x} V_{xl}$ and claim that the conclusion of the lemma holds with

$$U := \bigcup_{k\in K'} U_k\,,\ \text{with } V := \bigcap_{k\in K'} V_k\,,\ \text{and } \Lambda' := \big\{\lambda(k,l)\,\big|\, k\in K',\, l\in L_k\big\}\subset\Lambda.$$

By construction we have $K\subset U$ and $L\subset V_k$ for all $k\in K'$, hence $L\subset V$: thus $K\times L\subset U\times V$, which is the first half of the claim. As to the second consider any $(x,y)\in U\times V$. We find some $k\in K'$ with $x\in U_k$, and further an $l\in L_k$ such that $y\in V_{kl}$. Then

$$(x,y)\in U_k\times V_{kl}\subset U_{kl}\times V_{kl}\subset W_{\lambda(k,l)}$$

and this completes the proof of the claim.

9.5 Proposition Every closed subspace of a compact space is compact. Every compact subspace of a Hausdorff space is closed.

Proof Assume X compact and $F\subset X$ closed. For any given open covering $(U_\lambda)_{\lambda\in\Lambda}$ of F we choose open subsets $V_\lambda\subset X$ with $U_\lambda = F\cap V_\lambda$. Adding to Λ an extra index $\circ\notin\Lambda$ and putting $V_\circ = X\backslash F$ we obtain an open covering (V_λ) of X, indexed by the set $\{\circ\}+\Lambda$. Since X is compact there is a finite subcovering. After throwing away the index $\circ$ the corresponding subset of Λ defines a finite subfamily of (U_λ) which covers F. This proves that F is compact.

To prove the second part let X be a Hausdorff space, $F\subset X$ a compact subspace, and $x\in X\backslash F$ a point. Applying the key lemma to $\{x\}\times F\subset X\times X$ and the family consisting of the single open set $(X\times X)\backslash\Delta_X\subset X\times X$, we obtain open sets U,V with

$$\{x\}\times F\subset U\times V\subset (X\times X)\backslash\Delta_X.$$

In particular $x\in U\subset X\backslash F$, so $X\backslash F$ is open and F is closed in X.

9.6 Proposition The direct product of two compact topological spaces is compact.

Proof Read the key lemma with $K=X$ and $L=Y$.

Remark Of course the result extends to finite products of compact spaces. Harder to prove is the fact that the product of an arbitrary family of compact spaces is compact. This is known as Tychonov's theorem and has applications in functional analysis.

9.7 Question Explain the following: the topological sum of finitely many compact spaces is compact but the sum of an infinite family of non-empty spaces is never compact.

Using the previous propositions it is easy to give a handy characterisation of the compact subsets of $\mathbb{R}^n$.

9.7 Theorem of **Heine** and **Borel** A subspace $K \subset \mathbb{R}^n$ is compact if and only if it is bounded and closed as a subset of $\mathbb{R}^n$.

Proof Let $K \subset \mathbb{R}^n$ be compact. The sequence $\big(K \cap U_r(0)\big)_{r\in\mathbb{N}}$ is an open covering of K. It contains a finite subcovering: this comes down to the statement that K is contained in $U_r(0)$ for some $r \in \mathbb{N}$, and shows that K is bounded. On the other hand $K \subset \mathbb{R}^n$ is closed by Proposition 9.5.

Conversely assume that K is bounded and closed. Boundedness means that K is contained in some cube $[a, b]^n \subset \mathbb{R}^n$. By 9.3 and 9.6 this cube is a compact space, so as a closed subspace K likewise is compact, by the other part of Proposition 9.5.

Note that closedness of a subset $K \subset \mathbb{R}^n$ is a relative topological notion in the sense that it refers to the topology of the ambient $\mathbb{R}^n$. Boundedness $K \subset \mathbb{R}^n$ is not a topological notion at all. Nevertheless the conjunction of the two is equivalent to compactness, which, being a topological property of the subspace K involves no further reference to $\mathbb{R}^n$. This explains, for instance, the well-known fact that a continuous real function of one variable need not preserve closedness or boundedness of intervals but always takes compact intervals to compact ones.

The corresponding general topological statement does not surprise:

9.8 Proposition Let X and Y be topological spaces. If X is compact and if $f: X \longrightarrow Y$ continuous and surjective then Y is compact.

Proof Let $(V_\lambda)_{\lambda\in\Lambda}$ be an open covering of Y. Then $\big(f^{-1}(V_\lambda)\big)_{\lambda\in\Lambda}$ is an open covering of X so it contains a finite subcovering given by some finite $\Lambda' \subset \Lambda$. Since f is surjective one has

$$Y = f(X) = f\Big(\bigcup_{\lambda\in\Lambda'} f^{-1}(V_\lambda)\Big) = \bigcup_{\lambda\in\Lambda'} f\big(f^{-1}(V_\lambda)\big) = \bigcup_{\lambda\in\Lambda'} V_\lambda.$$

Therefore $(V_\lambda)_{\lambda\in\Lambda'}$ is a finite subcovering of the given one.

Compactness is a very strong topological property as we will see in many places. For the moment let me just record two applications.

9.9 Proposition Let $f: X \longrightarrow Y$ be a continuous bijection. If X is a compact, and Y a Hausdorff space then f is a homeomorphism.

Proof We show that $f^{-1}: Y \longrightarrow X$ pulls back closed subsets of X to closed subsets of Y. Thus let $F \subset X$ be closed. By 9.5, F is compact. Then $(f^{-1})^{-1}(F) = f(F)$ also is compact in view of 9.8, and is closed in Y by another application of 9.5.

9.10 Proposition In a Hausdorff space any two disjoint compact subsets can be separated by neighbourhoods. In particular every compact Hausdorff space is normal.

Proof This is another application of the key lemma 9.4. Let X be a Hausdorff space, and $K, L \subset X$ compact subsets with $K \cap L = \emptyset$. Then the open subset $(X\times X)\setminus\Delta_X \subset X\times X$ contains $K\times L$, and by 9.4 there are open sets N and P with

$$K \times L \subset N \times P \subset (X\times X)\setminus\Delta_X.$$

Thus $K \subset N$, $L \subset P$, and $N \cap P = \emptyset$.

If X is a compact Hausdorff space then all closed subsets are compact and the second part of the proposition becomes a special case of the first.

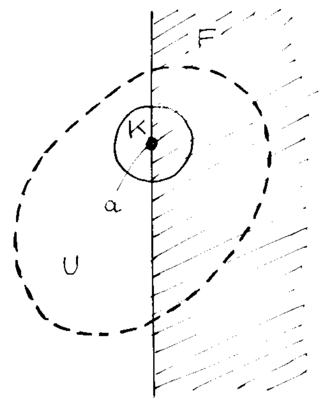
A topological space which is not compact may still be locally compact — recall the definition of local topological properties following 5.12. An important example of a locally compact space is $\mathbb{R}^n$ (for the closed balls $D_r(a)$ are compact). In practice local compactness often is established using one of the following simple facts.

9.11 Proposition Let X a Hausdorff space. If each point of X admits at least one compact neighbourhood then X is locally compact.

Proof Let $a \in X$ be a point, and $N \subset X$ a neighbourhood of a. We must show that N contains a compact neighbourhood of a. To this end choose a compact neighbourhood K of a and note that neighbourhoods of a in K are the same as neighbourhoods of a in X which are contained in K. By Proposition 9.10, K is normal, and applying Lemma 8.6 to $\{a\}$ (as closed subset of K) and $K \cap N$ (as neighbourhood of $\{a\}$) we find a neighbourhood of a which is contained in $K \cap N$ and is closed in K, hence compact.

9.12 Proposition Let X be a locally compact Hausdorff space. If a subspace of X can be written as the intersection of an open and a closed subset of X then it is locally compact.

Proof Let $U \subset X$ be an open and $F \subset X$ a closed subset, and consider a point $a \in S := U \cap F$. Thus U is a neighbourhood of a in X, and we can choose some compact neighbourhood K of a contained in U. Then $S \cap K = F \cap K$ is a compact neighbourhood of a in S, and in view of 9.10 this suffices to ensure that S is locally compact.



10 New Spaces from Old: Quotients

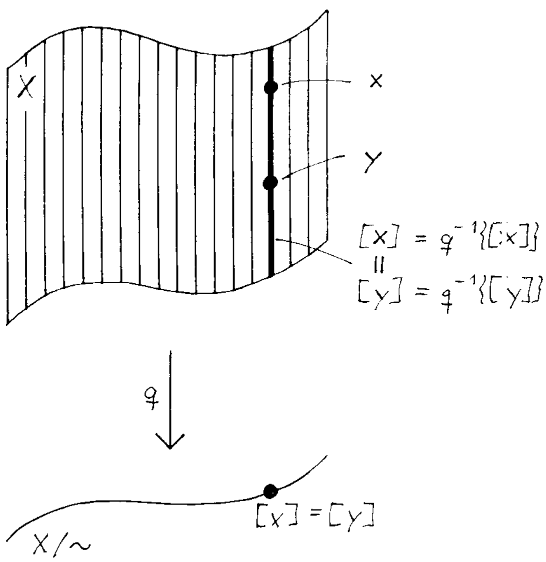
Beyond product, sum, and subspace topologies there is a fourth way of constructing new topological spaces, and it is by far the most interesting. Recall the following facts from set theory: every equivalence relation $\sim$ on a set X gives rise to a partition of X into the equivalence classes. The class of an element $x \in X$ is usually written

$$[x] = \{y \in X \mid x \sim y\}$$

and the set of all equivalence classes is called the *quotient set*

$$X/\sim \; = \big\{[x] \,\big|\, x \in X\big\}.$$

There seems to be no standard notation for the *quotient mapping* $X \longrightarrow X/\sim$ that sends x to $[x]$. We will preferably denote it by q and draw it vertically whenever it occurs in a diagram. Its fibres are, of course, just the equivalence classes: $q^{-1}\{[x]\} = [x] \subset X$.



10.1 Definition Let X be a topological space, and $\sim$ an equivalence relation on X. Then the quotient or identification topology on $X/\sim$ is defined by:

$$V \subset X/\sim \text{ is open} \quad :\Longleftrightarrow \quad q^{-1}(V) \text{ is open in } X$$

The resulting topological space is called the quotient (space) of X by the equivalence relation $\sim$.

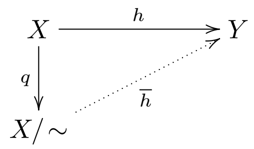
I have described in Section 3 how the notion of subobject, which sometimes is too narrow, can be extended to that of embeddings, which is equivalent but more flexible. With quotient objects one encounters quite a similar situation.

10.2 Definition Let X and Y be topological spaces. A surjective map $h\colon X \longrightarrow Y$ is an identification if

$$V \text{ is open in } Y \quad \Longleftrightarrow \quad h^{-1}(V) \text{ is open in } X$$

holds for all $V \subset Y$.

While it is clear that identifications are continuous it is not at once obvious that they are essentially the same as quotient mappings. Nevertheless it is true. Note first that every surjective map $h: X \longrightarrow Y$ determines an equivalence relation on X: the equivalence classes are just the fibres of h. Forming the quotient set $X/\sim$ with respect to that relation we clearly obtain a well-defined bijection $\overline{h}: X/\sim \longrightarrow Y$ sending the class $[x]$ to $h(x)$:

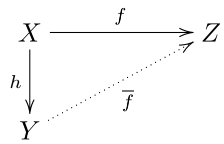


So far this is mere set theory. Assume now that X and Y carry a topology. Then Definition 10.2 may be rephrased by saying that h is an identification mapping if and only if the bijective map $\overline{h}$ is a homeomorphism, for by definition one has $h = \overline{h} \circ q$ hence

$$q^{-1}(\,\overline{h}\,)^{-1}(V) = h^{-1}(V) \quad \text{for any subset } V \subset Y.$$

The construction of continuous mappings defined on a topological quotient space is a purely formal affair, and relies on the following important proposition.

10.3 Proposition Let $h: X \longrightarrow Y$ be an identification map, and let $f: X \longrightarrow Z$ be a continuous mapping which is constant on each fibre of h. Then there is a uniquely determined continuous map $\overline{f}: Y \longrightarrow Z$ such that the diagram



commutes.

Proof $\overline{f}$ must send $h(x) \in Y$ to $f(x) \in Z$, and this rule does define a map from Y to Z because h is surjective, and f constant on the fibres of h. Continuity of $\overline{f}$ follows as before, from the equality $h^{-1}(\,\overline{f}\,)^{-1}(W) = f^{-1}(W)$ for all $W \subset Z$.

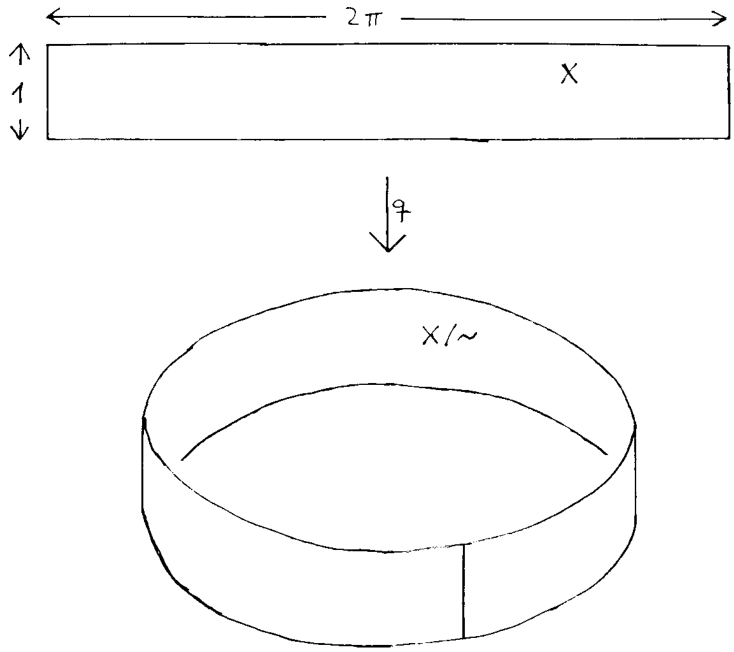
Remarks The converse is trivial: any continuous $g: Y \longrightarrow Z$ produces a continuous map $g \circ h: X \longrightarrow Z$ by composition. — Of course the proposition applies to the special case where $Y = X/\sim$ is a quotient and $h = q$ the quotient map: under the condition that the continuous map f is constant on each equivalence class there is exactly one continuous $\overline{f}: X/\sim \longrightarrow Z$ with $\overline{f} \circ q = f$. Thus continous maps defined on the quotient $X/\sim$ corrrespond to those continuous map defined on X which are constant on each equivalent classes, and this is how continuous functions on quotients are constructed in practice, without exception.

10.4 Question Explain the analogy with 6.6 and 6.8.

10.5 Examples (1) Imagine $X := [0, 2\pi] \times [0, 1]$ as a strip of paper. (From the topological point of view $[0, 1]$ would have been as good as $[0, 2\pi]$ but certainly visualizing X as a strip is easier if its length is reasonably large compared to its width. The particular choice of 2π will come in handy in a minute.) Consider the following equivalence relation on X:

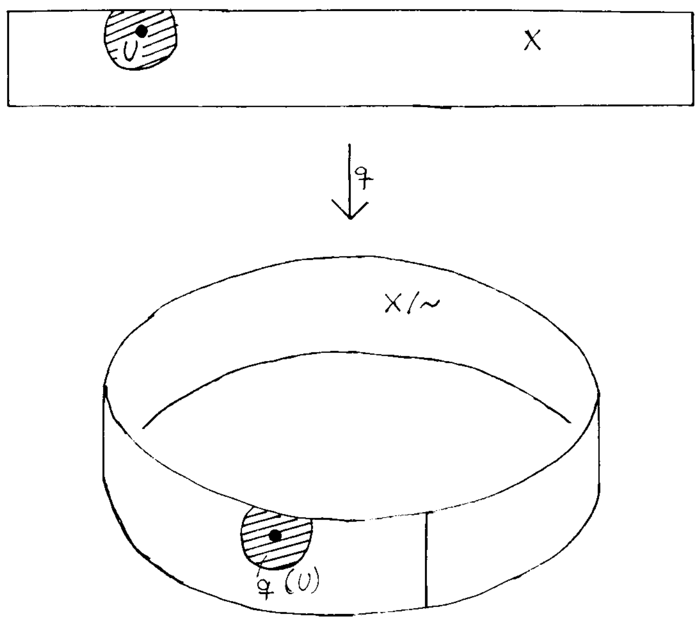
$$X \ni (0, t) \sim (2\pi, t) \in X \quad \text{for all } t \in [0, 1]$$

while each point $(s, t) \in X$ with $0 < s < 2\pi$ is equivalent to nothing but itself. Let us look at the resulting quotient $X \xrightarrow{q} X/\sim$. Since "most" equivalence classes consist of a single point the quotient map q would be bijective but for the fact that any two opposite points at the ends of the strip are mapped to the same point in the quotient. Thus it looks as if $X/\sim$ were obtained from the strip X by gluing the two short edges in the obvious way, and the following discussion will show that this indeed is the right idea.



What does $X/\sim$ locally look like? First let $q(s,t)$ be an "ordinary" point of $X/\sim$, that is, one with $0<s<2\pi$. For $\delta < \min\{s, 2\pi - s\}$ the open subset $U := X \cap U_\delta(s,t)$ of X has the property that

$$q^{-1}\big(q(U)\big) = U,$$

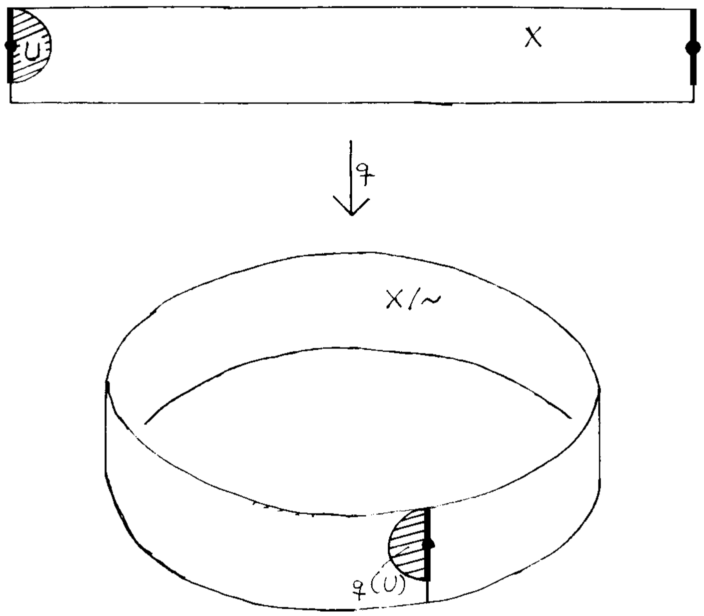


therefore $q(U)$ is open in $X/\sim$. This implies that q sends the subspace $(0,2\pi)\times[0,1]$ of X homeomorphically to its image in $X/\sim$. In particular, the latter locally looks just like X at the corresponding point.

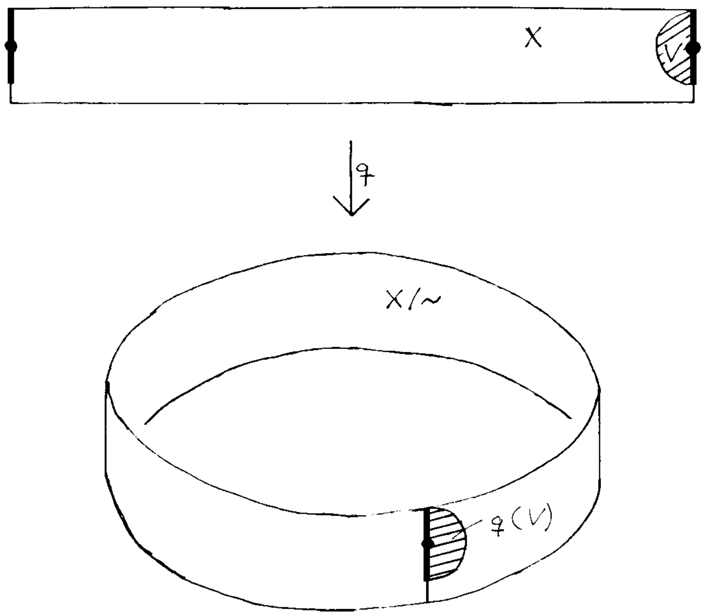
Consider now a "special" point $q(0,t) = q(2\pi,t)$, and choose any $\delta \le \pi$. While the quotient map q does send $U := X \cap U_\delta(0,t)$ homeomorphically to its image in $X/\sim$ it is crucial to realise that $q(U)$ is *not* an open subset of $X/\sim$ because its inverse image

$$q^{-1}\big(q(U)\big) = U \;\cup\; \Big(\{2\pi\} \times \big([0,1] \cap (t-\delta, t+\delta)\big)\Big)$$

is not open in X.



For the same reason, $V := X \cap U_\delta(2\pi, t)$ does not map to an open set in the quotient.



On the other hand the union $q(U) \cup q(V) = q(U \cup V)$ has inverse image

$$q^{-1}\big(q(U \cup V)\big) = \big(X \cap U_\delta(0,t)\big) \cup \big(X \cap U_\delta(2\pi,t)\big) = U \cup V$$

and therefore is an open neighbourhood of $q(0,t) = q(2\pi,t)$ in $X/\sim$.

Let us complete discussion of the example by a formal proof that $X/\sim$ is homeomorphic to the closed ribbon $S^1 \times [0,1]$: in view of the symmetry of S^1 this will show that there remains nothing special about the points $q(0,t) = q(2\pi,t)$ once the construction of $X/\sim$ from X is forgotten. Clearly one has a continuous mapping

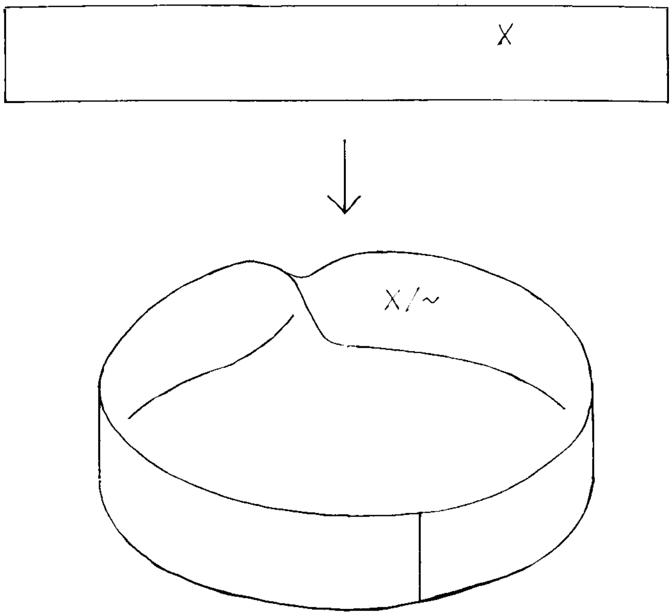
$$X \xrightarrow{f} S^1 \times [0,1]$$

sending (s,t) to $(\cos s, \sin s, t)$, and f is constant on equivalence classes. By Proposition 10.3 a continuous map $\overline{f}: X/\sim \longrightarrow S^1 \times [0,1]$ is induced, which is at once seen to be a bijection. The strip X is compact so $X/\sim = q(X)$ also is compact by Proposition 9.8. Since $S^1 \times [0,1]$ is a Hausdorff space Proposition 9.9 now implies that $\overline{f}: X/\sim \longrightarrow S^1 \times [0,1]$ is a homeomorphism.

(2) Thus the first topological quotient we have constructed has turned out to be a space that is easily described otherwise. Just a slight twist in the construction makes the result more interesting: let us change the rule for the equivalence relation to

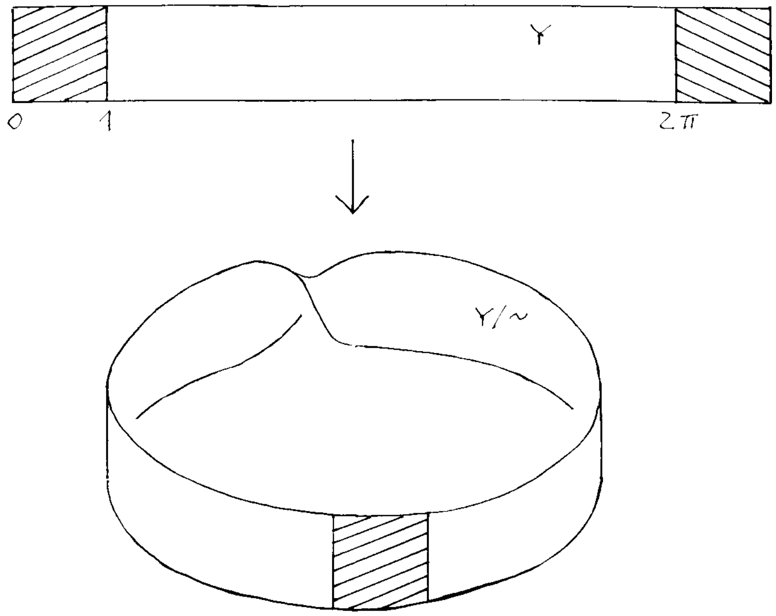
$$X \ni (0,t) \sim (2\pi, 1-t) \in X \quad \text{for all } t \in [0,1]$$

Formally this does not seem to change much but the resulting quotient is quite a different space, a *Moebius strip.*



(3) Real world Moebius strips are usually produced by gluing a strip of paper not just along the short edges but with some overlap. Let us prove that topologically that makes no difference — which incidentally suggests that topology *is* part of the real world. Thus let $Y := [0, 2\pi+1] \times [0,1]$ be a slightly longer strip and extend the gluing relation to Y by

$$Y \ni (s,\, t) \sim (2\pi+s,\, 1-t) \in Y \quad \text{for all } s,t \in [0,1].$$



We want to prove that $Y/\sim$ is homeomorphic to $X/\sim$ of the second example. We begin with the inclusion $i: X \hookrightarrow Y$ and note that the composition $X \xrightarrow{i} Y \xrightarrow{q} Y/\sim$ is constant on equivalence classes in X. Therefore a continuous mapping $h: X/\sim \longrightarrow Y/\sim$ is induced, and this map is easily seen to be bijective. Since $X/\sim$ is compact h must be a homeomorphism as soon as we know that $Y/\sim$ is a Hausdorff space. While this is true it is not obvious, and is a point that deserves careful attention. Let me first illustrate the problem by yet another example.

(4) An n-dimensional sphere can be constructed by gluing two copies of $\mathbb{R}^n$ as follows. Start from $X := X_1 + X_2$ with $X_1 = X_2 = \mathbb{R}^n$ and form the quotient space $X/\sim$ by identifying

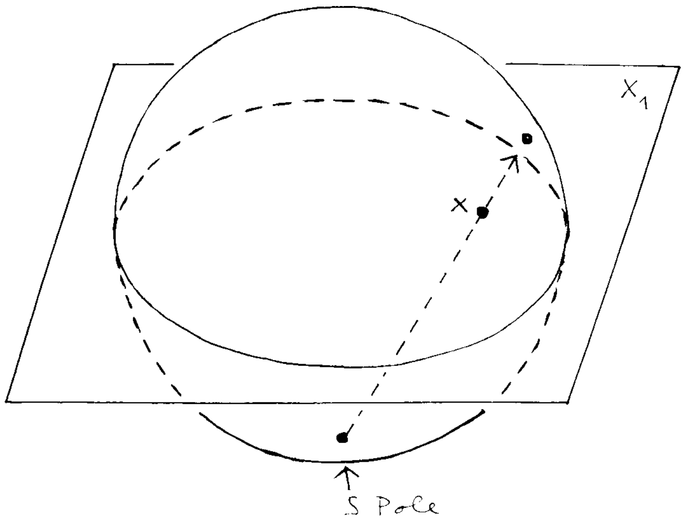
$$x \in X_1 \setminus \{0\} \quad \text{with} \quad \frac{1}{|x|^2}\, x \in X_2 \setminus \{0\}.$$

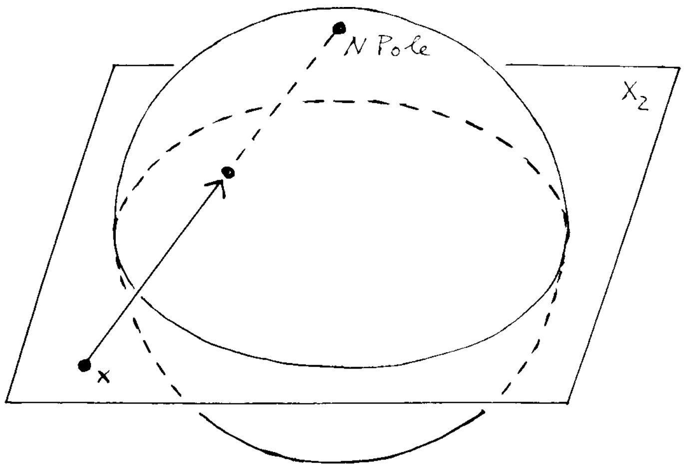
10.6 Question $X/\sim$ is a compact space though X certainly isn't. Explain why.

Arguing as in (1) it is easily proved that inverse stereographic projection

$$\left.\begin{aligned} X_1 \ni x &\longmapsto \left(\frac{2}{1+|x|^2}\, x,\ \frac{1-|x|^2}{1+|x|^2}\right) \\ X_2 \ni x &\longmapsto \left(\frac{2}{1+|x|^2}\, x,\ \frac{|x|^2-1}{|x|^2+1}\right) \end{aligned}\right\} \in S^n \subset \mathbb{R}^n \times \mathbb{R}$$

induces a homeomorphism from $X/\sim$ to the sphere S^n.



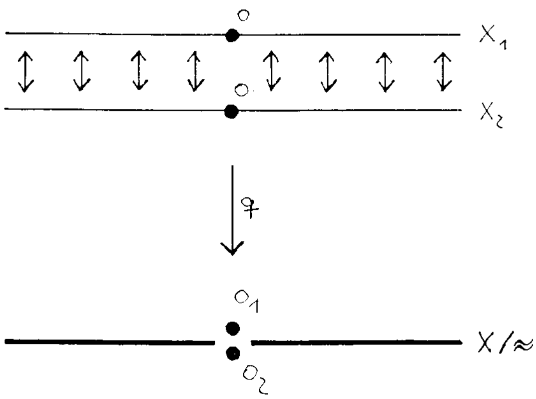


10.7 Question Exactly two points of X are not equivalent to any other point. Which? To which points of S^n are they sent? Prove that removing one of these points (strictly speaking, its class) from $X/\sim$ leaves a homeomorphic copy of $\mathbb{R}^n$.

Example (4) continued We construct another quotient of $X = X_1 + X_2$ using the gluing relation

$$X_1\backslash\{0\} \ni x \approx x \in X_2\backslash\{0\}.$$

The resulting $X/\approx$ is a strange space which is like $\mathbb{R}^n$ but for the fact that it contains two copies 0_1 and 0_2 of the origin, coming from $0 \in X_1$ and $0 \in X_2$ respectively.



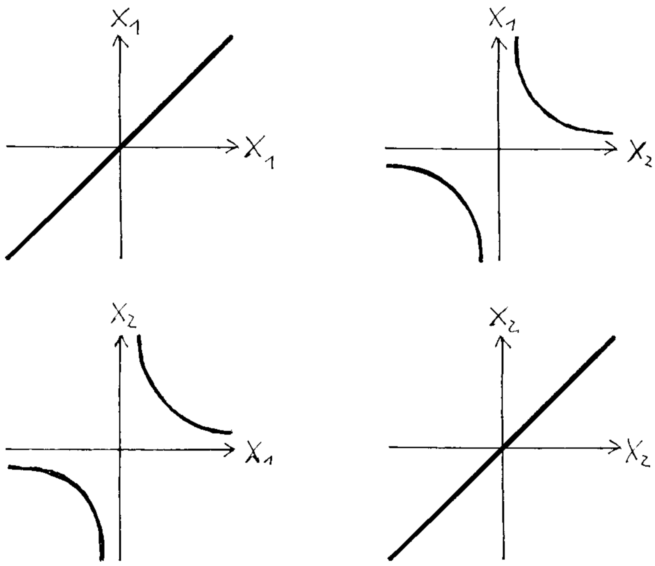
While removing either of them leaves a true copy of $\mathbb{R}^n$ as in case of $X/\sim$ the space $X/\approx$ cannot be a Hausdorff space because every neighbourhood of 0_1 in $X/\approx$ contains a punctured neighbourhood of 0_2 and vice versa.

We thus have seen some of the power of the quotient topology: it can produce nice and interesting spaces but it may also produce peculiar ones. There is no way to predict in every single case into which of these two categories a particular quotient construction will fall but some tools are available. A necessary condition for a quotient to be a Hausdorff space is provided by Corollary 7.5, applied to the maps $q\circ\mathrm{pr}_1$ and $q\circ\mathrm{pr}_2$ associated with any quotient mapping $q: X \longrightarrow X/\sim$: the quotient cannot be a Hausdorff space unless the set

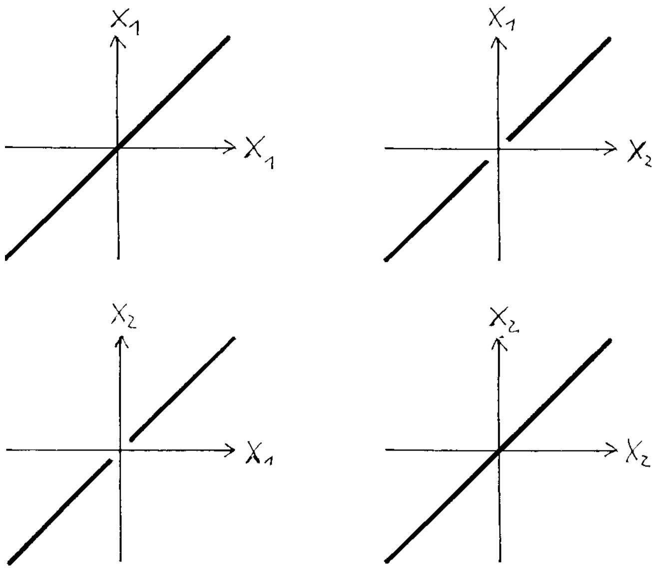
$$\big\{(x,y)\in X\times X \,\big|\, q(x)=q(y)\big\} = \big\{(x,y)\in X\times X \,\big|\, x\sim y\big\}$$

is a closed subset of $X\times X$.

Indeed in example (4) above this set is closed in case of $\sim$



but not so for $\approx$:



In practice the problem usually is to *establish* the Hausdorff property of a quotient, so one would look for the converse statement. It does not hold without additional assumptions, and while the following version is easy to prove it only applies to rather special situations.

10.8 Proposition Let $h: X \longrightarrow Y$ be an identification mapping and assume that h sends open sets to open sets. Then Y is a Hausdorff space if and only if

$$\big\{(x,y) \in X\times X \,\big|\, h(x) = h(y)\big\}$$

is closed in $X\times X$.

Proof Necessity of the condition has just been discussed. Conversely let $\{(x,y) \in X\times X \,|\, h(x) = h(y)\}$ be closed, so that $\{(x,y) \in X\times X \,|\, h(x) \neq h(y)\}$ is open in $Y\times Y$. The extra assumption on h at once implies that the cartesian product $h\times h: X\times X \longrightarrow Y\times Y$ also sends open sets to open sets. In particular

$$(Y\times Y)\backslash\Delta_Y = h\big\{(x,y) \in X\times X \,\big|\, h(x) \neq h(y)\big\}$$

is open in $Y\times Y$, and in view of Proposition 7.4 this means that Y is a Hausdorff space.

For quotients of a compact Hausdorff space the following result can be considered to settle the question.

10.9 Theorem Let X be a compact Hausfdorff space, and $h: X \longrightarrow Y$ an identification map. Then the following three statements are equivalent:

- $\{(x,y) \in X\times X \,|\, h(x) = h(y)\}$ is a closed subset of $X\times X$,
- h sends closed subsets to closed subsets,
- Y is a Hausdorff space.

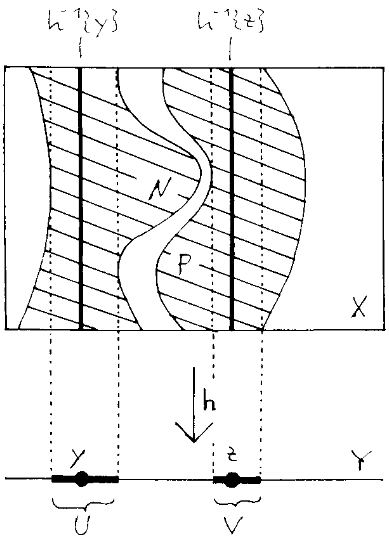
Proof First we assume that $D := \{(x,y) \in X\times X \,|\, h(x) = h(y)\}$ is closed, and show that for any closed $F \subset X$ the image $h(F)$ is closed in Y. Since h is an identification this means to prove that $h^{-1}\big(h(F)\big)$ is closed in X. Using cartesian projections we re-write

$$h^{-1}\big(h(F)\big) = \big\{x \in X \,\big|\, h(x) = h(y) \text{ for some } y \in F\big\} = \mathrm{pr}_1\big(\mathrm{pr}_2^{-1}(F) \cap D\big).$$

As pr_2 is continuous $\mathrm{pr}_2^{-1}(F)$ is closed. Therefore $\mathrm{pr}_2^{-1}(F) \cap D$ is a closed subset of the compact space $X \times X$ hence is compact itself. But then $\mathrm{pr}_1\left(\mathrm{pr}_2^{-1}(F) \cap D\right)$ is compact hence closed since X is a Hausdorff space. This proves that $h(F)$ is closed.

Next we make the assumption that h sends closed sets to closed sets and prove that Y is a Hausdorff space. Thus let $y \neq z$ be two distinct points of Y. As an identification h is surjective so we may write $y = h(x)$. Since X is a Hausdorff space the one point set $\{x\}$ is closed, thus $\{y\} = h\{x\}$ is closed and so is the fibre $h^{-1}\{y\}$, by continuity. The same reasoning applies to $h^{-1}\{z\}$ of course, and therefore $h^{-1}\{y\}$ and $h^{-1}\{z\}$ are disjoint closed subsets of X. Now recall Proposition 9.10: a compact Hausdorff space is normal. Thus we find disjoint open neighbourhoods $N, P \subset X$ of $h^{-1}\{y\}$ and $h^{-1}\{z\}$. Their complements $X \backslash N$ and $X \backslash P$ map to closed sets in Y, and finally

$$U := Y \setminus h(X \backslash N) \quad \text{and} \quad V := Y \setminus h(X \backslash P)$$



are open in Y. It is easily verified that these sets are disjoint and contain y and z, respectively.

If the quotient Y is a Hausdorff space then as we already know the set $\{(x,y) \in X \times X \mid h(x) = h(y)\}$ must be closed in $X \times X$. Thus the three statements of the theorem are linked by a cycle of implications and the proof is complete.

Examples (2) and (3) revisited By Theorem 10.9 both constructions yield Hausdorff quotients since with $x = (s,t)$ and $y = (u,v)$ the relevant sets $\{(x,y) \in X \times X \mid x \sim y\}$ and $\{(x,y) \in Y \times Y \mid x \sim y\}$ are given by the condition

$$(s,t) = (u,v) \quad \text{or} \quad (s,t) = (u \pm 2\pi, 1-t)$$

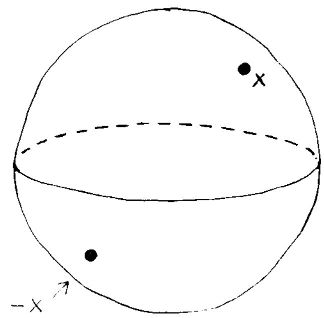
and therefore are closed. This also fills in the missing bit in the proof that $X/\sim$ and $Y/\sim$ are canonically homeomorphic.

11 More Quotients: Projective Spaces

Projective spaces play a prominent role in both algebra and geometry. From the latter point of view the real version is easiest to grasp.

11.1 Definition The topological quotient of the sphere S^n with respect to the equivalence relation

$$x \sim y \quad :\Longleftrightarrow \quad x = \pm y$$



is called the real projective n-space $\mathbb{R}P^n$.

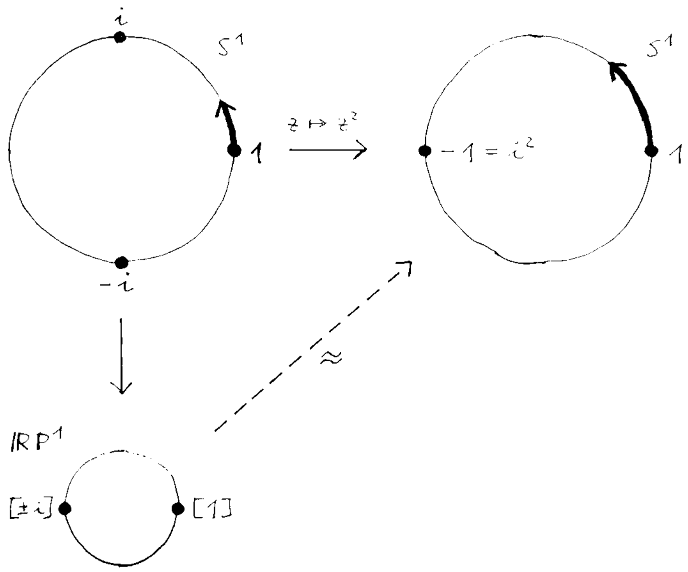
$\mathbb{R}P^n$ is not only compact, as a quotient of S^n, but also a Hausdorff space, for the set

$$\big\{(x,y) \in S^n \times S^n \,\big|\, x \sim y\big\} = \big\{(x,y) \in S^n \times S^n \,\big|\, x = \pm y\big\}$$

is closed in $S^n \times S^n$ and Theorem 10.9 applies.

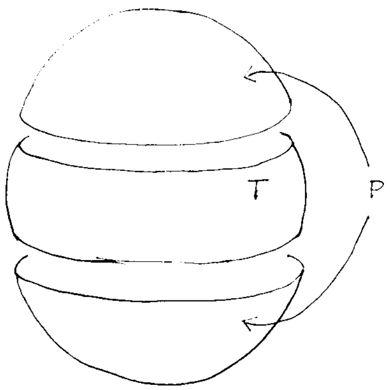
Let us try to visualize $\mathbb{R}P^n$ for small n. For $n = 0$ we just have the one-point space obtained from identifying the two points of $S^0 = \{-1, 1\}$ with each other. Less trivial is $\mathbb{R}P^1$: identifying opposite points of the circle S^1 yields another circle. More precisely, in terms of the complex coordinate z on $S^1 = \big\{z \in \mathbb{C} \,\big|\, |z| = 1\big\}$, the continuous mapping

$$S^1 \ni z \longmapsto z^2 \in S^1$$



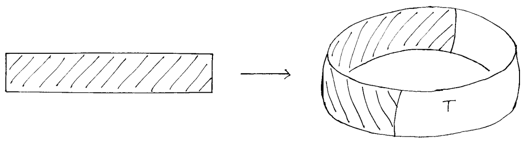
takes the same value on opposite points $\pm z$ and therefore induces a continuous map $\mathbb{R}P^1 \longrightarrow S^1$. This map from a compact to a Hausdorff space is bijective, therefore a homeomorphism.

The real projective plane is best visualized by first decomposing S^2 into two parts: the Tropics T which are homeomorphic to $S^1\times[-23.5°, 23.5°]$, and P, the closure of the complement which comprises the temperate and polar zones and is homeomorphic to the sum of two disks.

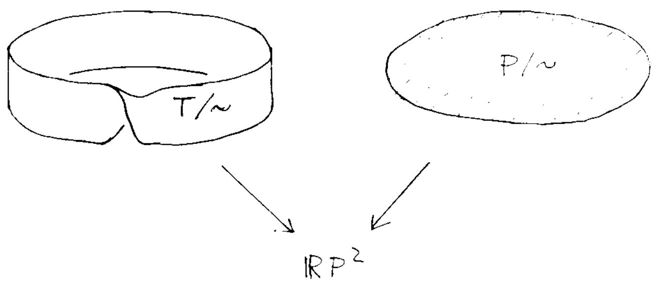


This decomposition is compatible with the equivalence relation so it makes sense to separately form the quotients $T/\!\sim$ and $P/\!\sim$.

11.2 Question Describe a mapping $[0, 2\pi] \times [0, 1] \longrightarrow S^2$ that induces a homeomorphism between $X/\!\sim$ of Example 10.5(2), and $T/\!\sim$.



Thus $T/\!\sim$ is a Moebius strip. On the other hand, $\sim$ on P homeomorphically identifies the two summands of P with each other and therefore $P/\!\sim$ is homeomorphic to a single disk. Finally the projective plane is obtained by gluing $T/\!\sim$ and $P/\!\sim$ along their respective boundary curves which are homeomorphic to the circle S^1.



11.3 Question Why does this gluing construction give the same topology on $\mathbb{R}P^2$ as the original one?

From a purely scientific point of view the construction of $\mathbb{R}P^2$ from a Moebius strip and a disk is, of course, quite awkward and arbitrary. On the intuitive level it has the advantage that both the strip and the disk can be realized as subspaces of the familar ambient $\mathbb{R}^3$ while the gluing process itself is easy to visualize, at least locally. If, nevertheless, you will find it hard to imagine the resulting projective plane it is for the simple reason that $\mathbb{R}P^2$ cannot be embedded in $\mathbb{R}^3$ as a topological subspace. A good argument in favour of abstract topology since for the construction of projective, and quotient spaces in general there is no need for an ambient space at all!

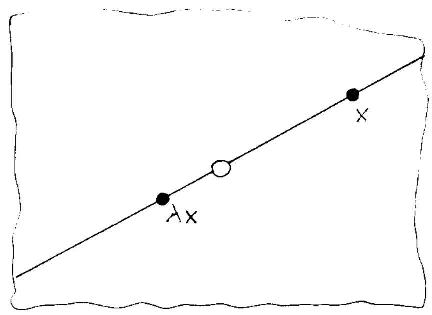
Whether for given integers n and p the projective n-space admits an embedding in $\mathbb{R}^p$ is an interesting topological question. It follows from general theorems that the answer always is yes if $2n \le p$ but for more precise results specific and rather deeper methods have to be used. If $n \ge 8$ is a power of 2 then it can be shown that $\mathbb{R}P^n$ is not embeddable in $\mathbb{R}^{2n-1}$. But this is not the whole story: if $n = 2^k + 2$ with $k \ge 3$ then $\mathbb{R}P^n$ can be embedded in $\mathbb{R}^{2n-2}$, to quote just one sample from a wealth of known results. "Good" embeddings (with relatively small p) are difficult to describe explicitly, and by no means as canonical as the standard embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$.

Algebraists prefer to define projective spaces — not surprisingly and with good reason — completely in terms of an algebraic structure, namely as the set of lines in a vector space V. Equivalently one may form a quotient of $V \setminus \{0\}$, identifying proportional vectors. Let us therefore re-cast the definition of projective spaces to make it compatible with the algebraic notion and at the same time, more general.

11.4 Definition Let V be a finite dimensional vector space over either $K = \mathbb{R}$ or $K = \mathbb{C}$ (if you like the exotic you may also envisage the skew field of quaternions, $K = \mathbb{H}$), and let V carry the standard topology induced by any linear isomorphism $V \xrightarrow{\simeq} K^n$. The topological quotient of $V \setminus \{0\}$ with respect to the equivalence relation

$$x \sim y \quad :\Longleftrightarrow \quad x = \lambda y \text{ for some } \lambda \in K^*$$

is the projective space $P(V)$ of V.



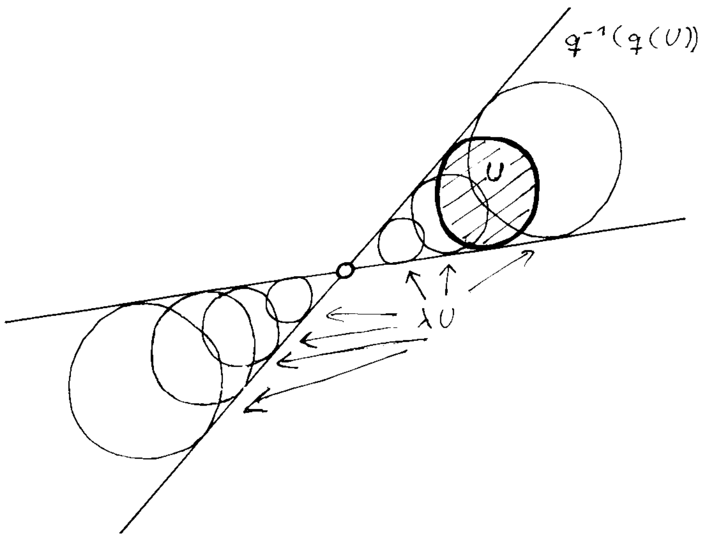
11.5 Question Explain why the topology used on V does not depend on the choice of the isomorphism between V and K^n.

Note that unlike Definition 11.1 this new one makes no reference to a euclidean structure on V. On the other hand euclidean structures in the real, and hermitian ones in the complex case do exist, and we may pick any of them as a tool to prove that $P(V)$ is compact: indeed the restriction of the quotient map $V \setminus \{0\} \longrightarrow P(V)$ to the corresponding compact sphere is still surjective.

For the proof that $P(V)$ is a Hausdorff space we can no longer appeal to Theorem 10.9 since $V \setminus \{0\}$ is not compact. Instead we use the fact that the quotient mapping $q: V \setminus \{0\} \longrightarrow P(V)$ sends open sets to open sets. Indeed, if $U \subset V \setminus \{0\}$ is open then

$$q^{-1}\big(q(U)\big) = \bigcup_{\lambda \in K^*} \lambda U$$

is open too, being a union of open sets. Thus $q(U)$ is open by definition of the quotient topology.



On the other hand the set

$$\{(x,y) \in V\backslash\{0\} \times V\backslash\{0\} \mid x \sim y\} = \{(x,y) \in V\backslash\{0\} \times V\backslash\{0\} \mid (x,y) \text{ linearly dependent}\}$$

is given in $V\backslash\{0\} \times V\backslash\{0\}$ by the vanishing of the 2×2-minors of (x,y) (with respect to any fixed basis of V), hence is closed. By Proposition 10.8 we conclude that $P(V)$ is a Hausdorff space.

11.6 Question It is now clear that Definition 11.1 is recovered as the special case $\mathbb{R}P^n = P(\mathbb{R}^{n+1})$ — why?

When working with points in one of the standard projective spaces $KP^n = P(K^{n+1})$ components are conveniently numbered not from 1 to $n+1$ but rather from 0 to n as we have already done, and it is customary to write the class represented by the non-zero vector $x = (x_0, x_1, \ldots, x_n)$ as

$$[x] = [x_0 : x_1 : \cdots : x_n] \in KP^n$$

since by definition it is exactly the *ratios* of the scalars x_i that make the point $[x] \in KP^n$.

11.7 Example The complex projective line $\mathbb{C}P^1$ is homeomorphic to the sphere S^2. To see this we put $L := L_1 + L_2$ with $L_1 = L_2 = \mathbb{C}$ and identify by

$$L_1\backslash\{0\} \ni z \sim \frac{1}{z} \in L_2\backslash\{0\}.$$

Sending

$$\left.\begin{array}{l} L_1 \ni z \longmapsto [z:1] \\ L_2 \ni z \longmapsto [1:z] \end{array}\right\} \in \mathbb{C}P^1$$

we obtain a continuous mapping from L to $\mathbb{C}P^1$ which induces a homeomorphism $L/\sim \longrightarrow \mathbb{C}P^1$. But the quotient $L/\sim$ is a 2-sphere as follows by comparison with $X/\sim$ of Example 10.5(4).

11.8 Question Give the details of a direct proof that $L/\sim$ and $X/\sim$ are homeomorphic (not using stereographic projection).

The subspace

$$\{[x_0 : x_1 : \cdots : x_n] \in KP^n \mid x_0 = 0\}$$

of the standard projective space clearly is closed and canonically homeomorphic to KP^{n-1}, with which it may be identified. By contrast the complement $KP^n\backslash KP^{n-1}$ is an open dense subset, and homeomorphic to K^n via

$$K^n \ni \left\{\begin{array}{ccc} (x_1, \ldots, x_n) & \longmapsto & [1 : x_1 : \cdots : x_n] \\ \left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) & \longleftarrow & [x_0 : x_1 : \cdots : x_n] \end{array}\right\} \in KP^n\backslash KP^{n-1}.$$

Thus one may say that KP^n *compactifies* K^n by adding a copy of KP^{n-1}.

As an application of projective spaces let us study the dependence of the zeros of a complex polynomial on its coefficients. Fix $n \in \mathbb{N}$ and let

$$V' := \left\{ p(X) = X^n + \sum_{j=0}^{n-1} c_j X^j \;\middle|\; c_j \in \mathbb{C} \text{ for } j = 0, \ldots, n-1 \right\}$$

be the affine space of unitary polynomials of that degree. The continuous mapping

$$h\colon \mathbb{C}^n \longrightarrow V', \quad (x_1, \ldots, x_n) \mapsto \prod_{j=1}^{n} (X - x_j)$$

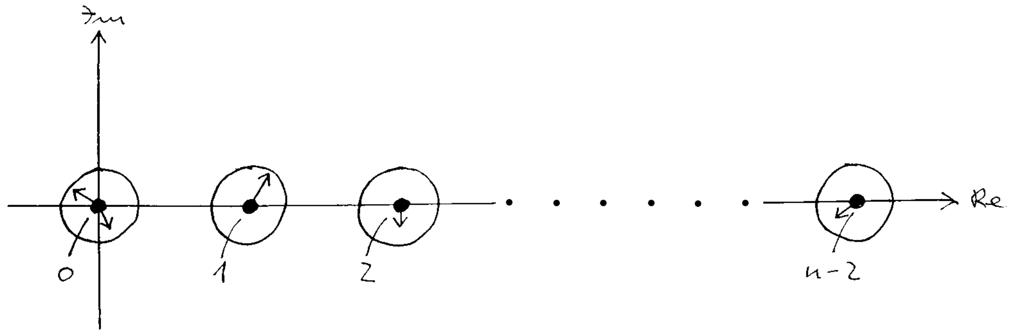
is invariant with respect to permutation of the x_j. Therefore it induces a continuous $\overline{h}\colon \mathbb{C}^n/\sim \longrightarrow V'$ if the equivalence relation $\sim$ on $\mathbb{C}^n$ is defined by

$$x \sim y \quad :\Longleftrightarrow \quad (x_1, \ldots, x_n) = (y_{\sigma 1}, \ldots, y_{\sigma n}) \text{ for some permutation } \sigma \in \mathrm{Sym}_n \,.$$

Since the class of x in $\mathbb{C}^n/\sim$ can be recovered from $h(x)$ as the unordered n-tupel of zeros of $h(x)$ the map $\overline{h}$ is injective. It is also surjective by the fundamental theorem of algebra. We will prove that $\overline{h}$ is, in fact, a homeomorphism.

Let us first discuss the possible use of such a result. It clearly implies that the so-called *symmetric product* $\mathbb{C}^n/\sim$ is, quite surprisingly, homeomorphic to the ordinary cartesian product $\mathbb{C}^n$. But it likewise contains a statement about continuous dependence of the zeros on the coefficients of a unitary polynomial. That *simple* zeros *locally* vary even differentiably with the coefficients is a familiar consequence of the implicit function theorem. But even to formulate a global analogue including multiple zeros is a non-trivial task since there is no natural way to number the zeros of a polynomial (in the real case they are naturally ordered but their number may jump). Topology provides a solution: if you think about it the statement we are looking for just is that $(\overline{h})^{-1}\colon V' \longrightarrow \mathbb{C}^n/\sim$ is a continuous map.

11.9 Question Put $p(X) = X^2 \cdot \prod_{j=1}^{n-2}(X-j)$ and work out what continuity of $(\overline{h})^{-1}\colon V' \longrightarrow \mathbb{C}^n/\sim$ at $p(X)$ means in terms of ε and δ.



The conclusion that $\overline{h}$ is a homeomorphism would be automatic if $\mathbb{C}^n/\sim$ were compact — which it is not. This is the point where projective spaces come in: the mapping h between affine spaces has a natural projective analogue H which extends it, and as projective spaces are compact . . .

In place of V' we now consider the vector space

$$V := \left\{ p(X) = \sum_{j=0}^{n} c_j X^j \;\middle|\; c_j \in \mathbb{C} \text{ for } j = 0, \ldots, n \right\}$$

of all polynomials of degree at most n and define $H\colon (\mathbb{C}P^1)^n \longrightarrow P(V)$ by

$$H\big([w_1 : x_1], \ldots, [w_n : x_n]\big) := \left[\prod_{j=1}^{n}(w_j X - x_j)\right] \in P(V)$$

(verify that this is a well-defined map!). Note that H sends $\big([w_1 : x_1], \ldots, [w_n : x_n]\big)$ to the class of a polynomial of degree d if exactly d among the w_j do not vanish. As we shall see in a minute the map H is continuous. It is also invariant under permutations, and induces a continuous bijection

$$\overline{H}\colon (\mathbb{C}P^1)^n/\!\sim\; \longrightarrow P(V)$$

from the corresponding symmetric product to $P(V)$. Since the quotient is a compact, and $P(V)$ a Hausdorff space $\overline{H}$ is a homeomorphism.

As to continuity of H the routine argument runs as follows. The map

$$\big(\mathbb{C}^2\setminus\{0\}\big)^n \ni \big((w_1, x_1), \ldots, (w_n, x_n)\big) \longmapsto \prod_{j=1}^{n}(w_j X - x_j) \in V\setminus\{0\}$$

obviously is continuous. Composing with the quotient map $V\setminus\{0\} \longrightarrow P(V)$ we obtain a continuous map from $\big(\mathbb{C}^2\setminus\{0\}\big)^n$ to $P(V)$ which is constant on equivalence classes with respect to the product relation

$$\big((w_1, x_1), \ldots, (w_n, x_n)\big) \approx \big((y_1, z_1), \ldots, (y_n, z_n)\big) \quad :\Longleftrightarrow \quad (w_j, x_j) \sim (y_j, z_j) \text{ for all } j \in \{1, \ldots, n\}.$$

Therefore a continuous mapping

$$\big(\mathbb{C}^2\setminus\{0\}\big)^n/\!\approx\; \longrightarrow P(V)$$

is induced. As a set, $\big(\mathbb{C}^2\setminus\{0\}\big)^n/\!\approx$ may be identified with $(\mathbb{C}P^1)^n$, and the resulting map from $(\mathbb{C}P^1)^n$ to $P(V)$ is just H. But careful: we have thus shown continuity of H with respect not to the product topology on $(\mathbb{C}P^1)^n$ but rather to the quotient topology inherited from $\big(\mathbb{C}^2\setminus\{0\}\big)^n$. Luckily, these topologies coincide. For by the standard argument the set-theoretic identity

$$\big(\mathbb{C}^2\setminus\{0\}\big)^n/\!\approx\; \longrightarrow (\mathbb{C}P^1)^n$$

is continuous, and as it takes a compact to a Hausdorff space it is a homeomorphism.

Returning to the affine version h of H it would now seem to follow from the commutative diagram

$$\begin{array}{ccc}
\mathbb{C}^n/\!\sim & \xrightarrow{\;\overline{h}\;} & V' \\
\big\downarrow & & \big\downarrow \\
(\mathbb{C}P^1)^n/\!\sim & \xrightarrow[\approx]{\;\overline{H}\;} & P(V)
\end{array}$$

that $\overline{h}$ likewise is a homeomorphism. But while this in itself is a correct argument the diagram already contains the implicit claim that the quotient topology of $\mathbb{C}^n/\!\sim$ coincides with the subspace topology induced from $(\mathbb{C}P^1)^n/\!\sim$. The point is best clarified by addressing the question in greater generality. Thus let us consider an equivalence relation on a topological space X and let $S \subset X$ be a subspace. The quotient spaces $X/\!\sim$ and $S/\!\sim$ (with respect to the restricted relation on S) may be formed and the set theoretic inclusion map i in

$$\begin{array}{ccc}
S & \hookrightarrow & X \\
\big\downarrow & & \big\downarrow \\
S/\!\sim & \xrightarrow{\;i\;} & X/\!\sim
\end{array}$$

is continuous. But some additional assumptions must be made to ensure that i is an embedding:

11.10 Proposition Let $\sim$ be an equivalence relation on the topological space X and let $S \subset X$ be a subspace which is a union of equivalence classes. If S is open or closed in X then $S/\sim$ is open, respectively closed in $X/\sim$, and the induced map $i: S/\sim \longrightarrow X/\sim$ is a topological embedding.

Proof Let $X \xrightarrow{q} X/\sim$ denote the quotient mapping, and consider a subset $U \subset S/\sim$. As S is a union of equivalence classes the inverse image $q^{-1}(U)$ is contained in S. Assume now that $S \subset X$ is open and that U is open in $S/\sim$: this means that $q^{-1}(U)$ is open in S, hence also in X, and therefore U is open in $X/\sim$. Thus we have shown that i sends open sets to open sets, and this proves the proposition in the open case. The case of closed S is handled exactly alike, using closed subsets.

The proposition clearly applies to

$$S = \mathbb{C}^n = \Big\{[w_0 : x_0], \ldots, [w_n : x_n] \in \big(\mathbb{C}P^n\big)^n \,\Big|\, w_j \neq 0 \text{ for } j = 1, \ldots, n\Big\} \subset (\mathbb{C}P^1)^n = X,$$

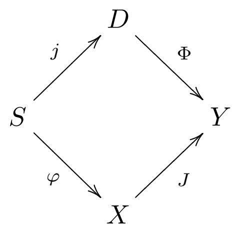
and this completes our discussion of continuous dependence of the roots of a polynomial.

Remark As we have seen in two instances, repeated application of standard constructions may suggest competing topologies on one and the same set, and the question of whether these coincide is ubiquitous. For the purpose it is useful to group the four basic topological constructions into product and subspaces on one hand (the so-called *categorical limits*) as opposed to sum and quotient spaces (*categorical colimits*) on the other. Within one of these groups constructions of the same or different kind may be freely interchanged: if, for instance, $S \subset X$ is a subspace and Y another space then the product topology makes $S \times Y$ also a subspace of $X \times Y$. By contrast, a construction from one group will not in general commute with those from the other, and a careful analysis will be required.

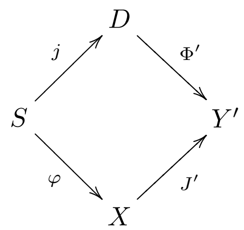
12 Attaching Things to a Space

One would often like to enlarge a given topological space by attaching extra material. In order to make this idea precise it is best to take a much more general point of view at first.

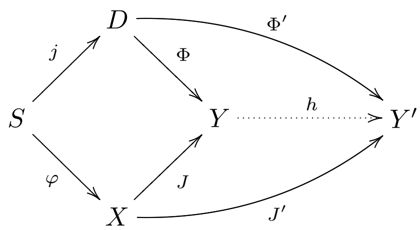
12.1 Definition Let **C** be a category. A commutative diagram



in **C** is called a *pushout diagram* or *co-cartesian square* if it has the following universal property: For any morphisms $\Phi' : D \longrightarrow Y'$ and $J' : X \longrightarrow Y'$ that let the diagram



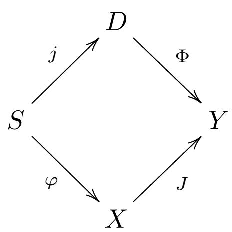
commute there exists a unique morhism $h: Y \longrightarrow Y'$ that renders the big diagram



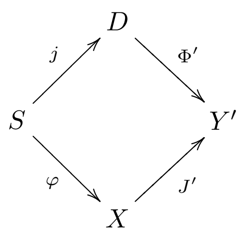
commutative: $h\Phi = \Phi'$ and $hJ = J'$.

It turns out that every pushout diagram is essentially determined by its left half:

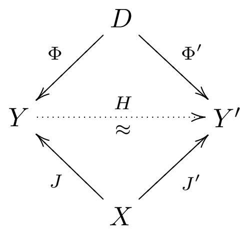
12.2 Proposition Let **C** be a category, and assume that both



and

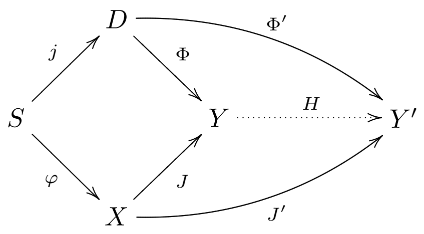


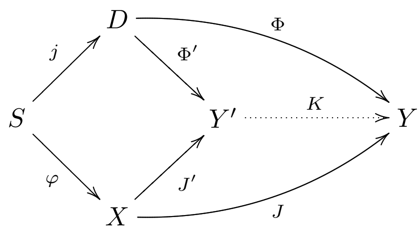
are pushout diagrams in **C**. The there exists a unique isomorphism $H: Y \longrightarrow Y'$ such that the diagram



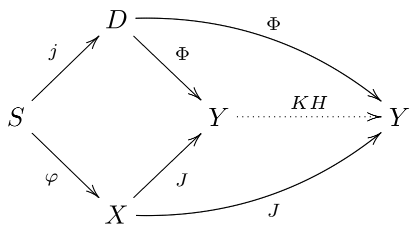
commutes.

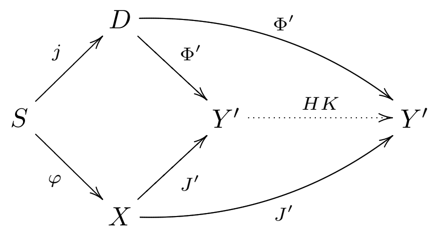
Proof The pushout property of the two diagrams defines morphisms H and K:





Since the compositions KH and HK render the diagrams





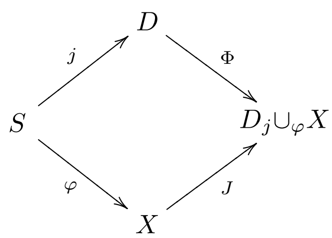
commutative we must have $KH = 1_Y$ and $HK = 1_{Y'}$ by the uniqueness part of the universal property. In particular H is an isomorphism.

In many categories every given pair of morphisms $D \longleftarrow S \longrightarrow X$ can be completed to a co-cartesian square. Among those cateogies are **Ens** and $\mathbf{Lin}_K$ for any field K, and also **Top**:

12.3 Proposition Let $j: S \longrightarrow D$ and $\varphi: S \longrightarrow X$ be continuous maps. On the topological sum $D + X$, introduce the smallest equivalence relation that satisfies

$$D \ni j(s) \sim \varphi(s) \in X \quad \text{for all } s \in S.$$

Let $D + X \xrightarrow{q} D_j\cup_\varphi X$ be the corresponding quotient, and let $\Phi: D \hookrightarrow D + X \xrightarrow{q} D_j\cup_\varphi X$ and $J: X \hookrightarrow D + X \xrightarrow{q} D_j\cup_\varphi X$ be the restrictions of q. Then



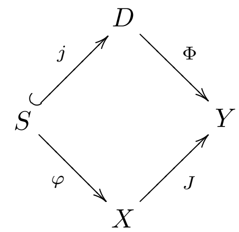
is a pushout diagram in **Top**.

Proof The diagram commutes by construction. According to Definition 12.1 we now must consider continuous maps $\Phi': D \longrightarrow Y'$ and $J': X \longrightarrow Y'$ into an arbitrary space Y', with $\Phi' \circ j = J' \circ \varphi$. Together these maps define a continuous mapping from $D + X$ to Y' which for each $s \in S$ takes equal values on $j(s)$ and $\varphi(s)$, and therefore induces a continuous map $h: D_j{\cup}_\varphi X \longrightarrow Y'$. The required identities $h \circ \Phi = \Phi'$ and $h \circ J = J'$ hold by construction. On the other hand it is clear that they leave no other choice for h even as a morphism in **Ens**.

Thus completing $D \longleftarrow S \longrightarrow X$ to a pushout diagram in **Top** geometrically means to glue the two spaces D and X along the respective images of S. Note the finer point of the uniqueness statement 12.2: the resulting space Y not only is homeomorphic to $D_j{\cup}_\varphi X$ but it is so by a particular, even unique homeomorphism that respects the relation of $D_j{\cup}_\varphi X$ and Y to the building blocks D and X.

The notion of co-cartesian square induced by $D \longleftarrow S \longrightarrow X$ has an obvious symmetry swapping (D, j) and (X, φ). This symmetry is broken by the particular application we now have in mind (and which, by the way, is the reason for my asymmetric choice of letters).

12.4 Definition Let X and D be topological spaces, and $\varphi: S \longrightarrow X$ a continuous map from a subspace $S \subset D$ to X. Then the pushout diagram

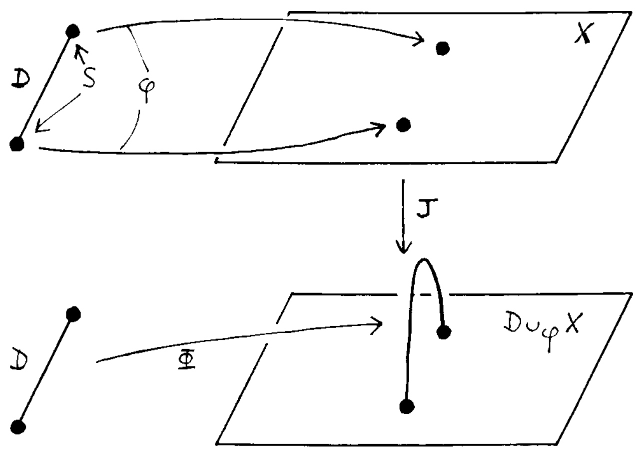


describes the process of <u>attaching</u> D to X via the <u>attaching map</u> φ. The map $\Phi: D \longrightarrow Y$ is called the <u>characteristic map</u> of the attachment.

In view of the preceding discussion the attaching process is essentially uniquely determined by the given data $D \supset S \xrightarrow{\varphi} X$, and whenever a concrete realisation of $D \xrightarrow{\Phi} Y \xleftarrow{J} X$ is desired one may read

$$Y = D \cup_\varphi X := D_j{\cup}_\varphi X$$

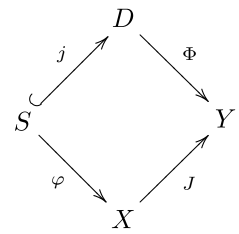
as in 12.3.



12.5 Question What happens if $S = \emptyset$?

After this rather formal discussion let us deal with some of the specific topological properties of the attaching process.

12.6 Proposition Let



be an attaching diagram and assume that $S \subset D$ is closed. Then

- J embeds X as a closed subspace of Y, and
- Φ sends $D\backslash S$ homeomorphically to the open subspace $Y \backslash J(X)$.

Proof We may assume that $Y = D \cup_\varphi X$ and that Φ and J are the restrictions of the quotient mapping $D + X \longrightarrow D \cup_\varphi X$. To prove the first part of the proposition we must verify that J sends closed subsets of X to closed subsets of Y. Thus let $F \subset X$ be closed. Then

$$\Phi^{-1}\big(J(F)\big) = \varphi^{-1}(F) \subset S \subset D \quad \text{and} \quad J^{-1}\big(J(F)\big) = F \subset X$$

are both closed, so $J(F)$ is closed by definition of the sum and quotient topologies.

As to the second statement it is clear that Φ restricts to a continuous bijection $D \backslash S \approx Y \backslash J(X)$. Since $S \subset D$ is closed any open subset $U \subset D\backslash S$ will be open in D, and it follows that $\Phi(U)$ is open in Y, for the inverse image of this set under the quotient map $D+X \longrightarrow Y$ is $U + \emptyset$.

In view of the proposition the "old" space X will usually be identified with its image in the new one, Y.

Unlike the previous proposition the next imposes topological conditions that are quite restrictive. Wherever it applies it makes sure that the attaching construction works very smoothly indeed, and in the following section we will heavily rely on it.

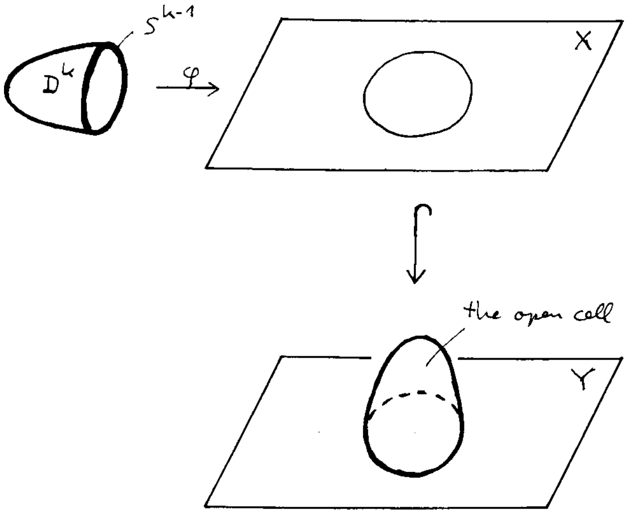
12.7 Proposition Let D and X be a compact Hausdorff spaces, and $S \subset D$ a closed subspace. Then the space Y formed using an attaching map $\varphi: S \longrightarrow X$ is a compact Hausdorff space too.

Proof Again we may assume that $Y = D \cup_\varphi X$. Two points of $D+X$ represent the same point in Y if and only if they are either $s, t \in S$ with $\varphi(s) = \varphi(t)$, or $s \in S$ and $\varphi(s) \in X$ or vice versa, or equal points of D or X. Thus the corresponding subset of $(D+X) \times (D+X)$ is the union of the set $\{(s,t) \in S \times S \,|\, \varphi(s) = \varphi(t)\}$, of the graph of $\varphi: S \longrightarrow X$ and its mirrored copy in $X \times S$, and the diagonal. As these four subsets are closed and $D+X$ is a compact Hausdorff space the result follows from Theorem 10.9.

13 Finite Cell Complexes

We now put the attaching construction to use and describe a pleasant class of synthetically defined topological spaces. Their infinite (non-compact) analogues are commonly known as *CW complexes*. While our restriction to the finite case is a serious limitation for certain applications it appears justified by the considerable amount of technical fuss it allows to eliminate from the exposition.

Cell complexes are constructed by successively attaching *k-cells*: this is the accepted terminology do denote the process of attaching D^k — the cell — to a space X via an attaching map $\varphi: S^{k-1} \longrightarrow X$. The difference between new and old, $Y \backslash X \approx D^k \backslash S^{k-1} = U^k$ is also referred to as the *open cell* attached to X.



More generally simultaneous attachment of a finite collection of k-cells can be useful, that is attaching a topological sum $\sum_{\lambda\in\Lambda} D^k$ along a continuous map $\varphi: \sum_{\lambda\in\Lambda} S^{k-1} \longrightarrow X$ with a finite index set Λ. Note that both cases are covered by Proposition 12.7.

13.1 Question Simultaneous attachment of finitely many k-cells may be replaced by successive attachment of single cells but not in general the other way round. Explain why.

13.2 Definition Let X be a topological space. A <u>(finite) cell filtration</u> of X is a finite sequence of subspaces

$$\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^{n-1} \subset X^n = X$$

such that for each $k \in \{0, \ldots, n\}$ the space X^k is obtained from X^{k-1} by attaching a finite collection $(D^k)_{\lambda\in\Lambda_k}$ of k-cells. X^k is called the k-<u>skeleton</u> of X with respect to the filtration. A collection of attaching data

$$S^{k-1} \xrightarrow{\varphi_\lambda} X^{k-1} \quad \text{for all } \lambda \in \bigcup_{k=0}^{n} \Lambda_k$$

is called a <u>cell structure</u> for X, making X a <u>(finite) cell complex</u>. A topological space will be called a <u>cell space</u> if it admits a cell structure.

If the number $n \in \mathbb{N}$ is chosen as small as possible — that is, if $X^{n-1} \neq X^n$ — then it is called the <u>dimension</u> of the cell filtration or structure. Finally, the dimension $\dim X$ of a non-empty cell space

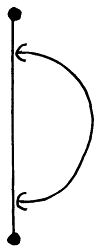
X is defined as the smallest possible dimension of some cell structure on X (we will not define $\dim \emptyset$ but allow $\dim \emptyset \le n$ as a statement that is true for all $n \in \mathbb{N}$ by convention).

13.3 Question This definition involves attaching cells to the empty space. Explain why this makes perfect sense.

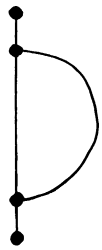
In view of the preceding the following are obvious

13.4 Facts Every cell space is a compact Hausdorff space (by Proposition 12.7, and thus even a normal space by Proposition 9.10). Every cell structure on X makes X the disjoint union of finitely many open cells whose dimension does not exceed $\dim X$. For each cell e the closure $\overline{e} \subset X$ is the union of e with open cells of smaller dimension.

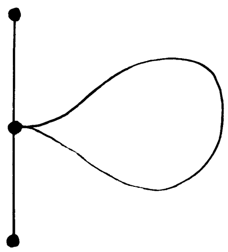
Let us illustrate the notion of cell complex by sketching a number of one-dimensional examples, some of which correspond to cell structures while others do not.



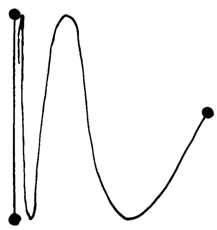
two 0-cells and two 1-cells: not a cell complex



four 0-cells and four 1-cells: a cell complex



three 0-cells and three 1-cells: a cell complex



three 0-cells and two 1-cells, one contained in the closure of the other: not a cell complex

To check all the details of Definition 13.2 could be quite cumbersome in practice but in fact there is no need to do so:

13.5 Theorem Let X be a compact Hausdorff space equipped with a finite partition

$$X = \bigcup_{\lambda \in \Lambda} e_\lambda = \bigcup_{k=0}^{n} \bigcup_{\lambda \in \Lambda_k} e_\lambda \quad \text{where } \Lambda = \sum_{k=0}^{n} \Lambda_k.$$

For each $\lambda \in \Lambda$ let $\dim \lambda$ be defined by $\lambda \in \Lambda_{\dim \lambda}$ and put

$$X^k := \bigcup_{\dim \mu \le k} e_\mu \subset X$$

for $k = -1, 0, \ldots, n$. Assume further that for every $\lambda \in \Lambda_k$ a continuous map $\Phi_\lambda \colon D^k \longrightarrow X$ is given that sends S^{k-1} into X^{k-1}, and $U^k = D^k \backslash S^{k-1}$ bijectively onto e_λ.

Then the restrictions

$$\varphi_\lambda := \Phi_\lambda | S^{k-1}, \text{ for all } \lambda \in \Lambda_k \text{ and } k = 0, \ldots, n$$

define a cell structure on X with the Φ_λ as the characteristic maps. In particular X^k is the k-skeleton of a cell filtration on X, and X is a cell space of dimension at most n.

Proof by induction on $k \in \{0, \dots, n\}$. Assuming that the conclusion holds for X^{k-1} in place of X we will show that it is true for X^k. First note that for $\lambda \in \Lambda_k$ the given mappings Φ_λ take values in X^k, therefore together with the inclusion $X^{k-1} \hookrightarrow X^k$ define a morphism

$$\sum_{\lambda \in \Lambda_k} D^k + X^{k-1} \longrightarrow X^k.$$

This morphism drops to a continuous mapping

$$\sum_{\lambda \in \Lambda_k} D^k \cup_\varphi X^{k-1} \xrightarrow{h} X^k$$

where $\varphi\colon \sum_{\lambda \in \Lambda_k} S^{k-1} \longrightarrow X^{k-1}$ is the map that restricts to φ_λ on the λth summand. h is bijective since each Φ_λ sends U^k bijectively to e_λ by assumption. As a continuous bijection between compact Hausdorff spaces h is a homeomorphism. Thus X^k is obtained from X^{k-1} by simultaneous attachment of k-cells, using the φ_λ ($\lambda \in \Lambda_k$) as attaching maps. This completes the induction step.

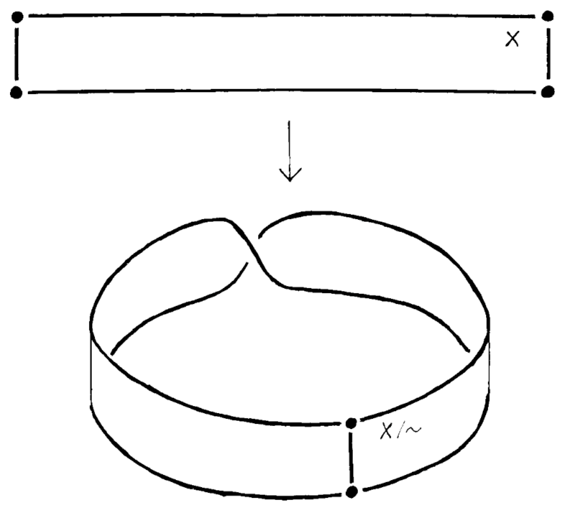
13.6 Examples of cell complexes (1) The Moebius strip $X/\!\sim$ of Example 10.5(2) is a 2-dimensional cell complex in the following way. The four corner points

$$(0,0),\ (0,2\pi),\ (1,0),\ (2\pi,1) \in X = [0,2\pi] \times [0,1]$$

represent two 0-cells in $X/\!\sim$, and the open intervals

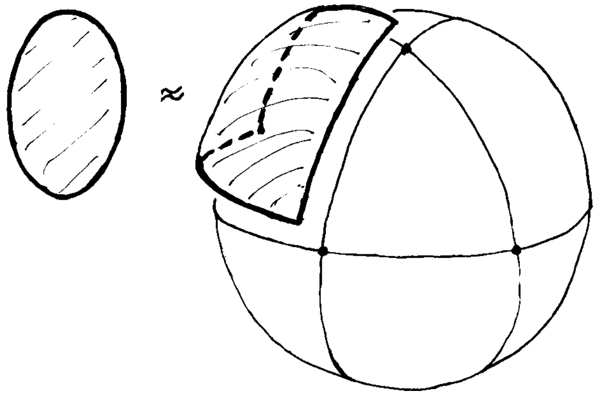
$$(0,2\pi)\times\{0\},\ (0,2\pi)\times\{1\},\ \{0\}\times(0,1) \text{ and } \{2\pi\}\times(0,1)$$

map to three 1-cells. Together with the open rectangle $(0,2\pi)\times(0,1)$ as the unique 2-cell these form a partition of $X/\!\sim$ into cells.

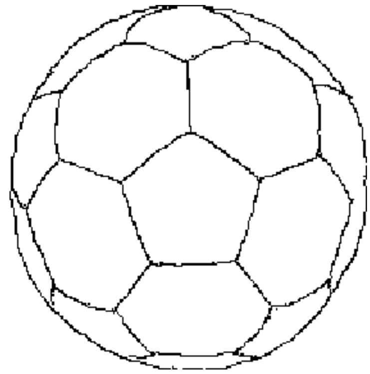


In order to apply Theorem 13.5, for each cell e_λ a characteristic mapping $\Phi_\lambda\colon D^{\dim \lambda} \longrightarrow X/\!\sim$ has yet to be specified. While the obvious choice will do for $\dim \lambda < 2$, a suitable mapping for the 2-cell is obtained by composing a homeomorphism $D^2 \approx X$ (sending S^1 to the boundary of the strip) with the quotient map $X \longrightarrow X/\!\sim$.

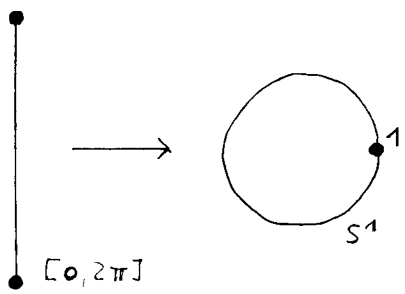
(2) Each Platonic solid determines a 2-dimensional cell filtration of the sphere S^2. Giving it a particular cell structure involves the choice of a homeomorphism between D^2 and the corresponding regular n-gon: the case of an octahedron is shown here.



There are, of course, many more cell structures on S^2, including one for soccer fans. In any case, adding a 3-cell in the obvious way will produce a cell structure on D^3.

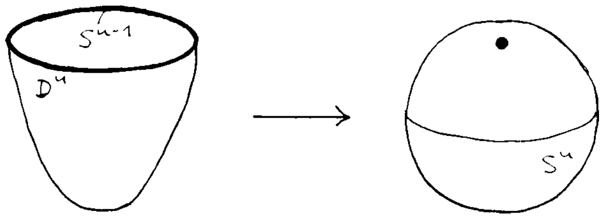


(3) The map $[0, 2\pi] \ni t \mapsto e^{it} \in S^1$ gives S^1 a cell structure with one 0-cell $\{1\} \subset S^1$, and its complement as the unique 1-cell.



More generally, for any $n \geq 0$ the sphere S^n admits a cell structure with one 0-cell and one n-cell. To specify such a structure just means to choose a homeomorphism between S^n and D^n/S^{n-1}, the space D^n with the subspace S^{n-1} collapsed to a point[1].

[1] This notion is introduced and studied in Aufgaben 17 and 18.



13.7 Question Which are the topological spaces that admit a cell structure with at most two cells?

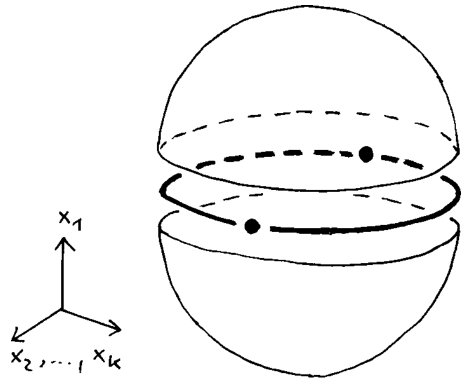
(4) There is another simple cell structure on S^n that uses exactly two cells of each dimension between 0 and n. The corresponding cell filtration is the natural one by subspheres

$$S^0 \subset S^1 \subset \cdots \subset S^{n-1} \subset S^n$$

and S^k is inductively obtained from $S^{k-1} = S^k \cap \big(\{0\} \times \mathbb{R}^k\big)$ by attaching two copies of D^k.

$$D^k \ni (x_1, \ldots, x_k) \longmapsto \Big(\pm\sqrt{1 - \textstyle\sum_j x_j^2}\,, x_1, \ldots, x_k\Big) \in S^k \subset \mathbb{R}^{k+1}$$

is a natural choice for the two characteristic mappings.



The principal interest in this cell structure lies in the obvious fact that it is invariant under the antipodal map $S^n \ni x \mapsto -x \in S^n$, and therefore induces a cell structure on $\mathbb{R}P^n$ comprising one cell $\mathbb{R}P^k \setminus \mathbb{R}P^{k-1}$ in each dimension $k \in \{0, 1, \ldots, n\}$.

(5) The same idea allows to put a cell structure on the complex projective spaces. The underlying filtration of $\mathbb{C}P^n$ is

$$\mathbb{C}P^0 \subset \mathbb{C}P^1 \subset \cdots \subset \mathbb{C}P^{n-1} \subset \mathbb{C}P^n$$

and for each $k \in \{0, 1, \ldots, n\}$

$$\mathbb{C}^k \supset D^{2k} \ni (z_1, \ldots, z_k) \longmapsto \Big[\sqrt{1 - \textstyle\sum_j |z_j|^2} \,:\, z_1 \,:\, \cdots \,:\, z_k\Big] \in \mathbb{C}P^k$$

is a characteristic mapping that presents $\mathbb{C}P^k$ as $\mathbb{C}P^{k-1}$ with an attached $2k$-cell.

Let us briefly discuss cell structures that arise from the standard categorical constructions in topology.

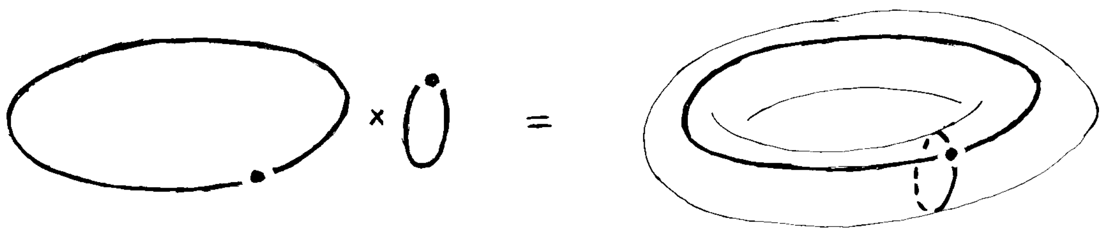
You will have little doubt as to what will be meant by a *subcomplex* of a cell complex X: a subspace $S \subset X$ that inherits a cell structure by restriction. This happens if and only if S is a union of (open) cells of X which is closed as a topological subspace (or, equivalently: if for each open cell $e \subset S$ the closure $\overline{e} \subset X$ is the union of e and other open cells that belong to S). It is clear that the intersection and the union of given subcomplexes of X are subcomplexes again. Simple examples of subcomplexes are provided by the union of

X^{k-1} with any collection of k-cells — such subcomplexes naturally arise in proofs using induction on the number of cells in X.

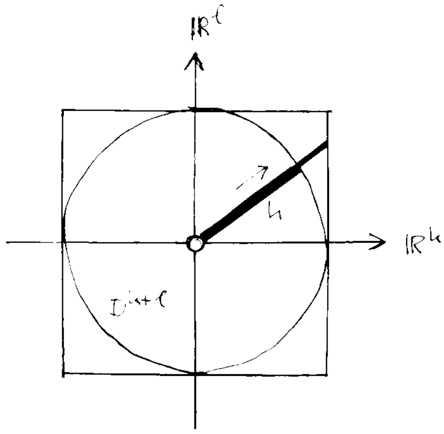
The situation with quotient spaces is similar in so far as a cell structure on X will descend to one on a quotient $X/\sim$ but in very special circumstances. We have encountered one of them in Example 13.6(4) where the equivalence relation comes from a symmetry of X that preserves the cell structure. Another common case is that of collapsing a subcomplex $S \subset X$ to a point.

Coming to topological sums, it is trivial that the sum of two (or finitely many) cell complexes again carries a canonical cell structure. On the other hand products require a bit of care. If X and Y are finite cell complexes then the *product filtration*

$$(X\times Y)^m := \bigcup_{k+l=m} X^k \times Y^l$$



is the obvious candidate for a cell filtration on $X\times Y$. But a choice has to be made in order to define the product of the corresponding cell structures. Let e be a k-cell of X, and f an l-cell of Y, with $k+l=m$, and let $\Phi\colon D^k \to X$ and $\Psi\colon D^l \to Y$ be their characteristic maps. The cartesian product $\Phi\times\Psi\colon D^k\times D^l \longrightarrow X\times Y$ does not qualify as a characteristic map since its domain is not D^m but only homeomorphic to it. In order to be able to talk about a well-defined product cell structure let us agree to compose $\Phi\times\Psi$ with a particular homeomorphism $h\colon D^m \approx D^k\times D^l$: that which for each $z \in S^{m-1}$ linearly stretches the line segment $\{tz \mid t \in [0,1]\} \subset D^m \subset \mathbb{R}^m$ onto the corresponding segment in $D^k\times D^l \subset \mathbb{R}^m$.



Using Theorem 13.5 it is easy to verify that the collection of all such $(\Phi\times\Psi)\circ h$ does determine a cell structure on $X\times Y$.

Remark This definition of the product cell structure is less than perfect, for it fails to make the product associative. A more radical solution would have been to use cubes rather than disks in the definition of cell complexes — equivalent of course but against all tradition in topology. For most purposes the notion of cell structure is unnecessarily fine anyway and the question of what the best definition of the product structure is will turn out to be largely irrelevant.

14 Homotopy

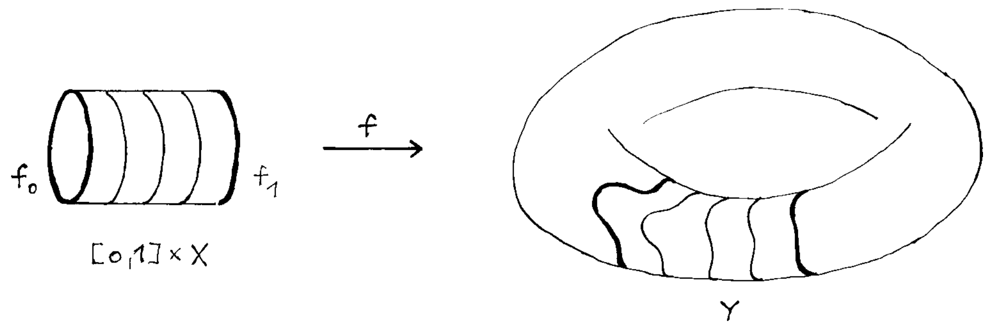
We now have been playing around with topological notions for quite some time, and I hope you agree that this often is great fun. On the other hand if you recall the discussion from the end of Section 2 we seem to have made little progress toward the goals stated there. These included the proposal of defining invariants of topological spaces that, ideally, would allow to classify them up to homeomorphy. The main problem already mentioned there persists: between two given spaces X and Y there tend to be just too many continuous mappings $f: X \longrightarrow Y$.

The notion of homotopy to be introduced in this section solves this problem, and is a decisive step towards the final goal. The idea is to group together, and eventually identify, all f that can be deformed one into another. The formal definition is quite simple and runs as follows.

14.1 Definition Let $f_0, f_1: X \longrightarrow Y$ be continuous maps.

- A homotopy from f_0 to f_1 is a continuous map $f: [0,1]\times X \longrightarrow Y$ with

$$f(0,x) = f_0(x) \text{ and } f(1,x) = f_1(x) \quad \text{for all } x \in X.$$

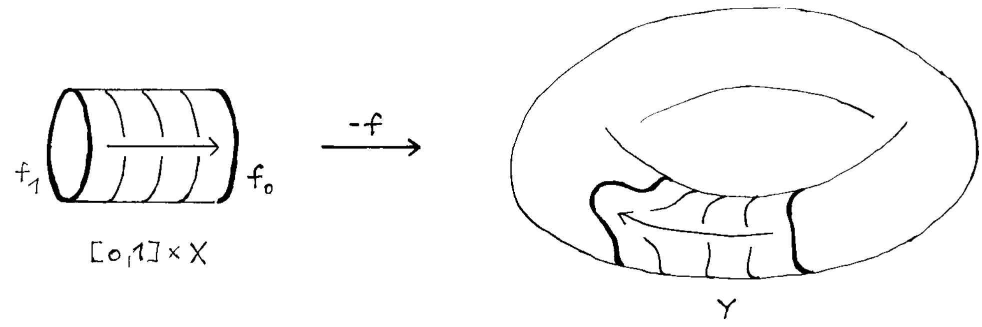


f_0 is called homotopic to f_1 if a homotopy from f_0 to f_1 exists, this is written $f_0 \simeq f_1$.

- If f is a homotopy from f_0 to f_1 then the inverse homotopy from f_1 to f_0 is defined by

$$[0,1]\times X \ni (t,x) \longmapsto f(1-t,x) \in Y,$$

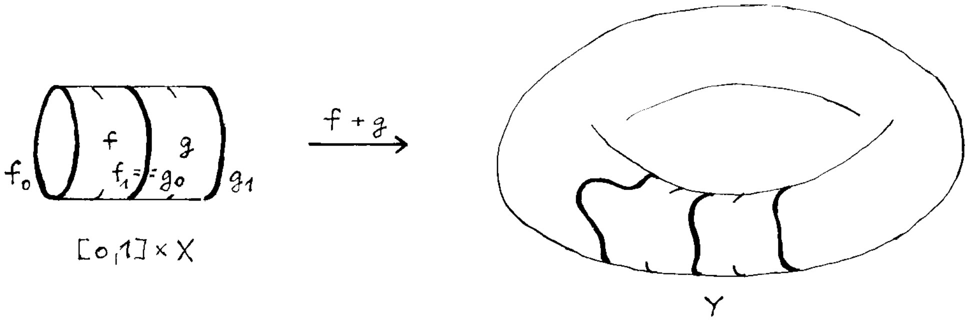
and usually written $-f$, provided this does not conflict with other uses of the minus sign in the context.



- If f is a homotopy from f_0 to f_1, and g one from $g_0 := f_1$ to a third continuous map $g_1: X \longrightarrow Y$ then the sum of f and g is the homotopy h from f_0 to g_1 that sends $(t,x) \in [0,1]\times X$ to

$$h(t,x) = \begin{cases} f(2t,x) & \text{for } t \leq 1/2, \\ g(2t-1,x) & \text{for } t \geq 1/2. \end{cases}$$

One may write $h = f+g$ if the context allows it.



No doubt you have encountered the idea of homotopy before, in connection with path integrals. While homotopies of paths will be discussed later in this section, let me point out here that the notion of path itself is a special case of homotopy: if $X = \{\circ\}$ is a one-point space then a homotopy from $f_0: \{\circ\} \longrightarrow Y$ to $f_1: \{\circ\} \longrightarrow Y$ is nothing but a path in Y from the point $f_0(\circ)$ to $f_1(\circ)$ while inversion and addition of homotopies are just straightforward generalisations of the corresponding notions for paths as explained in 5.3.

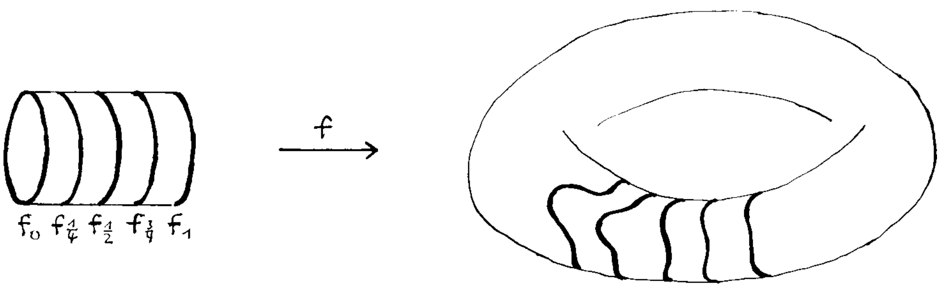
Before we proceed to study homotopy in more detail let as agree on a number of conventions that are commonly made in this field.

14.2 Conventions in homotopy theory (and beyond)

- Maps between topological spaces will always be understood to be continuous unless the contrary is stated explicitly.
- The compact unit interval $[0,1]$ is denoted by the letter I which will be reserved for this purpose.
- If $f: I\times X \longrightarrow Y$ is a homotopy and $t \in I$ then f_t denotes the map

$$f_t: X \longrightarrow Y; \quad f_t(x) = f(t,x).$$

The last convention suggests an alternative interpretation of a homotopy from f_0 to f_1: the assignment $t \longmapsto f_t$ may be thought of as a path in the space of functions from X to Y, joining the points f_0 and f_1. This point of view can often be made precise if a suitable topology is put on the space of continuous mappings, but even as it stands it is useful on the intuitive level.



For fixed topological spaces X and Y homotopy is an equivalence relation $\simeq$ on $\mathbf{Top}(X,Y)$, the set of continuous maps from X to Y. This follows at once from the existence of the inverse and the sum of homotopies, and, of course, of the "constant" homotopy $f_0 \circ \mathrm{pr}_2: I \times X \longrightarrow Y$ joining f_0 to itself. The relation $\simeq$ is compatible with composition:

14.3 Observation Let $f_0, f_1: X \longrightarrow Y$ and $g_0, f_1: Y \longrightarrow Z$ be continuous maps. Then $f_0 \simeq f_1$ and $g_0 \simeq g_1$ implies $g_0 \circ f_0 \simeq g_1 \circ f_1$.

Proof Choose homotopies $f: I \times X \longrightarrow Y$ and $g: I \times Y \longrightarrow Z$. The composition

$$I \times X \xrightarrow{(\mathrm{pr}_1, f)} I \times Y \xrightarrow{g} Z$$

is a homotopy from $g_0 \circ f_0$ to $g_1 \circ f_1$.

This suggests to make the equivalence classes of maps under homotopy — *homotopy classes* for short — the morphisms of a new category.

14.4 Definition There is a homotopy category **hTop** associated to the category **Top**. Its objects are the same as those of **Top**, but for any two topological spaces $X, Y \in |\mathbf{hTop}| = |\mathbf{Top}|$ the morphisms from X to Y in this new category are the homotopy classes of maps from X to Y:

$$\mathbf{hTop}(X, Y) = \mathbf{Top}(X, Y)/\simeq$$

$\mathbf{hTop}(X, Y)$ is also called a homotopy set and often written as $[X, Y]$. Units and the composition law in **hTop** are defined by

$$1_X = [1_X] \text{ and } [g][f] = [gf]$$

using representatives in **Top**.

A map $f: X \longrightarrow Y$ is called a homotopy equivalence if $[f] \in \mathbf{hTop}(X, Y)$ is an isomorphism, that is if there exists a $g: Y \longrightarrow X$ such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$: such a map g is called a homotopy inverse to f. If a homotopy equivalence from X to Y exists then X and Y are said to be homotopy equivalent, or to have the same homotopy type. Finally, X is called contractible if it is homotopy equivalent to a one-point space.

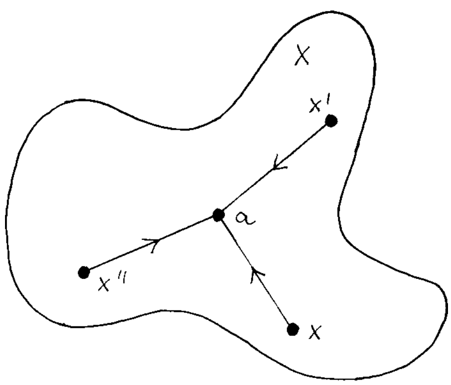
14.5 Examples (1) If $X \subset \mathbb{R}^n$ is non-empty and convex or, more generally, star-shaped with respect to some point $a \in X$ then X is contractible, for the inclusion of $\{a\}$ in X and the unique map from X to $\{a\}$

$$j: \{a\} \hookrightarrow X \quad \text{and} \quad p: X \longrightarrow \{a\}$$

are homotopy inverse to each other. Indeed $pj = 1_{\{a\}}$ holds even in **Top**, and

$$h: I \times X \longrightarrow X; \quad (t, x) \longmapsto ta + (1-t)x$$

is a homotopy that joins the identity of X to the constant map $jp: X \longrightarrow X$.

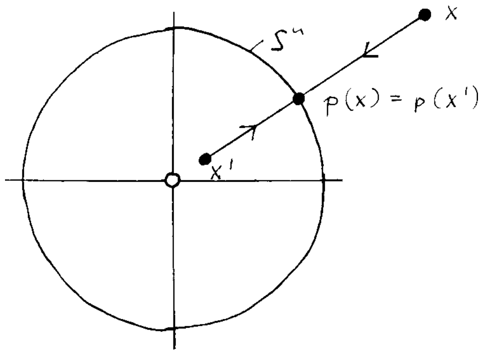


(2) The inclusion mapping $j\colon S^n \longrightarrow \mathbb{R}^{n+1}\setminus\{0\}$ admits the projection

$$\mathbb{R}^{n+1}\setminus\{0\} \ni x \overset{p}{\longmapsto} \frac{1}{|x|}x \in S^n$$

as a homotopy inverse. While again $pj = 1_{S^n}$ in **Top** the identity map of $\mathbb{R}^{n+1}\setminus\{0\}$ is homotopic to the composition jp by the homotopy

$$h\colon I\times\big(\mathbb{R}^{n+1}\setminus\{0\}\big) \longrightarrow \mathbb{R}^{n+1}\setminus\{0\}; \quad (t,x) \longmapsto \Big(\frac{t}{|x|} + 1-t\Big)\cdot x.$$



(3) Removing a point from an n-dimensional real or complex projective space leaves a topological space that is homotopy equivalent to a projective space of dimension $n-1$. This is easily seen if one considers the standard model

$$X := KP^n \setminus \big\{[1:0:\,\cdots\,:0]\big\}$$

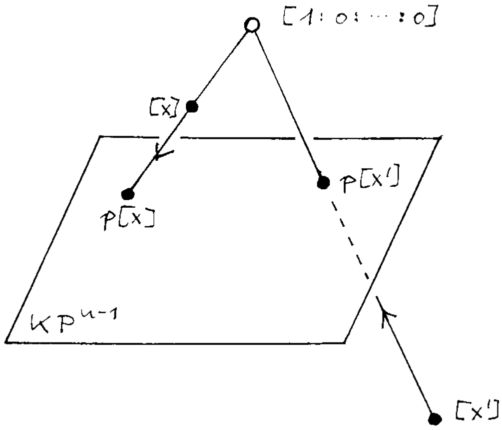
where the embedding

$$KP^{n-1} \ni [x_1:\,\cdots\,:x_n] \overset{j}{\longmapsto} [0:x_1:\,\cdots\,:x_n] \in X$$

and the projection

$$X \ni [x_0:x_1:\,\cdots\,:x_n] \overset{p}{\longmapsto} [x_1:\,\cdots\,:x_n] \in KP^{n-1}.$$

are homotopy inverse to each other. To algebraic geometers p is known as the linear projection from $[1:0:\,\cdots\,:0]$ to a complementary hyperplane.



(4) A more sophisticated example is provided by the polar decomposition of invertible matrices: every $x \in GL(n,\mathbb{R})$ can uniquely be written as $x = u\cdot s$ with an orthogonal matrix $u \in O(n)$ and a positive definite symmetric matrix $s \in \mathrm{Sym}^+(n,\mathbb{R})$. In other words the multiplication map

$$O(n) \times \mathrm{Sym}^+(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$$

is a bijection, and in fact a homeomorphism since the symmetric part s of $us = x$ can be computed from x by the formula $s = \sqrt{x^{t}x}$. In terms of this homeomorphism the inclusion $j: O(n) \hookrightarrow GL(n, \mathbb{R})$ becomes the embedding

$$O(n) \ni u \longmapsto (u, 1) \in O(n) \times \{1\} \subset O(n) \times \mathrm{Sym}^{+}(n, \mathbb{R}).$$

Since $\mathrm{Sym}^{+}(n, \mathbb{R})$ is a contractible space by example (1), this embedding and, a fortiori, j are homotopy equivalences. It can be shown that more generally every Lie group is homotopy equivalent to any one of its maximal compact subgroups.

As a consequence of example (1) the euclidean spaces $\mathbb{R}^n$ are all homotopy equivalent to each other, and in fact, being contractible spaces, are trivial objects in the homotopy category **hTop**. This seems to suggest that the homotopy category might rather be too coarse in order to be useful. If homotopy does not even respect such a basic intuitive notion as dimension, how can it ever be used to prove, for instance, our still pending claim that $\mathbb{R}^m$ is never homeomorphic to $\mathbb{R}^n$ for different m and n ? The remedy is already contained in the way we have solved the special case $m = 1$ using connectedness (see 5.6): if nothing can be gained by studying the homotopy type of $\mathbb{R}^n$ itself example (2) shows that the space obtained from $\mathbb{R}^n$ by removing an arbitrary point is homotopy equivalent to S^{n-1}. As it will turn out that two spheres of different dimension are not of the same homotopy type we will be able to conclude that euclidean spaces of different dimension cannot be homeomorphic.

In practice homotopy equivalences are often embeddings, as indeed has been the case in all our examples, or they can be factored into such. In this context the following terminology is common and useful.

14.6 Definition Let **C** be a category. If

$$X \underset{r}{\overset{s}{\rightleftarrows}} Y$$

are morphisms in **C** such that $rs = 1_X$ (but not necessarily $sr = 1_Y$) then s is called a section of r while r is called a retraction of s.

14.7 Observation Let Let X and Y be topological spaces. If $s: X \longrightarrow Y$ admits a retraction in **Top** then s is an embedding.

Proof Let $r: Y \longrightarrow X$ be a retraction for s. The composition $s(X) \hookrightarrow Y \xrightarrow{r} X$ is a continuous map which is inverse to $X \ni x \longmapsto s(x) \in s(X)$.

Thus in **Top** (and other categories which give sense to the notion of subobject) morphisms that admit a retraction are essentially inclusions of a certain type of subspaces:

14.8 Definition Let Y be a topological space. A retract of Y is a subspace $R \subset Y$ such that the inclusion $j: R \hookrightarrow Y$ admits a retraction, that is, a map $r: Y \longrightarrow R$ with $r|R = \mathrm{id}_R$. If furthermore the retraction r can be chosen to be a homotopy inverse to j then R is called a deformation retract of Y.

14.9 Question Which are the retracts in the category **Ens** of sets and mappings?

14.10 Question Can you think of an example of a topological space Y and a non-empty subspace $S \subset Y$ that is not a retract of Y ?

A retract R of a space Y still has no reason to be a deformation retract: take any one-point subspace R of a disconnected space Y. More interesting examples will have to wait until we know better methods to recognise spaces of non-trivial homotopy type. For positive examples of deformation retracts you need but look at any of those in 14.5.

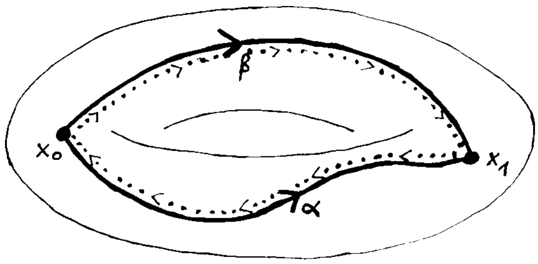
Let us return to the notion of homotopy as such. A well-known theorem from vector analysis states that integrals of a 1-form ω are invariant under homotopies of the path of integration if and only if ω is a closed form. But then this statement refers to a notion of homotopy which is narrower than that of our Definition

14.1, for it does not allow the ends of the paths to move. Indeed according to our notion of "free" homotopy, any two paths $\alpha, \beta: I \longrightarrow X$ with common end points

$$\alpha(0) = \beta(0) = x_0 \in X \quad \text{and} \quad \alpha(1) = \beta(1) = x_1 \in X$$

are homotopic to each other by $h: I \times I \longrightarrow X$,

$$h(s,t) := \begin{cases} \alpha\big((1-2s)\cdot t\big) & \text{for } s \le 1/2, \\ \beta\big((2s-1)\cdot t\big) & \text{for } s \ge 1/2 \end{cases}$$



(the first half contracts α to the constant path at x_0 while the second expands the latter to β). Clearly a notion of homotopy with extra conditions is needed and in order to accommodate a reasonably large variety of situations we make the following

14.11 Definition To every category $\mathbf{C}$ and every fixed object $A \in |\mathbf{C}|$ we assign a new category $\mathbf{C}^A$ of objects and morphisms of $\mathbf{C}$ under A. The objects of $\mathbf{C}^A$ are the morphisms

$$A \to X$$

from A to an arbitrary object $X \in |\mathbf{C}|$, and a morphism in $\mathbf{C}^A$ from $A \xrightarrow{j} X$ to $A \xrightarrow{k} Y$ is a commutative diagram

$$\begin{array}{ccc} & A & \\ {}^{j}\swarrow & & \searrow^{k} \\ X & \longrightarrow & Y \end{array}$$

in $\mathbf{C}$. Composition of two morphisms just means composition of the horizontal arrows, and of course

$$\begin{array}{ccc} & A & \\ {}^{j}\swarrow & & \searrow^{j} \\ X & = & X \end{array}$$

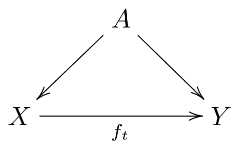
is the unit in $\mathbf{C}^A(A \xrightarrow{j} X, A \xrightarrow{j} X)$.

When dealing with objects under A the meaning of the arrow $A \to X$ is often clear from the context, and then no need for a label. We now specialise to the topological category:

14.12 Definition Fix $A \in |\mathbf{Top}|$, and let $A \to X$ and $A \to Y$ be spaces under A. Two maps under A

$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ X & \xrightarrow[f_0]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ X & \xrightarrow[f_1]{} & Y \end{array}$$

are said to be homotopic in $\mathbf{Top}^A$ if they can be joined by a homotopy under, or relative A: a map $f\colon I\times X \longrightarrow Y$ with the given f_0 and f_1 such that the triangle



commutes for every $t \in I$.

The usual arguments prove that for given $A \to X$ and $A \to Y$ in $\mathbf{Top}^A$, homotopy under A, written $\overset{A}{\simeq}$, is an equivalence relation on the set of morphisms and that therefore a homotopy category $\mathbf{hTop}^A$ with

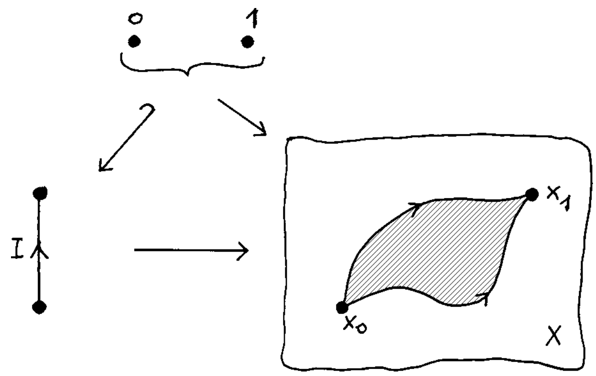
$$|\mathbf{hTop}^A| = |\mathbf{Top}^A|$$

and

$$\mathbf{hTop}^A(A \to X,\, A \to Y) = [A \to X,\, A \to Y]^A := \mathbf{Top}^A(A \to X,\, A \to Y)/\overset{A}{\simeq}$$

is defined. By complete analogy with 14.4 a morphism $f \in \mathbf{Top}^A(A \to X,\, A \to Y)$ is said to be a homotopy equivalence under, or relative A if there exists a morphism $g \in \mathbf{Top}^A(A \to Y,\, A \to X)$ with $gf \overset{A}{\simeq} 1_{A\to X}$ and $fg \overset{A}{\simeq} 1_{A\to Y}$.

14.13 Examples (1) Put $A = \{0,1\} \subset \mathbb{R}$. The objects of $\mathbf{Top}^{\{0,1\}}$ are topological spaces X with two distinguished points $x_0, x_1 \in X$ (the images of 0 and 1), and a morphism in $\mathbf{Top}^A$ from $\{0,1\} \hookrightarrow I$ to such an X is a path that joins x_0 to x_1. In this case homotopy under $\{0,1\}$ reduces to homotopy of paths with fixed end points.



(2) Closed paths or *loops* based at a single point of X are included in (1) as the special case $x_0 = x_1$. An alternative treatment is based on the following

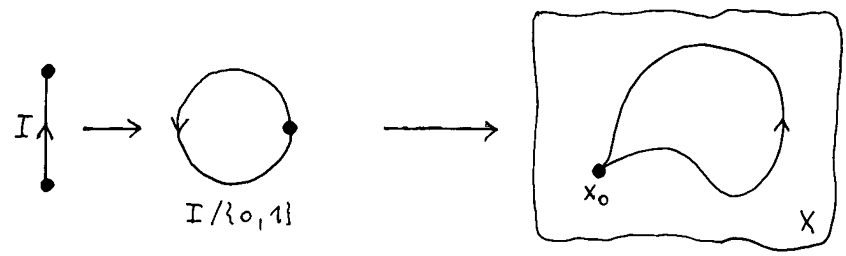
14.14 Terminology and **Notation** Fix once and for all an abstract one-point space $\{\circ\}$. The category $\mathbf{Top}^{\{\circ\}}$ is denoted by $\mathbf{Top}^\circ$ for short. Its objects $\circ \to X$ are called pointed topological spaces and may be considered as pairs (X, x_0) consisting of a space X and a base point $x_0 \in X$ (the image of $\circ$). When no particular name for the base point is needed — as is often the case — then by abuse of language the little circle may be used: $\circ \in X$. Morphisms in $\mathbf{Top}^\circ$ are called pointed or base point preserving maps, and written

$$f\colon (X, \circ) \longrightarrow (Y, \circ).$$

Such a morphism f is called null homotopic if it is homotopic in $\mathbf{Top}^\circ$ to the unique constant map $(X, \circ) \longrightarrow (Y, \circ)$.

Example (2) continued By definition, a loop based at $x_0 \in X$ is a map $I \longrightarrow X$ that sends both 0 and 1 to x_0. In view of the universal property of the quotient topology 10.3 such a loop is essentially just a pointed map

$$\big(I/\{0,1\}, \{0,1\}/\{0,1\}\big) \longrightarrow (X, x_0)$$



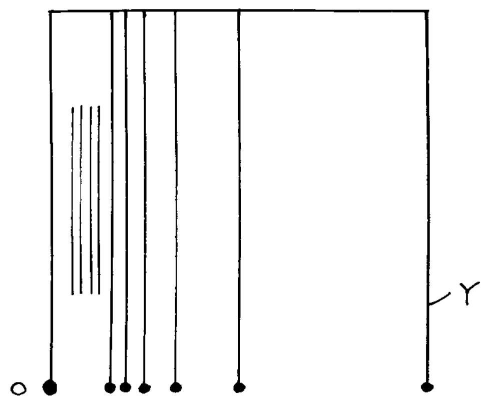
and the corresponding notion of pointed homotopy of loops is exactly what is desired for most applications. At this opportunity note that any space obtained from another by collapsing a subspace has a natural base point: the image of the collapsed space. In the case at hand $I/\{0,1\} \approx S^1$ is a pointed circle.

(3) Two pointed spaces may be homotopy equivalent in **Top** without being homotopy equivalent in **Top**°. As an example consider the (admittedly somewhat exotic) "comb"

$$Y = X \times I \;\cup\; I \times \{1\} \subset \mathbb{R}^2$$

with

$$X = \{0\} \cup \left\{ \frac{1}{k} \;\middle|\; 0 < k \in \mathbb{N} \right\} \subset \mathbb{R}.$$



Y is contractible, for clearly the backbone $I \times \{1\}$ is a deformation retract of Y, and is contractible itself. Thus Y is homotopy equivalent to the one-point space $\{\circ\}$. We now choose the origin $0 \in \mathbb{R}^2$ as the base point of Y. While there is a unique pointed map $j\colon (\{\circ\}, \circ) \longrightarrow (Y, 0)$ it is not too difficult to show that j does not admit a *pointed* homotopy inverse, and that therefore $(Y, 0)$ and $(\{\circ\}, \circ)$ are not homotopy equivalent in **Top**°.

We close this section with a refinement of the notion of deformation retract, suggested by the third example.

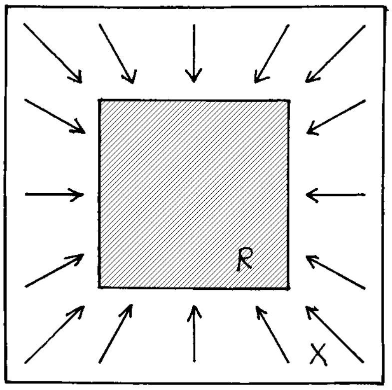
14.15 Definition Let X be a topological space. A <u>strong</u> <u>deformation</u> <u>retract</u> of X is a subspace $R \subset X$ such that

$$\begin{array}{ccc} & R & \\ {\scriptstyle \mathrm{id}} \swarrow & & \searrow \\ R & \hookrightarrow & Y \end{array}$$

is a homotopy equivalence in $\mathbf{Top}^R$. Explicitly, the existence of a homotopy $h: I \times X \longrightarrow X$ with

- $h_0 = \mathrm{id}_X$,
- $h_1(x) \in R$ for all $x \in X$, and
- $h_t(x) = x$ for all $x \in R$

is required.



14.16 Question Which of the examples in this section are about strong deformation retracts?

15 Homotopy Groups

A key observation in topology is that many homotopy sets have a natural group structure. Let us first briefly look at a situation where such a structure does not come as a surprise.

15.1 Definition A topological group is a topological space G that carries a group structure (or, for algebraists, rather a group that carries a topology, which comes down to the same). The two structures are required to be compatible in the sense that

- the multiplication $G \times G \longrightarrow G$, and
- the inversion $G \ni g \longmapsto g^{-1} \in G$

are continuous maps.

15.2 Examples (1) Putting the discrete topology on any group does give examples, but not very interesting ones.

(2) Lie groups are the most important class of topological groups. Their definition is quite analogous to 15.1, with topological space and continuity replaced by manifold and differentiability. Prominent examples are the groups that occur in linear algebra like K^n, $GL(n, K)$, the subgroup of upper triangular invertible $n \times n$-matrices (all with $K = \mathbb{R}$ or $K = \mathbb{C}$), $O(n)$, and others.

Let G be a topological group, and X an arbitrary topological space. It is easily seen that the set of continuous maps $\mathbf{Top}(X, G)$ is a group under pointwise multiplication of values, and that multiplication is compatible with the homotopy relation: for any maps $f_0 \simeq f_1$ and $g_0 \simeq g_1$ in $\mathbf{Top}(X, G)$ one has $f_0 \cdot g_0 \simeq g_0 \cdot g_1$. In other words, the group structure of $\mathbf{Top}(X, G)$ given by pointwise multiplication descends to the homotopy category $\mathbf{hTop}$, making the homotopy set $[X, G]$ a group via

$$[f] \cdot [g] = [f \cdot g] \text{ and, a fortiori } 1 = [1], \text{ and } [f]^{-1} = [f^{-1}]$$

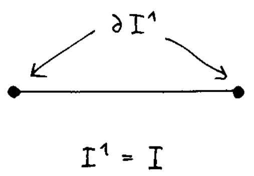
(where f^{-1} denotes, for once, not the inverse but the reciprocal of f). However this is hardly more than could have been expected, and little else can be said about it. Let me just add the one observation that the same construction makes sense in the pointed category $\mathbf{hTop}$ since every topological group G comes with the natural base point $1 \in G$ and thereby is a pointed space. (This is not to say that the two groups $[X, G]$ and $[X, G]^\circ$ were necessarily isomorphic to each other.)

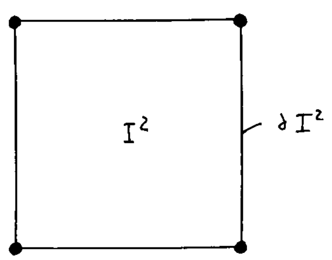
Much more interesting are group structures on homotopy sets that arise from special properties of the domain rather than the target space. They generally are defined in the pointed category, and in this section we consider the case of pointed spheres, an example of particular importance. For the purpose a particular one among the numerous possible representations of topological spheres is most convenient. We let

$$\partial I^n := I^n \setminus (I^n)^\circ \subset \mathbb{R}^n$$

denote the boundary of the n-dimensional standard cube. Thus for instance ∂I^0 is empty, $\partial I^1 = \{0, 1\}$, and

$$\partial I^2 = I \times \{0, 1\} \;\cup\; \{0, 1\} \times I.$$





For each $n \in \mathbb{N}$ the space $I^n/\partial I^n$ obtained from the cube by collapsing its boundary to a point is homeomorphic to D^n/S^{n-1} and hence to S^n. It will be our standard model of the pointed n-sphere, the base point $\circ \in I^n/\partial I^n$ being represented by ∂I^n. Note that pointed maps from $(I^n/\partial I^n, \circ)$ to $(X, \circ)$ correspond to, and may be identified with mappings $f\colon I^n \longrightarrow X$ with constant value $\circ$ on ∂I^n.

15.3 Definition Let $n \in \mathbb{N}$ be positive. For any two morphisms $f, g\colon (I^n/\partial I^n, \circ) \longrightarrow (X, \circ)$ their (homotopy) sum

$$f+g\colon (I^n/\partial I^n, \circ) \longrightarrow (X, \circ)$$

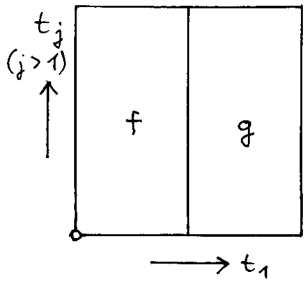
is defined by

$$(f+g)(t_1, t_2, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & \text{for } t_1 \leq 1/2, \\ g(2t_1-1, t_2, \ldots, t_n) & \text{for } t_1 \geq 1/2. \end{cases}$$

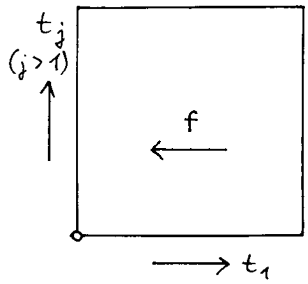
Likewise the formula

$$(-f)(t_1, t_2, \ldots, t_n) = f(1-t_1, t_2, \ldots, t_n)$$

defines the (homotopy) inverse of f.



Definition of $f+g$



and of $-f$

There is in fact nothing new in this definition, for if the first coordinate on I^n is singled out then a map defined on I^n becomes a homotopy of maps defined on I^{n-1}, and 15.3 just a remake of Definition 14.1 in a particular case.

Despite their suggestive names, addition and inversion of maps are a long way from defining a good algebraic structure on $\mathbf{Top}^\circ(I^n/\partial I^n, X)$, but the picture improves drastically as soon as we pass to homotopy classes.

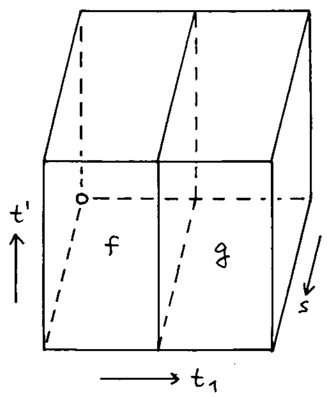
15.4 Theorem Let $n \in \mathbb{N}$ be positive, and $(X, \circ)$ a pointed space. Addition and inversion are compatible with pointed homotopy of maps, and induce a group structure on the homotopy set

$$\mathbf{hTop}^\circ(I^n/\partial I^n, X) = [I^n/\partial I^n, X]^\circ.$$

Proof We abbreviate $(t_1, t_2, \ldots, t_n) = (t_1, t')$ since the definitions really involve but the first coordinate on I^n. If $f\colon I \times I^n \longrightarrow X$ is a pointed homotopy from f_0 to f_1, and $g\colon I \times I^n \longrightarrow X$ one from g_0 to g_1 then

$$I \times I^n \ni (s, t_1, t') \longmapsto \begin{cases} f(s, 2t_1, t') & \text{for } t_1 \leq 1/2, \\ g(s, 2t_1-1, t') & \text{for } t_1 \geq 1/2 \end{cases}$$

defines a pointed homotopy from f_0+g_0 to f_1+g_1.



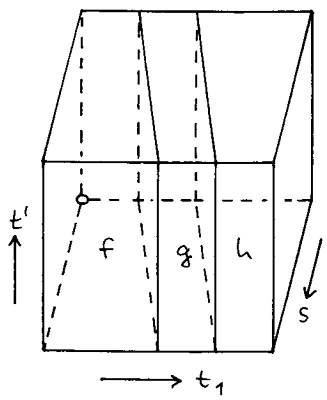
Similarly

$$I\times I^n \ni (s,t_1,t') \longmapsto f(s,1-t_1,t') \in X$$

joins $-f_0$ to $-f_1$. It remains to verify that the induced addition on $[I^n/\partial I^n, X]^\circ$ satisfies the group axioms.

Associativity: A pointed homotopy from $(f+g)+h$ to $f+(g+h)$ is given by

$$I\times I^n \ni (s,t_1,t') \longmapsto \begin{cases} f\big(\frac{4t_1}{1+s}, t'\big) & \text{for } t_1 \in \big[0, \frac{1+s}{4}\big], \\ g(4t_1-1-s, t') & \text{for } t_1 \in \big[\frac{1+s}{4}, \frac{2+s}{4}\big], \\ h\big(1-\frac{4(1-t_1)}{2-s}, t'\big) & \text{for } t_1 \in \big[\frac{2+s}{4}, 1\big]. \end{cases}$$

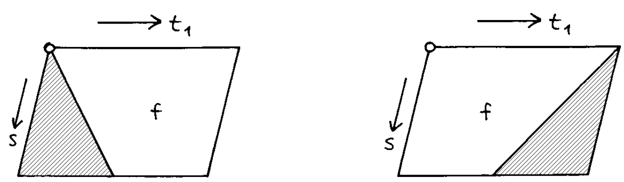


Neutral element: It is represented by the unique constant map into $(X,\circ)$. Pointed homotopies joining f to $\circ+f$, and to $f+\circ$ are realised by the respective formulae

$$I\times I^n \ni (s,t_1,t') \longmapsto \begin{cases} \circ & \text{for } t_1 \le s/2, \\ f\big(\frac{2t_1-s}{2-s}, t'\big) & \text{for } t_1 \ge s/2 \end{cases}$$

and

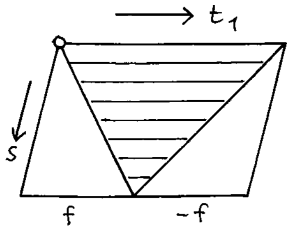
$$I\times I^n \ni (s,t_1,t') \longmapsto \begin{cases} f\big(\frac{2t_1}{2-s}, t'\big) & \text{for } t_1 \le 1-s/2, \\ \circ & \text{for } t_1 \ge 1-s/2. \end{cases}$$



The neutral element will of course be denoted by $0 = [\circ] \in [I^n/\partial I^n, X]^\circ$.

Inverse elements: We must verify that $-[f]$ is inverse to $[f]$ indeed, or that $f+(-f)$ and $(-f)+f$ are both null homotopic. In view of the fact that $f = -(-f)$ by definition, it suffices to consider the first sum, for which the homotopy

$$I \times I^n \ni (s, t_1, t') \longmapsto \begin{cases} f(1-2t_1, t') & \text{for } t_1 \in \left[0, \frac{s}{2}\right], \\ f(1-s, t') & \text{for } t_1 \in \left[\frac{s}{2}, 1-\frac{s}{2}\right], \\ f(2t_1-1, t') & \text{for } t_1 \in \left[1-\frac{s}{2}, 1\right] \end{cases}$$



does the job.

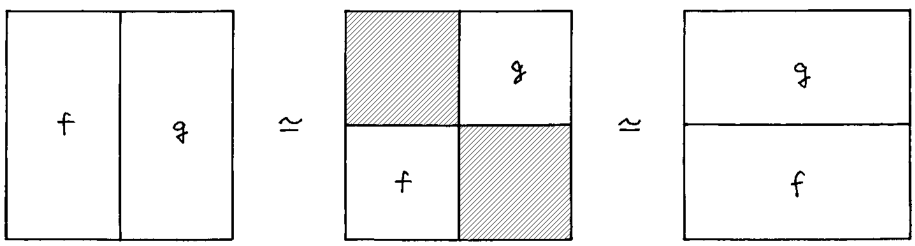
Let us now have a more critical look at Definition 15.3. The special role played there by the first coordinate might likewise have been assigned to any of the others, and we thus have in fact several competing group structures on $[I^n/\partial I^n, X]^\circ$. While it is clear that all these structures are isomorphic to each other, the following surprising theorem tells us much more.

15.5 Theorem Let $(X, \circ)$ be a pointed space, and assume $n \geq 2$. Then

- the group structure on $[I^n/\partial I^n, X]^\circ$ is always the same, whichever coordinate on I^n is used to define the addition, and
- $[I^n/\partial I^n, X]^\circ$ is an abelian group.

Proof We reserve $+$ for addition of maps as in 15.3, and temporarily use $*$ for that with respect to some fixed coordinate other than the first. Of course, the conclusions of Theorem 15.4 hold for both structures. Therefore, for all pointed maps $f, g\colon (I^n/\partial I^n, \circ) \longrightarrow (X, \circ)$ we have the relation

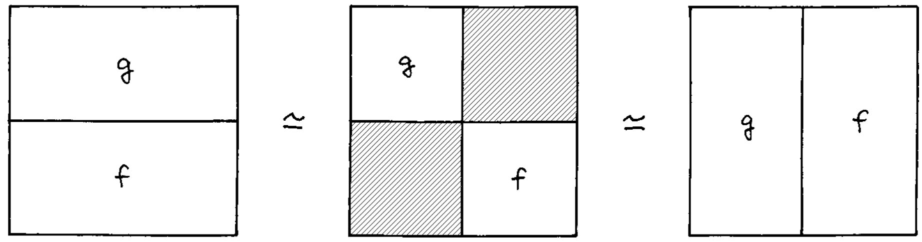
$$f + g \overset{\circ}{\simeq} (f * \circ) + (\circ * g) = (f + \circ) * (\circ + g) \overset{\circ}{\simeq} f * g$$



in $\mathbf{Top}^\circ$: note that the two inner terms are not just homotopic but truely the same. This proves the first half of the theorem.

Since the homotopy relation above may be continued as

$$f * g \overset{\circ}{\simeq} (\circ + f) * (g + \circ) = (\circ * g) + (f * \circ) \overset{\circ}{\simeq} g + f$$



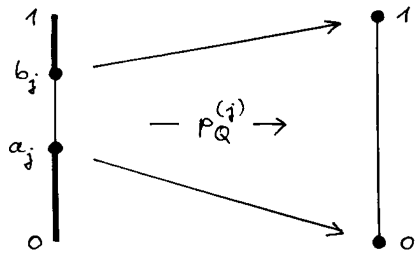
commutativity of the group structure now follows too.

It is instructive and useful to rewrite the addition of homotopy classes in a more general manner which makes its commutativity geometrically evident. Let, for given $n \in \mathbb{N}$,

$$Q = \prod_{j=1}^{n} [a_j, b_j] \subset I^n \quad \text{with } b_1 - a_1 = b_2 - a_2 = \cdots = b_n - a_n > 0$$

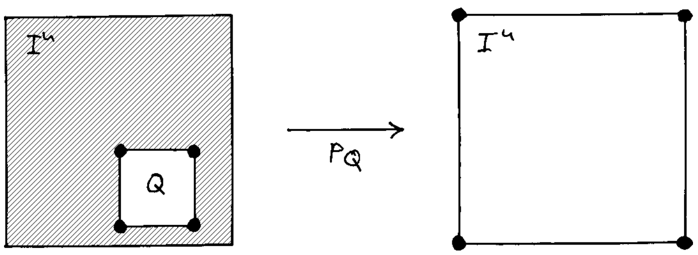
be a subcube of the unit cube I^n. For each $j \in \{1, \ldots, n\}$ denote by $p_Q^{(j)}: I \longrightarrow I$ the map that linearly stretches $[a_j, b_j]$ to the unit interval:

$$p_Q^{(j)}(t) = \begin{cases} 0 & \text{for } t \in [0, a_j] \\ \frac{t-a_j}{b_j-a_j} & \text{for } t \in [a_j, b_j] \\ 1 & \text{for } t \in [b_j, 1] \end{cases}$$



The cartesian product

$$p_Q := \prod_{j=1}^{r} p_Q^{(j)}: I^n \longrightarrow I^n$$



blows up Q to the standard cube and sends ∂I^n into itself, and we have the

15.6 Lemma The pointed map $\overline{p}_Q: (I^n/\partial I^n, \circ) \longrightarrow (I^n/\partial I^n, \circ)$ induced by p_Q is homotopic in the category **Top°**, to the identity of I^n.

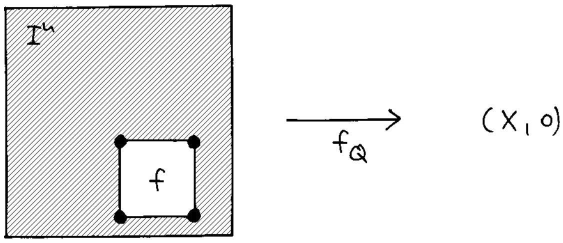
Proof p_Q sends each face of the cube Q into the corresponding face of I^n. Therefore the homotopy by linear connection, $I \times I^n \ni (s,t) \longmapsto s\,t + (1-s)\,p_Q(t) \in I^n$ still respects the boundary ∂I^n, and so induces a pointed homotopy $I \times (I^n/\partial I^n) \longrightarrow I^n/\partial I^n$.

.

15.7 Question Explain why this last, seemingly trivial conclusion depends on the fact that like the quotient map $q: I^n \longrightarrow I^n/\partial I^n$, the cartesian product $\mathrm{id} \times q: I \times I^n \longrightarrow I \times (I^n/\partial I^n)$ also is an identification.

15.8 Corollary Every pointed mapping $f: (I^n/\partial I^n, \circ) \longrightarrow (X, \circ)$ is homotopic in $\mathbf{Top}^\circ$ to the map $f_Q: (I^n/\partial I^n, \circ) \longrightarrow (X, \circ)$ defined by $f_Q = f \circ \overline{p}_Q$. In particular the class of f_Q in $[I^n/\partial I^n, X]^\circ$ does not depend on the choice of the cube $Q \subset I^n$.

The corollary shows how to compress the homotopy information carried by f to an arbitrarily prescribed (often small) subcube Q of I^n, for the representative f_Q sends everything outside Q to the base point of X.



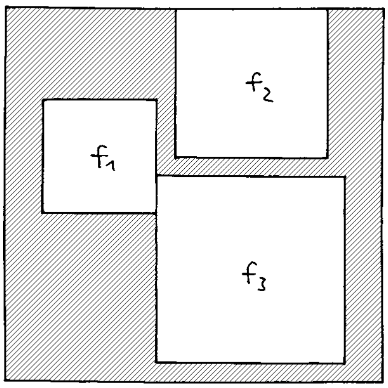
We are now ready to give the proposed description of sums of homotopy classes.

15.9 Proposition For a pointed space $(X, \circ)$ and $n \geq 2$ let r pointed maps

$$f_1, \ldots, f_r: (I^n/\partial I^n, \circ) \longrightarrow (X, \circ)$$

be given. Choose r subcubes $Q_1, \ldots, Q_r \subset I^n$ such that the interiors Q_j° are pairwise disjoint for $j = 1, \ldots, r$, and let $f: I^n \longrightarrow X$ be the map which for each j coincides with f_{Q_j} on Q_j, and sends $I^n \setminus \bigcup_{j=1}^r Q_r$ to the base point in X. Then in $[I^n/\partial I^n, X]^\circ$ one has

$$[f] = [f_1] + \cdots + [f_r].$$

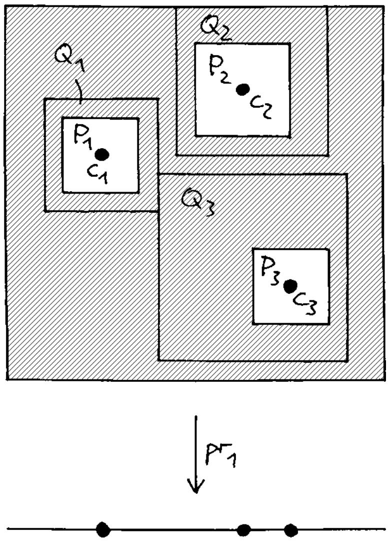


Proof For each j choose a subcube $P_j \subset I^n$ with

$$P_j \subset Q_j^\circ \quad \text{(interior with respect to } \mathbb{R}^n\text{)}$$

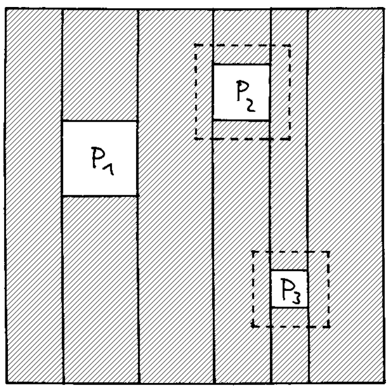
and let $c_j \in P_j$ denote the centre of P_j. Applying Lemma 15.6 to the inclusion $P_j \subset Q_j$ (in place of $Q \subset I^n$ as stated) we may replace f_{Q_j} in the definition of f by f_{P_j} without changing the homotopy class of f. We may also shift P_j within Q_j° so as to make the r projections $\mathrm{pr}_1(c_j) \in I$ all distinct. After renumbering the f_j we thus have

$$\mathrm{pr}_1(c_1) < \mathrm{pr}_1(c_2) < \cdots < \mathrm{pr}_1(c_r).$$



By yet another application of 15.6 we scale down each P_j to an even smaller cube with the same centre such that

$$\mathrm{pr}_1(P_i^\circ) \cap \mathrm{pr}_1(P_j^\circ) = \emptyset \quad \text{for all } i \neq j.$$



The resulting map $(I^n/\partial I^n, \circ) \longrightarrow (X, \circ)$ is still homotopy equivalent to $[f]$, and as should now be clear it represents the sum $[f_1] + \cdots + [f_r]$ formed with respect to the first coordinate on I^n.

The group structure on the homotopy sets $[I^n/\partial I^n, X]^\circ$ was discovered and investigated by Henri Poincaré in 1895 for $n = 1$, and by Witold Hurewicz in 1935 for general n. The traditional names of these groups are due to their discoverors:

15.10 Definition For any positive $n \in \mathbb{N}$ and any pointed topological space $(X, \circ)$ the group

$$\pi_n(X, \circ) := [I^n/\partial I^n, X]^\circ$$

is called the n-th homotopy group of $(X, \circ)$. Alternatively the first homotopy group $\pi_1(X, \circ)$ is called the fundamental group of $(X, \circ)$. The notation π_n is extended to the homotopy set $\pi_0(X, \circ)$ (which, in general, is not a group).

15.11 Question Explain the geometric meaning of $\pi_0(X, \circ)$.

Let us return to the theme discussed at the beginning of this section. If X happens to be a topological group then for $n \geq 1$ a second group structure is induced on the homotopy group $\pi_n(X, 1)$ and we may ask ourselves whether the two structures are related. Surprisingly, this situation turns out to be quite analogous to that of Theorem 15.5.

15.12 Theorem Let X be a topological group, and assume $n \geq 1$. Then

- the homotopy group structure on $\pi_n(X, 1)$ coincides with that induced from X, and
- $\pi_n(X, 1)$ is abelian even for $n = 1$.

Proof Re-read the proof of Theorem 5.5, interpreting the asterisk as multiplication induced from X, and $\circ$ as the unit element $1 \in X$. You will find that the formulae hold true even if you now must do without the illustrations.

Remark In general the fundamental group π_1 is definitely non-abelian, and most authors therefore switch to multiplicative notation in that particular instance. For the same reason the theory of the fundamental group has a flavor which is quite distinct from that of the higher homotopy groups and other algebraic aspects of topology. In this course I will not treat the fundamental group in any detail: while π_1 is interesting and important in its own right it is not really at the heart of the story. When you have the need, and find the time do feel encouraged to study the excellently readable book [Massey] which largely deals with this subject.

16 Functors

The introduction of the homotopy groups in the previous section is a good opportunity to discuss functors. Their relation to categories is similar to that of mappings to sets.

16.1 Definition Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A functor from $\mathbf{C}$ to $\mathbf{D}$, written

$$S\colon \mathbf{C} \longrightarrow \mathbf{D}, \quad \text{or} \quad \mathbf{C} \xrightarrow{S} \mathbf{D},$$

assigns to each object $X \in |\mathbf{C}|$ an object $SX \in |\mathbf{D}|$, and to each morphism $f \in \mathbf{C}(X,Y)$ a morphism $Sf \in \mathbf{D}(X,Y)$ in a way which is compatible with composition of morphisms:

$$S1_X = 1_{SX} \quad \text{and} \quad S(gf) = (Sg)(Sf) \text{ whenever } gf \text{ is defined in } \mathbf{C}.$$

16.2 Examples (1) For each $n \in \mathbb{N}$ with $n \geq 2$ there is a functor π_n from $\mathbf{Top}^\circ$ to the category $\mathbf{Ab}$ of abelian groups and homomorphisms which assigns to each pointed space its n-th homotopy group. To make the definition complete we must say how π_n acts on morphisms. Thus let $f\colon (X,\circ) \longrightarrow (Y,\circ)$ be a pointed map. Then $\pi_n f\colon \pi_n(X,\circ) \longrightarrow \pi_n(Y,\circ)$ is defined by

$$\pi_n(X,\circ) = [I^n/\partial I^n, X]^\circ \ni [\varphi] \longmapsto [f \circ \varphi] \in [I^n/\partial I^n, Y]^\circ = \pi_n(Y,\circ).$$

$\pi_n f$ is well-defined as a map (by 14.3), is then immediately seen to be a homomorphism of groups, and the functor axioms trivially hold. Furthermore, again by 14.3 the functor π_n is *homotopy invariant* in the sense that $\pi_n f$ does not depend on the choice of f within the homotopy class $[f] \in [X,Y]^\circ$. Therefore another functor

$$\overline{\pi}_n\colon \mathbf{hTop}^\circ \longrightarrow \mathbf{Ab}$$

is induced which one may prefer as an alternative to π_n itself.

Of course one also has functors π_1 and π_0 with respective values in the category of all groups, and in the category $\mathbf{Ens}^\circ$ of pointed sets.

(2) Many mathematical constructions that one would like to regard as "natural" can be phrased in terms of some functor, even if the latter's action may be less spectacular than that of the homotopy groups. So there clearly is a projection functor from $\mathbf{Top}$ to $\mathbf{hTop}$ which acts identically on objects and sends each map to its homotopy class. Of course there also is a version $Q^A\colon \mathbf{Top}^A \longrightarrow \mathbf{hTop}^A$ of this functor for spaces under a fixed space A, and, for instance, Q° relates the functors of the first example by $\overline{\pi}_n = \pi_n \circ Q^\circ$ — where the notion of composition of two functors is the obvious one.

The so-called *forgetful functors* act on objects and morphisms by stripping them of some or all their particular structures. Typical examples are defined on categories like $\mathbf{Top}$, or $\mathbf{Ab}$, or $\mathbf{Lin}_K$ and take values in $\mathbf{Ens}$, assigning to a topological space, a group, or a vector space its underlying set, and to a morphism just the underlying mapping. The functor from $\mathbf{Top}^\circ$ to $\mathbf{Top}$ which simply forgets about the base point is another, not quite so forgetful example. While forgetful functors usually are uninteresting by themselves they may be helpful or even indispensable for the proper formulation of a more substantial statement.

Other "trivial" functors are the inclusion functors of subcategories. The term is self-explaining once the notion of subcategory is made precise.

16.3 Definition Let $\mathbf{S}$ and $\mathbf{C}$ be categories. $\mathbf{S}$ is a subcategory of $\mathbf{C}$ if

- each object of **S** is an object of **C**,
- for any $X, Y \in |\mathbf{S}|$ one has $\mathbf{S}(X,Y) \subset \mathbf{C}(X,Y)$, and
- for each $X \in |\mathbf{S}|$ the identity $1_X \in \mathbf{C}(X,X)$ belongs to $\mathbf{S}(X,X)$.

S is a full subcategory of **C** if furthermore

- $\mathbf{S}(X,Y) = \mathbf{C}(X,Y)$ for all $X, Y \in |\mathbf{S}|$.

Thus the category of finite sets and mappings is a full subcategory of **Ens** while any given topological property (think of connectedness, being a Hausdorff, compact, or cell space) defines a full subcategory of **Top**. On the other hand the category of all sets and bijections is a subcategory of **Ens**, but not a full one.

Let us discuss some further examples of functors.

(3) Let **C** be an arbitrary category, and $A \in |\mathbf{C}|$ an object. Then there is a functor

$$\mathbf{C}(A, ?)\colon \mathbf{C} \longrightarrow \mathbf{Ens}$$

which sends $X \in |\mathbf{C}|$ to the set $\mathbf{C}(A,X)$, and the morphism $f \in \mathbf{C}(X,Y)$ to the map

$$\mathbf{C}(A,f)\colon \mathbf{C}(A,X) \longrightarrow \mathbf{C}(A,Y); \quad \varphi \longmapsto f\varphi.$$

The functors $\overline{\pi}_n$ are in fact of this type if the homotopy group structure is ignored: put $\mathbf{C} = \mathbf{hTop}^\circ$ and $A = (I^n/\partial I^n, \circ)$. To be quite precise: the functor $\mathbf{hTop}^\circ(I^n/\partial I^n, ?) = [I^n/\partial I^n, ?]^\circ$ equals $\overline{\pi}_n\colon \mathbf{hTop}^\circ \longrightarrow \mathbf{Ab}$ followed by the forgetful functor from **Ab** to **Ens**.

If the category **C** has products then for each object $A \in |\mathbf{C}|$ there is a functor

$$A \times ?: \mathbf{C} \longrightarrow \mathbf{C}$$

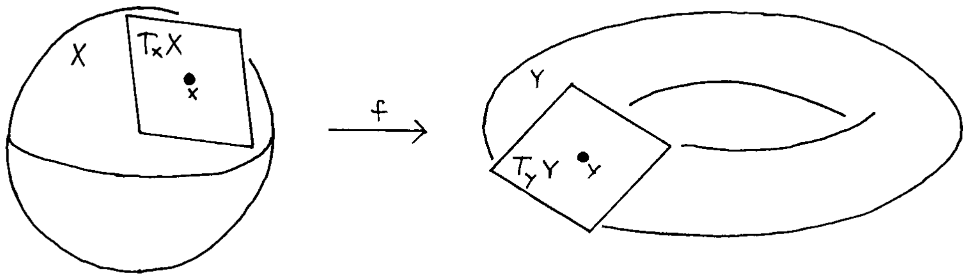
sending $X \in |\mathbf{C}|$ to the product $A \times X$, and the morphism $f \in \mathbf{C}(X,Y)$ to $1_A \times f \in \mathbf{C}(A \times X, A \times Y)$. The case of the sum with a fixed object A is similar. — There is, by the way, no general reason why functors should respect products or sums.

16.4 Question Let K be a field and consider the forgetful functor $\mathbf{Lin}_K \longrightarrow \mathbf{Ens}$. Does it respect products? Does it respect sums?

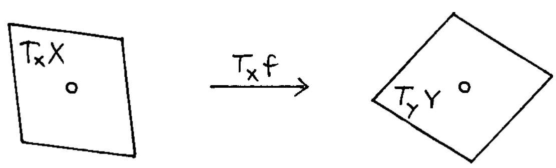
(4) Let **Dif** be the category of manifolds and differentiable mappings. Differentiation defines the *tangent functor*

$$T\colon \mathbf{Dif}^\circ \longrightarrow \mathbf{Lin}_{\mathbb{R}}$$

which sends the pointed manifold (X,x) to its tangent space T_xX, and the differentiable mapping $f\colon (X,x) \longrightarrow (Y,y)$



to its differential $T_xf\colon T_xX \longrightarrow T_yY$ at x.



Note that the functor axioms here reduce to the chain rule of differential calculus. Many other variants of the tangent functor may be derived from this basic version.

(5) Let K be a field. Assigning to a vector space V over K its dual $V^{\vee} := \mathbf{Lin}_K(V, K)$, and to $f \in \mathbf{Lin}_K(V, W)$ the dual linear mapping

$$f^{\vee}: W^{\vee} \longrightarrow V^{\vee}; \quad \varphi \longmapsto \varphi \circ f$$

does *not* define a functor of the category $\mathbf{Lin}_K$ to itself because dualizing reverts the order of composition:

$$(g \circ f)^{\vee} = f^{\vee} \circ g^{\vee}$$

The obvious remedy is to also dualize one copy of $\mathbf{Lin}_K$ by reversing all arrows:

$$|\mathbf{Lin}_K{}^{\vee}| := |\mathbf{Lin}_K| \quad \text{and} \quad \mathbf{Lin}_K{}^{\vee}(V, W) := \mathbf{Lin}_K(W, V)$$

so that dualisation of vector spaces and linear maps becomes a functor $D: \mathbf{Lin}_K \longrightarrow \mathbf{Lin}_K{}^{\vee}$. However this logically perfect solution is rather clumsy in practice and it is much more common to dualize the notion of functor instead:

16.5 Terminology Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A cofunctor S from $\mathbf{C}$ to $\mathbf{D}$ is a functor from $\mathbf{C}$ to the dual category $\mathbf{D}^{\vee}$ defined by $|\mathbf{D}^{\vee}| = |\mathbf{D}|$ and $\mathbf{D}^{\vee}(X, Y) = \mathbf{D}(Y, X)$. Thus in the original categories $\mathbf{C}$ and $\mathbf{D}$ the cofunctor axioms read

$$S1_X = 1_{SX} \quad \text{and} \quad S(gf) = (Sf)(Sg).$$

A more widespread and competing, but not quite compatible terminology refers to functors and cofunctors as co- respectively contravariant functors. Unless the type of a particular functor considered is evident from the context, or irrelevant it should always be clearly stated one or the other way as the notation $S: \mathbf{C} \longrightarrow \mathbf{D}$ is indifferently used for both.

Of course every future definition concerning functors will likewise apply to cofunctors, by dualizing the target category (or the domain, but it often does not matter which).

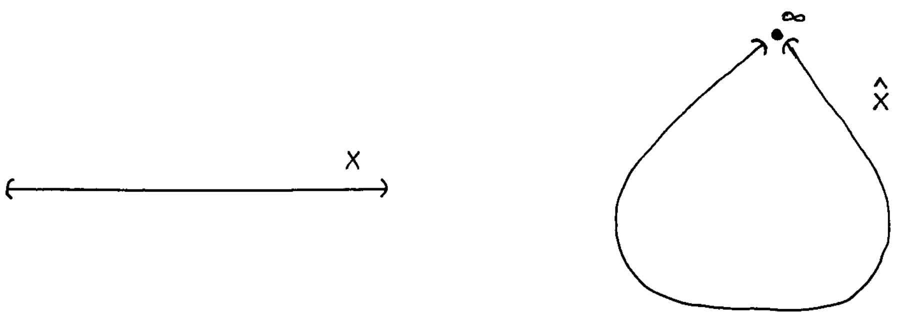
Other examples of functors arise from processes that add a point to a topological space, and these are of more specific interest to us.

(6) The silliest way of adding a single point to a topological space X is by forming the sum

$$X^{+} := \{\circ\} + X.$$

This clearly defines a functor $\mathbf{Top} \longrightarrow \mathbf{Top}^{\circ}$, which is an embedding in the obvious sense and will, despite its simplicity, turn out to be quite useful.

(7) A much more interesting construction that enlarges a space X by one point is the *Alexandroff* or *one point compactification* $\hat{X}$, which is defined for every locally compact Hausdorff space X. The extra point is usually denoted ∞, and the topology on $\{\infty\} \cup \hat{X}$ is such that $\hat{X}$ is a compact Hausdorff space that contains X as an open subspace[1].



[1] Aufgaben 13 and 14 deal with details of the Alexandroff compactification, in particular the fact that its topology is uniquely determined by the stated properties.

The assignment $X \longmapsto \hat{X}$ does not extend to arbitrary continuous maps because for given $f\colon X \longrightarrow Y$ the mapping of sets

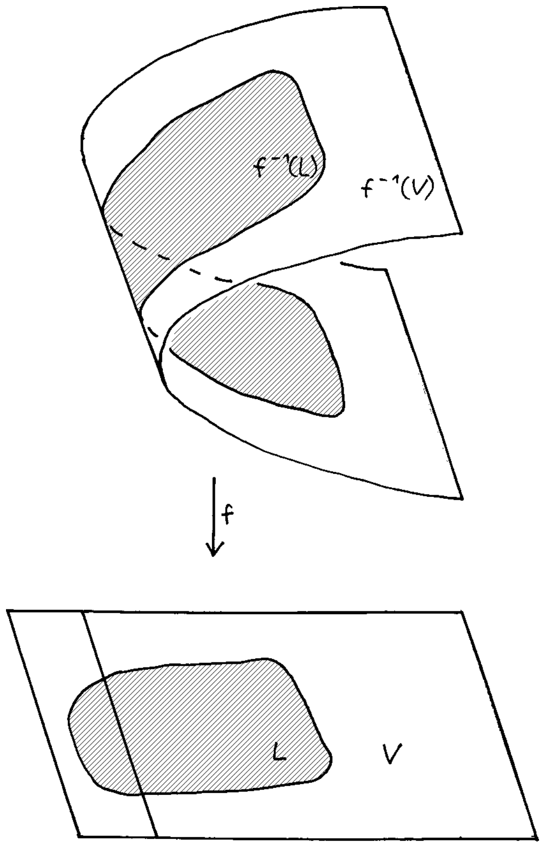
$$\hat{f}\colon \hat{X} \longrightarrow \hat{Y}; \quad x \longmapsto \begin{cases} \infty & \text{if } x = \infty \\ f(x) & \text{else} \end{cases}$$

is discontinuous at ∞ in general: take $X = \mathbb{R}$ and $Y = \{\circ\}$, hence $\hat{X} \approx S^1$ and $\hat{Y} = Y^+ = \{\infty\} + \{\circ\}$ as an example. In order to make Alexandroff compactification a functor we must envisage smaller sets of morphisms.

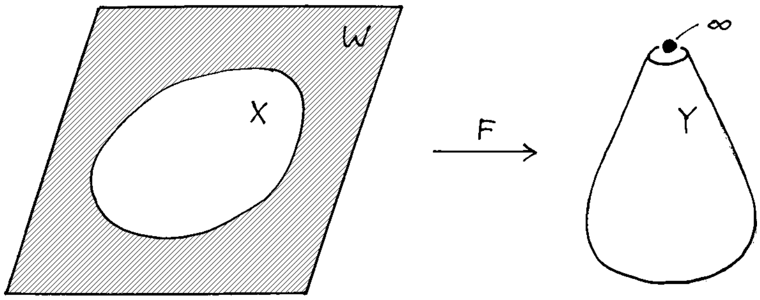
16.6 Definition Let X and Y be locally compact Hausdorff spaces. A continuous mapping $f\colon X \longrightarrow Y$ is called proper if

$$f^{-1}(L) \text{ is compact for each compact subspace } L \subset Y.$$

16.7 Question Verifying that $f^{-1}(L)$ is compact often comes down to a question of boundedness. Illustrate this using $\mathbb{R}^2 \ni (x, y) \longmapsto (x, y^2) \in \mathbb{R}^2$ as an example of a proper map.



16.8 Proposition Let W and Y be locally compact Hausdorff spaces. Let further $X \subset W$ be an open subspace, and $f\colon X \longrightarrow Y$ a proper map. Then the extension $F\colon W \longrightarrow \hat{Y}$ determined by $F|X = f$ and $F(W \backslash X) = \{\infty\}$ is continuous.



Proof Note that X is locally compact, by Proposition 9.12. Since X is open in W it is clear that F is continuous at each point of X, and it remains to show that for each open neighborhood V of ∞ in $\hat{Y}$ the inverse image $F^{-1}(V)$ is open in W. Consider the complement $L := \hat{Y} \setminus V$ which is a compact subspace of Y. Its inverse image $f^{-1}(L)$ is a compact subspace of X, therefore closed in W, and it follows that $F^{-1}(V) = W \setminus f^{-1}(L)$ is open indeed.

Example (7) continued Let $\mathbf{Top^{pr}}$ denote the category of locally compact Hausdorff spaces and proper mappings. Alexandroff compactification does define a functor from $\mathbf{Top^{pr}}$ to $\mathbf{Top}^\circ$ (or, if you prefer, to the category of pointed compact Hausdorff spaces). For if $f \in \mathbf{Top^{pr}}(X,Y)$ is a morphism then the extended mapping $\hat{f}: \hat{X} \longrightarrow \hat{Y}$ is continuous: apply Proposition 16.8 with $W = \hat{X}$.

(8) There also is a homotopy version of the Alexandroff functor. Two morphisms in $\mathbf{Top^{pr}}(X,Y)$ are considered homotopic to each other if they can be joined by a homotopy $h: I \times X \longrightarrow Y$ which itself is a proper map. Proposition 16.8 shows that any such h extends to a homotopy $\hat{h}: I \times \hat{X} \longrightarrow \hat{Y}$. Thus one point compactification takes homotopic morphisms in $\mathbf{Top^{pr}}$ to homotopic morphisms in $\mathbf{Top}^\circ$, and therefore induces a functor

$$\mathbf{hTop^{pr}} \longrightarrow \mathbf{hTop}^\circ$$

between the corresponding homotopy categories.

Let $\mathbf{C}$ and $\mathbf{D}$ be categories. If somebody tells you about a functor $S: \mathbf{C} \longrightarrow \mathbf{D}$ and specifies its action on objects you will often find it easy to guess what it does to morphisms because but one natural choice is at hand: go by yourself through our list of examples. There are of course exceptions to this rule but it is costumary, and usually safe, to omit the action on morphisms from the description of a functor altogether.

Remark Ironically, from a strictly logical point of view things are just the other way round. For in any category $\mathbf{C}$ the sets of morphisms $\mathbf{C}(X,Y)$ are disjoint for distinct $X, Y \in |\mathbf{C}|$, and consequently the object X is determined by the identity $1_X \in \mathbf{C}(X,X)$. Thus it is the objects, not the morphisms of a category that can be dispensed with, and the value $SX \in |\mathbf{D}|$ of a functor $S: \mathbf{C} \longrightarrow \mathbf{D}$ can be recovered from $S1_X \in \mathbf{D}(SX, SX)$.

The general notation $Sf \in \mathbf{D}(SX, SY)$ for the value of $S: \mathbf{C} \longrightarrow \mathbf{D}$ on a morphism $f \in \mathbf{C}(X,Y)$ will not always gracefully specialise to particular functors. While $f^\vee$ for the dual linear map, or f^+ and $\hat{f}$ for the extended continuous maps of Examples (6) and (7) are fine others can be quite cumbersome, like

$$\mathbf{C}(A,X) \xrightarrow{\mathbf{C}(A,f)} \mathbf{C}(A,Y) \quad \text{or} \quad \pi_n(W \times X) \xrightarrow{\pi_n(W \times f)} \pi_n(W \times Y).$$

In such cases a lighter notation is made possible by the following

16.9 Convention If $S: \mathbf{C} \longrightarrow \mathbf{D}$ is a functor and $f: X \longrightarrow Y$ a morphism then $Sf: SX \longrightarrow SY$ may be abbreviated

$$f_*: SX \longrightarrow SY \quad \text{respectively} \quad f^*: SY \longrightarrow SX$$

according to whether S is co- or contravariant.

Relations between functors with the same domain and target categories are also of interest. Roughly speaking, they are to functors what these are to categories.

16.10 Definition Let $S, T \colon \mathbf{C} \longrightarrow \mathbf{D}$ be two functors. A natural transformation $\zeta \colon S \longrightarrow T$ consists of morphisms $\zeta_X \in \mathbf{D}(SX, TX)$, one for each $X \in |\mathbf{C}|$, such that for each morphism $f \in \mathbf{C}(X, Y)$ the diagram in $\mathbf{D}$

$$\begin{array}{ccc} SX & \xrightarrow{\zeta_X} & TX \\ {\scriptstyle Sf}\downarrow & & \downarrow{\scriptstyle Tf} \\ SY & \xrightarrow{\zeta_Y} & TY \end{array}$$

commutes. Like morphisms and functors, natural transformations can be composed in the obvious way, and for each functor S there is an identical transformation $1_S \colon S \longrightarrow S$. A natural transformation $\zeta \colon S \longrightarrow T$ is called a natural equivalence between S and T if it admits an inverse with respect to composition: this happens if and only if ζ_X is an isomorphism for each $X \in |\mathbf{C}|$, and is written $\zeta \colon S \simeq T$.

Note When translating to cofunctors take care that S and T have the same kind of variance.

16.11 Examples (1) It is well-known that every vector space V over the field K has a canonical embedding j_V into its bidual $V^{\check{}\check{}}$:

$$V \ni v \longmapsto (V^{\check{}} \ni \varphi \mapsto \varphi(v) \in K) \in V^{\check{}\check{}}$$

The notion of natural transformation allows to pin down what the attribute "canonical" really means: the morphisms j_X define a natural transformation j from the identity functor $\mathbf{Lin}_K \longrightarrow \mathbf{Lin}_K$ to the bidualisation functor $D \circ D \colon \mathbf{Lin}_K \longrightarrow \mathbf{Lin}_K$. When both functors are restricted to the full subcategory of finite dimensional vector spaces then j becomes a natural equivalence of functors.

(2) Let $\mathbf{Inj}'_{\mathbb{R}}$ be the category of finite dimensional real vector spaces and injective linear maps. There is a functor P from this category to $\mathbf{Top^{cp}}$, the category of compact Hausdorff spaces, assigning to $V \in |\mathbf{Inj}'_{\mathbb{R}}|$ the projective space $P(V)$. Our example involves a slightly inflated version of this functor: it sends the vector space V to the projective space $P(\mathbb{R} \oplus V)$ of the same dimension. Recall from Section 11 that V may be identified with an open dense subspace of $P(\mathbb{R} \oplus V)$ via the map

$$V \ni v \longmapsto [1 \oplus v] \in P(\mathbb{R} \oplus V).$$

Thus the effect of the functor $P(\mathbb{R} \oplus ?)$ is a compactification, and it is interesting to compare it with the Alexandroff functor $A \colon \mathbf{Inj}'_{\mathbb{R}} \longrightarrow \mathbf{Top^{cp}}$ that takes V as a topological space and sends it to $\hat{V}$, the Alexandroff compactification. Indeed by Proposition 16.8 one has for each $V \in |\mathbf{Inj}'_{\mathbb{R}}|$ a unique commutative diagram

$$\begin{array}{ccccc} & \swarrow & V & \searrow & \\ P(\mathbb{R} \oplus V) & & \xrightarrow[\zeta_V]{} & & \hat{V} \end{array}$$

in $\mathbf{Top}$ such that ζ_V sends $P(\mathbb{R} \oplus V) \backslash V$ to $\infty \in \hat{V}$. The maps ζ_V form a natural transformation from $P(\mathbb{R} \oplus ?)$ to A.

Finally the notion of equivalence of categories is based on natural equivalences of functors too.

16.12 Definition Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A functor $S \colon \mathbf{C} \longrightarrow \mathbf{D}$ is an equivalence if there exist a functor $T \colon \mathbf{D} \longrightarrow \mathbf{C}$ and natural equivalences $TS \simeq 1_{\mathbf{C}}$ and $ST \simeq 1_{\mathbf{D}}$.

Why not rather $TS = 1_{\mathbf{C}}$ and $ST = 1_{\mathbf{D}}$? Because the notion of *isomorphy* of categories thus defined is too narrow to be of much use. By contrast there are interesting examples of equivalences:

16.13 Examples (1) For every field K the dualisation functor D is an equivalence from the category of finite dimensional vector spaces $\mathbf{Lin}'_K$ to its dual $\big(\mathbf{Lin}'_K\big)\check{}$. Note that even in this simple case D does not have an inverse because not every space in $\mathbf{Lin}'_K$ *truely is* the dual of another (even though it is canonically isomorphic to one).

(2) This example is much more radical: let $\mathbf{K}$ be the full subcategory of $\mathbf{Lin}'_K$ with objects just the spaces K^n for all $n \in \mathbb{N}$. Then the inclusion functor $\mathbf{K} \hookrightarrow \mathbf{Lin}'_K$ is an equivalence of categories. The proof depends on the axiom of choice for the class $|\mathbf{Lin}'_K|$: choose a basis $\underline{b}_V$ for each finite dimensional vector space over K and define the functor

$$R: \mathbf{Lin}'_K \longrightarrow \mathbf{K}$$

by sending the space V to $K^{\dim V}$, and the linear map $f: V \longrightarrow W$ to its matrix representation with respect to the bases $\underline{b}_V$ and $\underline{b}_W$. If, as we may assume, the basis chosen for K^n is the standard one then $RJ = \mathrm{id}_{\mathbf{K}}$ is the identity. On the other hand the collection of basis isomorphisms

$$\zeta_V: K^{\dim V} \xrightarrow{\simeq} V$$

corresponding to the bases $\underline{b}_V$ is a natural equivalence ζ from JR to the identity of $\mathbf{Lin}'_K$. — Of course no invertible functor between the categories $\mathbf{K}$ and $\mathbf{Lin}'_K$ can ever exist since the former is small (it has just countably many objects) while the latter is huge (though, as a category, not extraordinarily so).

To sum up this section let me make one more simple but basic observation about functors. If $S: \mathbf{C} \longrightarrow \mathbf{D}$ is a functor, and $f \in \mathbf{C}(X, Y)$ is an isomorphism then $Sf \in \mathbf{C}(SX, SY)$ must be an isomorphism too, for if $g \in \mathbf{C}(Y, X)$ inverts f then $Sg \in \mathbf{D}(SY, SX)$ is inverse to Sf. Thus on isomorphic objects $X, Y \in \mathbf{C}$ the functor S must take isomorphic values $SX, SY \in \mathbf{D}$, and in this sense the latter are invariants of the former. In topology most successful invariants are of this "category valued" type, often with values in $\mathbf{Ab}$ or in $\mathbf{Lin}_K$ for some field K. By their greater flexiblity and power they have largely replaced the older "naive" (for instance, integer valued) ones.

In this respect the homotopy group functors $\pi_n: \mathbf{Top}^\circ \longrightarrow \mathbf{Ab}$ appear most promising. So far, however, we do not know a single non-trivial value of them! The next sections will therefore be dedicated to the computation of homotopy groups.

17 Homotopy Groups of Big Spheres

The purpose of the present, and the following two, sections is to determine the homotopy groups $\pi_n(S^q)$ for $n \le q$. To this quite non-trivial task a number of different approaches are known. The one that should be mentioned first stays completely in the framework of homotopy theory and is described in [tom Dieck – Kamps – Puppe]. It could easily, if somewhat lengthily be presented on the basis of what we have already done in this course, building on Section 15 in particular. In spite of this apparent advantage I shall prefer a different method, which uses ideas from *differential topology*, the topological theory of manifolds and differentiable maps. While this does make the course less self-contained its by-results enrich homotopy theory by new methods which for calculations often are the most explicit and powerful ones. On the conceptual level the approach via differential topology will reveal a deep relation between topology and vector analysis.

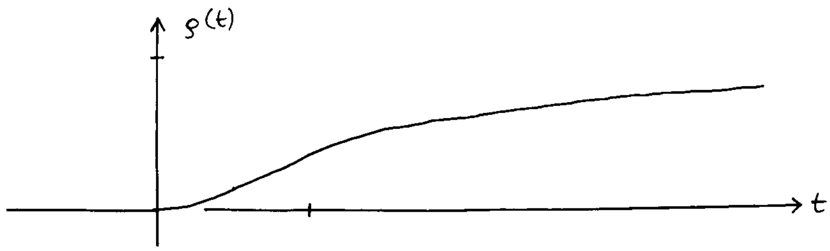
I will not, of course, assume that you are familiar with differential topology — indeed, if I knew you were, these three sections could be shrunk to a couple of hints about how to apply the differential topologist's standard tools. But the intended applications of these general tools are quite simple, and in most instances I have found it reasonable to substitute ad hoc proofs for general theory. The following presentation therefore is largely self-contained, apart from previous material of this course using but undergraduate mathematics, with one or possibly two exceptions: the use of Sard's theorem 17.5, and of the theory of integration of differential forms including Stokes's theorem in Section 18. The former is a result from measure theory used by differential topologists and functional analysts. It may here well be taken on faith since no relevant insight is gained from the way it is proved. On the other hand integration of differential forms, that is, vector analysis, is such a central part of mathematics that I trust every student will be, or become at some point acquainted with it.

In order to discuss differential topology in any generality one should, first of all, know what a differential manifold is. However the manifolds we consider here are mainly $\mathbb{R}^n$ and the n-sphere S^n, and the most complicated that will occur is the product $I \times S^n$, a manifold with *boundary* $\partial I \times S^n = \{0\} \times S^n + \{1\} \times S^n$. As long as you understand differentiable mappings between these few explicit objects there will be no need to be familiar with the notion of manifold in general. By the way, following established custom in differential topology, we will use the term "differentiable" as shorthand for "C^∞ differentiable".

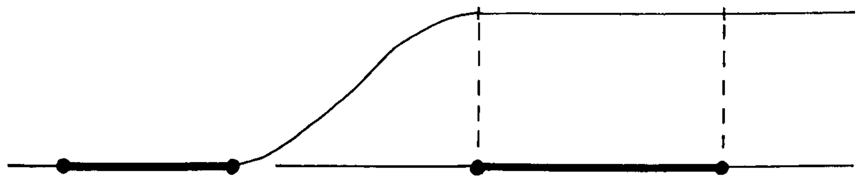
One elementary tool that will occur repeatedly is the use of differentiable separating functions. Its general and standard formulation is the differentiable analogue of Urysohn's theorem, stating that any two disjoint closed sets F and G can be separated by a *differentiable* real-valued function. Again, the sets F and G to be considered here will have a simple and perfectly explicit geometry, and there will be no need to refer to the general theorem since suitable separating functions are easily assembled using the well-known C^∞ function

$$\rho\colon \mathbb{R} \longrightarrow I; \quad \rho(t) = \begin{cases} 0 & \text{for } t \le 0 \\ e^{-1/t} & \text{for } t > 0 \end{cases}$$

as the starting point.



17.1 Question Explain how to separate two closed intervals on the real line, using $\int \rho(t)\rho(1-t)\,dt$.



Let us get down to work. The goal of this section is

17.2 Theorem $\pi_n(S^q, \circ) = 0$ for $0 \le n < q$.

Explanation Due to the symmetry of the spheres the exact choice of the base point $\circ$ does not matter, but for the sake of definiteness let it be

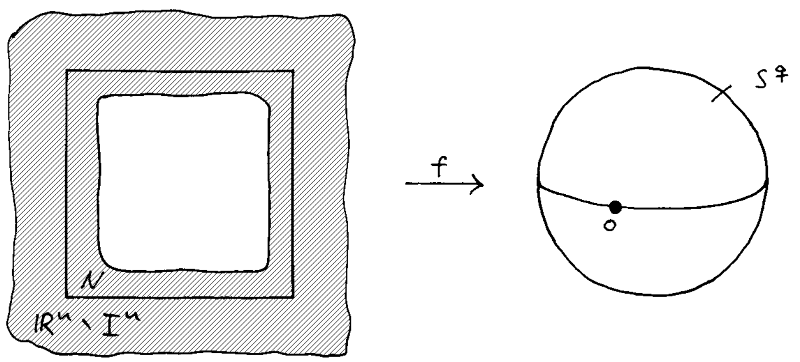
$$\circ = (-1, 0, \ldots, 0) \in S^q \subset \mathbb{R}^{q+1}.$$

— We use 0 to denote not only zero elements but also the trivial abelian group or vector space. — For $n = 0$ the theorem may be accepted as stating that the homotopy set $\pi_0(S^q, \circ)$ has just one element, which is clearly true since S^q is connected.

However from now on we will assume that we are in the non-trivial case $n > 0$. Let $f\colon (I^n/\partial I^n, \circ) \longrightarrow (S^q, \circ)$ be an arbitrary pointed map: we must prove that f is null homotopic. As usual it is convenient to re-write f as a mapping $f\colon I^n \longrightarrow S^q$ which takes the constant value $\circ \in S^q$ on ∂I^n. By Corollary 15.8 we may even assume that f is constant on some neighbourhood of ∂I^n in I^n, and by trivial extension we arrive at a mapping

$$f\colon \mathbb{R}^n \longrightarrow S^q \subset \mathbb{R}^{q+1}$$

defined on all $\mathbb{R}^n$ which takes constant value $\circ$ on some neighbourhood N of $\overline{\mathbb{R}^n \setminus I^n}$.



Note that homotopies of

$$\begin{array}{ccc} & \overline{\mathbb{R}^n \setminus I^n} & \\ \swarrow & & \searrow \circ \\ \mathbb{R}^n & \xrightarrow[f]{} & S^q \end{array}$$

relative the space $\overline{\mathbb{R}^n \setminus I^n}$ will induce pointed homotopies of the original map.

According to a general principle of differential topology two continuous maps between manifolds are homotopic if they are sufficiently close to each other. A precise formulation of the principle adapted to our situation runs as follows:

17.3 Lemma Let $g: \mathbb{R}^n \longrightarrow \mathbb{R}^{q+1}$ be a map with constant value $\circ$ on a neighbourhood of $\overline{\mathbb{R}^n \setminus I^n}$, and assume that the uniform distance between f and g is smaller than 1:

$$|f(x) - g(x)| < 1 \quad \text{for all } x \in \mathbb{R}^n$$

Let $p: \mathbb{R}^{q+1} \setminus \{0\} \longrightarrow S^q$ with $p(y) = \frac{1}{|y|} y$ be the projection of Example 14.5(2). Then

$$p \circ g: \mathbb{R}^n \xrightarrow{g} \mathbb{R}^{q+1} \setminus \{0\} \xrightarrow{p} S^q$$

is defined, and f and $p \circ g$ are homotopic under $\overline{\mathbb{R}^n \setminus I^n}$.

Proof The first statement follows from the estimate

$$|g(x)| \geq |f(x)| - |g(x) - f(x)| = 1 - |g(x) - f(x)| > 0 \text{ for all } x \in \mathbb{R}^n.$$

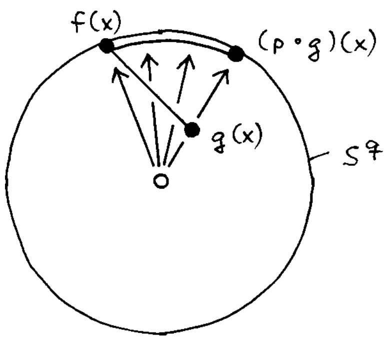
More generally,

$$|(1-t)f(x) + tg(x)| \geq |f(x)| - t\,|g(x) - f(x)| = 1 - t\,|g(x) - f(x)| > 0$$

holds for all $t \in I$ and $x \in \mathbb{R}^n$, and therefore the map

$$I \times \mathbb{R}^n \ni (t, x) \longmapsto p\big((1-t)f(x) + tg(x)\big) \in S^q$$

that projects the linear connection into S^q is a homotopy between f and $p \circ g$.



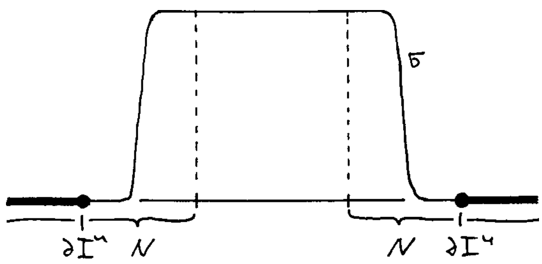
In order to put Lemma 17.3 to good use we turn to a second principle from differential topology: continuous maps between manifolds can be approximated by differentiable ones.

17.4 Lemma There exists a C^∞ map $g: \mathbb{R}^n \longrightarrow S^q$ which on a neighbourhood of $\overline{\mathbb{R}^n \setminus I^n}$ has constant value $\circ$, and whose uniform distance from f is smaller than 1.

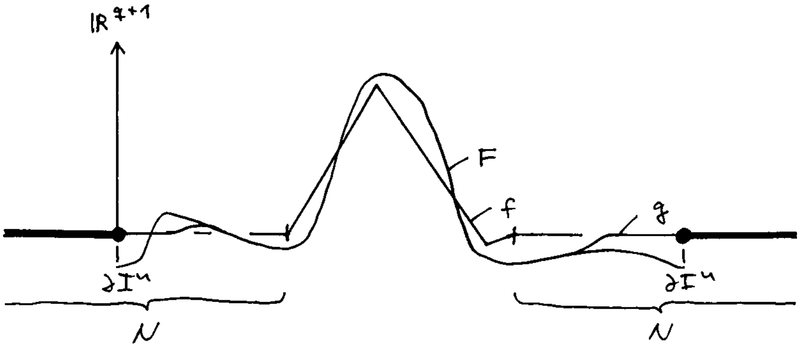
Proof By a classical theorem of Fourier analysis the restriction of f to the cube I^n may be uniformly approximated be a real Fourier polynomial

$$\begin{aligned} F(x) &= \mathsf{Re} \sum_k c_k \exp 2\pi i (k_1 x_1 + \cdots + k_n x_n) \\ &= \sum_k a_k \cos 2\pi (k_1 x_1 + \cdots + k_n x_n) + \sum_k b_k \sin 2\pi (k_1 x_1 + \cdots + k_n x_n), \end{aligned}$$

the sums being taken over a sufficiently large finite subset of $\mathbb{Z}^n$. We cannot use F as it stands since it will, of course, not be constant on the boundary ∂I^n. We therefore choose a C^∞ function $\sigma: \mathbb{R}^n \longrightarrow I$ with $\sigma = 1$ on $\mathbb{R}^n \setminus N$ but $\sigma = 0$ on some neighbourhood of $\overline{\mathbb{R}^n \setminus I^n}$ (necessarily contained in N).



The map $f+\sigma\cdot(F-f)$ then extends naturally and differentiably to all $\mathbb{R}^n$, and globally approximates f as well as F did on I^n. Projecting back into S^q we obtain a map $g = p\circ\bigl(f+\sigma\cdot(F-f)\bigr)$ that does all we have asked for.



A third principle from differential topology will be needed, and this time an ad hoc proof seems to make little sense. Recall that a *regular value* of a differentiable mapping $g\colon X \longrightarrow Y$ is a point $c \in Y$ such that for every $x \in g^{-1}\{c\}$ the differential $T_xg\colon T_xX \longrightarrow T_cY$ is surjective:

$$\operatorname{rank} T_xg = \dim Y \quad \text{for all } x \in g^{-1}\{c\}$$

Points of Y that are not regular values are called *critical values* of g. The principle is that most points of Y are regular values of g:

17.5 Sard's Theorem For every differentiable mapping $f\colon X \longrightarrow Y$ the set of critical values of f has measure zero in Y.

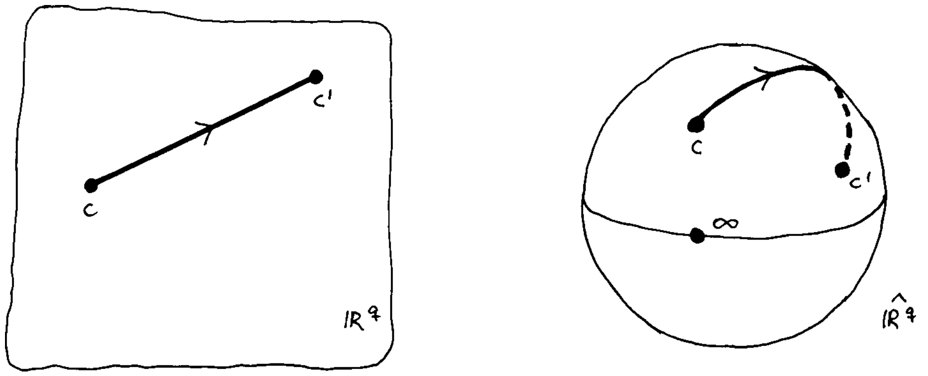
Remarks For a proof see any textbook of differential topology, like [Bröcker – Jänich]. Note that the infinitesimal version of Sard's theorem is a well-known fact of linear algebra: at a critical point $x \in X$, that is one with $\operatorname{rank} T_xg < \dim Y$, the image of the differential T_xg is a proper subspace of $T_{f(x)}Y$, and therefore has volume zero in it.

The proof of 17.1 will require one more ingredient, which is pure topology:

17.6 Lemma Let c and c' be two given points of a pointed sphere $(S^q, \circ)$, both distinct from the base point. Then there exists a self-homeomorphism of $(S^q, \circ)$ which is homotopic to the identity and takes c to c'.

Proof For the purpose we identify $(S^q, \circ)$ with the Alexandroff compactification $(\hat{\mathbb{R}^q}, \infty)$. Translation of $\mathbb{R}^q$ by $c'-c$ induces a pointed homeomorphism of S^q that moves c as required. In order to see that it is homotopic to the identity, just apply the Alexandroff functor 16.2(8) to the proper homotopy

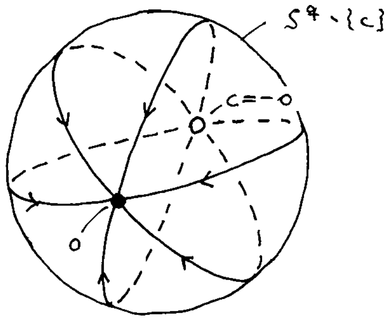
$$I\times\mathbb{R}^q \ni (t, y) \longmapsto y + t\,(c'-c) \in \mathbb{R}^q.$$



Proof of Theorem 17.1 According to 17.4 the given map $f: \mathbb{R}^n \longrightarrow S^q$ may be approximated by a differentiable one, g, and by 17.3 we may replace f by g. Now Sard's theorem guarantees the existence of some regular value $c \in S^q$, necessarily distinct from the base point. Since n is smaller than q the regular values of g are just the non-values, so c is not a value of g. Composing g with the appropriate homeomorphism from 17.6 we may move c to the point

$$-\circ = (1, 0, \ldots, 0) \in S^q \subset \mathbb{R}^{q+1}$$

opposite the base point. We thus have constructed a (continuous) map $h: (I^n/\partial I^n, \circ) \longrightarrow (S^q, \circ)$ which represents $[f] \in \pi_n(S^q, \circ)$ and takes values in the punctured sphere $S^q \setminus \{c\}$. But $(S^q \setminus \{c\}, \circ)$ clearly is contractible in $\mathbf{Top}^\circ$, and therefore f is null-homotopic.



This completes the proof.

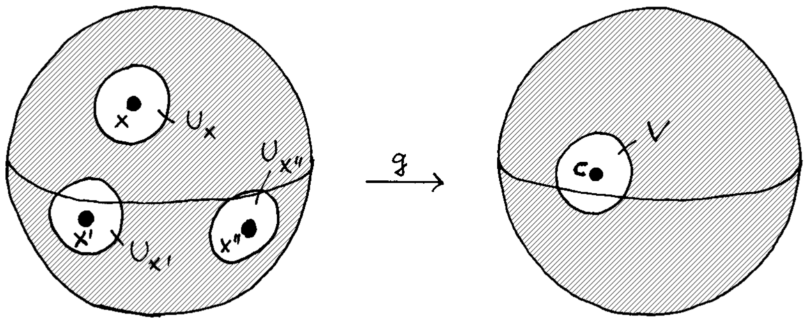
18 The Mapping Degree

Computing the equidimensional homotopy groups $\pi_n(S^n, \circ)$ is even better fun because unlike $\pi_n(S^q)$ for $n < q$, they will turn out to be non-trivial. Most of what we have learnt in the previous section still applies: in particular how to represent a homotopy class in $[I^n/\partial I^n, S^n]^\circ$ by a differentiable map. At the heart of the calculation will be the *mapping degree*, an integer which we shall define as an a priori invariant of differentiable maps $g: S^n \longrightarrow S^n$. Again there is more than one way to proceed. Even though there is no compelling reason to leave the framework of differential topology I have chosen an approach that also invokes vector analysis, thereby eliminating the need for a further reach into the differential topologist's toolbox. Furthermore the interplay between geometric and analytic aspects of the mapping degree is of interest in its own right, and, last not least, of striking mathematical beauty.

18.1 Proposition Let $c \in S^n$ be a regular value of the differentiable map $g: S^n \longrightarrow S^n$. Then the fibre $g^{-1}\{c\}$ is finite. There exist a connected open neighbourhood V of c such that $g^{-1}(V)$ splits as a topological sum

$$g^{-1}(V) = \sum_{x \in g^{-1}\{c\}} U_x$$

where each summand U_x is an open neighbourhood of x in S^n, and is sent by g diffeomorphically to V.



Proof Since c is a regular value the differential

$$T_x g: T_x S^n \longrightarrow T_c S^n$$

is a linear isomorphism at each $x \in g^{-1}\{c\}$. Therefore, by the inverse mapping theorem g is a local diffeomorphism there, and for each such x there are open neighbourhoods U_x of x and V_x of c such that g restricts to a diffeomorphism $g_x: U_x \approx V_x$. In particular x is the only point of $g^{-1}\{c\} \cap U_x$, and therefore $g^{-1}\{c\}$ a discrete space. Since it is also compact it must be finite.

We may now shrink the U_x and thereby the V_x to make the former pairwise disjoint. In view of the fact that

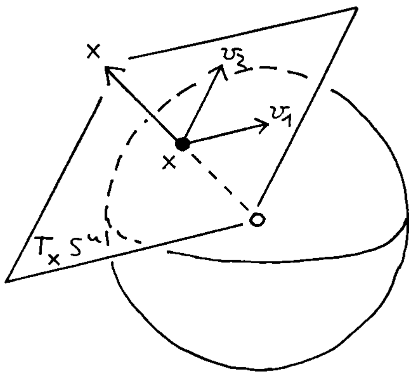
$$L := g\Big(S^n \setminus \bigcup_{x \in g^{-1}\{c\}} U_x\Big)$$

is compact, and does not contain c the set

$$V := \bigcap_{x \in g^{-1}\{c\}} V_x \setminus L$$

is a neighbourhood of c. We finally shrink V to a smaller neighbourhood which is connected and open, and replace U_x by $U_x \cap g^{-1}(V)$, thus establishing the stated conclusion.

The next step makes use of the fact that for each $n > 0$ the sphere S^n carries a standard orientation: by definition a basis $(v_1, \ldots, v_n)$ of the tangent space $T_x S^n = \{x\}^\perp \subset \mathbb{R}^{n+1}$ is positively oriented if $(x, v_1, \ldots, v_n)$ is a positively oriented basis of $\mathbb{R}^{n+1}$.



18.2 Definition Assume $n > 0$, and let $c \in S^n$ be a regular value of $g: S^n \longrightarrow S^n$. For each $x \in g^{-1}\{c\}$ define the *orientation character* $\varepsilon_x := \pm 1$ according to whether $T_x g: T_x S^n \longrightarrow T_c S^n$ respects or reverses orientation. The integer

$$\deg_c(g) := \sum_{x \in g^{-1}\{c\}} \varepsilon_x \in \mathbb{Z}$$

is called the mapping degree of g (with respect to the regular value c).

Our principal aim is to show that the mapping degree does not depend on the choice of the regular value c, and does not change under differentiable homotopies of g. We will achieve both by giving the degree an alternative interpretation in terms of vector analysis.

18.3 Proposition Assume $n > 0$, and let g_0 and g_1 be maps from S^n to itself which are differentiably homotopic to each other. Then for every C^∞ differential form ω of degree n on S^n one has

$$\int_{S^n} g_0^* \omega = \int_{S^n} g_1^* \omega.$$

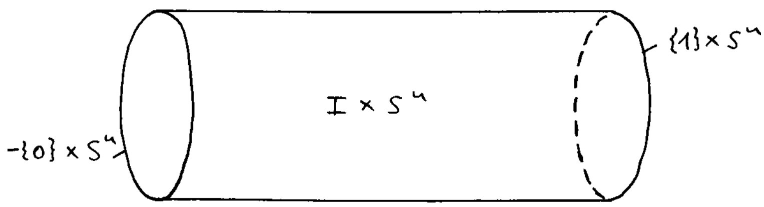
Proof Let $g: I \times S^n \longrightarrow S^n$ be a differentiable homotopy. Then

$$0 = \int_{I \times S^n} g^* 0 = \int_{I \times S^n} g^* d\omega = \int_{I \times S^n} d(g^* \omega) = \int_{\partial I \times S^n} g^* \omega = \int_{S^n} g_1^* \omega - \int_{S^n} g_0^* \omega$$

by the fact that there are no non-trivial $(n{+}1)$-forms on S^n, by Stokes' theorem, and by the identity

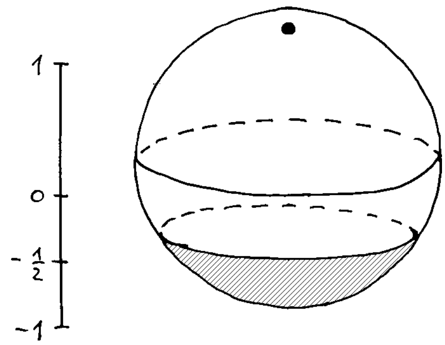
$$\partial I \times S^n = \{1\} \times S^n - \{0\} \times S^n$$

of *oriented* manifolds.



For the following let us agree upon a bit of temporary vocabulary: The set

$$\left\{(x_1,\ldots,x_{n+1})\in S^n \;\middle|\; x_{n+1}\geq -\tfrac{1}{2}\right\}$$



comprising the Northern hemisphere and (roughly) the Tropics will be referred to as a *greater hemisphere*, and this name will more generally be used for any congruent subset of S^n.

18.4 Lemma Let $V\subset S^n$ be a non-empty open subset. Then there exists a diffeomorphism $h\colon S^n\longrightarrow S^n$, differentiably homotopic to the identity, such that $h(V)$ contains a greater hemisphere.

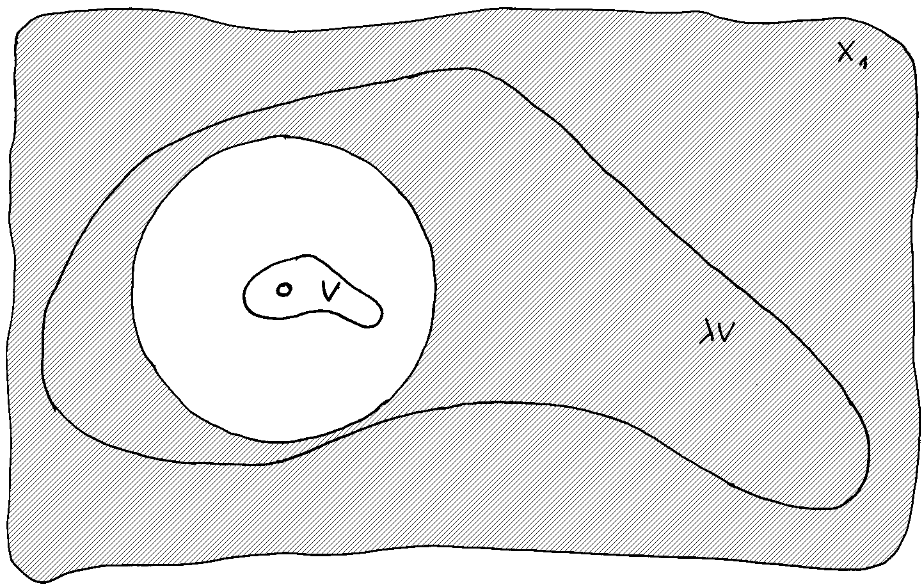
Proof Recall from Example 10.5(4) that S^n can be represented as the quotient of $X_1+X_2=\mathbb{R}^n+\mathbb{R}^n$ with respect to the relation

$$X_1\backslash\{0\}\ni x\sim\frac{1}{|x|^2}\,x\in X_2\backslash\{0\}.$$

The identification with S^n can be made in such a way that the origin of X_1 belongs to V. For any $\lambda>0$ the formulae

$$I\times X_1\ni(t,x)\longmapsto\big((1-t)+t\lambda\big)x\in X_1$$
$$I\times X_2\ni(t,x)\longmapsto\big((1-t)+t\lambda\big)^{-1}x\in X_2$$

define a differentiable homotopy $H\colon I\times S^n\longrightarrow S^n$ from id to a self-diffeomorphism h of S^n, and if λ is chosen sufficiently large then $h(V)\subset S^n$ — which corresponds to $\lambda V\subset\mathbb{R}^n=X_1$ — will contain a greater hemisphere.

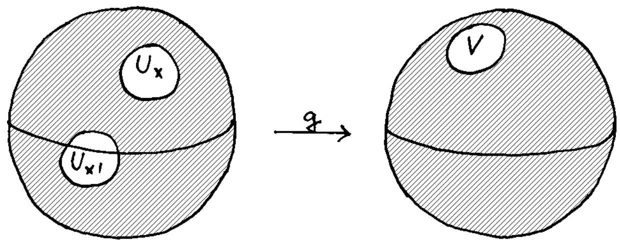


18.5 Theorem Assume $n > 0$, and let $c \in S^n$ be a regular value of $g\colon S^n \longrightarrow S^n$. Then

$$\int_{S^n} g^*\omega = \deg_c(g) \cdot \int_{S^n} \omega$$

holds for all n-forms ω on S^n.

Proof Choose neighbourhoods U_x of each $x \in g^{-1}\{c\}$, and V of c as supplied by Proposition 18.1:



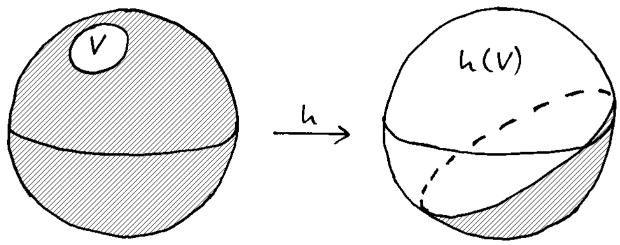
Since V is connected, for each $x \in g^{-1}\{c\}$ the diffeomorphism $U_x \overset{g}{\approx} V$ either preserves or reverses orientation, according to the value of ε_x. We therefore have

$$\int_{U_x} g^*\omega = \varepsilon_x \int_V \omega$$

by the integral transformation formula. Assume for the moment that ω vanishes identically outside V. Then $g^*\omega$ vanishes outside $\bigcup_x U_x$ and the stated formula follows:

$$\int_{S^n} g^*\omega = \int_{\bigcup_x U_x} g^*\omega = \sum_x \int_{U_x} g^*\omega = \sum_x \varepsilon_x \int_V \omega = \deg_c(g) \int_V \omega = \deg_c(g) \int_{S^n} \omega$$

Let us now prove the theorem under the much milder assumption that ω vanishes outside some greater hemisphere $H \subset S^n$. By Lemma 18.4 we find a diffeomorphism $h \simeq \mathrm{id}$ such that $h(V)$ contains a greater hemisphere H'.



Composing h with a suitable rotation in $SO(n{+}1)$ we may assume that $H' = H$, and therefore that ω vanishes outside $h(V)$. Then $h^*\omega$ vanishes outside V and since $h \simeq \mathrm{id}$ implies $g = \mathrm{id} \circ g \simeq h \circ g$ two applications of Proposition 18.3 once more yield the formula:

$$\int_{S^n} g^*\omega = \int_{S^n} (h \circ g)^*\omega = \int_{S^n} g^*(h^*\omega) = \deg_c(g) \int_{S^n} h^*\omega = \deg_c(g) \int_{S^n} \omega$$

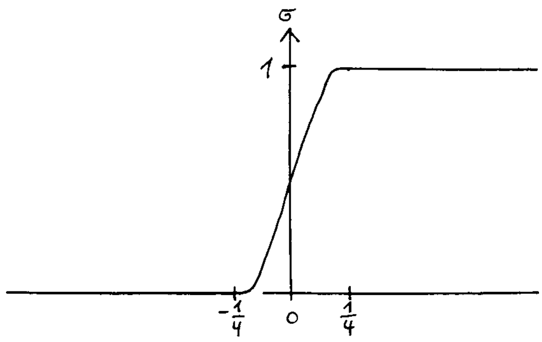
In order to remove the restriction on ω and thereby complete the proof it suffices to show the following simple

18.6 Lemma Let $H \subset S^n$ be a greater hemisphere. Every C^∞ differential form ω can be written as a sum

$$\omega = \omega_+ + \omega_-$$

of C^∞ forms ω_+, vanishing outside H, and ω_-, vanishing outside $-H$.

Proof There is no harm in assuming the H is the greater Northern hemisphere. Choose a differentiable function $\sigma\colon \mathbb{R} \longrightarrow I$ with $\sigma(t) = 0$ for $t \le -\frac{1}{4}$ and $\sigma(t) = 1$ for $t \ge \frac{1}{4}$,



and denote by $x_{n+1}\colon S^n \longrightarrow \mathbb{R}$ the restriction of the last coordinate function to the sphere. Then the differential forms

$$\omega_+ := (\sigma \circ x_{n+1}) \cdot \omega \quad \text{and} \quad \omega_- := \omega - \omega_+$$

do the trick.

Since there exist plenty of n-forms on S^n with non-vanishing integral Theorem 18.5, which is now proven, gives an alternative method to compute the mapping degree. In particular it is now clear that $\deg_c(g)$ does not depend on the choice of the regular value $c \in S^n$, and therefore c will be dropped from the notation. Note that each of the two descriptions of the degree has made an important contribution: integrality of the degree is due to the original one while its homotopy invariance is based on integration of forms.

Homotopy invariance here refers, of course, to differentiable rather than continuous homotopy; but then by principles already alluded to in the previous section this does not matter at all.

18.7 Proposition

- Every continuous mapping $S^n \longrightarrow S^n$ is homotopic to a differentiable one.
- If the differentiable maps $g_0, g_1\colon S^n \longrightarrow S^n$ are homotopic in **Top** then there exists a differentiable homotopy between them.

Proof Both statements are quite close to 17.3 and 17.4, and the proofs easily adapted. In order to approximate a given continuous mapping $f\colon S^n \longrightarrow S^n$ by a Fourier polynomial it should first be extended over some compact cube whose interior contains S^n: the homogeneous extension

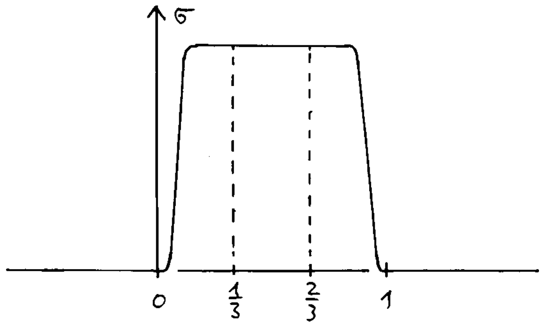
$$\tilde{f}\colon [-2,2]^{n+1} \longrightarrow \mathbb{R}^{n+1}; \quad \tilde{f}(0) = 0 \text{ and } \tilde{f}(x) = |x| \cdot f\Big(\frac{1}{|x|} \cdot x\Big) \text{ for } x \neq 0$$

will work nicely. Although $\tilde{f}$ need not have a (continuous) periodic extension over $\mathbb{R}^n$ a suitable Fourier polynomial F will approximate f on S^n with uniform distance smaller than 1. The restriction $F|S^n\colon S^n \longrightarrow \mathbb{R}^{n+1}\backslash\{0\}$ may be projected back into S^n, and gives a differentiable mapping $p \circ F|S^n$ that is homotopic to f.

Let now $g_0, g_1\colon S^n \longrightarrow S^n$ be two differentiable maps, and $f\colon I \times S^n \longrightarrow S^n$ a continuous homotopy between them. We wish to construct a differentiable homotopy g by approximation but first prepare

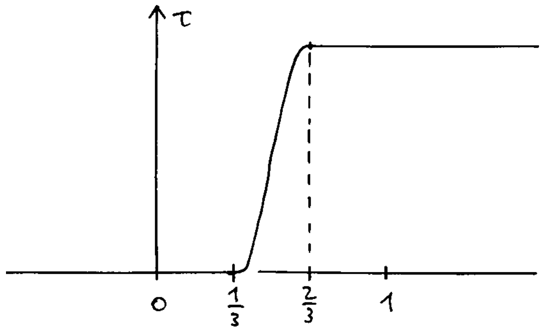
f, using another little trick from differential topology. Choose two C^∞ functions $\sigma, \tau: \mathbb{R} \longrightarrow I$ such that

$$\sigma(t) = \begin{cases} 0 & \text{for } t \in (-\infty, 0] \cup [1, \infty) \\ 1 & \text{for all } t \text{ in some neighbourhood of } [\frac{1}{3}, \frac{2}{3}], \end{cases}$$



and

$$\tau(t) = \begin{cases} 0 & \text{for } t \in (-\infty, \frac{1}{3}] \\ 1 & \text{for } t \in [\frac{2}{3}, \infty). \end{cases}$$



The function τ is used to define a new homotopy $f' := f \circ (\tau \times \mathrm{id})$ from g_0 to g_1 which still joins g_0 to g_1 but is inactive for all times before $\frac{1}{3}$ and after $\frac{2}{3}$:

$$f'_t = g_0 \text{ for } t \leq \tfrac{1}{3} \quad \text{and} \quad f'_t = g_1 \text{ for } t \geq \tfrac{2}{3}.$$

Now apply the usual argument: extend f' as a continuous map over $I \times [-2, 2]^{n+1}$ and approximate by a Fourier polynomial. While the resulting map $F: I \times S^n \longrightarrow \mathbb{R}^{n+1} \backslash \{0\}$ is differentiable it will no longer be a homotopy between g_0 and g_1. However, the function σ was designed to correct this fault:

$$g': I \times S^n \longrightarrow \mathbb{R}^{n+1} \backslash \{0\}; \quad g'(t, x) := f'(t, x) + \sigma(t) \cdot \big(F(t, x) - f'(t, x)\big)$$

is differentiable with $g'_0 = f'_0 = g_0$ and $g'_1 = f'_1 = g_1$ and it only remains to project the values of g' into S^n by putting $g = p \circ g'$. This completes the proof of the proposition.

Our results may be summed up as follows.

18.8 Theorem For each $n > 0$ the mapping degree is a well-defined function

$$\deg: [S^n, S^n] \longrightarrow \mathbb{Z}.$$

Its value on the homotopy class $[f] \in [S^n, S^n]$ may be calculated by applying to any differentiable representative of $[f]$ either Definition 18.2 or Theorem 18.5.

18.9 Question Explain why the mapping degree is multiplicative with respect to composition:

$$\deg(g \circ f) = \deg g \cdot \deg f$$

Remark More generally, a mapping degree

$$\deg\colon [X, Y] \longrightarrow \mathbb{Z}$$

can be defined under the following conditions: X and Y are compact oriented n-dimensional differential manifolds (without boundary), and Y is non-empty and connected. The arguments we have used in the case of spheres carry over to the general case, with little change.

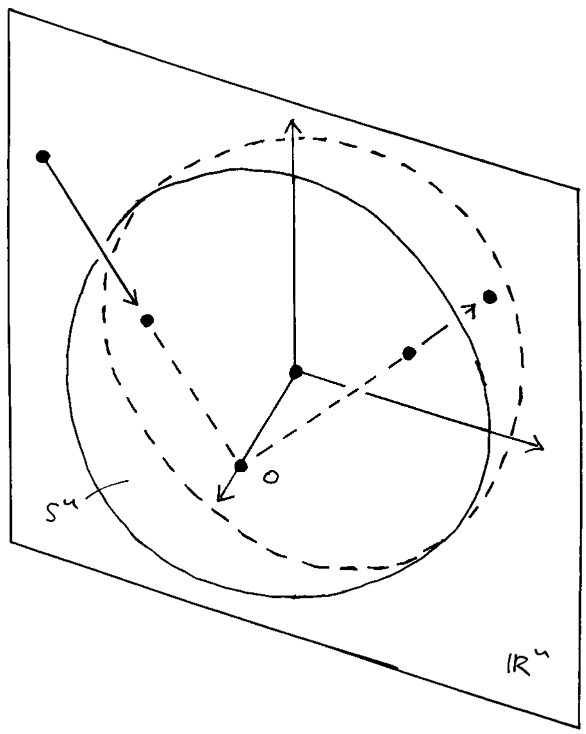
19 Homotopy Groups of Equidimensional Spheres

The mapping degree can be linked up with the homotopy groups $\pi_n(S^n,\circ) = [I^n/\partial I^n, S^n]^\circ$ since $I^n/\partial I^n$ is homeomorphic to the sphere S^n. In order to fix a particular homeomorphism let $\varphi\colon (0,1)^n \approx \mathbb{R}^n$ be the n-fold cartesian product of the homeomorphism

$$(0,1) \ni t \longmapsto \tan \pi\big(t-\tfrac{1}{2}\big) \in \mathbb{R}.$$

φ induces a homeomorphism $\hat{\varphi}\colon I^n/\partial I^n = \big((0,1)^n\big)\hat{} \longrightarrow \hat{\mathbb{R}^n}$, and composing with inverse stereographic projection from the base point $\circ = (-1,0,\ldots,0) \in S^n$,

$$\mathbb{R}^n \ni x \longmapsto \left(\frac{1-|x|^2}{1+|x|^2}, \frac{2}{1+|x|^2}\,x\right) \in S^n \subset \mathbb{R}\times\mathbb{R}^n$$



we finally obtain the homeomorphism

$$\Phi\colon I^n/\partial I^n \longrightarrow \hat{\mathbb{R}^n} \longrightarrow S^n$$

which will be used to identify $I^n/\partial I^n$ with S^n throughout. Note that Φ respects base points and, after removing them, becomes a diffeomorphism which preserves orientation.

19.1 Definition Let $n \in \mathbb{N}$ be positive, and let $u\colon [S^n,S^n]^\circ \longrightarrow [S^n,S^n]$ denote the map that forgets the base point. The mapping degree

$$\deg\colon \pi_n(S^n,\circ) \longrightarrow \mathbb{Z}$$

is defined as the dotted arrow that renders the diagram

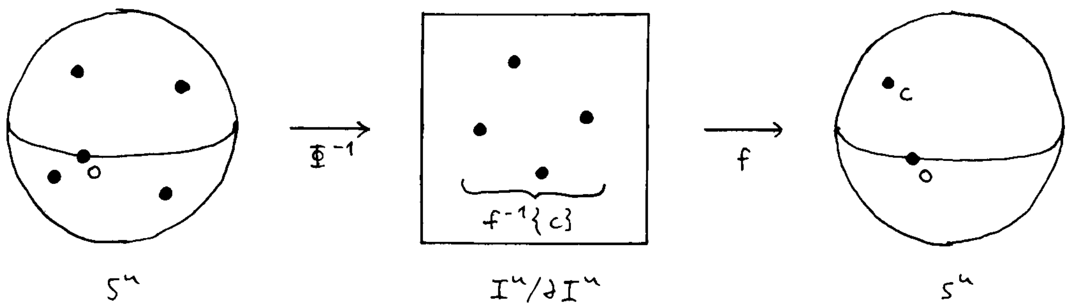
$$\begin{array}{ccc}
[S^n, S^n]^\circ & \xrightarrow{\quad u \quad} & [S^n, S^n] \\
{\scriptstyle \Phi^*}\downarrow & & \downarrow{\scriptstyle \deg} \\
[I^n/\partial I^n, S^n]^\circ & \cdots\cdots\cdots\cdots > & \mathbb{Z}
\end{array}$$

commutative.

We investigate the properties of this new version of the mapping degree. First we prove:

19.2 Proposition For every $n > 0$ the map $\deg\colon \pi_n(S^n, \circ) \longrightarrow \mathbb{Z}$ is surjective, and is a homomorphism of groups.

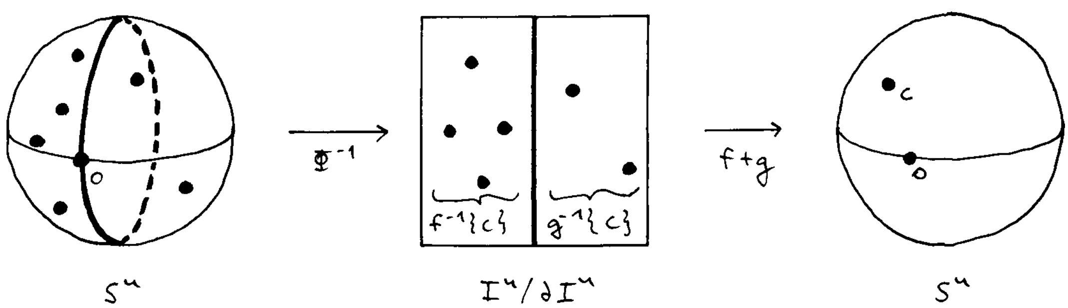
Proof Consider two homotopy classes $[f], [g] \in \pi_n(S^n, \circ)$. By Lemmas 17.2 and 17.3 — which, of course, hold for $n = q$ too — we may assume that f and g are represented by differentiable maps $\mathbb{R}^n \longrightarrow S^n$ which take constant value $\circ$ on some neighbourhood of $\overline{\mathbb{R}^n \setminus I^n}$. The compositions $\tilde{f} := f \circ \Phi^{-1}$ and $\tilde{g} := g \circ \Phi^{-1}$ are differentiable maps from S^n into itself, and according to Sard's theorem 17.5 we find a common regular value $c \in S^n$.



Let $\varepsilon_x \in \{\pm 1\}$ denote the orientation character of $T_x\tilde{f}$ at $x \in \tilde{f}^{-1}\{c\}$, and $\zeta_x \in \{\pm 1\}$ that of $T_x\tilde{g}$ at $x \in \tilde{g}^{-1}\{c\}$. Then

$$\deg f = \sum_{x \in \tilde{f}^{-1}\{c\}} \varepsilon_x \quad \text{and} \quad \deg g = \sum_{x \in \tilde{g}^{-1}\{c\}} \zeta_x$$

by definition. On the other hand c also is a regular value of the differentiable map $\tilde{h} := (f+g) \circ \Phi^{-1}$, and the fibre $\tilde{h}^{-1}\{c\}$ is in an obvious bijection with $\tilde{f}^{-1}\{c\} + \tilde{g}^{-1}\{c\}$ that respects the orientation characters.



Therefore the mapping degree of $f+g$ may be calculated as

$$\deg(f+g) = \sum_{x \in \tilde{f}^{-1}\{c\}} \varepsilon_x + \sum_{x \in \tilde{g}^{-1}\{c\}} \zeta_x.$$

This proves that the mapping degree is a homomorphism:

$$\deg[f+g] = \deg[f] + \deg[g]$$

Its surjectivity now follows easily: the pointed homeomorphism $\Phi: I^n/\partial I^n \longrightarrow S^n$ corresponds to the identity mapping of S^n, and therefore has degree 1. Since 1 generates $\mathbb{Z}$ as a group the image subgroup of the mapping degree must be the full group.

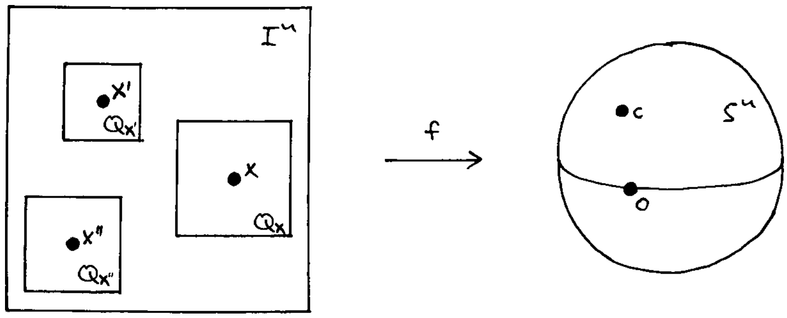
The following theorem is the principal result of this section, and completes our calculation of homotopy groups of spheres.

19.3 Theorem

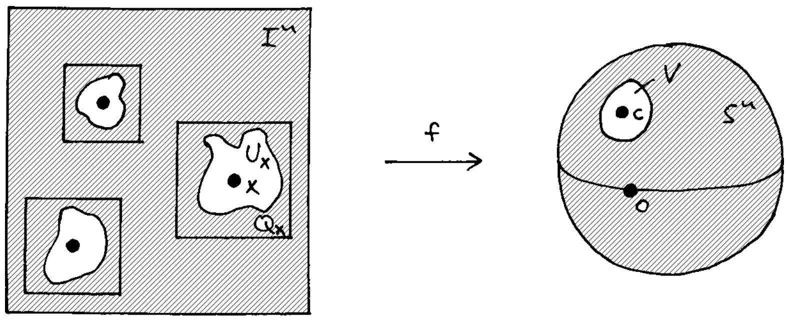
- $\pi_0(S^0, 1) = \{\pm 1\}$ consists of two points.
- For $n > 0$ the mapping degree $\deg: \pi_n(S^n, \circ) \longrightarrow \mathbb{Z}$ is an isomorphism.

Proof The statement about π_0 is trivial and has been included for the sake of completeness only.

For $n > 0$ it remains to prove the injectivity of the mapping degree. Thus let $[f] \in \pi_n(S^n, \circ)$ be an arbitrary homotopy class. We will show that f can be normalized, within its homotopy class, to a standard representative that only depends on the number $\deg f$. Again we may assume that f is represented by a differentiable map $I^n \longrightarrow S^n$ with constant value $\circ$ on some neighbourhood of ∂I^n. Let $c \in S^n$ be a regular value of f, and choose for each $x \in f^{-1}\{c\}$ a compact cube $Q_x \subset I^n$ centred at x, small enough so that these (finitely many) cubes are pairwise disjoint.



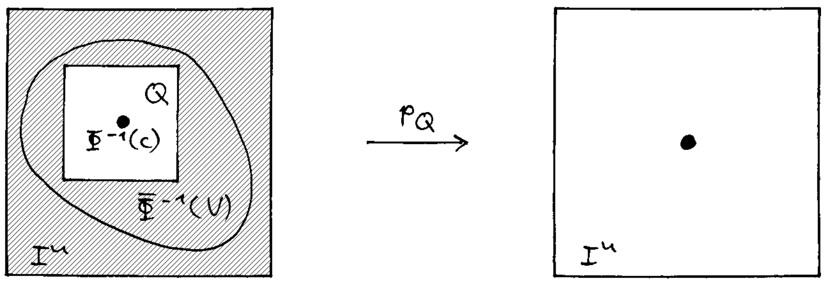
By Proposition 18.1 we find an open neighbourhood V of c such that $f^{-1}(V) = \sum_x U_x$ with open neighbourhoods U_x of x, each of which f sends diffeomorphically onto V. All these neighbourhoods may be made small, and we assume $U_x \subset Q_x$ for every x.



However, now that all choices have been made the sitiuation is similar to that in the proof of Theorem 18.5: we would rather like to see V large. A suitable modification of f will be based on the following lemma, which is similar to 18.4.

19.4 Lemma Let $c \in S^n$ be a point, not equal to the base point $\circ$, and let $V \subset S^n$ be a neighbourhood of c. Then there exist an open neighbourhood $V' \subset V$ of c and a map $h :: S^n \longrightarrow S^n$ with the following properties: h is homotopic in $\mathbf{Top}^\circ$ to the identity, it sends V' diffeomorphically onto $S^n \backslash \{\circ\}$, and the complement $S^n \backslash V'$ to the base point $\circ \in S^n$. It finally sends c to the opposite point $-\circ$.

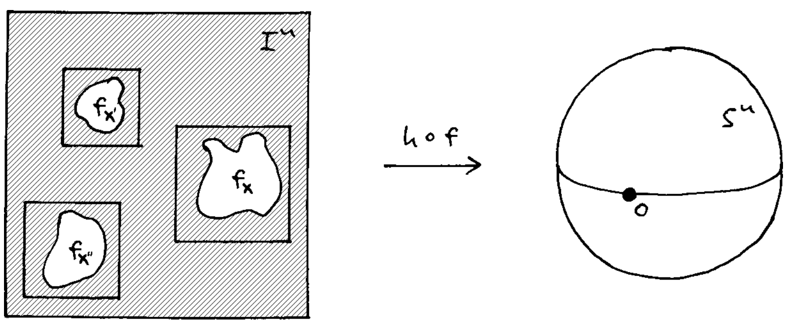
Proof The homeomorphism $\Phi: I^n/\partial I^n \approx S^n$ defines an identification mapping $\overline{\Phi}: I^n \longrightarrow S^n$. Make $\Phi^{-1}(c) \in (0,1)^n$ the centre of a compact cube Q which is completely contained in $\overline{\Phi}^{-1}(V) \subset I^n$. As we know from Lemma 15.6 the map $p_Q: I^n \longrightarrow I^n$ that linearly expands Q to I^n induces a map $\overline{p}_Q: (I^n/\partial I^n, \circ) \longrightarrow (I^n/\partial I^n, \circ)$ which is pointedly homotopic to the identity of I^n. The corresponding map $h: S^n \longrightarrow S^n$ therefore is homotopic to the identity of $(S^n, \circ)$. By construction h restricts to a diffeomorphism $V' := \overline{\Phi}(Q^\circ) \approx S^n \backslash \{\circ\}$, it sends $S^n \backslash V'$ to the base point $\circ$, and c to $-\circ$ as required.



Proof of 19.3 (continuation) We apply Lemma 19.4, with the given notation: the composition $h \circ f$ then represents the same homotopy class as f, it maps each of the open sets $U'_x := U_x \cap f^{-1}(V')$ diffeomorphically onto $S^n \backslash \{\circ\}$, and the complement $I^n \backslash \bigcup_x U'_x$ to the base point $\circ \in S^n$. In view of $U'_x \subset Q_x$ it now follows from Proposition 15.9 that

$$[f] = [h \circ f] = \sum_{x \in f^{-1}\{c\}} [f_x]$$

where $f_x: I^n \longrightarrow S^n$ is the map that coincides with $h \circ f$ on the cube Q_x, and sends the complement of that cube to the base point.



Let $\varepsilon_x \in \{\pm 1\}$ denote the orientation character of f at x, and define for each $x \in f^{-1}\{c\}$ a new map $g_x: I^n \longrightarrow S^n$ by

$$g_x|(I^n \backslash Q_x) = \circ \quad \text{and} \quad g_x|Q_x = \varepsilon_x \cdot (f_x|Q_x)$$

where $-(f_x|Q_x)$ denotes the homotopy inverse of $f_x|Q_x$ in the sense of Definition 14.4. If U'_x is replaced, in case $\varepsilon_x = -1$, by its reflected copy within Q_x then g_x restricts to a diffeomorphism

$g_x\colon U'_x \longrightarrow S^n\backslash\{\circ\}$ which now preserves orientation for all $x \in f^{-1}\{c\}$. Of course $[f]$ is expressed in terms of the $[g_x]$ by the formula

$$[f] = \sum_{x\in f^{-1}\{c\}} \varepsilon_x [g_x].$$

We shall prove that the various classes $[g_x]$ are in fact one and the same standard class that does not depend on $[f]$ at all. This will complete the proof since our equation in $\pi_n(S^n,\circ)$ then reduces to

$$[f] = \Big(\sum_x \varepsilon_x\Big)[g_x] = \deg f \cdot [g_x]$$

and thus describes $[f]$ in terms of $\deg f$ alone.

In order to analyze $g_x\colon I^n/\partial I^n \longrightarrow S^n$ we leave the cubic picture and once more represent the pointed spheres by the Alexandroff compactification $\hat{\mathbb{R}^n}$, making the identification on the left hand side by $\hat\varphi\colon I^n/\partial I^n \longrightarrow \hat{\mathbb{R}^n}$ from the beginning of this section, and using stereographic projection $S^n \approx \hat{\mathbb{R}^n}$ on the right hand side. Thereby g_x becomes a map $g\colon \hat{\mathbb{R}^n} \longrightarrow \hat{\mathbb{R}^n}$, and U'_x an open subset $U \subset \mathbb{R}^n$ such that g sends U diffeomorphically onto $\mathbb{R}^n$ preserving orientation, while the complement $\hat{\mathbb{R}^n}\backslash U$ is mapped to the base point $\infty \in \hat{\mathbb{R}^n}$.

We shall prove that g is homotopic to the identity of $(\hat{\mathbb{R}^n},\infty)$. We know that $0 \in \mathbb{R}^n \subset \hat{\mathbb{R}^n}$ has a unique inverse image point under g. Composing g with a translation as in 17.6 we move that point to the origin too: thus $g(0) = 0$. The principle underlying the next step is the fundamental one of differential calculus: the map g, which is differentiable at the origin, is approximated by, and therefore homotopic to its differential there. The following simple result from differential analysis is most suited to turn this rough notion into valid proof.

19.5 Proposition Let $C \subset \mathbb{R}^n$ be a convex open neighbourhood of the origin, and $g\colon C \longrightarrow \mathbb{R}^p$ a C^∞ map with $g(0) = 0$. Then there exist C^∞ mappings $u_1,\ldots,u_n\colon C \longrightarrow \mathbb{R}^p$ such that

$$g(x) = \sum_{j=1}^n x_j \cdot u_j(x) \quad \text{for all } x \in C.$$

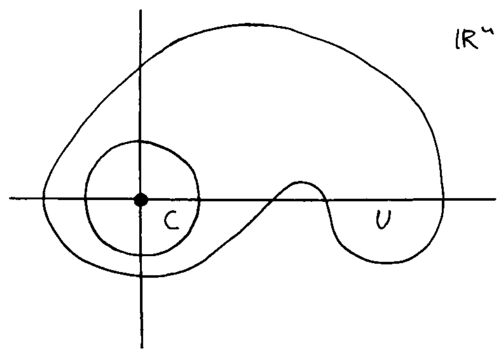
The differential of g at the origin is then given by $T_0 g(x) = \sum_{j=1}^n u_j(0)x_j$.

Proof Put $u_j(x) = \displaystyle\int_0^1 \frac{\partial g}{\partial x_j}(tx)\,dt$ and compute:

$$g(x) = g(x) - g(0) = \int_0^1 \frac{d}{dt} g(tx)\,dt = \int_0^1 \sum_{j=1}^n \frac{\partial g}{\partial x_j}(tx)x_j\,dt = \sum_{j=1}^n x_j \cdot u_j(x)$$

The formula for the differential follows from Leibniz's rule (or, if you prefer, from the very definition of the differential).

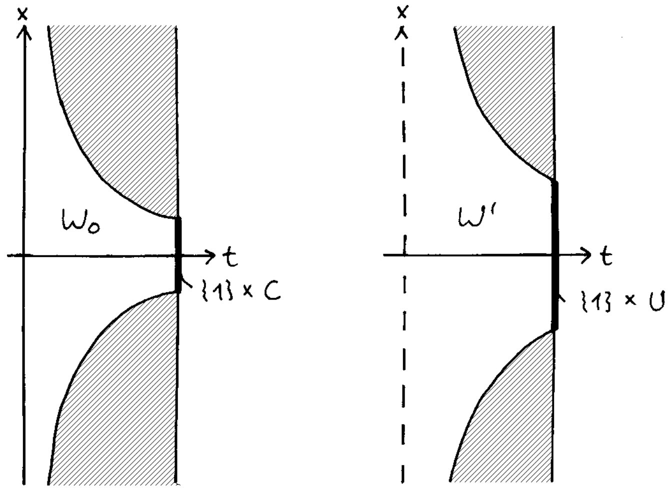
Proof of 19.3 (conclusion) Let $C \subset U$ be an open ball around 0;



then Proposition 19.5 allows to write $g(x) = \sum_j x_j \cdot u_j(x)$ for $x \in C$.

The sets

$$W_0 := \{(t,x) \in I\times\mathbb{R}^n \mid tx \in C\} \quad \text{and} \quad W' := \{(t,x) \in I\times\mathbb{R}^n \mid t > 0 \text{ and } tx \in U\}$$



are open in $I\times\mathbb{R}^n$, and their union is

$$W = \{(t,x) \in I\times\mathbb{R}^n \mid tx \in U\}.$$

The formula

$$I\times\mathbb{R}^n \ni (t,x) \longmapsto \begin{cases} \Big(t, \sum_j x_j \cdot u_j(tx)\Big) & \text{for } (t,x) \in W_0 \\ \Big(t, \dfrac{1}{t} g(tx)\Big) & \text{for } (t,x) \in W' \end{cases}$$

is at once seen to determine a well-defined mapping

$$\Phi: W \longrightarrow I\times\mathbb{R}^n$$

which in fact is a diffeomorphism, as we shall now prove. Since g sends U diffeomorphically onto $\mathbb{R}^n$ we may write

$$g^{-1}(y) = \sum_{j=1}^{n} y_j \cdot v_j(y)$$

with C^∞ mappings $v_j: \mathbb{R}^n \longrightarrow \mathbb{R}^n$. The map

$$I\times\mathbb{R}^n \ni (t,y) \longmapsto \Big(t, \sum_j y_j \cdot v_j(ty)\Big) \in W$$

is the inverse of Φ: by continuity this need only be verified on points with $t > 0$, where it is clear from the alternative formula

$$\sum_j y_j \cdot v_j(ty) = \frac{1}{t} g^{-1}(ty).$$

Being a homeomorphism, Φ is of course a proper map. Composing with the cartesian projection $\mathrm{pr}_2: I\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ we obtain a map

$$h: W \xrightarrow{\Phi} I\times\mathbb{R}^n \xrightarrow{\mathrm{pr}_2} \mathbb{R}^n$$

which is still proper. By Proposition 18.6, sending the complement of $W \subset I\times\hat{\mathbb{R}^n}$ to ∞ gives a continuous extension

$$H: I\times\hat{\mathbb{R}^n} \longrightarrow \hat{\mathbb{R}^n}.$$

Thus H is a base point preserving homotopy between the given map $H_1 = g\colon \hat{\mathbb{R}^n} \longrightarrow \hat{\mathbb{R}^n}$ and H_0, which according to 19.5 is the extended linear isomorphism $(T_0 g)\hat{}\colon \hat{\mathbb{R}^n} \approx \hat{\mathbb{R}^n}$.

The final step: since $T_0 g$ is an orientation preserving automorphism of $\mathbb{R}^n$, and since the group $GL^+(n,\mathbb{R})$ of such automorphisms is connected there is a path $\alpha\colon I \longrightarrow GL^+(n,\mathbb{R})$ from $\alpha(0) = 1$ to $\alpha(1) = T_0 g$. The proper map

$$I \times \mathbb{R}^n \ni (t, x) \longmapsto \alpha(t)\cdot x \in \mathbb{R}^n$$

extends to a homotopy $I \times \hat{\mathbb{R}^n} \longrightarrow \hat{\mathbb{R}^n}$ that connects the identity of $\hat{\mathbb{R}^n}$ to $(T_0 g)\hat{}$.

This completes the proof of Theorem 19.3.

19.6 Corollary For every $0 < n \le q$ the forgetful map

$$[S^n, S^q]^\circ \xrightarrow{u} [S^n, S^q]$$

is bijective.

Proof u is surjective: given $f\colon S^n \longrightarrow S^q$ pick a rotation $r \in SO(q+1)$ that moves $f(\circ)$ to $\circ$; then $r \circ f$ is homotopic to f and respects the base point. There is nothing else to show in case $n < q$ since $[S^n, S^q]^\circ = \pi_n(S^q, \circ)$ then is the trivial group. On the other hand u is injective for $n = q$ since even the composition

$$\deg\colon [S^n, S^n]^\circ \xrightarrow{u} [S^n, S^n] \xrightarrow{\deg} \mathbb{Z}$$

is injective by Theorem 19.3.

Under the hypotheses of Corollary 19.6 we thus have a canonical isomorphism between the groups $\pi_n(S^q, s)$ for various choices of the base point $s \in S^q$. In other words the base point, which was essential to the definition of homotopy groups, can be dispensed with in this particular case, and $\pi_n(S^q)$ is acceptable as a simplified notation.

19.7 Proposition Let n be positive, and let $f\colon S^n \to S^n$ be a map. Then the induced homomorphism

$$f_*\colon \pi_n(S^n) \longrightarrow \pi_n(S^n)$$

is multiplication by the integer $\deg f$.

Proof In view of $f_*[g] = [f \circ g]$ this is a mere restatement of 18.9.

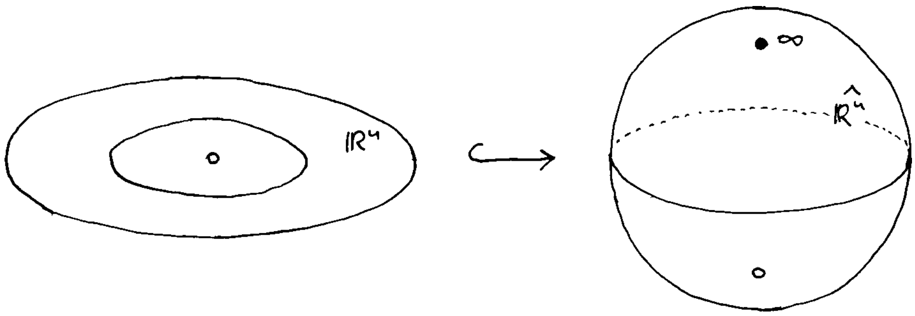
Remark The more general mapping degree $\deg\colon [X, Y] \longrightarrow \mathbb{Z}$ mentioned in 18.10 is bijective if X is non-empty and connected, and $Y = S^n$ is the sphere of the same (positive) dimension. This is known as Hopf's theorem, and can be derived like Theorem 19.3, adding a bit of further work. However it does not serve to compute more homotopy groups as the sphere is on the wrong side.

20 First Applications

Our efforts will now be rewarded. First of all there is the repeatedly mentioned but so far unproven

20.1 Theorem $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ unless $m = n$.

Proof We may assume that m and n are both positive. $\mathbb{R}^m \approx \mathbb{R}^n$ implies $S^m \approx \hat{\mathbb{R}^m} \approx \hat{\mathbb{R}^n} \approx S^n$,

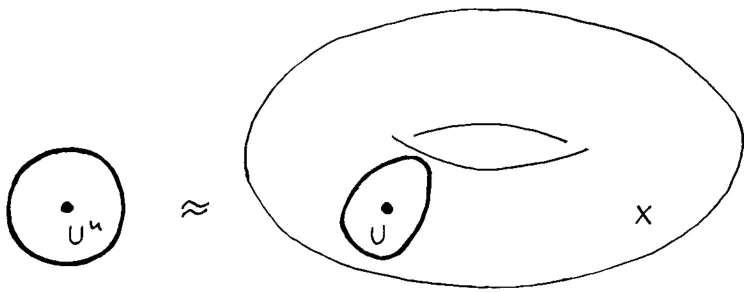


and therefore $\pi_m(S^m, \circ) \simeq \pi_m(S^n, \circ)$. Since $\pi_m(S^m, \circ)$ is isomorphic to $\mathbb{Z}$ while $\pi_m(S^n, \circ)$ is the trivial group for $m < n$ we must have $m \geq n$. By symmetry we also have the opposite inequality, and thus $m = n$.

While this proof would be difficult to beat for conciseness it is not quite satisfactory since it invokes a global argument in order to prove something which one feels to be of essentially local nature. We will therefore give an improved and more general version of it in the framework of topological manifolds.

20.2 Definition A topological space X is an n-dimensional topological manifold if

- it is a Hausdorff space,
- there exists a countable basis for the topology of X, and if
- it is locally euclidean: each $x \in X$ has an open neighbourhood U that is homeomorphic to the n-dimensional open ball U^n. (It follows at once that every given neighbourhood V contains such a neighbourhood U.)



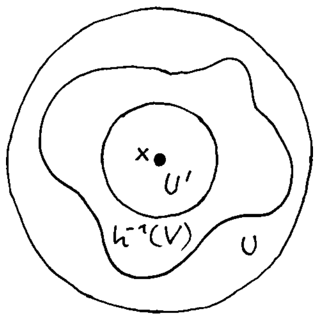
Remarks Thus an n-dimensional topological manifold locally looks like $\mathbb{R}^n$. It is a long-established custom among topologists to refer to the latter as n-dimensional *euclidean* space despite the fact that the euclidean structure plays no role whatsoever. — If you feel that the Hausdorff property might be a consequence of local euclideanness you should have another look at Example 10.5(4). — The most important examples of topological manifolds are obtained by stripping a submanifold of $\mathbb{R}^p$ of its differential structure. However a finer analysis reveals that the class of topological manifolds is wider than that, and the exact relation between the notions of topological and differential manifolds is quite subtle and a wide field of study.

20.3 Question Explain why every topological manifold is locally connected and locally compact.

The following result improves upon 20.1, and illustrates the diligent use of a functor invariant.

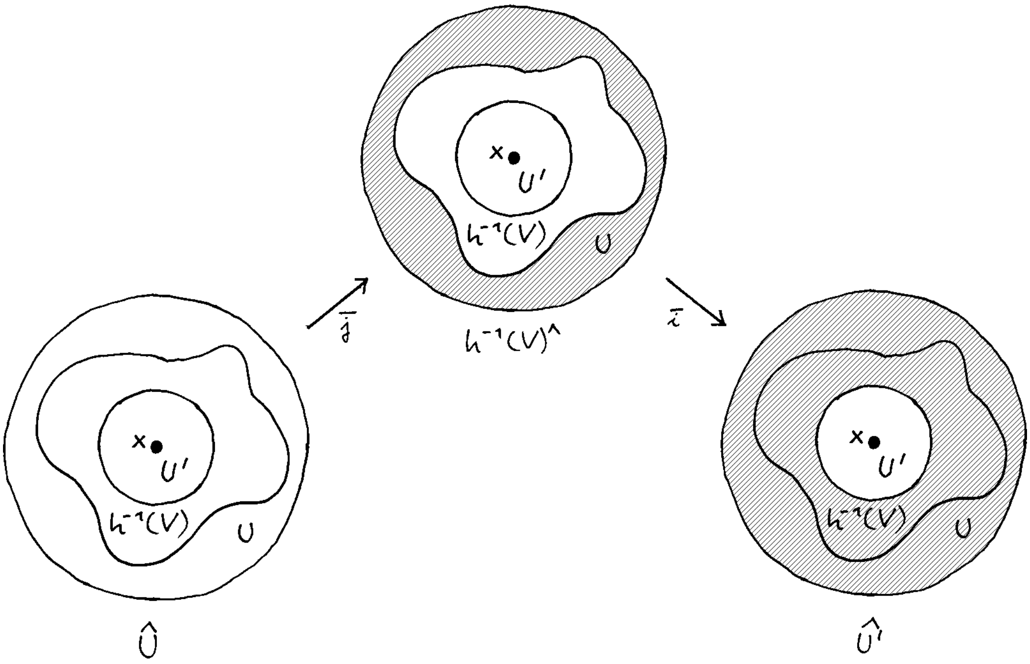
20.4 Theorem If two non-empty topological manifolds are homeomorphic then they have the same dimension. Therefore the dimension of a non-empty topological manifold is a well-defined invariant.

Proof Let X be an m-dimensional, Y, an n-dimensional topological manifold, and $h: X \approx Y$ a homeomorphism. Only 0-dimensional manifolds are discrete, so we may assume that both m and n are positive. Pick any $x \in X$ and let $U \subset X$ be an open neighbourhood of x that is homeomorphic to the ball U^m. Then $h(U) \subset Y$ is an open neighbourhood of $h(x)$, therefore contains an open neighbourhood V of $h(x)$ which is homeomorphic to U^n. By the same argument the pullback $h^{-1}(V) \subset U$ contains an open neighbourhood U' of x that is homeomorphic to U^m. We thus have a chain of inclusions $U' \overset{i}{\hookrightarrow} h^{-1}(V) \overset{j}{\hookrightarrow} U$ of open subsets of X.



Passing to Alexandroff compactifications Proposition 16.8 produces maps between spheres in the opposite direction:

$$S^m \approx \hat{U} \xrightarrow{\bar{j}} h^{-1}(V)\hat{} \xrightarrow{\bar{i}} \hat{U}' \approx S^m$$

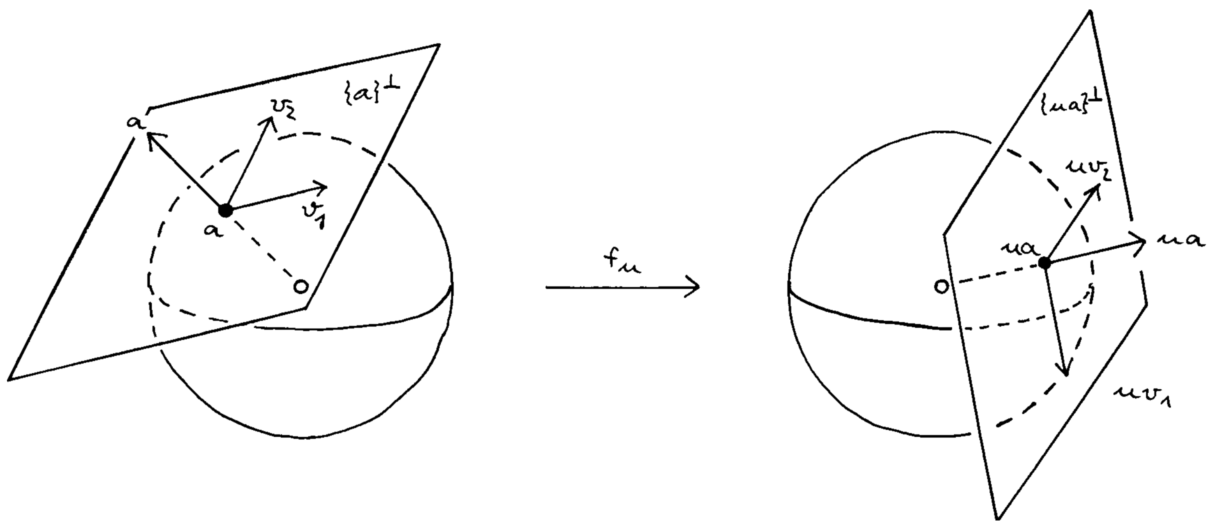


The composition $\bar{i}\,\bar{j}$ is the identity on the open subspace $U' \subset \hat{U}$, and collapses $\hat{U} \setminus U'$ to the base point ∞: therefore its mapping degree is $\deg \bar{i}\,\bar{j} = 1$, and the bottom arrow in the induced triangle of homotopy groups

$$\begin{array}{ccc} & \pi_m\big(h^{-1}(V)\hat{}\,\big) & \\ {\scriptstyle \bar{j}_*}\nearrow & & \searrow{\scriptstyle \bar{i}_*} \\ \pi_m(\hat{U}) & \xrightarrow[(\bar{i}\,\bar{j})_*]{} & \pi_m(\hat{U}') \end{array}$$

an isomorphism. This is only possible if the group $\pi_m\big(h^{-1}(V)\hat{}\,\big)$ is non-trivial, and since $h^{-1}(V)\hat{}$ is an n-sphere we conclude $m \geq n$. Since the argument is symmetric this implies $m = n$.

The simplest way of constructing maps of S^n into itself is by restricting an orthogonal map $u \in O(n+1)$ to $f_u: S^n \longrightarrow S^n$. Since f_u is a diffeomorphism the mapping degree is ± 1 depending on whether f_u preserves or reverses orientation. In fact one has $\deg f_u = \det u$ for if $(v_1, \ldots, v_n)$ is a basis of the tangent space $T_a S^n$ for some $a \in S^n$ then $(a, v_1, \ldots, v_n)$ is a basis of $\mathbb{R}^{n+1}$, and $(uv_1, \ldots, uv_n)$ a basis of $T_{ua} S^n = \{ua\}^{\perp}$.



In particular let us record:

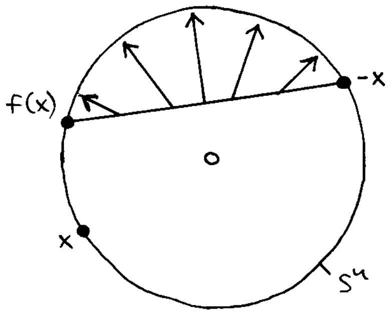
20.5 Fact The *antipodal* map $f_{-1}: S^n \longrightarrow S^n$, which sends x to $-x$, has degree $(-1)^{n+1}$.

More generally we prove:

20.6 Proposition Let $n \in \mathbb{N}$ be positive and $f: S^n \longrightarrow S^n$ be a continuous map without fixed points. Then $\deg f = (-1)^{n+1}$.

Proof The assumption is $f(x) \neq x$ for all $x \in S^n$. We construct a homotopy from f to the antipodal map f_{-1} by joining $f(x)$ to $-x$ in $\mathbb{R}^{n+1} \setminus \{0\}$ and radially projecting into the sphere as usual:

$$F: I \times S^n \longrightarrow S^n; \quad F(t,x) := \frac{(1-t)f(x) - tx}{\big|(1-t)f(x) - tx\big|}$$



Being homotopic to the antipodal map, f must have the same degree $(-1)^{n+1}$.

The proposition leads to a non-existence theorem if furthermore the homotopy class of f is known. The most important version is the following, which in the two-dimensional case sometimes is quoted as the combed hedgehog theorem.

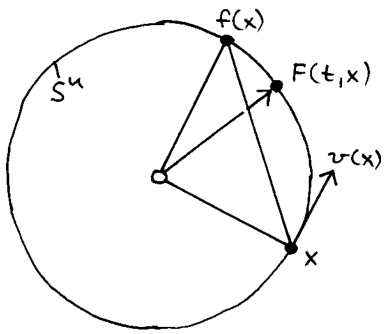
20.7 Theorem If $n \in \mathbb{N}$ is even then every vector field on S^n has at least one zero.

Proof There is nothing to show for $n = 0$. Consider now an arbitrary $n > 0$, and a vector field $v: S^n \longrightarrow TS^n$ without zeros. Since for each $x \in S^n$ the vector $v(x) \in T_xS^n$ is perpendicular to x the assignment

$$S^n \ni x \longmapsto \frac{1}{|v(x)|} \cdot v(x) \in S^n$$

defines a fixed point free mapping $f: S^n \longrightarrow S^n$. This map is homotopic to the identity via

$$F: I \times S^n \longrightarrow S^n; \quad F(t,x) := \frac{(1-t)f(x) + tx}{\big|(1-t)f(x) + tx\big|}$$



and therefore $\deg f = 1$. On the other hand $\deg = (-1)^{n+1}$ by Proposition 20.6, so that n must be odd.

The two-dimensional mapping degree is at the heart of the following geometric proof of the fundamental theorem of algebra.

Proof of the fundamental theorem of algebra Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be a unitary polynomial of degree $d \in \mathbb{N}$, say

$$f(z) = z^d + \sum_{j=0}^{d-1} a_j z^j .$$

The projectivized version of f is

$$F_1\colon \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1; \quad F_1[w:z] = \left[w^d : \Big(z^d + \sum_{j=0}^{n-1} a_j w^{d-j} z^j \Big) \right].$$

Now recall Example 11.7: the projective line $\mathbb{C}P^1$ is homeomorphic to S^2, and therefore the mapping degree of F_1 is defined. In order to compute it consider the homotopy

$$F\colon I \times \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1; \quad F\big(t, [w:z]\big) = \left[w^d : \Big(z^d + t \sum_{j=0}^{n-1} a_j w^{d-j} z^j \Big) \right];$$

it joins F_1 to $F_0\colon [w:z] \longrightarrow [w^d : z^d]$. This very simple map has mapping degree d, for $[1:1]$ is a regular value with exactly d inverse image points corresponding to the d-th roots of unity, and the complex differential respects the orientation at each of them. Therefore $\deg F_1 = d$.

Assume now that f has no zero: then $[1:0]$, being a non-value of F_1 is a regular value, and $\deg F_1 = 0$. Thus $d = 0$, and the proof is done.

Given our knowledge of homotopy groups it goes without saying that the sphere S^{n-1} can never be a deformation retract of the disk D^n. In fact it is not even an ordinary retract:

20.8 Theorem Let $n \in \mathbb{N}$ be arbitrary. Then S^{n-1} is not a retract of D^n.

Proof For $n = 0$ the statement is trivial. For $n > 0$ choose a base point $\circ \in S^{n-1}$ and let $j\colon S^{n-1} \longrightarrow D^n$ denote the inclusion. Assuming that there exists a retraction $r\colon D^n \longrightarrow S^{n-1}$, which necessarily preserves the base point, we obtain a commutative diagram

$$\begin{array}{ccccc} & & \pi_{n-1}(D^n, \circ) & & \\ & {\scriptstyle j_*}\nearrow & & \searrow{\scriptstyle r_*} & \\ \pi_{n-1}(S^{n-1}, \circ) & & = = = = = & & \pi_{n-1}(S^{n-1}, \circ) \end{array}$$

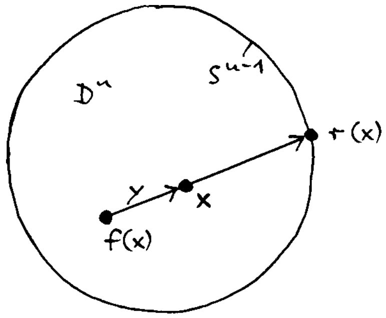
of groups and homomorphisms, or if $n = 1$, of sets and mappings. In either case this is impossible since $\pi_{n-1}(D^n, \circ)$ is trivial while $\pi_{n-1}(S^{n-1}, \circ)$ is not.

Thus the assertion is proved by contradiction.

Theorem 20.8 has an easy corollary which is known as Brouwer's fixed point theorem:

20.9 Theorem Let $n \in \mathbb{N}$ be arbitrary. Every continuous mapping of D^n to itself has at least one fixed point.

Proof Assume to the contrary that there exists an $f\colon D^n \longrightarrow D^n$ without fixed points: $f(x) \neq x$ for all $x \in D^n$. Then a retraction $r\colon D^n \longrightarrow S^{n-1}$ may be constructed by projecting $x \in D^n$ along the ray spanned by $y := x - f(x)$.



If you insist on a formula in order to verify continuity then $r(x) = x + t\cdot y$ where

$$t = \frac{-\langle x,y\rangle + \sqrt{\langle x,y\rangle^2 + \big(1-|x|^2\big)|y|^2}}{|y|^2}$$

is determined as the unique non-negative solution of the quadratic equation $|r(x)|^2 = 1$. By 20.8 such a retraction r does not exist, and therefore our assumption was false.

You will remember from first year calculus how the special case $n = 1$ of Theorem 20.9 is solved using the intermediate value theorem. Indeed connectedness is a one-dimensional notion, and one way to generalize it to higher dimensions is by homotopy groups. Therefore it does not surprise that these are the appropriate tool to treat the general case.

Rather than further add to our list of applications I would like to halt at this point in order to put in perspective what we have so far achieved. Firstly the idea of homotopy has been brilliantly confirmed: we have seen that passing to homotopy classes does reduce the vastness of sets of mappings to discreteness of homotopy sets. Yet at least in one interesting situation we have found homotopy sets to be non-trivial, and it is the conjunction of these two facts that makes all those applications possible.

The second point of importance is the discovery, already made in Section 15 that some homotopy sets carry a natural group structure. We certainly had no right to ask for that, however it comes as a bonus that greatly facilitates the computation of homotopy sets.

The further programme seems to be clear: one should try and generalise our computation of homotopy groups of spheres to many more spaces, perhaps also investigate other homotopy sets with natural algebraic structures, and apply these tools to general topological spaces in the same manner that we have applied them to spheres in this section.

Surprisingly, the programme does not work. Indeed, it comes to a virtual stop at the very next step to be taken, the calculation of the homotopy groups $\pi_n(S^q)$ for $n > q$. The obvious first guess that these groups might be trivial as are those for $n < q$, turns out to be quite off the mark. In the 1930s H. Hopf found that the canonical projection

$$\mathbb{C}^2 \setminus \{0\} \supset S^3 \longrightarrow \mathbb{C}P^1 \approx S^2$$

sending the pair (w, z) to its class $[w : z]$, is not homotopic to a constant map. Introducing his henceforth famous *Hopf invariant* he showed that more generally the groups $\pi_{4k-1}(S^{2k})$ are infinite for all $k \geq 1$. The determination of the homotopy groups $\pi_n(S^q)$ for arbitrary $n > q$ is considered an extremely difficult problem. There are some general results, for instance it is known that apart from Hopf's examples all these groups are finite, and that $\pi_{p+q}(S^q)$ is, up to canonical isomorphism, independent of q for $q \geq p+2$. The groups $\pi_n(S^q)$ have been explicitly determined for particular values of n and q but nobody has yet been able to recognise the general pattern. Pending a dramatic breakthrough these groups must be considered as kind of universal constants of topology. Naturally even less is known in general about the homotopy groups of more complicated spaces.

Due to their simple and natural definition, homotopy groups are interesting objects of study, but they are quite unsuitable as a tool to investigate topological spaces since they lack computability. Let us, however, dream a bit of what properties one might want a series of more accessible functors than the π_n to have:

- They should carry comparable information, let us say should be able at least to distinguish between spheres of different dimension.
- Simple operations on topological spaces should be reasonably well reflected in the values of those functors. Such operations might comprise taking sums and products, collapsing subspaces, gluing spaces along a common subspace, or attaching things to a space.
- Ideally, there should be an algorithm that allows to compute these functors for explicitly given topological spaces and mappings.

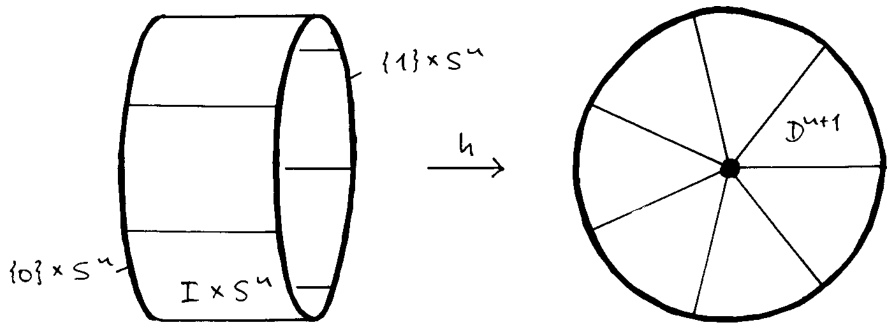
It is a central objective of my course to make this dream a reality.

21 Extension of Mappings and Homotopies

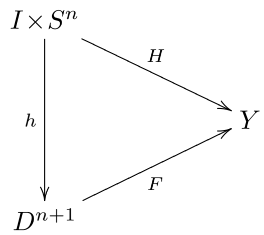
In this section we study aspects of the following *extension problem*: let X and Y be topological spaces, $A \subset X$ a subspace and $f: A \longrightarrow Y$ a map. Does there exist a (continuous) mapping $F: X \longrightarrow Y$ with $F|A = f$? An example is given by Tietze's extension theorem with gives an affirmative answer if X is normal, A is closed in X, and $Y = \mathbb{R}$ or, as an immediate generalisation, $Y = \mathbb{R}^q$. While the first two conditions just express the appropriate kind of well-behavedness of the inclusion $A \subset X$ the last one makes the situation quite special. We will now see that the validity of Tietze's theorem has much to do with the fact that $\mathbb{R}^q$ is a contractible space.

21.1 Proposition Let $n \in \mathbb{N}$ be arbitrary, Y a space, and let $f: S^n \longrightarrow Y$ be a pointed map. An extension $F: D^{n+1} \longrightarrow Y$ of f exists if and only if f is homotopic to a constant map.

Proof The map $h: I \times S^n \longrightarrow D^{n+1}$ sending (t, x) to tx collapses the subspace $\{0\} \times S^n$ to a point and is an identification.



By Proposition 10.3, in the triangle



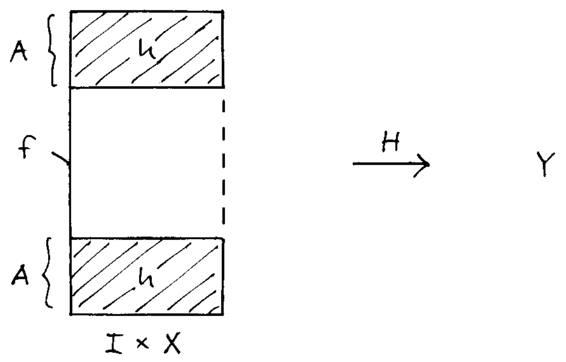
mappings F with $F|S^n = f$ therefore correspond to homotopies H between a constant map, and f. This proves the proposition.

Choosing $Y = S^n$ we see that Tietze's theorem does not hold with a sphere as target space: for $n > 0$ a map $f: S^n \longrightarrow S^n$ cannot be extended over D^{n+1} unless its mapping degree is zero.

For all its simplicity Proposition 21.1 is striking in that it allows to restate a map extension problem as an equivalent question of homotopy. Even more surprising is the fact that such a relation persists in quite general circumstances, as we will now see.

21.2 Definition Let X be a space and $A \subset X$ a subspace. The inclusion $A \hookrightarrow X$ or, equivalently, the pair (X, A) is said to have the homotopy extension property if the following holds. Given any homotopy

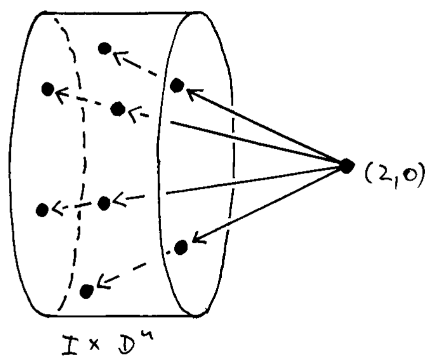
$h: I\times A \longrightarrow Y$ and a map $f: X \longrightarrow Y$ with $h_0 = f|A$ there exists a homotopy $H: I\times X \longrightarrow Y$ with $H|I\times A = h$ and $H_0 = f$.



Remark Inclusions with the homotopy extension property are more concisely called *cofibrations*. An explanation of this at first sight surprising name would take us too far afield.

21.3 Examples (1) For all $n \in \mathbb{N}$ the pair (D^n, S^{n-1}) has the homotopy extension property. The basic observation to make is that the space $I\times S^{n-1} \cup \{0\}\times D^n$ is a retract of the cylinder $I\times D^n$: a retraction

$$r: I\times D^n \longrightarrow I\times S^{n-1} \cup \{0\}\times D^n$$



is obtained by projecting from the point $(2,0) \in \mathbb{R}\times\mathbb{R}^n$. From the given data $h: I\times S^{n-1} \longrightarrow Y$ and $f: D^n \longrightarrow Y$ we assemble a map $I\times S^{n-1} \cup \{0\}\times D^n \longrightarrow Y$ sending $(t,x) \in I\times S^{n-1}$ to $h(t,x)$, and $(0,x)$ to $f(x)$. Composition with r gives a homotopy

$$I\times D^n \longrightarrow I\times S^{n-1} \cup \{0\}\times D^n \longrightarrow Y$$

that does everything we want.

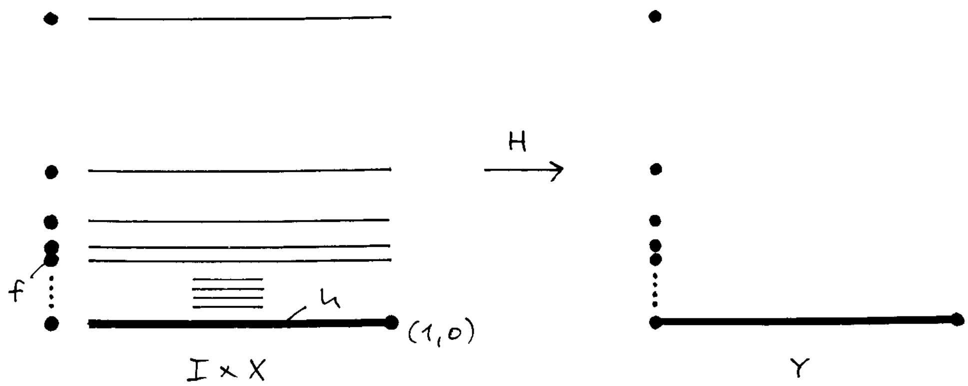
(2) One would not expect the homotopy extension property for (X,A) unless $A \subset X$ is a closed subspace, and this can be proved provided X is a Hausdorff space. A more interesting negative example is

$$X = \{0\} \cup \left\{ \frac{1}{k} \;\middle|\; 0 \neq k \in \mathbb{N} \right\} \subset \mathbb{R} \quad \text{and} \quad A = \{0\}.$$

In order to test it for the homotopy extension property we put $Y = \{0\}\times X \cup I\times A \subset \mathbb{R}^2$ and consider the extension problem

$$h: I\times A \longrightarrow Y; \quad h(t,0) = (t,0)$$
$$f: X \longrightarrow Y; \quad f(x) = (0,x).$$

There can be no solution $H: I\times X$ since for each $k>0$ the image $H\big(I\times\{\frac{1}{k}\}\big)$ must be connected, hence be equal to $\{0\}\times\{\frac{1}{k}\}$, and this clearly contradicts continuity of H at the point $(1,0)$.

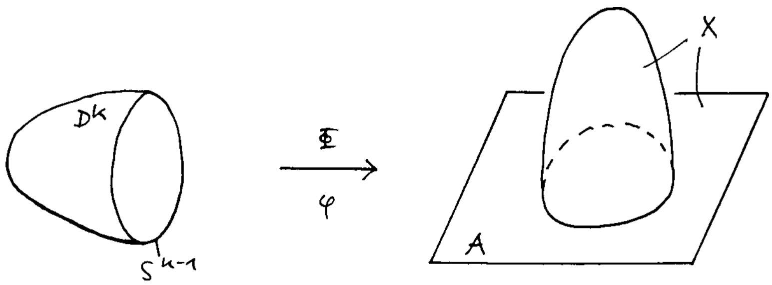


As the second example suggests inclusions of closed subspaces that lack the homotopy extension property tend to be pathological — an impression supported by the following positive result for cell complexes. Recall from Section 13 that in this course, cell complexes always are finite by convention.

21.4 Proposition Let X be a cell complex, and $A\subset X$ a subcomplex. Then (X,A) has the homotopy extension property.

Proof We may think of X as obtained from A by successive attachment of cells. By induction on the number of cells in $X\setminus A$ the problem of extending a homotopy from $I\times A$ to $I\times X$ at once is reduced to the case that X is built from A by attaching a single cell.

We thus may assume $X = D^k\cup_\varphi A$, and as usual we denote by $\Phi: D^k \longrightarrow X$ the corresponding characteristic map.



Let $h: I\times A\longrightarrow Y$ and $f: X\longrightarrow Y$ with $h_0 = f|A$ be given. The compositions

$$\tilde h: I\times S^{k-1} \xrightarrow{\mathrm{id}\times\varphi} I\times A \xrightarrow{h} Y \quad\text{and}\quad \tilde f: D^k \xrightarrow{\Phi} X \xrightarrow{f} Y$$

satisfy $\tilde h_0 = \tilde f|S^{k-1}$ and thereby define a homotopy extension problem for (D^k, S^{k-1}). By 21.3(1) there is a solution $\tilde H: I\times D^k\longrightarrow Y$. Together with the given mapping h it induces a homotopy

$$I\times\big(D^k\cup_\varphi A\big) = (I\times D^k)\cup_{\mathrm{id}\times\varphi}(I\times A)\xrightarrow{H} Y$$

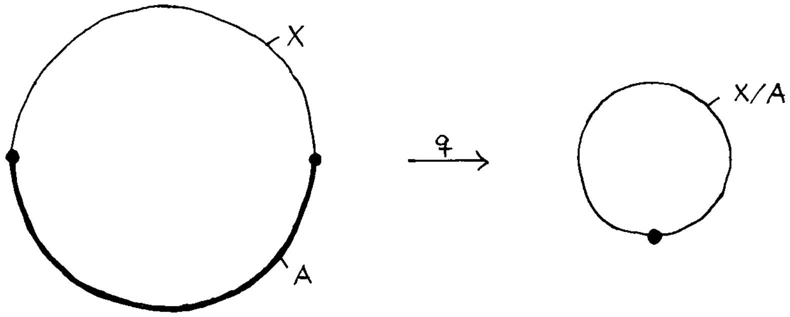
that solves the original problem.

21.5 Question At one point in this proof two a priori competing topologies on a cell space have been switched. Explain where, and why this is justified (also see 15.7).

It is clear how Proposition 21.4 can be applied to the map extension problem: if X is a finite cell complex, $A\subset X$ a subcomplex, and Y an arbitrary space then the question of whether $f: A\longrightarrow Y$ admits an extension over X depends not on f itself but only upon its homotopy class $[f]\in[A,Y]$.

Here is another, more explicit application.

21.6 Theorem Let X be a cell complex, and $A \subset X$ a contractible subcomplex. Then the quotient map $q: X \longrightarrow X/A$ is a homotopy equivalence.

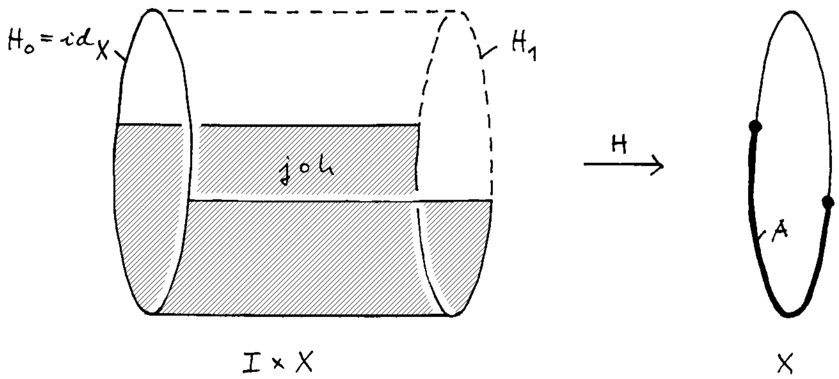


If furthermore A has a base point, and is contractible in **Top**° then q likewise is a homotopy equivalence in that category.

Proof Let $j: A \hookrightarrow X$ denote the inclusion and choose a homotopy $h: I{\times}A \longrightarrow A$ from $h_0 = \mathrm{id}_A$ to a constant map h_1. The mappings

$$j{\circ}h: I{\times}A \longrightarrow X \quad \text{and} \quad \mathrm{id}_X: X \longrightarrow X$$

pose a homotopy extension problem which by Proposition 21.4 has a solution $H: I{\times}X \longrightarrow X$.



Since $H_1: X \longrightarrow X$ is constant on A it induces a map $p: X/A \longrightarrow X$. We prove that p is homotopy inverse to q.

Firstly,

$$p{\circ}q = H_1 \simeq H_0 = \mathrm{id}_X$$

holds by definition. On the other hand H sends $I{\times}A$ into A so that $q{\circ}H$, being constant on $I{\times}A$ induces a map $\overline{H}$ which renders the diagram

$$\begin{array}{ccc} I{\times}X & \xrightarrow{\quad H \quad} & X \\ {\scriptstyle \mathrm{id}\times q}\downarrow & & \downarrow{\scriptstyle q} \\ I{\times}(X/A) & \xrightarrow{\quad \overline{H} \quad} & X/A \end{array}$$

commutative. We see that

$$\overline{H}_1{\circ}q = q{\circ}H_1 = q{\circ}p{\circ}q,$$

and this implies $\overline{H}_1 = q{\circ}p$ because q is surjective. Therefore we have

$$q{\circ}p = \overline{H}_1 \simeq \overline{H}_0 = \mathrm{id}_{X/A}$$

which concludes the proof of the main statement.

The pointed version of the proposition follows by simple inspection of the same proof.

We have observed in 19.6 that for $0 < n \le q$ the homotopy group $\pi_n(S^q, \circ)$ is essentially independent of the choice of the base point $\circ \in S^q$. This phenomenon is in fact much more general. We take the opportunity to introduce the following bit of useful vocabulary.

21.7 Definition For given $n \in \mathbb{N}$ a pointed topological space $(Y, \circ) \in |\mathbf{Top}^\circ|$ is called n-connected if $\pi_j(Y, \circ) = \{0\}$ is trivial for all $j \le n$. Alternatively 1-connected spaces are called simply connected.

21.8 Theorem Let X be a cell complex, pointed by one of its 0-cells $\circ$, and let Y be a simply connected space. Then the forgetful mapping

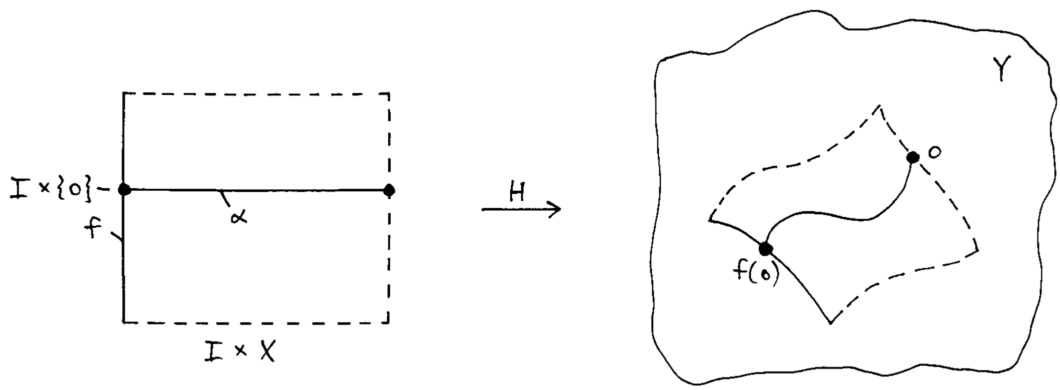
$$[X, Y]^\circ \xrightarrow{u} [X, Y]$$

is bijective.

Proof We first prove that u is surjective. Given an arbitrary map $f: X \to Y$ we must show that f is homotopic to a map that preserves base points. Since Y is connected we find a path $\alpha: I \longrightarrow Y$ from $f(\circ)$ to $\circ$. The maps

$$\alpha{\circ}\mathrm{pr}_1: I \times \{\circ\} \longrightarrow Y \quad \text{and} \quad f: X \longrightarrow Y$$

define a homotopy extension problem for the inclusion of complexes $\{\circ\} \subset X$ which, by Proposition 21.4 has a solution $H: I \times X \longrightarrow Y$.

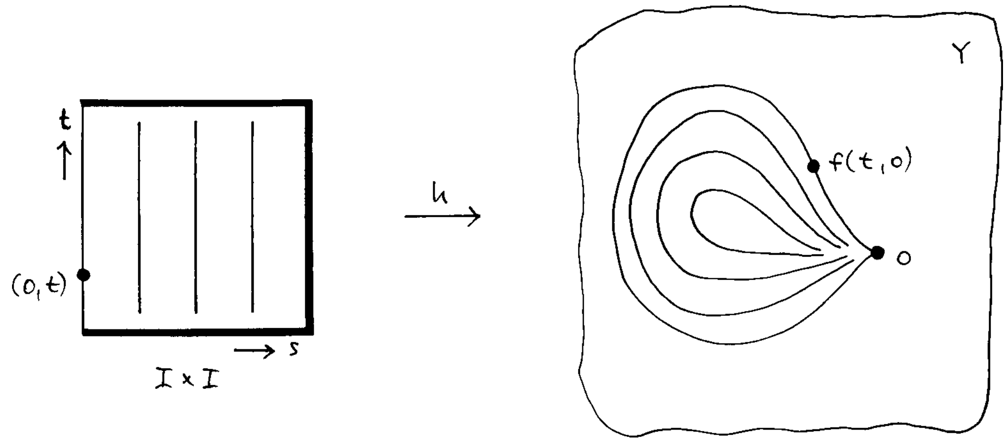


Thus H is a homotopy from $H_0 = f$ to $H_1: X \longrightarrow Y$, and $H_1(\circ) = \alpha(1) = \circ$.

The proof of injectivity of u follows the same lines but is more involved. Consider two pointed mappings $f_0, f_1: (X, \circ) \longrightarrow (Y, \circ)$ which are homotopic in **Top**, say via $f: I{\times}X \longrightarrow Y$. The restriction

$$I \ni t \longmapsto f(t, \circ) \in Y$$

is a closed path, so represents an element of the fundamental group $\pi_1(Y, \circ)$. Since this group is trivial by assumption we find a homotopy $h: I \times I \longrightarrow Y$ to the path with constant value $\circ \in Y$.

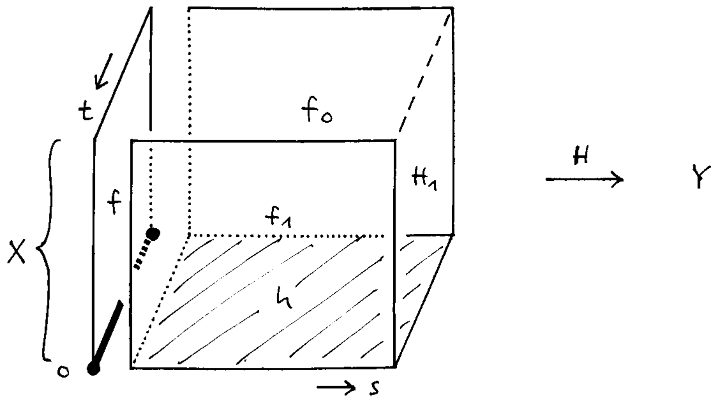


Putting $\tilde{h}(s,0,x) := f(0,x)$, $\tilde{h}(s,1,x) := f(1,x)$, and $\tilde{h}(s,t,\circ) := h(s,t)$ we assemble a mapping

$$\tilde{h}\colon I\times\big(I\times\{\circ\}\ \cup\ \{0,1\}\times X\big) \longrightarrow Y.$$

Together with $f\colon I\times X \longrightarrow Y$ it defines a homotopy extension problem with respect to the inclusion

$$I\times\{\circ\}\ \cup\ \{0,1\}\times X\ \subset\ I\times X.$$



It follows from our assumptions on X that this is an inclusion of cell complexes, so the extension problem has a solution $H\colon I\times(I\times X) \longrightarrow Y$. The resulting map $H_1\colon I\times X \longrightarrow Y$ is a pointed homotopy from f_0 to f_1. This completes the proof.

If, unlike the assumptions of Theorem 21.8, no condition is imposed on the space Y at all the method used in the proof still gives positive results, including this one: every path $\alpha\colon I \longrightarrow Y$ defines a bijection, indeed for $n > 0$ an isomorphism

$$\pi_n\big(Y,\alpha(0)\big) \simeq \pi_n\big(Y,\alpha(1)\big)$$

between the homotopy groups corresponding to different base points. This implies that the property of Y being n-connected makes sense even if no base point is specified. We insist however on $Y \neq \emptyset$, so that a space is 0-connected if and only if it is non-empty and connected. Take careful note that the isomorphic groups $\pi_n\big(Y,\alpha(0)\big)$ and $\pi_n\big(Y,\alpha(1)\big)$ may *not* generally be identified with each other, because the isomorphism depends on the choice of the path α.

This is different if Y is simply connected, for then the path α joining two given points y_0, y_1 is unique up to homotopy, and the resulting isomorphism between homotopy groups turns out to be the composition of the bijective forgetful maps of Theorem 21.8:

$$\pi_n(Y,y_0) = \big[(I^n/\partial I^n,\circ),(Y,y_0)\big]^\circ \longrightarrow [I^n/\partial I^n, Y] \longleftarrow \big[(I^n/\partial I^n,\circ),(Y,y_1)\big]^\circ = \pi_n(Y,y_1)$$

Thus for simply connected spaces one may drop the base point from the notation of homotopy groups as we already did in the case of spheres.

You can work ot the details yourself if you wish, or consult [tom Dieck – Kamps – Puppe].

22 Cellular Mappings

22.1 Definition Let X and Y be cell complexes. A continuous map $f: X \longrightarrow Y$ is cellular if it respects the cell filtrations in the sense that

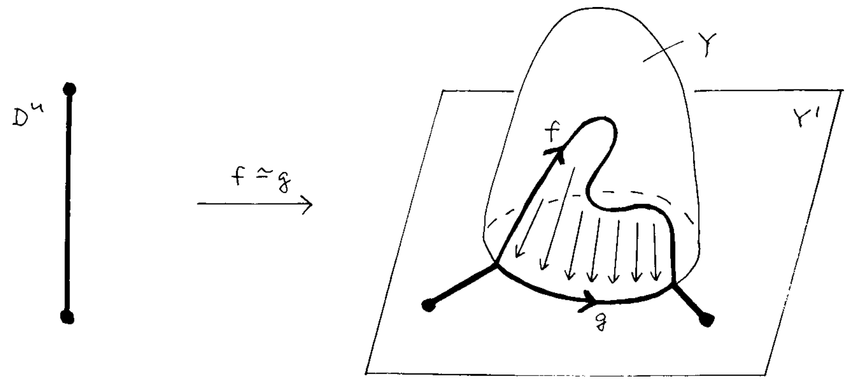
$$f(X^k) \subset Y^k \quad \text{for } k = 0, 1, \dots, \dim X.$$

It goes without saying that this notion does not depend on the full cell structures of X and Y but just on the filtrations determined by them. The importance of cellular maps lies in two facts: firstly, only cellular maps give rise to relations between individual cells of X and of Y, while secondly from a homotopy point of view, cellular maps are as general as arbitrary continuous maps. This is the conclusion of the following very important theorem, which (together with its corollary 22.6) parallels Proposition 18.7 and is sometimes referred to as the theorem of *cellular approximation*.

22.2 Theorem Let X, Y be cell complexes, and $A \subset X$ a subcomplex. If $f: X \longrightarrow Y$ is a map such that $f|A: A \longrightarrow Y$ is cellular then there exists a cellular map $g: X \longrightarrow Y$ that is homotopic to f relative A (so in particular $f|A = g|A$).

The geometric core of this theorem may be isolated as follows.

22.3 Proposition Let Y' be a topological space and let Y be obtained from Y' by attaching a q-cell. For some $n < q$ let $f: D^n \longrightarrow Y$ be a map such that $f(S^{n-1}) \subset Y'$. Then f is homotopic relative S^{n-1} to a map $g: D^n \longrightarrow Y$ with $g(D^n) \subset Y'$.



Postponing the proof of the proposition let us see how it implies Theorem 22.2. The first intermediate step is

22.4 Lemma Let Y be a cell complex, and $n \in \mathbb{N}$. Let $f: D^n \longrightarrow Y$ be a map such that $f(S^{n-1}) \subset Y^n$. Then f is homotopic relative S^{n-1} to a map $g: D^n \longrightarrow Y$ with $g(D^n) \subset Y^n$.

Proof Y is obtained from Y^n by attaching a finite number of cells, say $e_1, \dots, e_s$ in this order, and we work by induction on $s \in \mathbb{N}$. There is nothing to be shown if $s = 0$, and for $s > 0$ Proposition 22.3 yields a map $f': D^n \longrightarrow Y$ which is homotopic to f relative S^{n-1}, and takes values in

$$Y' := Y^n \cup e_1 \cup \dots \cup e_{s-1} \subset Y.$$

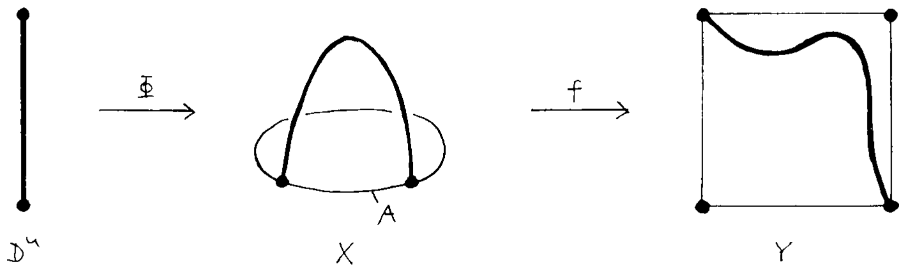
Since only s–1 cells are needed to build Y' from Y^n the inductive assumption applies to $f'\colon D^n \longrightarrow Y'$. The resulting $g\colon D^n \longrightarrow Y'$ takes values in Y^n, and read as a map with target space Y, is homotopic to f relative S^{n-1}. This completes the inductive step.

The next lemma already states a special case of Theorem 22.2.

22.5 Lemma Let X, Y be cell complexes, and $A \subset X$ a subcomplex such that X is obtained from A by attaching a single cell. If $f\colon X \longrightarrow Y$ is a map such that $f|A\colon A \longrightarrow Y$ is cellular then there exists a cellular map $g\colon X \longrightarrow Y$ that is homotopic to f relative A.

Proof Let $\Phi\colon D^n \longrightarrow X$ be the characteristic map of the cell $X \setminus A$. The composition

$$f \circ \Phi\colon D^n \longrightarrow X \longrightarrow Y$$



sends S^{n-1} to $Y^{n-1} \subset Y^n$, and by Lemma 22.4 admits an approximation $g'\colon D^n \longrightarrow Y$, with values in Y^n and homotopic to $f \circ \Phi$ relative S^{n-1}. The maps g' and $f|A$ together define a cellular mapping $g\colon X \longrightarrow Y$, and a homotopy from f to g is obtained similarly, by gluing a relative homotopy from $f \circ \Phi$ to g' with the trivial homotopy of $f|A$. This completes the proof of the lemma.

Proof of Theorem 22.2 Let X be obtained from A by successively attaching cells $e_1, \ldots, e_r$. Working by induction on $r \in \mathbb{N}$ we have nothing to do in case $r = 0$. If $r > 0$ is positive we consider the subcomplex

$$X' := A \cup e_1 \cup \cdots \cup e_{r-1} \subset X.$$

By inductive assumption we find a cellular map $g'\colon X' \longrightarrow Y$ and a homotopy $G'\colon I \times X' \longrightarrow Y$ from $f|X'$ to g' relative A. Since the pair (X, X') has the homotopy extension property G' may be extended to a homotopy $G\colon I \times X \longrightarrow Y$, automatically relative A, from f to some extension $G_1\colon X \longrightarrow Y$ of g'. The difference $X \setminus X'$ consists of the single cell e_r, and we may therefore apply 22.5 to the map G_1, with X' in place of A: the result is a cellular map $g\colon X \longrightarrow Y$ that is homotopic to G_1 relative X', hence to f relative A. This completes the inductive step.

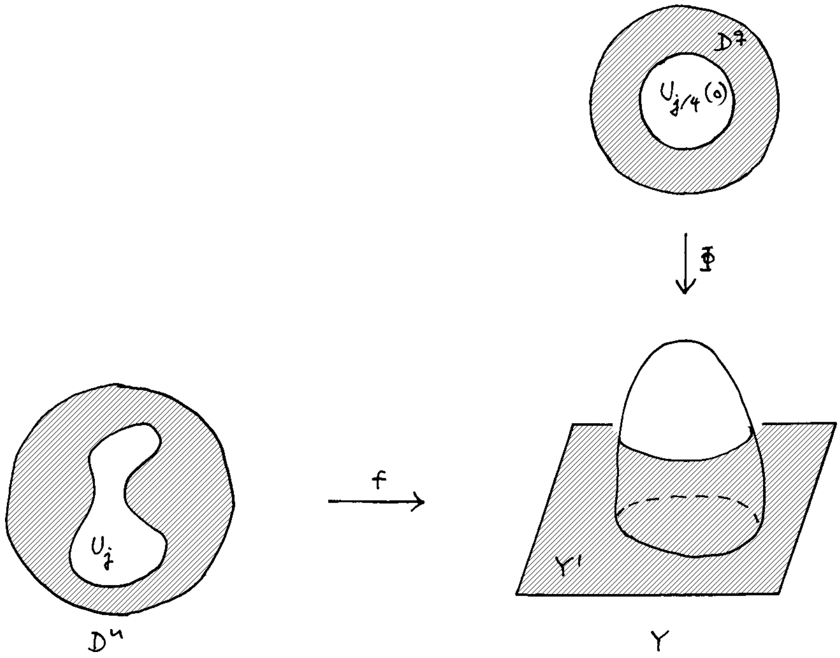
We thus have reduced Theorem 22.2 to Proposition 22.3, using mere formalities. By contrast the proof of Proposition 22.3 deals with the substance of the theorem, and is much more interesting.

Proof of Proposition 22.3 If Y' consisted of one point then Y would be a q-sphere and we could simply appeal to Theorem 17.2. The general situation however is not quite so neat and we have no choice but to take up and carefully adapt the arguments used in Section 17. Since for maps from D^n into Y differentiability does not make sense globally our first task is to isolate suitable subsets of D^n where it does. Thus let $\Phi\colon D^q \longrightarrow Y$ be the characteristic map for the attached q-cell. Taking the standard balls

$$U_{j/4}(0) \subset D_{j/4}(0) \subset U^q \subset D^q \quad \text{with } j = 1, 2, 3$$

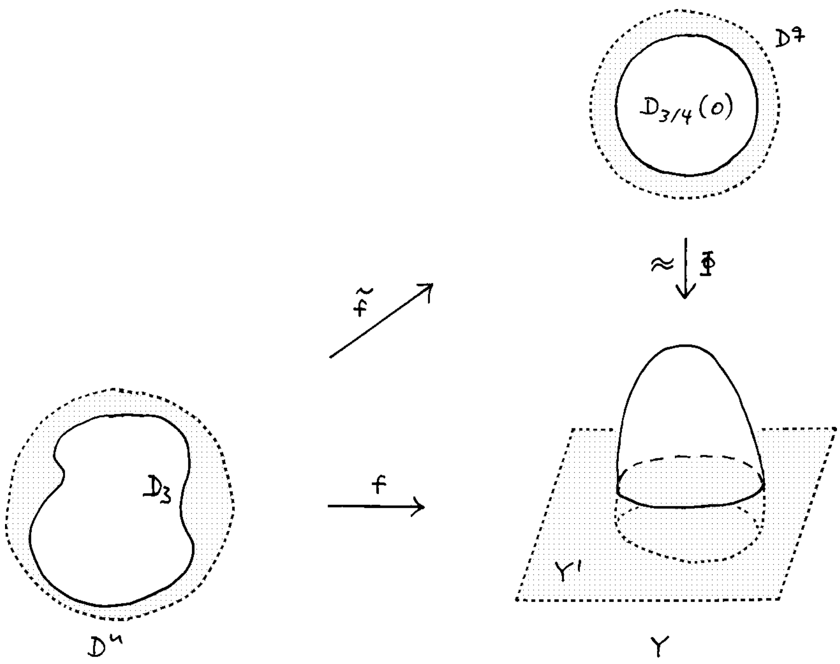
as our starting point we define open, respectively closed subsets of D^n,

$$U_j = f^{-1}\Big(\Phi\big(U_{j/4}(0)\big)\Big) \quad \text{and} \quad D_j = f^{-1}\Big(\Phi\big(D_{j/4}(0)\big)\Big).$$

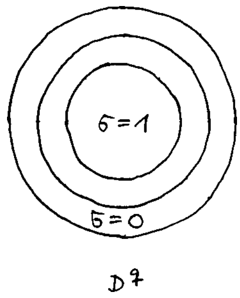


From the given map $f: D^n \longrightarrow Y$ we build

$$\tilde{f} := \Phi^{-1} \circ f: D_3 \longrightarrow D_{3/4}(0) \hookrightarrow U^q,$$



and as a further tool we choose a continuous function $\sigma\colon D^q \longrightarrow I$ which restricts to 1 on $D_{2/4}(0)$ but vanishes identically outside $U_{3/4}(0)$.



We now proceed as in 17.4: a suitable Fourier polynomial provides an approximation $F\colon D_3 \longrightarrow \mathbb{R}^q$ of $\tilde f$ with uniform distance smaller than $\frac{1}{4}$. Then

$$\tilde g := \tilde f + (\sigma\circ\tilde f)\cdot(F - \tilde f)$$

defines a map $\tilde g\colon D_3 \longrightarrow U^q$ with the following properties. Firstly, $\tilde g$ coincides with $\tilde f$ on $D_3\backslash U_3$, and even is homotopic to $\tilde f$ relative $D_3\backslash U_3$: to write down a homotopy just throw in a variable factor $t \in I$ before $\sigma\circ\tilde f$. Secondly, the restriction of $\tilde g$ to the open set U_2 is the same as that of F, in particular it is a differentiable map. Finally

$$\tilde g^{-1}\big(U_{1/4}(0)\big) \subset \tilde f^{-1}\big(U_{2/4}(0)\big) = U_2$$

since the distance between $\tilde f$ and $\tilde g$ is smaller than $\frac{1}{4}$.

We now assemble a map $g'\colon D^n \longrightarrow Y$ putting

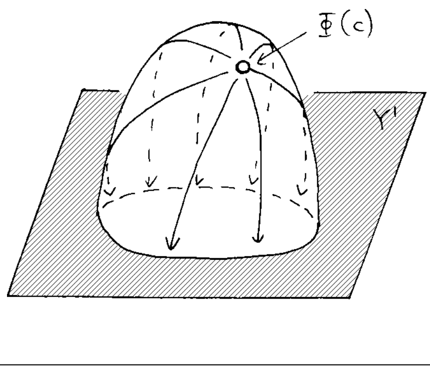
$$g' = \Phi\circ\tilde g\colon D_3 \xrightarrow{\tilde g} U^q \xrightarrow{\Phi} Y \quad \text{on } D_3,$$

and

$$g' = f\colon D^n\backslash U_3 \xrightarrow{f} Y \quad \text{on } D^n\backslash U_3.$$

Similarly a homotopy from f to g' may be constructed, proving that these maps are homotopic to each other relative S^{n-1}. Sard's theorem allows to choose a $c \in U_{1/4}(0)$ which is a regular value of the differentiable map $\tilde g|U_2$. In view of the assumption $n < q$ — which has not been used up to this point — c is a non-value of $\tilde g|U_2$. On the other hand we have seen that c cannot have inverse images under $\tilde g$ outside U_2, nor has $\Phi(c)$ inverse images under f outside U_3: therefore $\Phi(c)$ is not a value of g'.

The rest is simple: the projection from the point $c \in U^q$ to S^{q-1} induces a retraction $r\colon Y\backslash\{\Phi(c)\} \longrightarrow Y'$ which is homotopy inverse relative Y' to the inclusion $j\colon Y' \longrightarrow Y\backslash\{\Phi(c)\}$.



Thus the composition $g := j \circ r \circ g' : D^n \longrightarrow Y$ is homotopic relative Y' to f, and takes values in Y' as desired. This completes the proof of 22.3, and thereby that of Theorem 22.2.

Cellular approximation is not much of an approximation in the analytic sense. As an example think of maps from an n-dimensional complex into S^{n+1} with its two cell structure: the only cellular approximation is a constant map. Nevertheless let us record the following, albeit weak statement, which follows by mere inspection of the proofs.

22.6 Addendum The cellular approximation g of $f: X \longrightarrow Y$ supplied by Theorem 22.2 can be chosen such that for every subcomplex Δ of X and every subcomplex Θ of Y

$$f(\Delta) \subset \Theta \quad \text{implies} \quad g(\Delta) \subset \Theta.$$

The treatment of homotopies requires no extra effort.

22.7 Corollary Let X, Y be cell complexes, and $A \subset X$ a subcomplex. If $f_0, f_1: X \longrightarrow Y$ are cellular maps that are homotopic to each other relative A then there exists a cellular homotopy relative A between them.

Proof Let f be a homotopy from f_0 to f_1 relative A. Considering $f: I \times X \longrightarrow Y$ as a mapping in its own right its restriction to the subcomplex

$$I \times A \;\cup\; \{0,1\} \times X \;\subset\; I \times X$$

is cellular. Applying Theorem 22.2 to f we obtain a cellular map $g: I \times X \longrightarrow Y$ which has the same restriction, and thus still is a homotopy from f_0 to f_1 relative A.

Here is a simple direct application:

22.8 Theorem Let X be a cell complex, pointed by one of its 0-cells $\circ$. The mapping

$$j_*: \pi_n(X^q, \circ) \longrightarrow \pi_n(X, \circ)$$

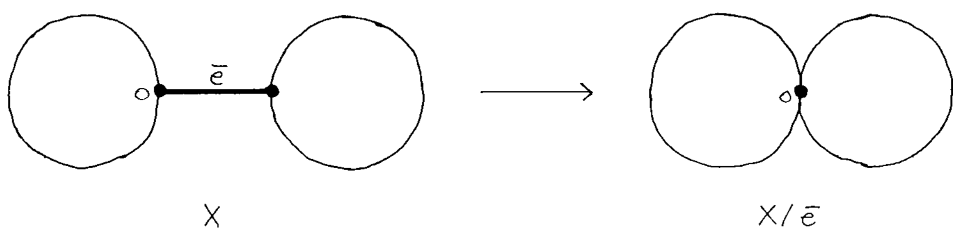
induced by the inclusion $j: X^q \hookrightarrow X$ of the q-skeleton is bijective for $n < q$, and still surjective for $n = q$.

Proof The sphere $I^n/\partial I^n$ is, of course, a pointed cell space of dimension n, so the previous results may be applied to pointed maps from $I^n/\partial I^n$ to X. Theorem 22.2 then yields the surjectivity, and Corollary 22.7 the injectivity statement.

In particular a cell complex is connected if and only if its 1-skeleton is connected. A nice further application that complements this fact is the following.

22.9 Theorem Let X be a connected cell complex, pointed by one of its 0-cells $\circ$. Then there exists another cell complex Y which has just one 0-cell and is homotopy equivalent to X in $\mathbf{Top}^\circ$.

Proof We argue by induction on the number of 0-cells in X. There is nothing to show in case there is but one. If there are several then there must be at least one 1-cell e that connects the base point $\circ$ to another 0-cell of X: otherwise X^1 would be disconnected, in contradiction to 22.8. The characteristic map of e is an injection $[-1,1] \longrightarrow X$, so it embeds $[-1,1]$ as the subcomplex $\overline{e} \subset X$. By Proposition 21.6 the quotient mapping $X \longrightarrow X/\overline{e}$ is a pointed homotopy equivalence. On the other hand $X/\overline{e}$ is a cell complex with one 0-cell (and one 1-cell) less than X, and this completes the inductive step.



23 The Boundary Operator

Consider a cell complex X. A complete description of X is inductive by its very definition, based on the filtration

$$\emptyset = X^{-1} \subset X^0 \subset \cdots \subset X^{n-1} \subset X^n = X$$

by the skeletons of X. Thus a blueprint for building X must specify, for each q, the way X^q is obtained from X^{q-1} by attaching q-cells: in other words it must specify the attaching map for each of these cells.

Of course the collection of all that data comprises an immense amount of detailed information, and you may ask whether really that much is needed to get a satisfactory picture of X. The answer will obviously depend on one's conception of what is satisfactory. So instead let us look at this question from the other end and see how much of the blueprint we are able to phrase in purely combinatorial terms.

An obvious idea is to record the number of cells in each dimension. This being done, one would want to add information concerning the way each q-cell is attached to the $(q-1)$-skeleton of X. Assume that for some fixed $q > 0$ there are exactly r cells of that dimension, and s cells of dimension $q-1$ in X, labelled

$$e_1, \ldots, e_r \quad \text{and} \quad d_1, \ldots, d_s$$

respectively. Given two indices $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, r\}$ we construct a map d_{ij} from the $(q-1)$-dimensional sphere to itself as follows.

The complement of the open $(q-1)$-cell d_i in X^{q-1} is obtained from X^{q-2} by attaching the remaining $(q-1)$-cells, so it is a subcomplex $X^{q-1} \backslash d_i \subset X^{q-1}$. Passing to quotient spaces the characteristic mapping of d_i

$$\Psi_i \colon D^{q-1} \longrightarrow X^{q-1}$$

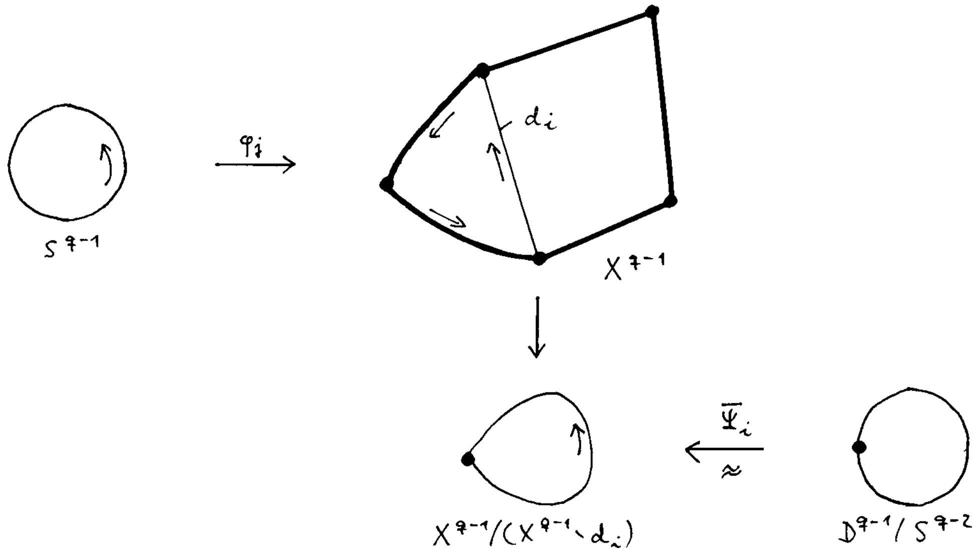
induces a map

$$\overline{\Psi}_i \colon D^{q-1}/S^{q-2} \longrightarrow X^{q-1}/(X^{q-1} \backslash d_i)$$

which clearly is bijective and therefore a homeomorphism. Using $\overline{\Psi}_i$ we distill from the attaching map $\varphi_j \colon S^{q-1} \longrightarrow X^{q-1}$ of the cell e_j a new map from S^{q-1} to D^{q-1}/S^{q-2}:

$$\begin{array}{ccccc}
S^{q-1} & \xrightarrow{\varphi_j} & X^{q-1} & & \\
 & & \downarrow & & \\
 & & X^{q-1}/(X^{q-1}\backslash d_i) & \xleftarrow[\approx]{\overline{\Psi}_i} & D^{q-1}/S^{q-2}
\end{array}$$

This composition is the map d_{ij} we wished to construct.



The action of d_{ij} is easily visualized. In fact d_{ij} is just a simplified version of the attaching map φ_j, with the target space X^{q-1} slimmed down to a $(q-1)$-sphere by collapsing the complement of the open cell d_i to a point. In short, one could say that d_{ij} is φ_j as seen by the cell d_i.

23.1 Definition Fix a cell complex X as above. We define the number

$$\partial_{ij} = \deg d_{ij} \in \mathbb{Z}$$

as the mapping degree if $q > 1$. For $q = 1$, where the relevant sphere is $S^{q-1} = S^0 = \{\pm 1\} \subset \mathbb{Z}$, the degree would not make sense. As a substitute in that case we use the formula

$$\partial_{ij} = \sum_{x \in d_{ij}^{-1}\{1\}} x \in \mathbb{Z}$$

which comes down to

$\partial_{ij} = 0$ $\quad$ $\partial_{ij} = 1$ $\quad$ $\partial_{ij} = -1$ $\quad$ $\partial_{ij} = 0$

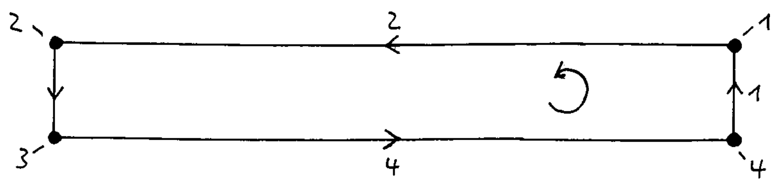
in the four possible cases.

For any value of q the matrix

$$\partial_q = \Big(\partial_{ij}\Big) \in \operatorname{Mat}(s \times r, \mathbb{Z})$$

corresponds to a linear mapping $\mathbb{Q}^r \longrightarrow \mathbb{Q}^s$ which is also denoted by ∂_q, and called the q-th <u>boundary operator</u> of the cell complex X.

23.2 Examples (1) Let us begin with a simple paper strip. In the figure

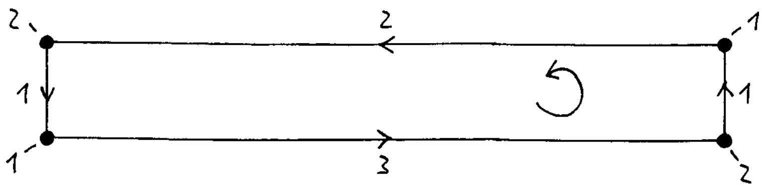


the four 1-cells (edges), and the four 0-cells (vertices) have already been numbered and oriented, and you may have fun reading off that

$$\partial_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \partial_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

are the matrices of the boundary operators in $\mathbb{Q} \xrightarrow{\partial_2} \mathbb{Q}^4 \xrightarrow{\partial_1} \mathbb{Q}^4$.

(2) Passing from the strip to a Moebius band the picture becomes



with the boundary operators

$$\partial_1 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \partial_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

in $\mathbb{Q} \xrightarrow{\partial_2} \mathbb{Q}^3 \xrightarrow{\partial_1} \mathbb{Q}^2$.

(3) In Example 13.6(4) we have studied a cell structure on S^n with two cells e_+ and e_- in each dimension $q \in \{0, \ldots, n\}$, and characteristic maps

$$D^q \ni (x_1, \ldots, x_q) \longmapsto \Big(\pm\sqrt{1 - \textstyle\sum_j x_j^2}\, , x_1, \ldots, x_q \Big) \in S^q \subset \mathbb{R}^{q+1}.$$

For each $q \in \{1, \ldots, n\}$ four numbers $\partial_{\pm\pm}$ must be computed. The diagram

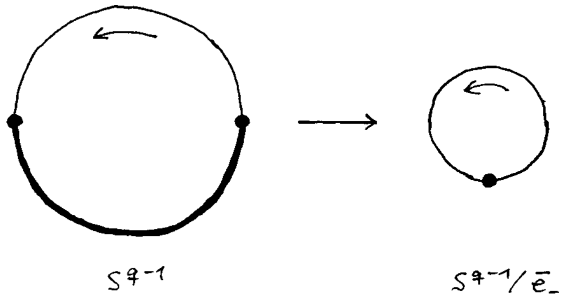
$$\begin{array}{ccc} S^{q-1} & \longrightarrow & S^{q-1} & & \\ & & \downarrow & & \\ & & S^{q-1}/\overline{e}_- & \xleftarrow[\approx]{} & D^{q-1}/S^{q-2} \end{array}$$

that defines the map d_{++} comes out as

$$\begin{array}{ccccc} (x_1, \ldots, x_q) & \longmapsto & (x_1, \ldots, x_q) & & & & \\ & & \downarrow & & & & \\ & & [x_1, \ldots, x_q] & = & \Big[\sqrt{1 - \textstyle\sum_j y_j^2}\, , y_1, \ldots, y_{q-1}\Big] & \longleftarrow & [y_1, \ldots, y_{q-1}] \end{array}$$

so that $d_{++}\colon S^{q-1} \longrightarrow D^{q-1}/S^{q-2}$ acts by

$$(x_1, \ldots, x_q) \longmapsto \begin{cases} [x_2, \ldots, x_q] & \text{if } x_1 \geq 0, \quad \text{and} \\ \circ & \text{if } x_1 \leq 0. \end{cases}$$



Thus d_{++} collapses the lower hemisphere $x_1 \leq 0$, and sends the upper one, $x_1 > 0$ diffeomorphically to the complement of the base point. In order to determine the sign of $\partial_{++} = \deg d_{++} = \pm 1$ it remains to check orientations at one point, say at $(1, 0, \dots, 0) \in S^{q-1}$. According to the definition of the standard orientation explained in Section 18, the system of coordinates $(x_2, \dots, x_q)$ is positively oriented, and it follows at once that d_{++} *preserves* orientation. Therefore $\partial_{++} = +1$.

23.3 Question Explain why, then, the full boundary operator is

$$\partial_q = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{for } q = 1, \dots, n.$$

(4) As explained in 13.6(5) it is a small step from the spheres of the previous example to real projective spaces $\mathbb{R}P^n$. In each dimension there remains only one cell to consider, say e_+, but the calculation of ∂_{++} must be revised for the extra identifications made. Reading the diagram

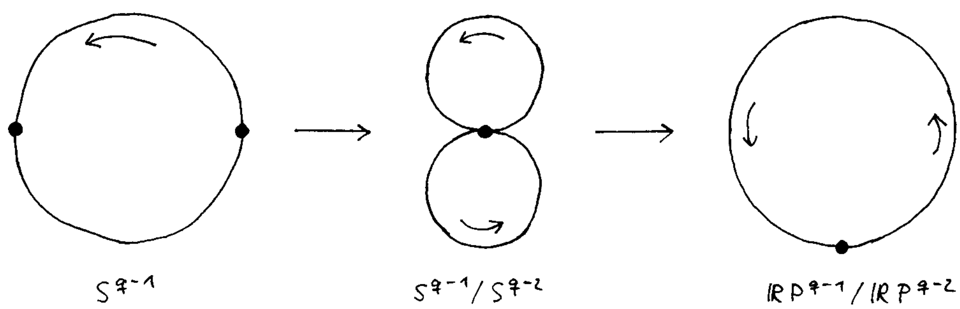
$$\begin{array}{ccccc} S^{q-1} & \longrightarrow & \mathbb{R}P^{q-1} & & \\ & & \downarrow & & \\ & & \mathbb{R}P^{q-1}/\mathbb{R}P^{q-2} & \xleftarrow[\approx]{} & D^{q-1}/S^{q-2} \end{array}$$

with

$$\begin{array}{ccccc} (x_1, \dots, x_q) & \longmapsto & [x_1 : \cdots : x_q] & & \\ & & \downarrow & & \\ & & [x_1 : \cdots : x_q] = \left[\sqrt{1 - \sum_j y_j^2} : y_1 : \cdots : y_{q-1}\right] & \longleftarrow & [y_1, \dots, y_{q-1}] \end{array}$$

we have to take into account that $[x_1 : \cdots : x_q]$ is the same as $[-x_1 : \cdots : -x_q]$. Therefore the action of $d_{++} : S^{q-1} \longrightarrow S^{q-1}$ now is

$$(x_1, \dots, x_q) \longmapsto \begin{cases} [x_2, \dots, x_q] & \text{if } x_1 \geq 0, \quad \text{and} \\ [-x_2, \dots, -x_q] & \text{if } x_1 \leq 0. \end{cases}$$



We thus have a second contribution to the homotopy class of d_{++} which differs from the original one by previous application of the antipodal map of S^{q-1}. As we know from 20.5 that the latter has mapping degree $(-1)^q$ the contributions add up for even, and cancel for odd q. We conclude that

$$\partial_q = 2 \text{ for even, and } \partial_q = 0 \text{ for odd } q \in \{0, \ldots, n\}.$$

(5) What about complex projective spaces? The cell structure on $\mathbb{C}P^n$ described in Example 13.6(5) uses but even cells, so all boundary operators must vanish. While this is a trivial, and at first sight quite uninteresting observation it does illustrate the fact that the descriptive value of the boundary operators has its limits. For clearly the bouquet[1]

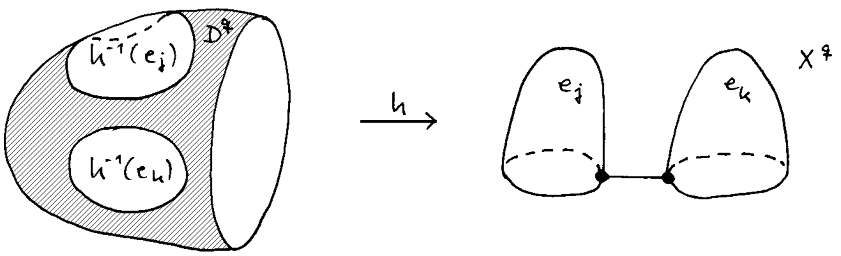
$$S^2 \vee S^4 \vee \cdots \vee S^{2n}$$

with its obvious cell structure has not only the same number of cells in each dimension as $\mathbb{C}P^n$ but equally trivial boundary operators: on the basis of this collection of data alone it is therefore impossible to distinguish between these two spaces, which at least intuitively seem quite different from each other.

Let us return to the general theory, and consider once more an arbitrary cell complex X. Given numberings $e_1, \ldots, e_r$ and $d_1, \ldots, d_s$ of the cells in dimensions q and $q-1$ we have defined mapping degrees ∂_{ij} and thereby a matrix $\partial_q \in \mathrm{Mat}(s \times r, \mathbb{Z})$. While the idea of arranging these integers as a matrix is perfectly natural the interpretation of ∂_q as a linear map so far seems quite arbitrary. It could be justified by an alternative conceptual definition of ∂_q on the basis of so-called *relative* homotopy groups — an approach I have not taken since it would lead us too far away from our main theme. Nevertheless a careful look at the proposition below also will reveal the intrinsically geometric nature of the linear mapping ∂_q.

Consider a map $h: D^q \longrightarrow X^q$ with one of the following properties:

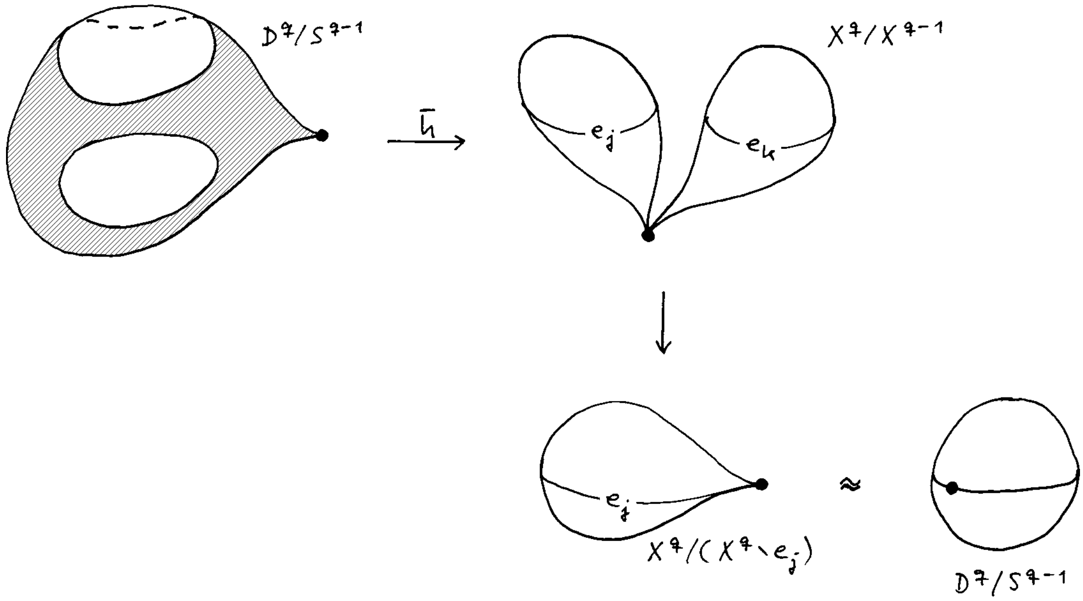
- $h(S^{q-1}) \subset X^{q-1}$, or
- $h|S^{q-1}$ is a constant map.



In both cases h induces a map $\overline{h}: D^q/S^{q-1} \longrightarrow X^q/X^{q-1}$, and for each q-cell e_j in X we construct an analogue of the map d_{ij} by substituting $\overline{h}$ for the attaching map φ_j, and the characteristic map Φ_j of the cell e_j for that of d_i. The composition

$$\begin{array}{ccccc} D^q/S^{q-1} & \xrightarrow{\overline{h}} & X^q/X^{q-1} & & \\ & & \downarrow & & \\ & & X^q/(X^q \backslash e_j) & \xleftarrow[\approx]{\overline{\Phi}_j} & D^q/S^{q-1} \end{array}$$

[1] The bouquet of a collection of pointed topological spaces was introduced in Problem 6.



thus defined will be denoted $\Delta_{h,j}\colon D^q/S^{q-1} \longrightarrow D^q/S^{q-1}$.

23.4 Terminology The integral vector

$$[h] := \begin{pmatrix} \deg \Delta_{h,1} \\ \vdots \\ \deg \Delta_{h,r} \end{pmatrix} \in \mathbb{Q}^r$$

will be referred to as the q-chain represented by the map h.

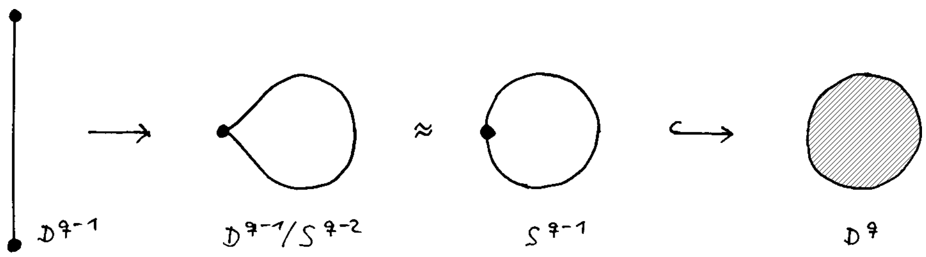
While the definition of the degree, and therefore of $[h]$ makes immediate sense for $q > 0$ we extend it to the case $q = 0$ as in Definition 23.1: $\Delta_{h,j}$ being a pointed map only two cases are possible here, of degrees 1 and 0. We will sometimes write $[h]_q$ in order to indicate the dimension. In any case $[h]$ only depends on the homotopy class of h relative S^{q-1}.

In order to formulate our next result it is best to agree upon a particular homeomorphism $D^q/S^{q-1} \approx S^q$ as the standard one. That induced by

$$D^q \ni x \longmapsto \Big(\cos \pi|x|, \frac{\sin \pi|x|}{|x|}\cdot x\Big) \in S^q$$

is a natural choice; it sends the rays emanating from the origin isometrically to meridians, is even diffeomorphic off the base points, and preserves orientation. It occurs as a factor of the composition

$$b_q\colon D^{q-1} \longrightarrow D^{q-1}/S^{q-2} \approx S^{q-1} \hookrightarrow D^q$$



referred to in the following proposition.

23.5 Proposition For any $q > 1$ and any $h: D^q \longrightarrow X^q$ as above one has

$$\partial_q[h] = [h \circ b_q]_{q-1}.$$

Explanation The map b_q collapses S^{q-2}, and therefore $[h \circ b]_{q-1}$ makes sense. — The characteristic map Φ_k of a q-cell e_k qualifies as a special choice of h. In this case $\Delta_{h,j}$ is the identity map of D^q/S^{q-1} if $j = k$, and is null homotopic if $j \neq k$: thus $[h]_q$ is the k-th standard base vector. On the other hand $\Delta_{h\circ b,i}: D^{q-1}/S^{q-2} \longrightarrow D^{q-1}/S^{q-2}$ is essentially $d_{ij}: S^{q-1} \longrightarrow D^{q-1}/S^{q-2}$, and therefore

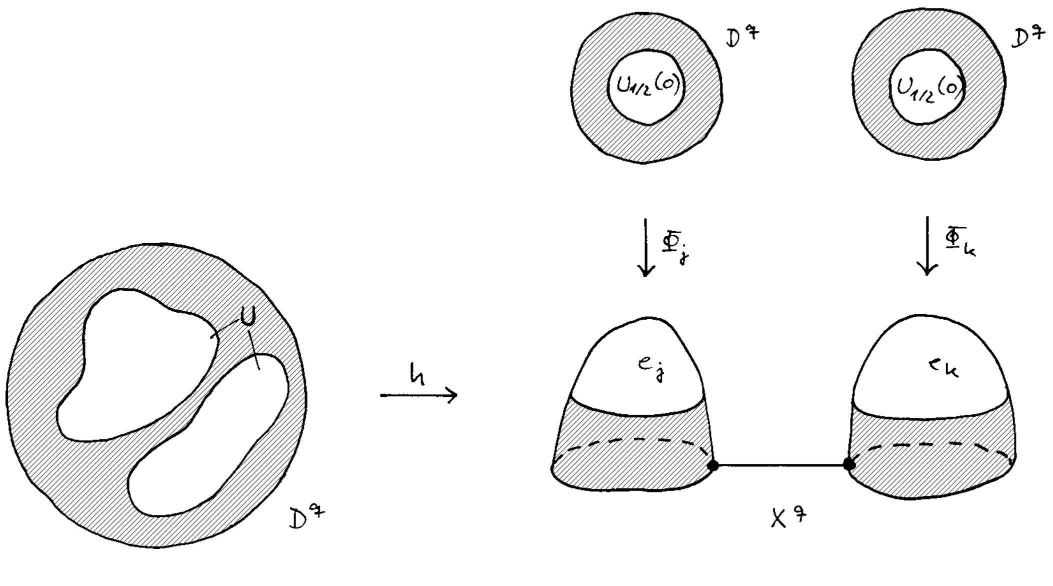
$$[h \circ b]_{q-1} = \begin{pmatrix} \partial_{1k} \\ \vdots \\ \partial_{rk} \end{pmatrix}$$

is the k-th column of ∂_q. So in this special case Proposition 23.5 just restates the definition of ∂_q. Do not be tempted to think that the general statement might follow by linear extension: in a sense the very point of the proposition is that treating the matrix ∂_q as a linear map is geometrically meaningful.

Proof We assume that we are in the more interesting case that $h(S^{q-1}) \subset X^{q-1}$. Recall that the proof of the theorem of cellular approximation was by refining arguments originally used to prove that $\pi_n(S^q)$ is trivial for $q < n$. In a similar way the present proof will elaborate on the calculation of the homotopy groups $\pi_n(S^n) \simeq \mathbb{Z}$ from Section 18.

To begin with recall that $e_1, \ldots, e_r$ are the q-cells of X, and that the characteristic map of e_j is denoted by $\Phi_j: D^q \longrightarrow X^q$. It sends the open disk U^q homeomorphically to e_j, and as in the proof of Proposition 22.3 we define an open subset

$$U = h^{-1}\Big(\bigcup_{j=1}^{r} \Phi_j\big(U_{1/2}(0)\big)\Big) \subset D^q.$$



The proof of 22.3 also shows how to construct a map $g: D^q \longrightarrow X^q$ which is homotopic to h relative S^{q-1}, such that the composition

$$\tilde{g}: U \xrightarrow{g} \bigcup_j e_j \xrightarrow{\Phi_j^{-1}} \sum_{j=1}^{r} U^q$$

is defined and a differentiable map, and such that the inverse image

$$g^{-1}\Big(\bigcup_{j=1}^{r}\Phi_j\big(U_{1/4}(0)\big)\Big)$$

is completely contained in U. In view of $[g]_q = [h]_q$ and $[g\circ b_q]_{q-1} = [h\circ b_q]_{q-1}$ we will work with g rather than h (but keep the old definition of U).

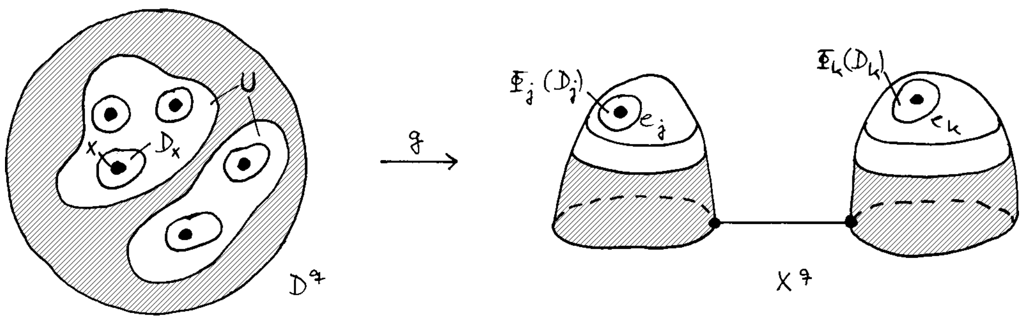
For each $j\in\{1,\dots,r\}$ choose a regular value c_j of $\tilde g$ in the j-th copy of $U_{1/4}(0)\subset U^q$. The fibre $g^{-1}\{\Phi_j(c_j)\}$ is contained in U; the reasoning leading to Proposition 18.1 shows that it is finite, and that we can choose small compact disks

$$D_j = D_\rho(c_j)\subset U_{1/4}(0)\subset U^q$$

such that

$$g^{-1}\big(\Phi_j(D_j)\big) = \sum_{x\in g^{-1}\{\Phi_j(c_j)\}} D_x$$

splits as a topological sum where each summand D_x is a neighbourhood of x in U, and is sent by $\tilde g$ diffeomorphically to D_j.



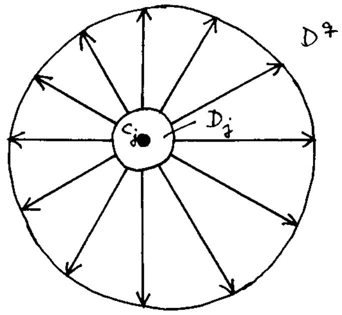
Let ε_x be the orientation character of $\tilde g$ at $x\in g^{-1}\{\Phi_j(c_j)\}$, then

$$\sum_{x\in g^{-1}\{\Phi_j(c_j)\}}\varepsilon_x = \deg\Delta_{g,j}\quad\text{for each } j,$$

essentially by the very definition of the mapping degree. Put

$$X' = X^q\setminus\bigcup_{j=1}^{r}\Phi_j(D_j^\circ)$$

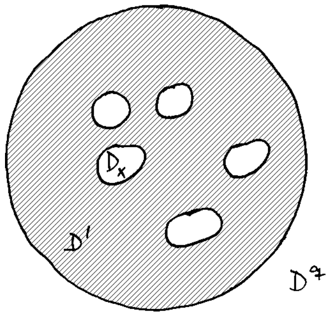
and let $r_j\colon D^q\backslash D_j^\circ \longrightarrow S^{q-1}$ denote the map obtained by projection from the centre c_j:



Together these maps define a retraction $R\colon X' \longrightarrow X^{q-1}$. In a similar way we form the difference

$$D' := D^q \setminus \bigcup_x D_x^\circ$$

where the union is taken over all $x \in g^{-1}\{\Phi_1(c_1), \ldots, \Phi_r(c_r)\}$. Note that $D' \subset D^q$ is a compact equidimensional submanifold with boundary components S^{q-1} and ∂D_x.



The $(q-1)$-cells of X are $d_1, \ldots, d_s$, and the characteristic map of d_i is $\Psi_i\colon D^{q-1} \longrightarrow X^{q-1}$. We now fix some $i \in \{1, \ldots, s\}$ and obtain a mapping $f\colon D' \longrightarrow S^{q-1}$ as the composition:

$$\begin{array}{ccc}
D' & \xrightarrow{\;g\;} & X' \\
 & & \big\downarrow R \\
 & & X^{q-1} \\
 & & \big\downarrow \\
 & & X^{q-1}/(X^{q-1}\setminus d_i) \xleftarrow[\approx]{\;\overline{\Psi}_i\;} D^{q-1}/S^{q-2} \approx S^{q-1}
\end{array}$$

Temporarily replacing f by a differentiable approximation, and choosing a volume form ω on S^{q-1} we may apply Stokes' formula:

$$0 = \int_{D'} f^*0 = \int_{D'} f^*d\omega = \int_{D'} d(f^*\omega) = \int_{\partial D'} f^*\omega = \int_{S^{q-1}} f^*\omega - \sum_x \int_{\partial D_x} f^*\omega$$

We have put in a minus sign since we prefer to give ∂D_x its orientation as the boundary of D_x, which is opposite to the one it carries as a boundary component of D'. Stated in terms of mapping degrees the resulting formula

$$\sum_x \deg f|\partial D_x = \deg f|S^{q-1}$$

makes sense and holds for the original map f, and we have no further need for a differentiable version of the latter. In order to conclude the proof of Proposition 23.5 it remains to evaluate both sides of the equation.

First look at $\deg f|\partial D_x$. In the commutative diagram

$$\begin{array}{ccccc}
 & & \partial D_j & \xrightarrow{\;r_j\;} & S^{q-1} \\
 & \overset{\tilde g}{\nearrow} & \big\downarrow \Phi_j & & \big\downarrow \varphi_j \\
\partial D_x & \xrightarrow[\;g\;]{} & X' & \xrightarrow[\;R\;]{} & X^{q-1}
\end{array}$$

$\tilde g$ and r_j are diffeomorphisms. The former has orientation character ε_x while the latter preserves orientation. Comparing with the diagram that defines d_{ij} we see that $\deg f|\partial D_x = \partial_{ij}\varepsilon_x$, so that

$$\sum_x \deg f|\partial D_x = \sum_x \partial_{ij}\varepsilon_x = \sum_{j=1}^r \partial_{ij} \sum\nolimits_{x \in g^{-1}\{\Phi_j(c_j)\}} \varepsilon_x = \sum_{j=1}^r \partial_{ij} \deg \Delta_{g,j}$$

is just the i-th component of the chain $\partial_q[g]$.

On the other hand, up to the identification $D^{q-1}/S^{q-2} \approx S^{q-1}$ the restriction $(R\circ g)|S^{q-1}$ is the same as $\overline{g\circ b}: D^{q-1}/S^{q-2} \longrightarrow X^{q-1}$, and therefore

$$\deg f|S^{q-1} = \deg \Delta_{g\circ b,i}$$

is the i-th component of $[g\circ b]_{q-1}$. This completes the proof under the assumption that h sends S^{q-1} into X^{q-1}.

The other case, of a constant restriction $h|S^{q-1}$ is rather simpler as h then essentially is a map of a q-sphere into X^q. We omit the details.

23.6 Question The formula of Proposition 23.5, valid in dimensions $q > 1$, does not extend to $q = 1$. Explain by testing it on a suitable example.

In fact the case $q = 1$ requires a minor

23.7 Modification For any $h: D^1 \longrightarrow X^1$ as above one has

$$\partial_1[h] = \big[h|\{1\}\big]_0 - \big[h|\{-1\}\big]_0.$$

Proof After you have gone through the proof of 23.5 this one should be straightforward.

In spite of its technical appearance Proposition 23.5 is of utmost importance. Its first application will be via the following corollary, which a close look at the examples may already have suggested to you.

23.8 Corollary $\partial_{q-1}\circ\partial_q = 0$ for all $q \geq 2$.

Proof It suffices to check the value of the composition on q-chains $[h]_q$, for these include the vectors of the standard base. Applying Proposition 23.5 twice we obtain for $q > 2$

$$(\partial_{q-1}\circ\partial_q)[h]_q = [h\circ b_q\circ b_{q-1}]_{q-2}.$$

But the composition

$$b_q\circ b_{q-1}: D^{q-2} \longrightarrow D^{q-2}/S^{q-3} \approx S^{q-2} \hookrightarrow D^{q-1} \longrightarrow D^{q-1}/S^{q-2} \approx S^{q-1} \hookrightarrow D^q$$

is a constant map and so $[h\circ b_q\circ b_{q-1}] = 0$.

In the case $q = 2$, for the second application of the proposition the modified version 23.7 must be used:

$$(\partial_1\circ\partial_2)[h]_2 = \big[h\circ b_2|\{1\}\big]_0 - \big[h\circ b_2|\{-1\}\big]_0$$

Since $b_2: D^1 \longrightarrow D^2$ takes one and the same value on the points ± 1 the difference vanishes, and this concludes the proof.

The geometric idea underlying the corollary is very simple: in our setup something which is a boundary should itself have no boundary. At least for the sphere S^{q-1} this is an obvious fact on which the proof of the corollary proper is based. Still as you can see, considerable efforts have been necessary to reach a generally valid conclusion.

24 Chain Complexes

There is an established algebraic formalism around the equation $\partial_{q-1}\circ\partial_q = 0$. For what follows, an arbitrarily chosen field k will be kept fixed.

24.1 Definition A graded vector space over k is a sequence

$$V = (V_q)_{q\in\mathbb{Z}}$$

of vector spaces V_q over k. Vectors in V_q are said to have degree q. A homomorphism between graded vector spaces, or graded homomorphism $f\colon V \longrightarrow W$ is a sequence of linear maps

$$V_q \xrightarrow{f_q} W_q.$$

More generally for each $d \in \mathbb{Z}$ there is a notion of homomorphism of degree d: a sequence of linear maps

$$V_q \xrightarrow{f_q} W_{q+d}.$$

So the true homomorphisms are those of degree 0 in this wider sense.

Note Sometimes an alternative way to describe the concept of gradedness is preferable. It makes not the sequence (V_q) but the direct sum

$$\bigoplus_{q=-\infty}^{\infty} V_q$$

the basic object, and its decomposition into the subspaces V_q — called a *grading* — a complementary piece of data. Every $v \in \bigoplus V_q$ has a unique representation as a sum

$$v = \sum_{q\in Q} v_q$$

with finite $Q \subset \mathbb{Z}$ and $0 \neq v_q \in V_q$ for all $q \in Q$, and the summands v_q are called the homogeneous components of v. From this point of view graded homomorphisms are linear maps that are compatible with the gradings.

24.2 Definition A chain complex $V_\bullet$ over k consists of a graded vector space V and an endomorphism $\partial\colon V \longrightarrow V$ of degree -1, which is called the differential of $V_\bullet$ and must satisfy

$$\partial \circ \partial = 0.$$

If $V_\bullet$ is a chain complex then vectors in V_q are also called q-chains. A homomorphism of chain complexes, or chain map $f\colon V_\bullet \longrightarrow W_\bullet$ is a homomorphism of graded vector spaces that commutes with the differential: $f\circ\partial = \partial\circ f$. Thus explicitly, homomorphisms are commutative ladder diagrams

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & V_{q+1} & \xrightarrow{\partial_{q+1}} & V_q & \xrightarrow{\partial_q} & V_{q-1} & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle f_{q+1}} & & \big\downarrow{\scriptstyle f_q} & & \big\downarrow{\scriptstyle f_{q-1}} & & \\
\cdots & \longrightarrow & W_{q+1} & \xrightarrow{\partial_{q+1}} & W_q & \xrightarrow{\partial_q} & W_{q-1} & \longrightarrow & \cdots
\end{array}$$

in the category $\mathbf{Lin}_\mathsf{k}$. The chain complexes and maps over k in turn form a new category which is denoted $\mathbf{Ch}_\mathsf{k}$.

24.3 Definition Let $V_\bullet$ be a chain complex over k, and $q \in \mathbb{Z}$. The property $\partial_{q-1} \circ \partial_q = 0$ means that the subspace

$$\partial_{q+1}(V_{q+1}) \subset V_q$$

of boundaries is contained in the subspace

$$\ker \partial_q \subset V_q$$

of so-called cycles. Therefore the quotient vector space

$$H_q(V_\bullet) := (\ker \partial_q)/\partial_{q+1}(V_{q+1})$$

is defined; it is called the q-th homology (vector) space of $V_\bullet$. By tradition the underlying equivalence relation is also called homology: two q-cycles which are congruent modulo $\partial_{q+1}(V_{q+1})$ are called homologous to each other.

Homology is functorial:

24.4 Proposition If $f: V_\bullet \longrightarrow W_\bullet$ is a homomorphism of chain complexes then f_q sends cycles to cycles, and boundaries to boundaries, so that a linear map $H_q(f): H_q(V_\bullet) \longrightarrow H_q(W_\bullet)$ is induced.

Proof Looking at the commutative diagram

$$\begin{array}{ccccc}
V_{q+1} & \xrightarrow{\partial_{q+1}} & V_q & \xrightarrow{\partial_q} & V_{q-1} \\
\downarrow{\scriptstyle f_{q+1}} & & \downarrow{\scriptstyle f_q} & & \downarrow{\scriptstyle f_{q-1}} \\
W_{q+1} & \xrightarrow{\partial_{q+1}} & W_q & \xrightarrow{\partial_q} & W_{q-1}
\end{array}$$

take a cycle $y \in \ker \partial_q$. Then

$$\partial_q\big(f_q(y)\big) = f_{q-1}\big(\partial_q(y)\big) = 0$$

and therefore $f_q(y) \in \ker \partial_q$ also is a cycle. Similarly consider a boundary $y \in V_q$, say $y = \partial_{q+1}(x)$ for some $x \in V_{q+1}$. Then

$$f_q(y) = f_q\big(\partial_{q+1}(x)\big) = \partial_{q+1}\big(f_{q+1}(x)\big)$$

and so $f_q(y)$ is a boundary too.

We thus have for each $q \in \mathbb{Z}$ a homology functor

$$H_q: \mathbf{Ch}_\mathsf{k} \longrightarrow \mathbf{Lin}_\mathsf{k}.$$

If you prefer, you may consider the collection of all these functors a single functor with values in the category of graded vector spaces over k.

Our interest in chain complexes is, of course, based on the fact that in the previous section we have for any given cell complex X effectively constructed a chain complex $V_\bullet$ over the field $\mathbb{Q}$. For each $q \in \{0, \ldots, \dim X\}$ the chain space V_q is defined as $\mathbb{Q}^r$ with r the number of q-cells in X, while the differentials are the boundary operators. To make the definition complete we only have to put $V_q = 0$ for all other $q \in \mathbb{Z}$. Let us briefly revisit the examples of 23.2, computing their homology spaces.

(1) In the chain complex of the paper strip

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Q} \xrightarrow{\left[\begin{smallmatrix}1\\1\\1\\1\end{smallmatrix}\right]} \mathbb{Q}^4 \xrightarrow{\left[\begin{smallmatrix}1&-1&0&0\\0&1&-1&0\\0&0&1&-1\\-1&0&0&1\end{smallmatrix}\right]} \mathbb{Q}^4 \longrightarrow 0 \longrightarrow \cdots$$

the square matrix ∂_1 has rank 3, and we obtain $H_0(V_\bullet) \simeq \mathbb{Q}$ while all other homology spaces are trivial.

(2) The chain complex of the Moebius strip

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Q} \xrightarrow{\left[\begin{smallmatrix} 2 \\ 1 \\ 1 \end{smallmatrix}\right]} \mathbb{Q}^3 \xrightarrow{\left[\begin{smallmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \end{smallmatrix}\right]} \mathbb{Q}^2 \longrightarrow 0 \longrightarrow \cdots$$

gives $H_0(V_\bullet) \simeq H_1(V_\bullet) \simeq \mathbb{Q}$ for the non-trivial homology.

(3) Let $X = S^n$ with the structure comprising two cells in each dimension. From the chain complex with non-zero terms in degrees $0, \ldots, n$

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Q}^2 \xrightarrow{\left[\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right]} \mathbb{Q}^2 \xrightarrow{\left[\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right]} \cdots$$

$$\cdots \xrightarrow{\left[\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right]} \mathbb{Q}^2 \xrightarrow{\left[\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right]} \mathbb{Q}^2 \longrightarrow 0 \longrightarrow \cdots$$

we read off that $H_0(V_\bullet) \simeq H_n(V_\bullet) \simeq \mathbb{Q}$ for $n > 0$, and $H_0(V_\bullet) \simeq \mathbb{Q}^2$ for $n = 0$, all other homology being trivial.

(4) Passing to real projective spaces the picture becomes more varied. For even n the relevant piece of the chain complex of $X = \mathbb{R}P^n$ is

$$0 \longrightarrow \mathbb{Q} \xrightarrow{[\,2\,]} \mathbb{Q} \xrightarrow{[\,0\,]} \mathbb{Q} \xrightarrow{[\,2\,]} \cdots$$

$$\cdots \xrightarrow{[\,0\,]} \mathbb{Q} \xrightarrow{[\,2\,]} \mathbb{Q} \xrightarrow{[\,0\,]} \mathbb{Q} \longrightarrow 0$$

and the only non-trivial homology space is $H_0(V_\bullet) \simeq \mathbb{Q}$. But if n is odd then the complex

$$0 \longrightarrow \mathbb{Q} \xrightarrow{[\,0\,]} \mathbb{Q} \xrightarrow{[\,2\,]} \mathbb{Q} \xrightarrow{[\,0\,]} \cdots$$

$$\cdots \xrightarrow{[\,0\,]} \mathbb{Q} \xrightarrow{[\,2\,]} \mathbb{Q} \xrightarrow{[\,0\,]} \mathbb{Q} \longrightarrow 0$$

starts differenty and there is another non-trivial homology space $H_n(V_\bullet) \simeq \mathbb{Q}$.

(5) If all differentials of the chain complex $V_\bullet = (V_q, \partial_q)$ vanish then clearly $H_q(V_\bullet) = V_q$ for all $q \in \mathbb{Z}$. In particular this case occurs if no two consecutive V_q are non-trivial. The chain complex $V_\bullet$ associated with $\mathbb{C}P^n$ is of this type, and therefore has

$$H_0(V_\bullet) \simeq H_2(V_\bullet) \simeq \cdots \simeq H_{2n}(V_\bullet) \simeq \mathbb{Q}$$

for its non-trivial homology spaces.

If we want to make the assignment of a chain complex to a cell complex X functorial our construction must be polished a bit. The definition of the boundary operator $\partial_q \colon \mathbb{Q}^r \longrightarrow \mathbb{Q}^s$ formally depended on the choice of a numbering of the cells in dimensions q and $q-1$. In fact there is no need for such a choice. Given the cell complex X consider, for any $q \in \mathbb{Z}$, the finite set E_q of cells of dimension q in X: so in particular $E_q = \emptyset$ if $q < 0$ or $q > \dim X$. We define

$$C_q(X) := \mathbb{Q}^{E_q}$$

as the vector space of all maps from the finite set E_q to the field $\mathbb{Q}$. Note that the classical correspondence between linear maps and matrices extends to linear maps from $\mathbb{Q}^R$ to $\mathbb{Q}^S$ for arbitrary finite sets R, S: all one has to do is use the elements of R and S as the column and row indices. An ordering of these is neither inherent in the concept of matrix nor does it play a role in matrix calculus — it rather has to do with the desire to write out matrices on a piece of paper. Likewise the mapping degrees ∂_{ij} of Definition 23.1 should be properly labelled ∂_{de} as they are assigned to each pair of cells $e \in E_q$ and $d \in E_{q-1}$ and not to their places with respect to a numbering. The matrix

$$\partial_q = \Big(\partial_{de}\Big) \in \mathbb{Q}^{E_{q-1}\times E_q}$$

then corresponds to the boundary operator $\mathbb{Q}^{E_q} \longrightarrow \mathbb{Q}^{E_{q-1}}$.

24.5 Question What, by the way, is a square matrix?

Let now X and Y be two cell complexes, and consider a cellular map $f\colon X \longrightarrow Y$. Let $e \in E_q(X)$ and $d \in E_q(Y)$ be q-cells of X and Y, with characteristic maps $\Phi\colon D^q \longrightarrow X^q$ and $\Psi\colon D^q \longrightarrow Y^q$. The maps f, Φ, and Ψ descend to quotients and by composition give a self-map f_{de} of the q-sphere:

$$\begin{array}{ccccc}
D^q/S^{q-1} & \xrightarrow{\overline{\Phi}} & X^q/X^{q-1} & \xrightarrow{\overline{f}} & Y^q/Y^{q-1} \\
 & & & & \downarrow \\
 & & Y^q/(Y^q\backslash d) & \xleftarrow[\approx]{\overline{\Psi}} & D^q/S^{q-1}
\end{array}$$

We define the linear map $C_q(f)\colon C_q(X) \longrightarrow C_q(Y)$ as the matrix

$$C_q(f) = \Big(\deg f_{de}\Big) \in \mathbb{Q}^{E_q(Y)\times E_q(X)}$$

where in case $q = 0$ we use the substitute for the degree as in 23.1.

24.6 Proposition Let $f\colon X \longrightarrow Y$ be a cellular map, and let $h\colon D^q \longrightarrow X^q$ be a map with $h(S^{q-1}) \subset X^{q-1}$, or with $h|S^{q-1}$ a constant. Then $C_q(f)$ sends $[h] \in C_q(X)$ to $[f \circ h] \in C_q(X)$.

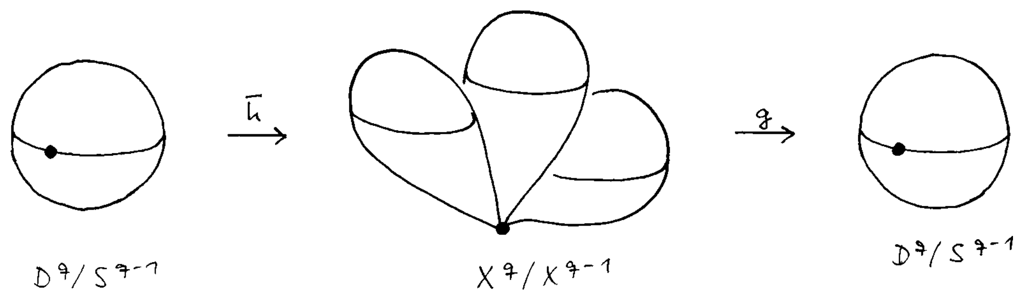
Proof Fix a cell $d \in E_q(Y)$ with characteristic map Ψ. The given maps induce a diagram

$$\begin{array}{ccccc}
D^q/S^{q-1} & \xrightarrow{\overline{h}} & X^q/X^{q-1} & \xrightarrow{\quad g \quad} & D^q/S^{q-1} \\
 & & \searrow^{\overline{f}} & & \approx\big\downarrow\overline{\Psi} \\
 & & Y^q/Y^{q-1} & \longrightarrow & Y^q/(Y^q\backslash d)
\end{array}$$

which defines g. Note that

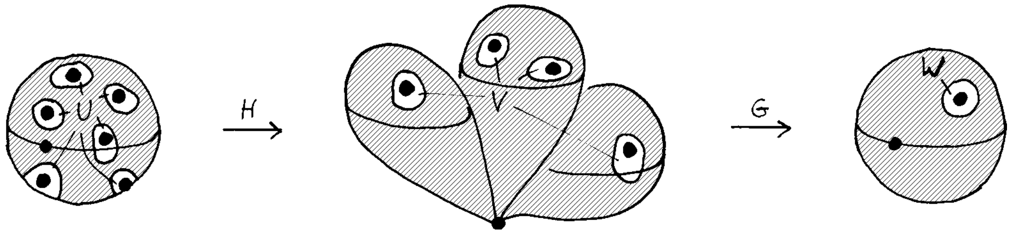
$$D^q/S^{q-1} \xrightarrow{\overline{h}} X^q/X^{q-1} \xrightarrow{g} D^q/S^{q-1}$$

is a composition of pointed maps passing through a bouquet of q-spheres.



The technique which by now has become standard allows to construct approximations G of g and H of $\overline{h}$, and an open disk $W \subset U^q \subset D^q/S^{q-1}$ such that G is differentiable on $V := G^{-1}(W)$, and H is differentiable on $U := H^{-1}(V)$. Choosing a regular value $c \in W$ of $G \circ H|U$ and counting oriented inverse image points we obtain in terms of the notation introduced before 23.3:

$$\deg(g \circ \overline{h}) = \deg(G \circ H) = \sum_{e \in E_q(X)} \deg f_{de} \cdot \deg \Delta_{h,e}$$



This is just the d-th component of the equation we wanted to prove.

It is now easy to see that $C_q(?)$ indeed is functorial.

24.7 Notation and **Proposition** Assigning to each cell complex its chain complex over $\mathbb{Q}$ gives a functor

$$C_\bullet\colon \mathbf{Cell} \longrightarrow \mathbf{Ch}_{\mathbb{Q}}$$

from the category **Cell** of cell complexes (finite, as always) and cellular maps, to the category of chain complexes over the field $\mathbb{Q}$.

Proof The action of $C_\bullet$ on objects $X \in |\mathbf{Cell}|$ and morphisms $f \in \mathbf{Cell}(X, Y)$ has already been described. We must verify that for each $q > 0$ the diagram

$$\begin{array}{ccc} C_q(X) & \xrightarrow{\partial_q} & C_{q-1}(X) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ C_q(Y) & \xrightarrow{\partial_q} & C_{q-1}(Y) \end{array}$$

commutes, and it suffices to test this on the chain $[h] \in C_q(X)$ represented by a map $h\colon D^q \longrightarrow X^q$. Propositions 23.5 and 24.6 give at once what is needed in case $q > 1$ for they imply that both $f_*\partial_q[h]$ and $\partial_q f_*[h]$ are represented by $f \circ h \circ b$. Similarly the result for $q = 1$ follows from 23.7 and 24.6:

$$f_*\partial_1[h] = [f \circ h|\{1\}] - [f \circ h|\{-1\}] = \partial_1 f_*[h]$$

So $f_* = C_\bullet(f)$ is a morphism of chain complexes. — Of the functor laws one, $\mathrm{id}_* = \mathrm{id}$, is trivial while $(g \circ f)_* = g_* \circ f_*$ once more follows from Proposition 24.6.

For the sake of definiteness I have chosen to introduce the chain complex associated with a cell complex X as a complex over the field $\mathbb{Q}$ of rational numbers. Clearly the same construction yields a complex for any other choice of the base field k: it will be denoted by $C_\bullet(X; \mathsf{k})$ unless k is obvious in the context, or irrelevant. While passing to other fields of characteristic zero adds nothing of interest the chain complexes over fields of varying characteristic may reflect quite different aspects of the geometry of X.

24.8 Example This is superbly illustrated by the real projective space $\mathbb{R}P^n$ studied above, for its chain complex

$$0 \longrightarrow \mathsf{k} \xrightarrow{[2]} \mathsf{k} \xrightarrow{[0]} \mathsf{k} \xrightarrow{[2]} \cdots$$

$$\cdots \xrightarrow{[0]} \mathsf{k} \xrightarrow{[2]} \mathsf{k} \xrightarrow{[0]} \mathsf{k} \longrightarrow 0$$

for even n respectively

$$0 \longrightarrow \mathsf{k} \xrightarrow{[\,0\,]} \mathsf{k} \xrightarrow{[\,2\,]} \mathsf{k} \xrightarrow{[\,0\,]} \cdots$$

$$\cdots \xrightarrow{[\,0\,]} \mathsf{k} \xrightarrow{[\,2\,]} \mathsf{k} \xrightarrow{[\,0\,]} \mathsf{k} \longrightarrow 0$$

for n odd, has all differentials equal to zero if k is a field of characteristic two like $\mathbb{F}_2 = \mathbb{Z}/2$. In particular the homology spaces

$$H_q\big(C_\bullet(\mathbb{R}P^n;\mathbb{F}_2)\big) \simeq \begin{cases} \mathbb{F}_2 & \text{if } 0 \le q \le n \\ 0 & \text{else} \end{cases}$$

now look much more like those obtained for complex projective spaces.

In fact k need not be a field at all: a commutative ring with unit will work equally well. While one would not usually talk about vector spaces over a mere ring the notion as such carries over unchanged, and defines what is called a *module* over the ring k, or k-module for short. Thus a k-module V is an additive group together with a scalar multiplication

$$\mathsf{k} \times V \ni (\lambda, x) \longmapsto \lambda x \in V$$

satisfying the well known axioms $1x = x$ and $(\lambda\mu)x = \lambda(\mu x)$, as well as the two distributive laws. The established terminology is not quite fortunate in that it renders the older term 'vector space' superfluous: given the base ring k and a k-module V it makes sense to ask whether V is a vector space, but the answer does not depend on V as it is already determined by the nature of k. While it would have been useful in many circumstances to allow the word 'vector' to denote an element of an arbitrary module custom restricts its use to the case of true vector spaces.

Many elementary notions from linear algebra of vector spaces carry over to modules, including those of linear mappings, sub and quotient modules, direct sums and products, dual modules, and others. In particular for any base ring k there is a category of k-modules and linear maps, and there is no harm in keeping the earlier notation $\mathbf{Lin}_\mathsf{k}$ for it. By contrast most *results* on vector spaces, even elementary ones fail for modules over an arbitrary ring since in general it is not possible to divide by non-zero scalars. As an example consider a *finitely generated* k-module, that is, one generated by a finite set of its elements, or, what comes down to the same, a homomorphic image of the k-module k^m for some $m \in \mathbb{N}$. Such a module need not admit a base, in other words need not be isomorphic to k^n for any $n \in \mathbb{N}$. Let us illustrate this point by a case of particular interest, that of the ring $\mathsf{k} = \mathbb{Z}$.

Comparing a $\mathbb{Z}$-module V with its underlying additive group it turns out that in this case the scalar multiplication is not an additional structure at all. Indeed for $\lambda \in \mathbb{Z}$ and $x \in V$ the axioms leave no choice for λx but the sum of $|\lambda|$ copies of $\pm x$ — a rule that on the other hand defines a scalar multiplication $\mathbb{Z} \times V \longrightarrow V$ for any given abelian group V. Thus $\mathbb{Z}$-modules are the same as abelian groups or, to be absolutely precise, the forgetful functor $\mathbf{Lin}_\mathbb{Z} \longrightarrow \mathbf{Ab}$ is an isomorphism of categories.

The submodules of the k-module k are generally called the *ideals* of the ring k. By what we just have observed the ideals of the ring $\mathbb{Z}$ are the additive subgroups of $\mathbb{Z}$, and each of them is easily seen to be generated by a unique number $a \in \mathbb{N}$. Though for $a > 0$ the submodule $a\mathbb{Z} \subset \mathbb{Z}$ clearly is isomorphic to $\mathbb{Z}$ itself the quotient module $\mathbb{Z}/a = \mathbb{Z}/a\mathbb{Z}$, being finite, cannot be isomorphic to $\mathbb{Z}^n$ for any n.

k-modules that do admit a (finite or infinite) base are called *free* modules. In the finite case linear maps between them can be decribed by matrices in the familiar way, and this is all we need in order to write down the functor $C_\bullet(?;\mathsf{k})$ assigning to each cell complex its chain complex with *coefficients in* k. Note that in order to compute the homology of this or indeed any chain complex over k Gauss' algorithm may be helpful but will not in general be sufficient.

Example 24.8 continued The integral chain complex of the cell space $\mathbb{R}P^n$ looks like the rational one:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{[\,2\,]} \mathbb{Z} \xrightarrow{[\,0\,]} \mathbb{Z} \xrightarrow{[\,2\,]} \cdots$$

$$\cdots \xrightarrow{[\,0\,]} \mathbb{Z} \xrightarrow{[\,2\,]} \mathbb{Z} \xrightarrow{[\,0\,]} \mathbb{Z} \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{[0]} \mathbb{Z} \xrightarrow{[2]} \mathbb{Z} \xrightarrow{[0]} \cdots$$

$$\cdots \xrightarrow{[0]} \mathbb{Z} \xrightarrow{[2]} \mathbb{Z} \xrightarrow{[0]} \mathbb{Z} \longrightarrow 0$$

for even respectively odd $n \in \mathbb{N}$. Nevertheless its homology is quite different:

$$H_q\big(C_\bullet(\mathbb{R}P^n;\mathbb{Z})\big) \simeq \begin{cases} \mathbb{Z} & \text{if } q=0, \text{ or } q=n \text{ is odd} \\ \mathbb{Z}/2 & \text{if } q \text{ is odd, and } 0<q<n \\ 0 & \text{else} \end{cases}$$

As this example suggests a large variety of base rings to choose from may be of considerable topological interest as it allows to extract different pieces of information from a given cell complex.

The ring $\mathbb{Z}$ is a first example of a *principal ideal domain*, another being the polynomial ring in one indeterminate over a field. For principal ideal domains k the structure of finitely generated k-modules is completely understood, and I will briefly list the known results, details of which can be found in any textbook of algebra, like [Lang] to name but one.

24.9 Structure of finitely generated modules over a principal ideal domain k

If V is such a module then every submodule of V is finitely generated: in particular the homology modules $H_q\big(C_\bullet(X;\mathsf{k})\big)$ are finitely generated for any cell complex X.

For every k-module V its *torsion submodule* is defined as

$$T(V) := \big\{ v \in V \,\big|\, \lambda v = 0 \text{ for some } \lambda \in \mathsf{k}\backslash\{0\}\big\}\,.$$

If V is finitely generated then $V/T(V)$ is a free module and there exists a complementary submodule $F \subset V$, and so a direct sum decomposition

$$V = F \oplus T(V).$$

Though in general there is no canonical choice for $F \simeq \mathsf{k}^r$ the *rank* r of F is well defined and an invariant of V.

It remains to study finitely generated *torsion modules* V: those with $T(V) = V$. They canonically split as direct sums

$$V = \bigoplus_{(p)} V(p)$$

indexed by the non-zero prime ideals $(p) \subset \mathsf{k}$, with

$$V(p) = \big\{ v \in V \,\big|\, p^e v = 0 \text{ for some } e \in \mathbb{N}\big\}$$

non-trivial for but finitely many prime ideals (p).

Finally each module $V(p)$ is isomorphic (in a non-canonical way) to a direct sum

$$V(p) = \mathsf{k}/(p^{e_1}) \oplus \mathsf{k}/(p^{e_2}) \oplus \cdots \oplus \mathsf{k}/(p^{e_r})$$

of *cyclic* modules: those that can be generated by one element. The r-tuple of ordered exponents $1 \le e_1 \le e_2 \le \cdots \le e_r$ is an invariant of $V(p)$.

Two further observations in the case $\mathsf{k} = \mathbb{Z}$ of finitely generated abelian groups. Firstly, a finitely generated torsion group is just a finite abelian group. Secondly, every non-zero prime ideal of $\mathbb{Z}$ is spanned by a unique prime number p, and therefore up to isomorphism a finitely generated abelian group V can be completely described by numerical invariants: the rank of its free quotient, and the series of prime powers that determine the non-trivial subgroups $V(p) \subset V$.

25 Chain Homotopy

The homology modules of the chain complexes associated to a cell complex do not depend of the cell structure but only on the underlying cell space: this will be the central result of the section.

Recall that in Definition 13.2 we have carefully distinguished between the notion of cell complex, which comprises the cell structure and so is a blueprint of the successive attachments, and the notion of cell space as a topological space that admits at least one structure as a cell complex. The latter notion determines a full subcategory $\mathbf{Top}^{\mathbf{cel}} \subset \mathbf{Top}$ of the category of all topological spaces and continuous maps. The chain complexes we have studied in the previous section are functors defined on $\mathbf{Cell}$ but evidently not on $\mathbf{Top}^{\mathbf{cel}}$.

We now will investigate the effect the functor $C_\bullet(?;\mathsf{k})$ has on a homotopy in $\mathbf{Cell}$. Thus let X and Y be cell complexes, and $f: I\times X \longrightarrow Y$ a cellular homotopy from $f_0: X \longrightarrow Y$ to $f_1: X \longrightarrow Y$. Like any cellular map f induces a homomorphism

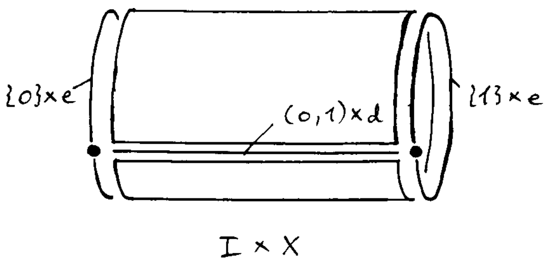
$$C_\bullet(f;\mathsf{k}): C_\bullet(I\times X;\mathsf{k}) \longrightarrow C_\bullet(Y;\mathsf{k})$$

of chain complexes over k. Let us have a closer look at the product complex $I\times X$. Of course the unit interval is understood to carry the cell decomposition

$$I = \{0\} \cup (0,1) \cup \{1\}$$

with the natural characteristic map $D^1 \longrightarrow I$. Therefore a $(q+1)$-cell of $I\times X$ is either

- $\{0\}\times e$, with $e \subset X$ a $(q+1)$-cell, or
- $(0,1)\times d$ with a q-cell d of X, or
- $\{1\}\times e$ for a $(q+1)$-cell $e \subset X$.



The chain module $C_{q+1}(I\times X;\mathsf{k}) = \mathsf{k}^{E_{q+1}(I\times X)}$ accordingly splits as a direct sum

$$C_{q+1}(I\times X) = C_{q+1}(X) \oplus C_q(X) \oplus C_{q+1}(X)$$

of chain modules of X. We define a new homomorphism

$$C'(f): C(X) \longrightarrow C(Y)$$

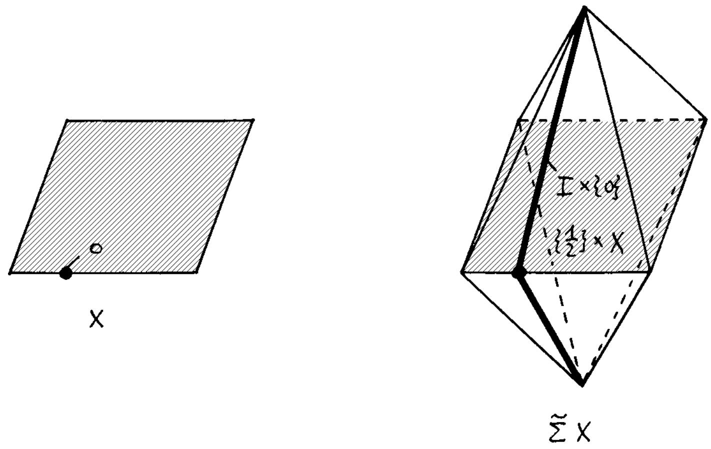
of graded k-modules of degree one so that $C'_q(f): C_q(X) \longrightarrow C_{q+1}(Y)$ is the restriction of $C_{q+1}(f)$ to the middle summand $C_q(X)$.

25.1 Proposition The formula

$$\partial \circ C'(f) + C'(f) \circ \partial = C(f_1) - C(f_0)$$

holds for any X, Y, and f as above.

The proof of this proposition will make use of the *suspension functors*[1]: the unreduced suspension $\tilde{\Sigma}X$ of a topological space X is the double pyramid obtained from $I\times X$ by collapsing each of the subspaces $\{0\}\times X$ and $\{1\}\times X$ to one point while the reduced suspension ΣX of a pointed space $(X,\circ)$ is defined by further collapsing $I\times\{\circ\}$.



25.2 Lemma Let X and Y be compact Hausdorff spaces.

- Homotopic maps $f\colon X\longrightarrow Y$ have homotopic suspensions $\tilde{\Sigma}f\colon\tilde{\Sigma}X\longrightarrow\tilde{\Sigma}Y$ and, in the pointed case, $\Sigma f\colon\Sigma X\longrightarrow\Sigma Y$.
- Both the reduced and the unreduced suspensions of the sphere S^n are homeomorphic to S^{n+1}.
- For $n>0$ and any map $f\colon S^n\longrightarrow S^n$ the suspensions $\tilde{\Sigma}f$ and Σf have the same mapping degree as f itself.

Proof The maps $\tilde{\Sigma}f$ and Σf are, of course, induced by $\mathrm{id}\times f\colon I\times S^n\longrightarrow I\times S^n$. If $F\colon I\times X\longrightarrow Y$ is a homotopy then

$$I\times(I\times X)\ni(s,t,x)\longmapsto\big(t,F(s,x)\big)\in I\times Y$$

induces homotopies $I\times\tilde{\Sigma}X\longrightarrow\tilde{\Sigma}Y$ respectively $I\times\Sigma X\longrightarrow\Sigma Y$: switching product and quotient topologies is justified by the assumptions we have made on X and Y (for the sake of convenience; they are quite unnecessary in fact). — That the unreduced suspension of a sphere is a sphere should be obvious enough. The analogous statement in the reduced case is proved by taking $I^n/\partial I^n$ as the model for the pointed sphere:

$$\Sigma(I^n/\partial I^n)=\big(I\times I^n\big)/\big(I\times\partial I^n\;\cup\;\{0,1\}\times I^n\big)=I^{n+1}/\partial I^{n+1}$$

Finally equality of the degrees follows by using a differentiable approximation of the map f and, for the reduced case, the fact that the quotient map $\tilde{\Sigma}S^n\longrightarrow\Sigma S^n$ is a homotopy equivalence by Theorem 21.6.

Proof of Proposition 25.1 We know that the diagram

$$\begin{array}{ccc}
C_{q+1}(I\times X) & \xrightarrow{\;\partial_{q+1}\;} & C_q(I\times X)\\
{\scriptstyle C_{q+1}(f)}\downarrow & & \downarrow{\scriptstyle C_q(f)}\\
C_{q+1}(Y) & \xrightarrow{\;\partial_{q+1}\;} & C_q(Y)
\end{array}$$

[1] The notion of suspension was introduced in Problem 23.

commutes. Restricting to the submodule $C_q(X) \subset C_{q+1}(I \times X)$ we obtain

$$\begin{array}{ccc} C_q(X) & \xrightarrow{\partial_{q+1}} & C_q(X) \oplus C_{q-1}(X) \oplus C_q(X) \\ {\scriptstyle C'_q(f)} \downarrow & & \downarrow {\scriptstyle C_q(f)} \\ C_{q+1}(Y) & \xrightarrow{\partial_{q+1}} & C_q(Y) \end{array}$$

and the proof will be mere evaluation of the homomorphisms involved in this diagram. The arrow at the top is, in (block) matrix notation

$$\partial_{q+1} = \begin{pmatrix} -1 \\ -\partial_q \\ 1 \end{pmatrix}$$

as we will show first. Consider two cells $e \in E_q(X)$ and $d \in E_{q-1}(X)$ with characteristic maps Φ and Ψ respectively. Writing $\varphi = \Phi | S^{q-1}$ as usual, the entry ∂_{de} of the boundary operator ∂_q is defined as the mapping degree of the composition d_{de}:

$$\begin{array}{ccccc} S^{q-1} & \xrightarrow{\varphi} & X^{q-1} & & \\ & & \downarrow & & \\ & & X^{q-1}/(X^{q-1} \backslash d_i) & \xleftarrow[\approx]{\overline{\Psi}} & D^{q-1}/S^{q-2} \end{array}$$

As discussed at the end of Section 13 the product complex $I \times X$ is more easily described using $I \times D^q$ rather than D^{q+1} as a model of the disk; the product has the boundary

$$\partial(I \times D^q) = I \times S^{q-1} \cup \{0, 1\} \times D^q.$$

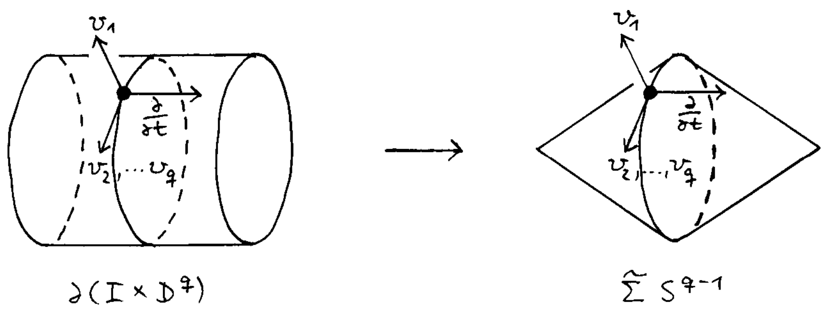
The characteristic map of the cell $(0,1) \times e$ then is $\mathrm{id} \times \Phi$, and similarly $\mathrm{id} \times \Psi$ is that of $(0,1) \times d$. Thus the diagram which defines the entry of $\partial_{q+1} \colon C_q(X) \longrightarrow C_{q-1}(X)$ corresponding to these cells is

$$\begin{array}{ccccc} \partial(I \times D^q) & \xrightarrow{(\mathrm{id} \times \Phi) | \partial(I \times D^q)} & I \times X^{q-1} & & \\ & & \downarrow & & \\ & & \Big(I \times X^{q-1}\Big) \Big/ \Big((I \times X^{q-1}) \backslash \big((0,1) \times d_i\big)\Big) & \xleftarrow[\approx]{\overline{\mathrm{id} \times \Psi}} & (I \times D^{q-1})/\partial(I \times D^{q-1}) \end{array}$$

and the composition comes out as

$$\begin{array}{cccc} \partial(I \times D^q) \longrightarrow \tilde{\Sigma} S^{q-1} & & & \\ & \searrow^{\tilde{\Sigma} d_{de}} & & \\ & & \tilde{\Sigma}(D^{q-1}/S^{q-2}) \longrightarrow \Sigma(D^{q-1}/S^{q-2}) & \end{array}$$

where the unlabelled arrows are quotient maps. Note that the first of them has mapping degree -1. For consider a positively oriented base $(v_1, \ldots, v_q)$ of $\mathbb{R}^q$ at $a \in S^{q-1}$, with v_1 pointing outward and $v_2, \ldots, v_q$ tangent to S^{q-1}: then $(\frac{\partial}{\partial t}, v_2, \ldots, v_q)$ is negatively oriented for $\partial(I \times D^q)$ at the point $(\frac{1}{2}, a)$ but positively oriented for $\tilde{\Sigma} S^{q-1}$ at the image point $[\frac{1}{2}, a]$.



Since $\deg \tilde{\Sigma} d_{de} = \deg d_{de} = \partial_{de}$ the middle entry of the matrix ∂_{q+1} is $-\partial_q$ as was claimed. The stated values of the top and bottom entries follow at once from the fact that restricting the attaching map of $(0,1)\times e$ to $\{0\}\times D^q$ or $\{1\}\times D^q$ essentially gives the characteristic map of e.

Thus the diagram from the beginning of the proof has become

$$\begin{array}{ccc}
C_q(X) & \xrightarrow{\left[\begin{smallmatrix} -1 \\ -\partial_q \\ 1 \end{smallmatrix}\right]} & C_q(X)\oplus C_{q-1}(X)\oplus C_q(X) \\
{\scriptstyle C'_q(f)}\Big\downarrow & & \Big\downarrow{\scriptstyle [\, C_q(f_0)\;\; C'_{q-1}(f)\;\; C_q(f_1)\,]} \\
C_{q+1}(Y) & \xrightarrow{\quad\partial_{q+1}\quad} & C_q(Y)
\end{array}$$

wherein the matrix for $C_q(f)$ is read off directly from the definitions. Commutativity of the diagramn means

$$-C_q(f_0) - C'_{q-1}(f)\partial_q + C_q(f_1) = \partial_{q+1} C'_q(f)$$

and this proves the proposition.

One would be hard pressed if asked to give a purely algebraic motivation of the following definition, but in the light of Proposition 25.1 it is quite natural.

25.3 Definition Let $V_\bullet$ and $W_\bullet$ be chain complexes, and $f_0, f_1: V_\bullet \longrightarrow W_\bullet$ two homomorphisms. A (chain) homotopy from f_0 to f_1 is a homomorphism $f: V \longrightarrow W$ of graded k-modules of degree 1 with $\partial f + f\partial = f_1 - f_0$. If such an f exists then f_0 and f_1 are called homotopic chain maps.

Of course, Proposition 25.1 can now be restated saying that every cellular homotopy of continuous maps induces a homotopy of chain maps. — It is a trivial exercise to verify that the algebraic version of homotopy is an equivalence relation like the geometric one. Much more important is a simple

25.4 Observation If $f_0, f_1: V_\bullet \longrightarrow W_\bullet$ are homotopic chain maps then the induced homomorphisms f_{0*} and f_{1*} of homology modules $H_q(V_\bullet) \longrightarrow H_q(W_\bullet)$ are equal for all $q \in \mathbb{Z}$.

Proof Let f be a chain homotopy from f_0 to f_1, and $x \in C_q(V_\bullet)$ a cycle. The difference

$$f_1(x) - f_0(x) = \partial f(x) + f\partial(x) = \partial f(x)$$

is a boundary, so $f_0(x)$ and $f_1(x)$ represent the same class in $H_q(W_\bullet)$.

At this point everything readily falls into place.

25.5 Theorem Let $X \in |\mathbf{Top^{cel}}|$ be a cell space, and consider two cell structures λ and μ on X. Writing $(X,\lambda) \in |\mathbf{Cell}|$ and $(X,\mu) \in |\mathbf{Cell}|$ for the corresponding cell complexes we have for each $q \in \mathbb{Z}$ a canonical isomorphism

$$h_{\mu\lambda}: H_q\big(C_\bullet(X,\lambda)\big) \xrightarrow{\simeq} H_q\big(C_\bullet(X,\mu)\big)$$

of homology modules (over a given, and fixed base ring).

Proof By Theorem 22.2 the identity map of X is homotopic to cellular maps

$$f: (X,\lambda) \longrightarrow (X,\mu) \quad \text{and} \quad g: (X,\mu) \longrightarrow (X,\lambda),$$

and we define

$$h_{\mu\lambda} = f_*: H_q\big(C_\bullet(X,\lambda)\big) \longrightarrow H_q\big(C_\bullet(X,\mu)\big) \quad \text{and} \quad h_{\lambda\mu} = g_*: H_q\big(C_\bullet(X,\mu)\big) \longrightarrow H_q\big(C_\bullet(X,\lambda)\big).$$

The compositions $g\circ f$ and $f\circ g$ are homotopic to the identity of X, and by Corollary 22.6 they are so by homotopies which are cellular with respect to the relevant cell structure λ or μ. Thus according to 25.1 and 25.4 the induced mappings on the homology level are the identities, and we conclude

$$h_{\lambda\mu}h_{\mu\lambda} = g_*f_* = (gf)_* = \mathrm{id} \quad \text{and} \quad h_{\mu\lambda}h_{\lambda\mu} = f_*g_* = (fg)_* = \mathrm{id}\,.$$

In particular $h_{\mu\lambda}$ is an isomorphism. Furthermore it is canonical in the sense that it does not depend on any particular choice: different cellular approximations f are homotopic by a cellular homotopy and so induce the same $h_{\mu\lambda}$, again by 25.1 and 25.4.

The theorem strongly suggests that homology of cell spaces can be defined as a functor on the category $\mathbf{Top^{cel}}$, and this is what we will now do. An obvious approach would be to single out one preferred cell structure for each cell space: you will agree that this should at once be discarded for sheer ugliness. But how else can we proceed?

We borrow an idea from physicists, who often are unwilling to talk about (three dimensional physical space) vectors as abstract entities. They would rather maintain that a vector consists of any three real components, with the (implicit) proviso that these must transform the right way under coordinate changes. From a mathematician's point of view they quote the representations of a vector $v \in V$ not with respect to a single but all possible bases of V.

Let us apply the same principle to the definition of $H_q(X)$ for a given cell space $X \in |\mathbf{Top^{cel}}|$, and given base ring k. Denoting by Λ the set of all cell structures on X we obtain one homology module $H_q\big(C_\bullet(X,\lambda)\big)$ for each $\lambda \in \Lambda$, and for any two structures $\lambda, \mu \in \Lambda$ the isomorphism $h_{\mu\lambda}\colon H_q\big(C_\bullet(X,\lambda)\big) \xrightarrow{\simeq} H_q\big(C_\bullet(X,\mu)\big)$ of Theorem 25.5. The direct product of k-modules

$$\prod_{\lambda\in\Lambda} H_q\big(C_\bullet(X,\lambda)\big)$$

is potentially huge but the submodule

$$H_q(X) := \Big\{(x_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} H_q\big(C_\bullet(X,\lambda)\big) \;\Big|\; x_\mu = h_{\mu\lambda}(x_\lambda) \quad \text{for all } \lambda,\mu\in\Lambda\Big\}$$

is not, as is implied by

25.6 Lemma For each $\kappa \in \Lambda$ the cartesion projection $\mathrm{pr}_\kappa\colon \prod H_q\big(C_\bullet(X,\lambda)\big) \longrightarrow H_q\big(C_\bullet(X,\kappa)\big)$ restricts to an isomorphism

$$\mathrm{pr}_\kappa\,|H_q(X)\colon H_q(X) \xrightarrow{\simeq} H_q\big(C_\bullet(X,\kappa)\big).$$

Proof The restriction is injective since x_κ determines all other components of $x \in H_q(X)$ via $x_\lambda = h_{\lambda\kappa}(x_\kappa)$. To prove surjectivity we must show that for any given element $x_\kappa \in H_q\big(C_\bullet(X,\kappa)\big)$ this formula determines an element $x \in H_q(X)$. In any case the family $(x_\lambda)_{\lambda\in\Lambda}$ is a well-defined element of $\prod H_q\big(C_\bullet(X,\lambda)\big)$ because $h_{\kappa\kappa}$ is the identity map. Next we consider arbitrary $\lambda,\mu \in \Lambda$. Since the isomorphisms $h_{\mu\lambda}$ of Theorem 25.5 furthermore satisfy $h_{\kappa\lambda}h_{\lambda\mu} = h_{\kappa\mu}$ we have

$$x_\mu = h_{\mu\kappa}(x_\kappa) = h_{\mu\lambda}h_{\lambda\kappa}(x_\kappa) = h_{\mu\lambda}(x_\lambda)$$

and so $x \in H_q(X)$ indeed.

25.7 Definition The k-module $H_q(X)$ is called the q-th homology module of X with coefficients in k. The base ring may be included in the notation by writing $H_q(X;\mathsf{k})$, and the special case $H_q(X;\mathbb{Z})$ is simply referred to as the q-th homology group of X. Homology is a functor

$$H_q(?;\mathsf{k})\colon \mathbf{Top^{cel}} \longrightarrow \mathbf{Lin_k}$$

which acts on a morphism $f \in \mathbf{Top^{cel}}(X,Y)$ as follows. Let λ be a cell structure on X, and μ one on Y: then f allows a cellular approximation $g\colon (X,\lambda) \longrightarrow (Y,\mu)$. The induced homomorphism

$g_*: H_q\big(C_\bullet(X,\lambda)\big) \longrightarrow H_q\big(C_\bullet(Y,\mu)\big)$ does not depend on the choice of the approximation g, and for an alternative pair of cell structures λ', μ' and an approximation g' the diagram

$$\begin{array}{ccc} H_q\big(C_\bullet(X,\lambda)\big) & \xrightarrow{\quad g_* \quad} & H_q\big(C_\bullet(Y,\mu)\big) \\ {\scriptstyle h_{\lambda'\lambda}}\big\downarrow & & \big\downarrow{\scriptstyle h_{\mu'\mu}} \\ H_q\big(C_\bullet(Y,\lambda')\big) & \xrightarrow{\quad g'_* \quad} & H_q\big(C_\bullet(Y,\mu')\big) \end{array}$$

commutes. The linear map

$$H_q(f): H_q(X) \longrightarrow H_q(Y)$$

is defined as the unique map that makes the diagram

$$\begin{array}{ccc} H_q(X) & \xrightarrow{\quad H_q(f) \quad} & H_q(Y) \\ {\scriptstyle \mathrm{pr}_\lambda}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_\mu} \\ H_q\big(C_\bullet(Y,\lambda)\big) & \xrightarrow{\quad g_* \quad} & H_q\big(C_\bullet(Y,\mu)\big) \end{array}$$

commutative for some, and then every choice of λ and μ.

25.8 Question Has the construction of H_q for cell spaces rendered the diagram of functors

$$\begin{array}{ccc} \mathbf{Cell} & \xrightarrow{\quad C_\bullet \quad} & \mathbf{Ch}_{\mathsf{k}} \\ {\scriptstyle \text{forget}}\big\downarrow & & \big\downarrow{\scriptstyle H_q} \\ \mathbf{Top}^{\mathbf{cel}} & \xrightarrow{\quad H_q \quad} & \mathbf{Lin}_{\mathsf{k}} \end{array}$$

commutative? If not, does it commute in some weaker sense?

It goes without saying that for an explicit calculation of homology there is no need to take into account all cell structures on a given space: one is enough, and may be arbitrarily picked. — Due to the very definition of the induced homomorphism we also can note the important

25.9 Fact The homology functors are homotopy invariant, so one may prefer to think of them as a series of functors $H_q(?;\mathsf{k}): \mathbf{hTop}^{\mathbf{cel}} \longrightarrow \mathbf{Lin}_{\mathsf{k}}$ defined on the homotopy category of $\mathbf{Top}^{\mathbf{cel}}$.

Homology is the best known and most widely used of all topological invariants. Certainly the efforts made to define them have been altogether considerable — but once constructed the functors H_q are easy to understand, and their calculation on the base of cell structures and cellular maps is by a simple algorithm. As the examples we have studied indicate it is also true that many interesting topological spaces (though necessarily only compact ones) admit cell structures for which the calculation can be explicitly performed. By contrast, to determine the homomorphism induced by a continuous mapping $f: X \longrightarrow Y$ may pose more of a problem because for given cell structures on X and Y it will often be quite cumbersome to construct an explicit cellular approximation of f. Usually a better strategy is to try and choose cell structures compatible with f and thereby avoid the approximation step.

25.10 Example Let $a \in \mathrm{Mat}(2\times 2, \mathbb{Z})$ be an integral matrix. We identify a with its associated linear map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$; this mapping respects the subgroup $\mathbb{Z}^2 \subset \mathbb{R}^2$ and therefore induces a self-map f of the torus $\mathbb{R}^2/\mathbb{Z}^2 = (S^1)^2$. Thus putting $X := Y := S^1$ we have the commutative diagram

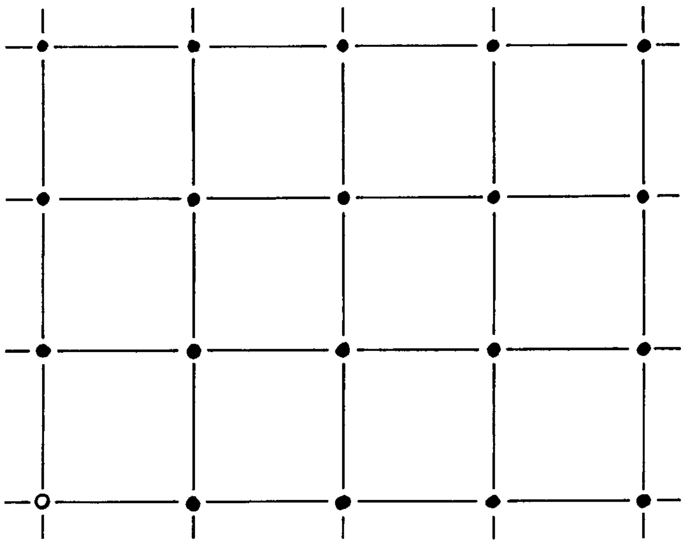
$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\quad a \quad} & \mathbb{R}^2 \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

where q is the obvious identification mapping built from the exponential.

We give X the standard cell structure. It produces an integral chain complex

$$0 \longrightarrow C_2(X) \longrightarrow C_1(X) \longrightarrow C_0(X) \longrightarrow 0$$

with $C_2(X) = C_0(X) = \mathbb{Z}$ and $C_1(X) = \mathbb{Z}^2$, and with vanishing differentials[2], so that we can identify $H_q(X)$ with $C_q(X)$. Since f will almost never preserve the cell filtration of X we force cellularity by putting a finer cell structure on Y as follows. Let us temporarily call a *cell* of $\mathbb{R}^2$ any connected component of an inverse image under q of an open cell of X.

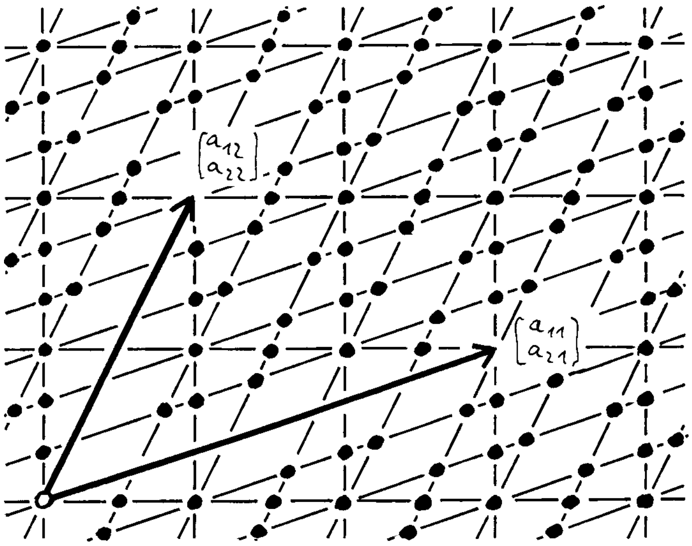


These cells form, of course, a partition P of $\mathbb{R}^2$ which is periodic with respect to the translation lattice $\mathbb{Z}^2$. A second periodic partition Q of $\mathbb{R}^2$ is obtained from the sets

$$\mathbb{Z}^2 + \mathbb{R}\left[\begin{smallmatrix} a_{11} \\ a_{21} \end{smallmatrix}\right] \subset \mathbb{R}^2 \quad \text{and} \quad \mathbb{Z}^2 + \mathbb{R}\left[\begin{smallmatrix} a_{12} \\ a_{22} \end{smallmatrix}\right] \subset \mathbb{R}^2,$$

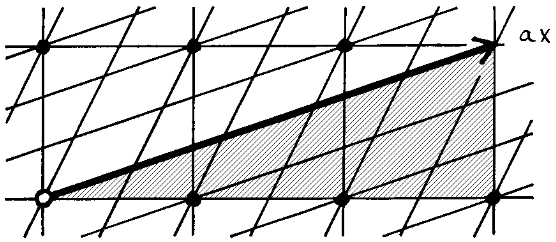
by taking the connected components of their intersection, of their mutual set theoretic differences, and of the complement of their union. Let R be the smallest partition of $\mathbb{R}^2$ into connected subsets that refines both P and Q: so the cells of R are the connected components of intersections of a cell of P with a cell of Q.

2 This is a special case of Problem 32.

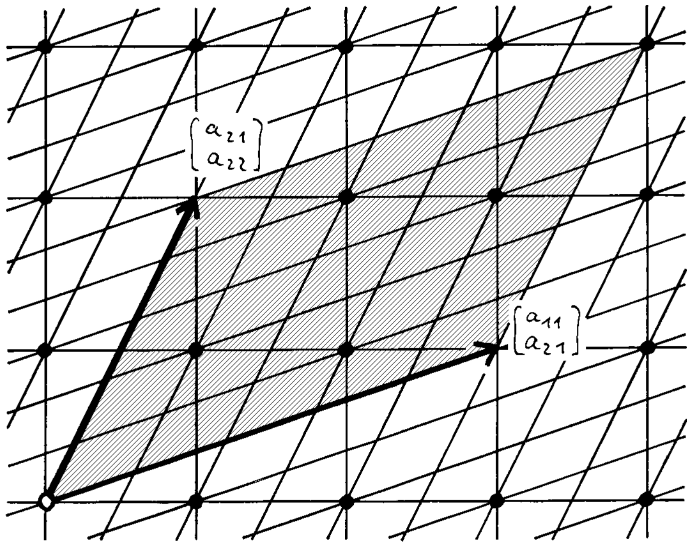


Each cell of R is a convex open subset of the affine space spanned by it, and it is not difficult to see that R projects to a partition of Y which corresponds to a cell structure of Y. Note that since this structure refines that of X we have a canonical embedding of the chain complex $C_\bullet(X)$ in $C_\bullet(Y)$, and so may consider the former a subcomplex of the latter.

The map $f\colon X \longrightarrow Y$ now is cellular. The image of a vector $x \in C_1(X)$ is represented in $\mathbb{R}^2$ by the oriented segment from the origin to ax, and is therefore homologous to $ax \in C_1(X) \subset C_1(Y)$:



Thus a is the matrix of $f_*\colon H_1(X) \longrightarrow H_1(X)$. Similarly f maps the unique 2-cell of X to the 2-chain represented by the oriented parallelogram spanned by $\left[\begin{smallmatrix} a_{11} \\ a_{21} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} a_{12} \\ a_{22} \end{smallmatrix}\right]$.



Expressed in terms of the cell structure of X this chain is just the $(\det a)$-fold of the base vector. Therefore $f_*: H_2(X) \longrightarrow H_2(X)$ is multiplication by $\det a$. Since it is clear that f_* is the identity on $H_0(X)$ this completes the calculation of f_*. The result at once carries over to an arbitrary base ring, a and $\det a$ acting as their reductions if the base ring has non-zero characteristic.

25.11 Question The example can be extended to $(S^1)^n$ for all $n \in \mathbb{N}$. Can you guess the result?

For arbitrary $n \in \mathbb{N}$ there is little point in determining f_* beyond its component of degree one: due to the product structure of $(S^1)^n$ the higher degree components can then be formally derived using the multiplicative structure of homology which we have not yet studied.

26 Euler and Lefschetz Numbers

26.1 Definition Let X be a (finite) cell complex, and for any $q \in \{0, \ldots, \dim X\}$ denote by $E_q(X)$ the set of q-cells in X. The Euler number of X is the number

$$e(X) := \sum_{q=0}^{\dim X} (-1)^q |E_q(X)| \in \mathbb{Z}.$$

The point of this notion is that it is a topological invariant. Using homology the proof will be quite easy, but it is convenient to first introduce a bit of algebraic vocabulary.

26.2 Definition Let k be a field, and $V_\bullet$ a chain complex over k so that $\bigoplus_q H_q(V_\bullet)$ has finite dimension. Then for any $q \in \mathbb{Z}$ the integer $\dim H_q(V_\bullet)$ is called the q-th Betti number of $V_\bullet$, and the alternating sum

$$\sum_{q=-\infty}^{\infty} (-1)^q \dim H_q(V_\bullet) \in \mathbb{Z}$$

the Euler characteristic of $V_\bullet$.

The finiteness condition imposed on the homology of $V_\bullet$ makes sure that the sum is but formally infinite. It is always satisfied if $V_\bullet = C_\bullet(X; \mathsf{k})$ comes from a cell complex $X \in |\mathbf{Cell}|$: in this case we obtain topological invariants of X, simply called the Betti numbers and the Euler characteristic of the cell space X, with coefficient filed k.

For chain complexes with a stronger finiteness property there is an alternative way to calculate the Euler characteristic.

26.3 Lemma Let k be a field, and $V_\bullet$ a chain complex over k so that $\bigoplus_q V_q$ has finite dimension. Then

$$\sum_{q=-\infty}^{\infty} (-1)^q \dim H_q(V_\bullet) = \sum_{q=-\infty}^{\infty} (-1)^q \dim V_q.$$

Proof For each $q \in \mathbb{Z}$ abbreviate $h_q = \dim H_q(V_\bullet)$, as well as $z_q = \dim \ker \partial_q$ and $b_q = \dim \partial_q(V_q)$. Then $\dim V_q = z_q + b_q$ by linear algebra and $h_q = z_q - b_{q+1}$ by the definition of homology. Adding up the formula follows:

$$\sum_q (-1)^q h_q = \sum_q (-1)^q (z_q - b_{q+1}) = \sum_q (-1)^q z_q + \sum_q (-1)^q b_q = \sum_q (-1)^q \dim V_q$$

26.4 Theorem Let X be a cell complex. Then

$$e(X) = \sum_{q=-\infty}^{\infty} (-1)^q \dim H_q(X; \mathsf{k})$$

for any field k. In particular the Euler number $e(X)$ is a topological homotopy invariant of X, and unlike the individual Betti numbers of X their alternating sum does not depend on the choice of the coefficient field.

Proof The obvious application of Lemma 26.3.

The famous formula found by Euler in 1750 states that for every polyhedron (implicitly assumed convex and compact) the number of vertices plus the number of faces is equal to the number of edges plus two. It may be considered the earliest among all topological discoveries and is a special case of Theorem 26.4: the surface of such a polyhedron is homeomorphic to S^2, and $e(S^2) = 2$.

Let X be a cell space. Theorem 26.4 shows that there are restrictions on the number of cells a cell structure on X can have in each dimension. Surprisingly strong further restrictions are obtained from a completely trivial observation: $|E_q(X)|$, the number of q-cells in such a structure cannot be smaller than the Betti number $\dim H_q(X;\mathsf{k})$ with coefficients in an arbitrarily chosen field k.

26.5 Application The standard cell structures of real and complex projective spaces are minimal in the sense that every cell structure on $\mathbb{R}P^n$ or $\mathbb{C}P^n$ must comprise at least one cell in each dimension between 0 and n, respectively each even dimension between 0 and $2n$.

Proof Take care to use $\mathsf{k} = \mathbb{F}_2$ in the real case.

Better estimates on the number of cells are possible: the following so-called *Morse inequalities* imply both Theorem 26.4 and the naive inequality $\dim H_q(X) \geq |E_q(X)|$.

26.6 Theorem Let X be a cell complex, and k a field. Then the inequality

$$\sum_{q=0}^{l}(-1)^{l-q}\dim H_q(X;\mathsf{k}) \leq \sum_{q=0}^{l}(-1)^{l-q}|E_q(X)|$$

holds for each $l \in \mathbb{N}$.

Proof Substituing for X the l-skeleton X^l we even get an equality by Theorem 26.4. The substitution does not affect the right hand side, and the inclusion $j\colon X^l \hookrightarrow X$ clearly induces isomorphisms

$$j_*\colon H_q(X^l) \xrightarrow{\simeq} H_q(X)$$

for all $q < l$. In degree l

$$j_*\colon H_l(X^l) = \ker\partial_l \longrightarrow \big(\ker\partial_l\big)Big/\partial_{l+1}\big((C_{l+1}(X)\big) = H_l(X)$$

is still surjective, and the l-th Morse inequality follows:

$$\begin{aligned}\sum_{q=0}^{l}(-1)^{l-q}\dim H_q(X;\mathsf{k}) &\leq \sum_{q=0}^{l}(-1)^{l-q}\dim H_q(X^l;\mathsf{k})\\ &= \sum_{q=0}^{l}(-1)^{l-q}|E_q(X^l)|\\ &= \sum_{q=0}^{l}(-1)^{l-q}|E_q(X)|.\end{aligned}$$

A very interesting idea is to try and define an analogue of the Euler characteristic for continuous maps rather than topological spaces. On the algebraic level we would look for an integral invariant of linear maps that reduces to the dimension in case of the identity map. If the characteristic of k is zero then the trace of an endomorphism is such an invariant, and we use it to mimic 26.2 and 26.3:

26.7 Definition Fix a field k, and let $V_\bullet$ be a chain complex over k so that $\bigoplus_q H_q(V_\bullet)$ has finite dimension. A chain endomorphism $f\colon V_\bullet \longrightarrow V_\bullet$ induces endomorphisms $H_q(f)\colon H_q(V_\bullet;\mathsf{k}) \longrightarrow H_q(V_\bullet;\mathsf{k})$ in homology, and

$$\Lambda(f) := \sum_{q=-\infty}^{\infty} (-1)^q \operatorname{trace} H_q(f) \in \mathsf{k}$$

is called the Lefschetz characteristic of f.

26.8 Lemma If $\bigoplus_q V_q$ has finite dimension then in the definition of the Lefschetz characteristic $f_q\colon V_q \to V_q$ may be substituted for $H_q(f)$.

Proof For each q put $Z_q = \ker \partial_q$ and $B_q = \partial_q(V_q)$. By Proposition 24.4 the chain map f_q respects the flag

$$\{0\} \subset B_q \subset Z_q(V_q) \subset V_q$$

and so induces endomorphisms

$$b_q\colon B_q \longrightarrow B_q, \quad h_q\colon Z_q/B_q \longrightarrow Z_q/B_q, \text{ and } \quad c_q\colon V_q/Z_q \longrightarrow V_q/Z_q.$$

On the other hand the differential ∂_q induces a commutative diagram

$$\begin{array}{ccc} V_q/Z_q & \xrightarrow[\approx]{d_q} & B_{q-1} \\ {\scriptstyle c_q}\downarrow & & \downarrow{\scriptstyle b_{q-1}} \\ V_q/Z_q & \xrightarrow[\approx]{d_q} & B_{q-1} \end{array}$$

which proves that c_q and b_{q-1} have equal trace. In the alternating sum of the identities

$$\operatorname{trace} f_q = \operatorname{trace} b_q + \operatorname{trace} h_q + \operatorname{trace} c_q \quad (q \in \mathbb{Z})$$

the terms containing b_q and c_q therefore cancel, and the lemma follows.

26.9 Definition If $f\colon X \longrightarrow X$ be a continuous map then the rational number

$$\Lambda(f) := \sum_{q=-\infty}^{\infty} (-1)^q \operatorname{trace} H_q(f;\mathbb{Q})$$

is called the Lefschetz number of f. It clearly is a homotopy invariant, and is in fact an integer since by Lemma 26.8 it may be computed as the Lefschetz characteristic of a chain endomorphism $C_\bullet(f;\mathbb{Q})\colon C_\bullet(X;\mathbb{Q}) \longrightarrow C_\bullet(X;\mathbb{Q})$ which is described by integral matrices.

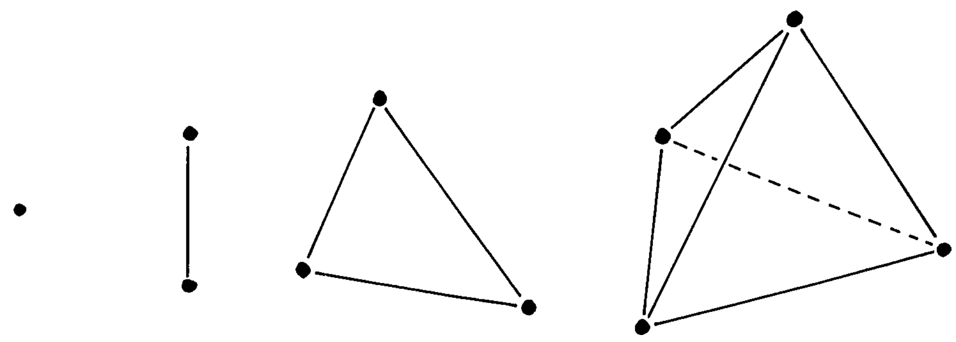
Remark For the definition of $\Lambda(f)$ there is no point here in using other fields than $\mathbb{Q}$ since the result will be the same at best. Nevertheless a field of prime characteristic p may be used in order to calculate $\Lambda(f) \bmod p$.

The Lefschetz number of a continuous self-map f is geometrically interesting because it tells about the fixed point set of f. In order to state Theorem 26.14 below, a classical result of this type, I shall first introduce a certain class of special cell complexes.

26.10 Definition Let integers $q, n \in \mathbb{N}$ and a subset $T \subset \mathbb{R}^n$ be given. Assume that T consists of $q+1$ points *in general position*, which means that the affine subspace $\operatorname{Aff} T \subset \mathbb{R}^n$ spanned by them has dimension q. The set

$$\Delta(T) := \left\{ \sum_{t\in T} \lambda_t \cdot t \;\middle|\; \lambda_t > 0 \text{ for all } t, \text{ and} \sum_{t\in T} \lambda_t = 1 \right\}$$

is an open subset of Aff T, and called the <u>open</u> q-<u>simplex</u> spanned by T. Note that the corresponding closed simplex $\overline{\Delta(T)}$ coincides with the convex hull of T, and is the disjoint union of all open <u>facets</u> of $\Delta(T)$: these are, by definition, the simplices spanned by non-empty subsets of T.

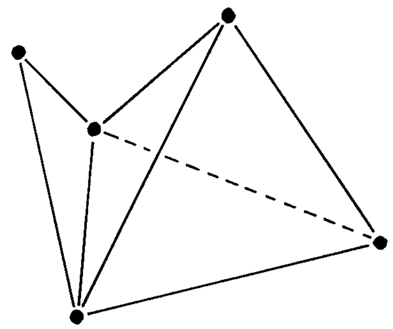


$q = 0$ $q = 1$ $q = 2$ $q = 3$

A <u>simplicial</u> <u>complex</u> is a subspace X of some euclidean space $\mathbb{R}^n$ together with a finite partition of X into open simplices, such that if $\Delta(T)$ belongs to the partition then so does $\Delta(T')$ for all non-empty subsets $T' \subset T$. A <u>polyhedron</u> is a topological space that is homeomorphic to some simplicial complex (more precisely, to its underlying topological space).

Note The notion is usually generalised so as to allow for infinite complexes but in keeping with our treatment of cell complexes we stay with the finite version.

26.11 Proposition If two closed simplices of a complex X meet then their intersection is a closed facet of each.



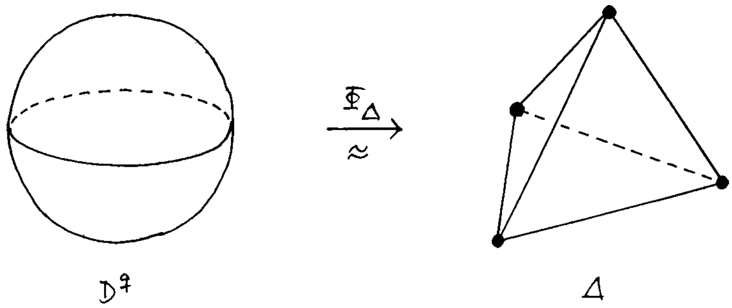
Proof Let $\Delta(S)$ and $\Delta(T)$ be two simplices of X. We shall show that the intersection $\overline{\Delta(S)} \cap \overline{\Delta(T)}$ is either empty or equal to $\overline{\Delta(S \cap T)}$.

We may assume $\overline{\Delta(S)} \cap \overline{\Delta(T)} \neq \emptyset$. Consider any point $x \in \overline{\Delta(S)} \cap \overline{\Delta(T)}$. It must lie in some open simplex $\Delta(R)$ of X. On the other hand x belongs to an open facet of $\Delta(S)$ as well as some open facet of $\Delta(T)$: since both these facets belong to the simplicial structure of X they must <u>coincide</u> with $\Delta(R)$. Thus we have $R \subset S \cap T$ so that $\Delta(S \cap T)$ makes sense. In view of $x \in \Delta(R) \subset \overline{\Delta(S \cap T)}$ the inclusion

$$\overline{\Delta(S)} \cap \overline{\Delta(T)} \subset \overline{\Delta(S \cap T)}$$

follows, and the opposite inclusion is clear.

Our simplicial complexes are compact spaces, and can easily be given cell structures: all one has to do is choose for each q-simplex Δ a homeomorphism $\Phi_\Delta\colon D^q \approx \Delta$, which then makes Δ an open q-cell, and Φ_Δ its characteristic map.



Thus every polyhedron is a cell space, and you will easily convince yourself that all our examples of cell spaces admit a simplicial structure, and so are polyhedra. While presumably there exist cell spaces which are not polyhedra it is the difference between simplicial and cell structures on a given polyhedron X which is of more practical interest. The point is that a simplicial structure on X often requires many more simplices than a cell structure need have cells. Look at the case of D^n as an example: a simplicial structure must comprise at least one n-simplex and thereby a total of at least $2^{n+1}-1$ simplices — by contrast for any $n \geq 1$ there is a cell structure with just three cells. The other side of the coin is that cells are often quite big subsets of a given space while the simplices of a simplicial structure tend to be small. It is for exactly this reason that simplicial structures are more suitable for the application to fixed point sets we have in mind. In fact we will resort to a method that allows to pass from a given simplicial structure to one with arbitrarily small simplices, naturally at the cost of increasing their number.

26.12 Definition Let $\Delta(T) \subset \mathbb{R}^n$ be a q-simplex, so that the closed simplex $\overline{\Delta(T)}$ is a simplicial complex in the obvious (minimal) way. We define a second simplicial structure on $\overline{\Delta(T)}$, which will be called the barycentric subdivision. For each non-empty subset $T' \subset T$ let

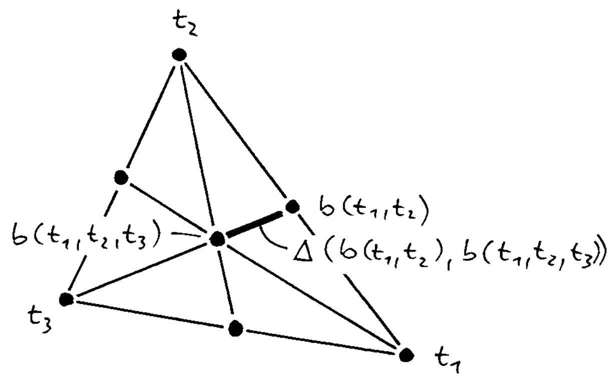
$$b(T') = \frac{1}{|T'|} \sum_{t \in T'} t \in \mathbb{R}^n$$

be the barycentre of $\Delta(T')$. The simplices of the new structure are indexed by the strictly increasing chains of length $p \geq 0$

$$T_0 \subset T_1 \subset \cdots \subset T_p$$

of non-empty subsets of T, the corresponding simplex being

$$\Delta\big(b(T_0), b(T_1), \ldots, b(T_p)\big).$$



Justification It is verified at once that the points $b(T_0), b(T_1), \ldots, b(T_p)$ are in general position. Each of the newly defined simplices is contained in $\overline{\Delta(T)}$, and it remains to prove that they form a partition of $\overline{\Delta(T)}$. Thus let $x \in \overline{\Delta(T)}$ be an arbitrary point; it has the form

$$x = \sum_{t \in T} \lambda_t \cdot t$$

with uniquely determined $\lambda_t \in [0,\infty)$ such that $\sum_t \lambda_t = 1$. Arrange the distinct positive ones among the λ_t according to size and relabel them $\mu_0 > \mu_1 > \cdots > \mu_p$. Then $T_j := \{t \in T \mid \lambda_t \geq \mu_j\}$ defines a strictly increasing chain $T_0 \subset T_1 \subset \cdots \subset T_p$ of non-empty subsets of T and the expression for x may be rewritten as

$$\begin{aligned} x &= \sum_{t\in T_0} \mu_0\cdot t + \sum_{t\in T_1\setminus T_0} \mu_1\cdot t + \cdots + \sum_{t\in T_p\setminus T_{p-1}} \mu_p\cdot t \\ &= \sum_{t\in T_0} \mu_0\cdot t + \sum_{t\in T_1} (\mu_1-\mu_0)\cdot t + \cdots + \sum_{t\in T_p} (\mu_p-\mu_{p-1})\cdot t \\ &= \mu_0|T_0|\cdot b(T_0) + (\mu_1-\mu_0)|T_1|\cdot b(T_1) + \cdots + (\mu_p-\mu_{p-1})|T_p|\cdot b(T_p). \end{aligned}$$

Therefore $x \in \Delta\big(b(T_0), b(T_1), \ldots, b(T_p)\big)$. Conversely let $\Delta\big(b(S_0), b(S_1), \ldots, b(S_r)\big)$ be any one of the open simplices containing x. Then for some positive numbers $\kappa_1, \ldots, \kappa_r$ with $\sum_j \kappa_j = 1$ we have

$$\begin{aligned} x &= \kappa_0\cdot b(S_0) + \kappa_1\cdot b(S_1) + \cdots + \kappa_p\cdot b(S_r) \\ &= \frac{\kappa_0}{|S_0|}\sum_{t\in S_0} t \;+\; \frac{\kappa_1}{|S_1|}\sum_{t\in S_1} t \;+\; \cdots \;+\; \frac{\kappa_r}{|S_r|}\sum_{t\in S_r} t \\ &= \left(\frac{\kappa_0}{|S_0|}+\cdots+\frac{\kappa_r}{|S_r|}\right)\sum_{t\in S_0} t \;+\; \left(\frac{\kappa_1}{|S_1|}+\cdots+\frac{\kappa_r}{|S_r|}\right)\sum_{t\in S_1\setminus S_0} t \;+\; \cdots \;+\; \frac{\kappa_r}{|S_r|}\sum_{t\in S_r\setminus S_{r-1}} t, \end{aligned}$$

and comparing with $x = \sum_t \lambda_t\cdot t$ we conclude that $r = p$ and $(\kappa_j+\cdots+\kappa_r)/|S_0| = \mu_j$ for $j = 0, \ldots, r$. This implies $S_j = T_j$ for all j, and we have thus proved that x belongs to exactly one open simplex of the barycentric subdivision.

It is clear that the barycentric subdivision of a closed simplex induces the barycentric subdivision on each of its facets. Therefore the notion of barycentric subdivision generalises to an arbitrary simplicial complex X: the barycentric subdivision of X is the unique simplicial structure on X that for each simplex Δ of X restricts to the barycentric subdivision of $\overline{\Delta}$.

As one would expect barycentric subdivision reduces the size of the simplices. The following proposition tells by how much, and its proof is left to you as an exercise in concrete geometry.

26.13 Proposition If the diameter of a q-simplex $\Delta \subset \mathbb{R}^n$ is d then that of each simplex of the barycentric subdivision does not exceed $\frac{q}{q+1}\cdot d$.

We now have the tools at hand to prove what has become known as the fixed point theorem of Lefschetz.

26.14 Theorem Let X be a (compact) polyhedron, and $f: X \longrightarrow X$ a map with $\Lambda(f) \neq 0$. Then f has a fixed point.

Proof We may assume that $X \subset \mathbb{R}^n$ is a simplicial complex, and will prove the opposite implication: if $f: X \longrightarrow X$ is fixed point free then $\Lambda(f) = 0$. By compactness we first find an $\varepsilon > 0$ such that

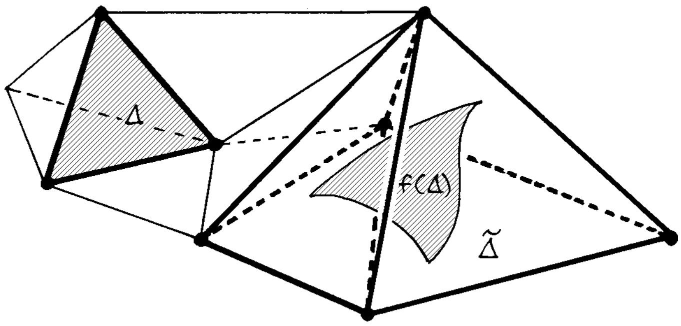
$$|f(x) - x| \geq 2\varepsilon \quad \text{for all } x \in X.$$

Next we apply Proposition 26.13: repeated barycentric subdivision gives a new simplicial structure on X such that the diameter of each simplex is smaller than ε.

Consider now any closed simplex $\Delta \subset X$. Let $\tilde{\Delta} \subset X$ be the smallest subcomplex that contains $f(\Delta)$: it is the union of all closed simplices that meet $f(\Delta)$. For any $x \in \Delta$ and any $z \in \tilde{\Delta}$ we therefore find a point $y \in \Delta$ such that $f(y)$ and z belong to one and the same closed simplex, so we can estimate

$$|x - z| \geq |y-f(y)| - |x-y| - |f(y)-z| > 2\varepsilon - \varepsilon - \varepsilon = 0$$

and conclude that $\Delta \cap \tilde{\Delta} = \emptyset$.



From this point onwards we consider the simplicial partition of X as just the cell partition of a cell structure. By 22.6 we can choose a cellular approximation g of f such that $g(\Delta) \subset \tilde{\Delta}$, and therefore

$$\Delta \cap g(\Delta) = \emptyset \quad \text{for every simplex } \Delta \text{ of } X.$$

Calculating the induced homomorphism $C(g)\colon C(X) \longrightarrow C(X)$ we obtain that the mapping degree $\deg g_{\Delta\Delta}$ is zero. Thus the diagonal entries of all the matrices $C_q(g)$ vanish, so their traces vanish too, and by Lemma 26.8 the proof is concluded: $\Lambda(f) = \Lambda\big(C_\bullet(g;\mathbb{Q})\big) = 0$.

26.15 Examples (1) If the polyhedron X is contractible then every self-map f of X is homotopic to the identity of X, so that $\Lambda(f) = e(X) = 1$. Theorem 26.14 thus implies that f has a fixed point, and thereby generalizes Theorem 20.9.

(2) A map $f\colon S^n \longrightarrow S^n$ induces, for $n > 0$, the identity map of $H_0(S^n;\mathbb{Q}) = \mathbb{Q}$, and multiplication by $\deg f$ on $H_n(S^n;\mathbb{Q}) = \mathbb{Q}$. Its Lefschetz number therefore is related to the degree via

$$\Lambda(f) = 1 + (-1)^n \deg f.$$

It must, of course, vanish if f is the antipodal map, and indeed we know from 20.6 that then $\deg f = (-1)^{n+1}$.

(3) If $n \in \mathbb{N}$ is even then the only non-trivial rational homology space of $\mathbb{R}P^n$ is $H_0(\mathbb{R}P^n;\mathbb{Q}) = \mathbb{Q}$. Since every map $f\colon \mathbb{R}P^n \longrightarrow \mathbb{R}P^n$ induces the identity of that space we have $\Lambda(f) = 1$, and f must have a fixed point.

Remarks Lefschetz' fixed point theorem can be generalised to spaces X that are more general than polyhedra, but compactness remains essential as is shown by the example of translations of a real vector space X.— The theorem is not the last word on the subject of fixed point sets but rather a coarse first step. More advanced fixed point theorems usually relate global invariants of f like the Lefschetz number, to quantities whose definition is local along the fixed point set $F = \{x \in X \mid f(x) = x\}$ in the sense that they can be computed from the restriction of f to an arbitrary neighbourhood of F. For further discussion see the book [Dold].

27 Reduced Homology and Suspension

The natural notion of a pointed cell complex $(X, \circ)$ is that which requires the base point $\circ$ to be one of the 0-cells. The notion defines the category $\mathbf{Cell}^\circ$, and correspondingly $\mathbf{Top}^{\mathbf{cel},\circ}$ will be used to denote the category of pointed topological spaces obtained by stripping the objects of $\mathbf{Cell}^\circ$ of their cell structure. For such pointed cell spaces a slightly modified version of homology is often convenient. As before, a base ring k will be held fixed.

27.1 Definition Let $X \in |\mathbf{Cell}^\circ|$ be a pointed cell space, and k. The reduced chain complex $\tilde{C}_\bullet(X;\mathsf{k})$ of X is defined by $\tilde{C}_q(X) = C_q(X)$ for $q \neq 0$ and

$$\tilde{C}_0(X;\mathsf{k}) = \mathsf{k}^{E_0(X)\setminus\{\circ\}}.$$

Thus in the reduced complex the base point is ignored as a 0-cell. Likewise the differentials of the reduced complex are those of the ordinary one, with the exception of ∂_1 which is adapted using the cartesian projection from $C_0(X;\mathsf{k})$ to $\tilde{C}_0(X;\mathsf{k})$:

$$\begin{array}{ccc} C_1(X;\mathsf{k}) & \xrightarrow{\partial_1} & C_0(X;\mathsf{k}) \\ \Vert & & \big\downarrow{\scriptstyle \mathrm{pr}} \\ \tilde{C}_1(X;\mathsf{k}) & \xrightarrow{\tilde{\partial}_1} & \tilde{C}_0(X;\mathsf{k}) \end{array}$$

The homology modules

$$\tilde{H}_q(X;\mathsf{k}) := H_q\big(\tilde{C}_\bullet(X;\mathsf{k})\big)$$

constitute the reduced homology of X. It is functorial by the reasoning detailed in Section 25, and yields a sequence of homotopy invariant functors

$$\tilde{H}_q(?;\mathsf{k})\colon \mathbf{Top}^{\mathbf{cel},\circ} \longrightarrow \mathbf{Lin}_\mathsf{k}$$

on the pointed category.

As is obvious from the definitions unreduced homology commutes with finite sums:

$$H_q(X+Y) = H_q(X) \oplus H_q(Y)$$

Note that reduced homology behaves in perfect analogy with respect to the bouquet, which is the sum in the pointed category:

$$\tilde{H}_q(X \vee Y) = \tilde{H}_q(X) \oplus \tilde{H}_q(Y)$$

It will not surprise that the relation between reduced and unreduced homology is very simple. Nevertheless, if we wish to formulate it in an adequately functorial way a few general remarks about the category $\mathbf{Ch}_\mathsf{k}$ are in order. From its parent category $\mathbf{Lin}_\mathsf{k}$ this category inherits a large variety of notions, including

- pointwise addition and scalar multiplication of morphisms,
- direct sums and products,
- sub and quotient objects in general, and in particular
- kernels and images of morphisms.

Any of these notions can be transferred from $\mathbf{Lin}_{\mathsf{k}}$ to $\mathbf{Ch}_{\mathsf{k}}$ by simply applying it separately to each homogeneous component, and we thus have no need to take a more general point of view (as is often done in this context, introducing the concept of *abelian category*).

Note that $\mathbf{Lin}_{\mathsf{k}}$ may be considered as a full subcategory of $\mathbf{Ch}_{\mathsf{k}}$ using the functor that sends $V \in |\mathbf{Lin}_{\mathsf{k}}|$ to the chain complex

$$\cdots \longrightarrow 0 \longrightarrow V \longrightarrow 0 \longrightarrow \cdots$$

with V in the zero position.

27.2 Proposition For every pointed cell complex X there is a canonical isomorphism

$$C_\bullet(X;\mathsf{k}) \xrightarrow{\simeq} \mathsf{k} \oplus \tilde{C}_\bullet(X;\mathsf{k}),$$

and these isomorphisms define a natural equivalence of functors $\mathbf{Cell}^\circ \longrightarrow \mathbf{Ch}_{\mathsf{k}}$.

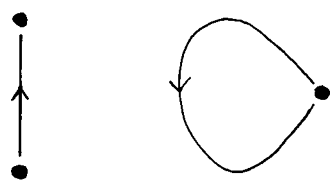
Proof For $q \neq 0$ the isomorphism $C_q(X;\mathsf{k}) \simeq 0 \oplus \tilde{C}_q(X;\mathsf{k})$ is the identity map, and its component of degree zero is given in terms of the spanning cells $\circ$ and $\circ \neq e \in E_0(X)$ by

$$C_0(X;\mathsf{k}) \ni \left\{ \begin{matrix} \circ & \longmapsto & 1 \oplus 0 \\ e & \longmapsto & 1 \oplus e \end{matrix} \right\} \in \mathsf{k} \oplus \tilde{C}_0(X;\mathsf{k}).$$

To see that this graded isomorphism is a chain map the commutativity of

$$\begin{array}{ccc} C_1(X;\mathsf{k}) & \xrightarrow{\partial_1} & C_0(X;\mathsf{k}) \\ \| & & \downarrow \simeq \\ 0 \oplus \tilde{C}_1(X;\mathsf{k}) & \xrightarrow{0 \oplus \tilde{\partial}_1} & \mathsf{k} \oplus \tilde{C}_0(X;\mathsf{k}) \end{array}$$

must be proved: this is immediate, using the fact that the two boundary points of a 1-cell carry opposite orientations.



27.3 Corollary There are natural equivalences

$$H_0(X;\mathsf{k}) \simeq \mathsf{k} \oplus \tilde{H}_0(X;\mathsf{k}) \quad \text{and} \quad H_q(X;\mathsf{k}) \simeq \tilde{H}_q(X;\mathsf{k}) \text{ for } q \neq 0$$

which allow to identify the corresponding functors $\mathbf{Cell}^\circ \longrightarrow \mathbf{Lin}_{\mathsf{k}}$, or $\mathbf{Top}^{\mathbf{cel},\circ} \longrightarrow \mathbf{Lin}_{\mathsf{k}}$.

Proof Homology of chain complexes commutes with direct sums.

Of course for $0 \neq q \neq 1$ this conclusion has been clear right from the beginning. — It is often preferable to express the relation between reduced and unreduced homology directly in geometric terms, bypassing the level of chain complexes. This is done by

27.4 Proposition For all cell spaces X one has identities

$$H_q(X) = \tilde{H}_q(X^+)$$

and for pointed X there are a natural equivalences

$$\tilde{H}_q(X) \simeq \ker\big(H_q(X) \xrightarrow{H_q(\circ)} H_q\{\circ\}\big)$$

of functors $\mathbf{Top}^{\mathbf{cel},\circ} \longrightarrow \mathbf{Lin}_{\mathsf{k}}$.

Proof The identities are clear from the definitions: recall from 16.2(6) that X^+ is made from X by adding a discrete base point. The equivalences for pointed X result from those of Corollary 27.3: the compositions

$$\mathsf{k} \oplus \tilde{H}_0(X;\mathsf{k}) \simeq H_0(X;\mathsf{k}) \longrightarrow H_0\{\circ\}, \quad \text{and}$$
$$0 \oplus \tilde{H}_q(X;\mathsf{k}) \simeq H_q(X;\mathsf{k}) \longrightarrow H_q\{\circ\} \text{ for } q \neq 0$$

are injective on the first summand, and vanish identically on the second.

Remark If $\circ$ denotes a universal one-point space then every topological space admits, of course, exactly one map into it. Reduced homology of X could thus have been defined as the kernel of the induced homomorphism $H_q(X) \longrightarrow H_q\{\circ\}$, without the need to specify a base point for X. This point of view is often adopted in the literature, though excluding the empty space X as an awkward special case. We prefer to think of $\tilde{H}_q$ as a functor on the pointed category; only as such it gives the splitting described in 27.2.

An advantage of reduced homology is that it behaves in a particularly nice way with respect to suspension.

27.5 Theorem and **Terminology** Let X be a pointed cell complex. Assigning to each cell $e \neq \circ$ of X the product cell $(0,1)\times e$ of the reduced suspension ΣX induces a natural equivalence of functors

$$\sigma_q\colon \tilde{H}_q(X) \simeq \tilde{H}_{q+1}(\Sigma X)$$

for each $q \in \mathbb{Z}$, called the suspension isomorphism.

Proof By definition of ΣX the assignment is bijective on the level of cells, so it induces isomorphisms $C_q(X) \simeq C_{q+1}(\Sigma X)$ for all $q \in \mathbb{Z}$. The resulting diagrams

$$\begin{array}{ccc}
\tilde{C}_q(X) & \xrightarrow{\partial_q} & \tilde{C}_{q-1}(X) \\
\simeq\downarrow & & \downarrow\simeq \\
\tilde{C}_{q+1}(\Sigma X) & \xrightarrow{\partial_{q+1}} & \tilde{C}_q(\Sigma X)
\end{array}$$

are not strictly commutative but they do commute up to a sign -1: this is a matter of orientation, and precisely the point discussed in the proof of Proposition 25.1. Thus if one would like to see a chain isomorphism from $C_\bullet(X)$ to $C_\bullet(\Sigma X)$ one would not only have to shift indices but also change the differentials of one complex by a factor -1. We do not bother to do so since such factors clearly have no effect on homology: we thus may safely conclude that isomorphisms $\sigma_q\colon H_q\big(\tilde{C}_\bullet(X)\big) \simeq H_{q+1}\big(\tilde{C}_\bullet(\Sigma X)\big)$ are induced as claimed.

Two points remain to be verified: that for a given space X the isomorphism $\sigma_q\colon \tilde{H}_q(X) \simeq \tilde{H}_{q+1}(\Sigma X)$ is defined independently of the chosen cell structure on X, and secondly that σ_q is indeed natural with respect to morphisms in $\mathbf{Top}^{\mathbf{cel},\circ}$. Following the ideas of Section 25 both are a routine matter.

The suspension isomorphisms are of obvious interest for the calculation of homology, and even where the result is already known they may provide a pretty alternative: for example, starting from $\tilde{H}_0(S^0;\mathsf{k}) \simeq \mathsf{k}$ and $\tilde{H}_q(S^0;\mathsf{k}) = 0$ for $q \neq 0$ repeated use of the isomorphisms

$$\tilde{H}_q(S^n) \simeq \tilde{H}_{q+1}(\Sigma S^n) \simeq \tilde{H}_{q+1}(S^{n+1})$$

renders the determination of

$$\tilde{H}_q(S^n;\mathsf{k}) \simeq \begin{cases} \mathsf{k} & \text{if } q = n \\ 0 & \text{else} \end{cases}$$

a mere formality.

Another application of suspension has already been implicit in the definition of homotopy groups in Section 15. Recall that the addition of homotopy classes in $\pi_n(Y) = [I^n/\partial I^n, Y]^\circ$ is defined in terms of just one of the n coordinates of the cube — by convention the first. In view of $I^n = \Sigma I^{n-1}$ this suggests the generalisation to arbitrary suspensions, and a quick review of Section 15 will confirm the claims contained in the following definition:

27.6 Definition Let $X \in |\mathbf{Top}^{\mathsf{cel},\circ}|$ be a pointed cell space, and $Y \in |\mathbf{Top}^\circ|$ an arbitary pointed space. Then the homotopy addition on the set of homotopy classes $[\Sigma X, Y]^\circ$ is defined by the formula

$$[f] + [g] = [f+g], \quad \text{with } (f+g)[t,x] = \begin{cases} f[2t,x] & \text{for } t \le 1/2, \\ g[2t-1,x] & \text{for } t \ge 1/2. \end{cases}$$

It makes $[\Sigma X, Y]^\circ$ a group; if X itself happens to be a suspension, or if $(Y, 1)$ is a topological group then the structure on $[\Sigma X, Y]^\circ$ is commutative.

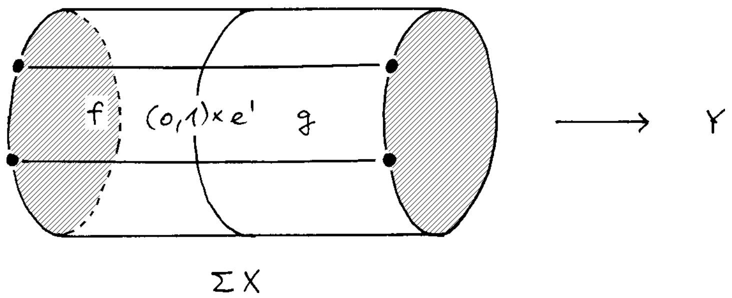
The homotopy addition thus defined turns out to be compatible with the homology group addition, a fact which is by no means self-evident since the original definition of the latter structure was, or at least seemed to be, quite ungeometric.

27.7 Proposition Let X and Y be pointed cell spaces. The map

$$[\Sigma X, Y]^\circ \ni [f] \longmapsto \tilde{H}_q(f) \in \mathrm{Hom}_{\mathsf{k}}\big(\tilde{H}_q(\Sigma X; \mathsf{k}), \tilde{H}_q(Y; \mathsf{k})\big)$$

is a homomorphism of additive groups.

Proof Choose cell structures on X and Y. We must consider two homotopy classes in $[\Sigma X, Y]^\circ$, and may assume they are represented by cellular maps $f, g\colon \Sigma X \longrightarrow Y$: then the homotopy sum $f{+}g$ is cellular too. Computing $(f+g)_*$ on the chain level we obtain for every pair of cells $e = (0,1)\times e' \in E_q(\Sigma X)$ and $d \in E_q(Y)$ that $(f+g)_{de}$ is the sum $f_{de}+g_{de}$ in the homotopy group $[D^q/S^{q-1}, D^q/S^{q-1}]^\circ$.



This implies the proposition: $(f+g)_* = f_*+g_*$.

The simplest example of interest is given by the choice $X = I^n/\partial I^n = \Sigma(I^{n-1}/\partial I^{n-1})$ with $n > 0$, and $\mathsf{k} = \mathbb{Z}$. In view of $H_n(I^n/\partial I^n; \mathbb{Z}) = \mathbb{Z}$ we obtain a homomorphism of abelian groups

$$\pi_n(Y) = [I^n/\partial I^n, Y]^\circ \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\big(H_n(I^n/\partial I^n; \mathbb{Z}), H_n(Y; \mathbb{Z})\big) = H_n(Y; \mathbb{Z})$$

called the *Hurewicz homomorphism*. Under certain (very restrictive) conditions it is known to an isomorphism by a classical result of Hurewicz.

27.8 Question The Hurewicz map can be expressed in terms of 23.4, which was previously used as a technical tool, and assigns to certain mappings $h\colon D^n \longrightarrow X$ the q-chain $[h] \in C_n(X; \mathbb{Z})$. Explain the details.

In principle the suspension isomorphisms allow to give a geometric interpretation of the additive structure of all homology groups. Let me briefly sketch a situation where this fact is put to good use.

27.9 Application Let X and A be the cell spaces underlying a cell complex X and a pointed subcomplex $A \subset X$. Assume that a map $f: X \longrightarrow X$ is given which restricts to the identity map of A. For each $q \in \mathbb{Z}$ the situation produces a commutative diagram

$$\begin{array}{ccc} \tilde{H}_q(X) & \xrightarrow{f_*-1} & \tilde{H}_q(X) \\ \downarrow & \overset{\mathrm{var}_q(f)}{\nearrow} & \downarrow \\ \tilde{H}_q(X/A) & \xrightarrow[\overline{f}_*-1]{} & \tilde{H}_q(X/A) \end{array}$$

where $\overline{f}: X/A \longrightarrow X/A$ is the induced map and 1 stands for the identity homomorphism. The dotted arrow is, so far, but an intuitive idea: in some sense the difference f_*-1 should be trivial on A, so one might hope for a homomorphism $\tilde{H}_q(X/A) \longrightarrow \tilde{H}_q(X)$ that fits into the diagram. Such a homomorphism, to be called the variation of f, would determine both f_*-1 and $\overline{f}_*-1$ by composition and could therefore be thought of as a more meaningful homological representation of f than either f_* or $\overline{f}_*$.

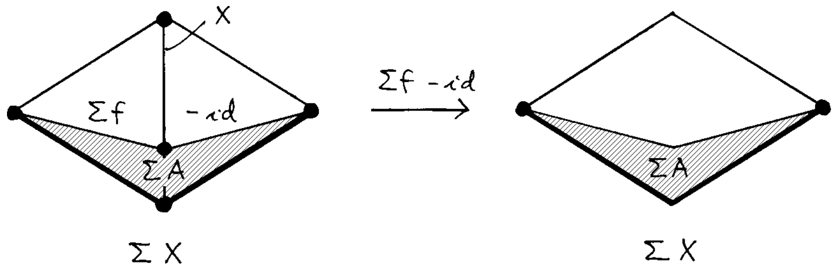
While it is possible to construct the variation homomorphism by going back to the chain level the suspension isomorphims provide an elegant alternative. Indeed they transform the diagram above into an equivalent one involving but suspensions:

$$\begin{array}{ccc} \tilde{H}_{q+1}(\Sigma X) & \xrightarrow{f_*-1} & \tilde{H}_{q+1}(\Sigma X) \\ \downarrow & \nearrow & \downarrow \\ \tilde{H}_{q+1}\big(\Sigma(X/A)\big) & \xrightarrow[\overline{f}_*-1]{} & \tilde{H}_{q+1}\big(\Sigma(X/A)\big) \end{array}$$

In this new context the variation homomorphism can be defined geometrically. For the difference

$$\Sigma f - \mathrm{id}: \Sigma(X/A) \longrightarrow \Sigma(X/A)$$

makes sense as the homotopy sum of Σf and the map that inverts the suspension coordinate.



In view of the canonical homeomorphism $\Sigma(X/A) = \Sigma X/\Sigma A$ the map $\Sigma f - \mathrm{id}$ may be restricted to a self-map of ΣA, and this map is null homotopic. If $h: I \times \Sigma A \longrightarrow \Sigma A$ is a null homotopy then $\Sigma f - \mathrm{id}$ and h constitute a homotopy extension problem with respect to the inclusion $\Sigma A \subset \Sigma X$. A solution $H: I \times \Sigma X \longrightarrow \Sigma X$ provides a pointed map H_1 which is homotopic to $\Sigma f - \mathrm{id}$ and constant on ΣA, so it drops to a map $\overline{H}_1: X/A \longrightarrow X$. The variation

$$\mathrm{var}_q(f): \tilde{H}_q(X/A) \longrightarrow \tilde{H}_q(X)$$

finally is defined as the desuspension of $(\overline{H}_1)_*: \tilde{H}_{q+1}\big(\Sigma(X/A)\big) \longrightarrow \tilde{H}_{q+1}(\Sigma X)$.

28 Exact Sequences

The formal properties of homology modules are most conveniently phrased in terms of exact sequences, a purly algebraic that I will explain first. An arbitrary base ring k is understood throughout.

28.1 Definition A sequence $(V_q, f_q)_{q\in\mathbb{Z}}$ of k-modules and linear maps

$$\cdots \longrightarrow V_{q+1} \xrightarrow{f_{q+1}} V_q \xrightarrow{f_q} V_{q-1} \xrightarrow{f_{q-1}} V_{q-2} \longrightarrow \cdots$$

is called exact if it has the property

$$f_{q+1}(V_{q+1}) = \ker f_q \quad \text{for all } q \in \mathbb{Z}.$$

Yes, it is formally true that an exact sequence is the same as a chain complex with vanishing homology. Nevertheless one would not usually think of an exact sequence as a graded object, and as you will shortly see such sequences often arise naturally with a different kind of indexing system. Nor are the homomorphisms that make up an exact sequence referred to as differentials as they would in the context of a chain complex.

28.2 Question Make sure you are aware of the following simple facts, which are often used: in the exact sequence above

- f_q is the zero map if and only if
- f_{q+1} is surjective, or again if and only if
- f_{q-1} is injective.

In particular $V_{q-2} = 0$ implies surjectivity of f_q, and $V_{q+1} = 0$, injectivity.

Exact sequences with just three non-trivial terms play a special role.

28.3 Definition A short exact sequence is an exact sequence $(V_q, f_q)_{q\in\mathbb{Z}}$ with $V_q = 0$ for all q with $|q| > 1$:

$$\cdots \longrightarrow 0 \longrightarrow V_1 \xrightarrow{f_1} V_0 \xrightarrow{f_0} V_{-1} \longrightarrow 0 \longrightarrow \cdots$$

For such sequences a shortened and index free notiation like

$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$$

usually is preferred.

Let us analyze the concrete meaning of such a short exact sequence. Exactness at U and W means that f is injective while g is a surjection. In view of exactness at V we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \xrightarrow{f} & V & \xrightarrow{g} & W & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow\simeq & & \| & & \simeq\downarrow{\scriptstyle g^{-1}} & & \\
0 & \longrightarrow & f(U) & \hookrightarrow & V & \xrightarrow{g} & V/f(U) & \longrightarrow & 0
\end{array}$$

which, in the obvious sense, constitutes an isomorphism between the two exact sequences at the top and the bottom. Thus a short exact sequence essentially means a pair consisting of a module and either a sub or, equivalently, a quotient module. While this may seem to make the notion of short exact sequence redundant the fact that U need to be a subset in V, nor W a quotient set of V is what makes the language of exact sequences a more flexible and often better choice.

The notion of exactness is quite formal: it makes sense in other categories, so one can speak of exact sequences in $\mathbf{Ch}_k$ rather than $\mathbf{Lin}_k$. In every case the exact sequences in one category themselves are the objects of a new category, with morphisms supplied by commutative ladder type diagrams as in the case of chain complexes described in 24.2.

The following algebraic construction is basic.

28.4 Theorem and **Terminology** Let

$$0 \longrightarrow U_\bullet \xrightarrow{\quad f \quad} V_\bullet \xrightarrow{\quad g \quad} W_\bullet \longrightarrow 0$$

be a short exact sequence of chain complexes. Then there is a natural (long) exact sequence

$$\begin{array}{l} \ldots \xrightarrow{\partial_{q+1}} H_q(U_\bullet) \xrightarrow{\quad f_* \quad} H_q(V_\bullet) \xrightarrow{\quad g_* \quad} H_q(W_\bullet) \xrightarrow{\partial_q} \\ \xrightarrow{\partial_q} H_{q-1}(U_\bullet) \xrightarrow{\quad f_* \quad} H_{q-1}(V_\bullet) \xrightarrow{\quad g_* \quad} H_{q-1}(W_\bullet) \xrightarrow{\partial_{q-1}} \quad \ldots \end{array}$$

of homology modules and linear maps, the corresponding homology sequence. It contains the maps ∂_q called the connecting homomorphisms: they are defined in terms of the differentials of $V_\bullet$ by the formula

$$\partial_q[z] = [f_{q-1}^{-1}\partial_q g_q^{-1} z].$$

The exact homology sequence is functorial with respect to morphisms of short exact sequences.

Proof The central point is the construction of the connecting homomorphism ∂_q, that is, the correct interpretation of the symbolic inverses in the formula given for it. We first write out the relevant part of the sequence $U_\bullet \longrightarrow V_\bullet \longrightarrow W_\bullet$, suppressing some indices for better readability:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U_{q+1} & \xrightarrow{\quad f \quad} & V_{q+1} & \xrightarrow{\quad g \quad} & W_{q+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
0 & \longrightarrow & U_q & \xrightarrow{\quad f \quad} & V_q & \xrightarrow{\quad g_q \quad} & W_q & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial_q} & & \downarrow{\scriptstyle \partial} & & \\
0 & \longrightarrow & U_{q-1} & \xrightarrow{\quad f_{q-1} \quad} & V_{q-1} & \xrightarrow{\quad g \quad} & W_{q-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
0 & \longrightarrow & U_{q-2} & \xrightarrow{\quad f \quad} & V_{q-2} & \xrightarrow{\quad g \quad} & W_{q-2} & \longrightarrow & 0
\end{array}$$

Remembering that the rows of the diagram are exact we now can realise the formula for ∂_q, arguing as follows. A homology class in $H_q(W_\bullet)$ is represented by a cycle $z \in W_q$. Since g is surjective we find a $y \in V_q$ such that $g_q y = z$. By commutativity of the middle right hand square, $g\partial_q y = \partial g_q y = \partial z = 0$, so by exactness we can choose a chain $x \in U_{q-1}$ with $f_{q-1}x = \partial_q y$. This chain is in fact a cycle, for by commutativity of the lower left hand square we have $f\partial x = \partial f_{q-1} x = \partial\partial_q y = 0$, and f is injective. Therefore $[x] \in H_{q-1}(U_\bullet)$ is a homology class.

Of course this definition of $\partial_q\colon H_q(W_\bullet) \longrightarrow H_{q-1}(U_\bullet)$ involves several arbitrary choices, and we must verify that none of them affects the final outcome.

• The last choice made, that of $x \in U_{q-1}$ with given $f_{q-1}x = \partial_q y$ involves no ambiguity since f is injective.

• The choice of $y \in V_q$ with given $g_q y = z$ is ambiguous up to an element fu for some $u \in U_q$. The commutative middle left hand square of the diagram shows that $f_{q-1}^{-1}\partial_q fu = \partial u$, so that the ambiguity maps to a boundary in U_{q-1}.

• Finally the choice of $z \in [z]$ is not unique but may be changed by a boundary in W_q, say ∂w for some $w \in W_{q+1}$. The lifting $g_q^{-1}\partial w$ is realised by ∂v for any $v \in V_{q-1}$ with $gv = w$, and then $\partial_q g_q^{-1}\partial w = \partial_q \partial v$ vanishes.

This proves that the connecting homomorphism is well-defined. — Functoriality of the homology sequence means that given a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U_\bullet & \xrightarrow{\;f\;} & V_\bullet & \xrightarrow{\;g\;} & W_\bullet & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle v} & & \big\downarrow{\scriptstyle w} & & \\
0 & \longrightarrow & U'_\bullet & \xrightarrow{\;f'\;} & V'_\bullet & \xrightarrow{\;g'\;} & W'_\bullet & \longrightarrow & 0
\end{array}$$

the induced ladder diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H_q(U_\bullet) & \xrightarrow{\;f_*\;} & H_q(V_\bullet) & \xrightarrow{\;g_*\;} & H_q(W_\bullet) & \xrightarrow{\;\partial_q\;} & H_{q-1}(U_\bullet) & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle u_*} & & \big\downarrow{\scriptstyle v_*} & & \big\downarrow{\scriptstyle w_*} & & \big\downarrow{\scriptstyle u_*} & & \\
\cdots & \longrightarrow & H_q(U'_\bullet) & \xrightarrow{\;f'_*\;} & H_q(V'_\bullet) & \xrightarrow{\;g'_*\;} & H_q(W'_\bullet) & \xrightarrow{\;\partial_q\;} & H_{q-1}(U'_\bullet) & \longrightarrow & \cdots
\end{array}$$

commutes. Commutativity is clear for the left hand and middles squares, and follows for the third one from

$$\begin{aligned}
\partial_q w_*[z] &= [(f')^{-1}\partial {g'}^{-1} wz] = [(f')^{-1}\partial v g^{-1} z] \\
&= [(f')^{-1} v \partial g^{-1} z] = [u f^{-1} \partial g^{-1} z] = u_* \partial_q [z].
\end{aligned}$$

It remains to prove exactness of the long sequence: given its (logical, not factual) periodicity it comprises six single statements. The verification is easy, and left to the reader.

Remarks If you wish to deal with symbolic inverses of linear maps in a more formal setting, you should consider so-called *linear correspondences*. A linear correspondence from V to W is given by its graph, which may be any linear subspace $\Gamma \subset V \times W$. Just as with the graph of a mapping, kernel and image of the second projection $\Gamma \longrightarrow W$ measure the degree of injectivity and surjectivity. Correspondences differ from mappings in that the first projection $\Gamma \longrightarrow V$ need not be bijective: its kernel and image measure the degree of well-definedness (in the sense of a mapping). Linear correspondences may not only be composed but also inverted, simply by swapping V and W. — The type of argument employed to construct the connecting homomorphism, and needed to prove exactness is known as *diagram chasing*. It can be quite amusing and may even fascinate beginners, but not for long.

We shall apply Theorem 28.4 to two situations which come from topology and are quite simple. The first concerns a pointed or unpointed cell complex X and a subcomplex $A \subset X$. Note that the cells of the quotient complex X/A are, apart from the base point, precisely the cells of X that do not belong to A: therefore the inclusion $j: A \hookrightarrow X$ and the quotient map $p: X \longrightarrow X/A$ induce a short exact sequence

$$0 \longrightarrow \tilde{C}_\bullet(A) \xrightarrow{\;j_*\;} \tilde{C}_\bullet(X) \xrightarrow{\;p_*\;} \tilde{C}_\bullet(X/A) \longrightarrow 0.$$

From 28.4 we obtain at once:

28.5 Theorem and **Terminology** Let X and A be the cell spaces underlying a cell complex X and a pointed subcomplex $A \subset X$. Then the maps $A \overset{j}{\hookrightarrow} X \overset{p}{\to} X/A$ induce a natural exact sequence

$$\cdots \longrightarrow \tilde{H}_q(A) \xrightarrow{j_*} \tilde{H}_q(X) \xrightarrow{p_*} \tilde{H}_q(X/A) \xrightarrow{\partial_q}$$

$$\xrightarrow{\partial_q} \tilde{H}_{q-1}(A) \xrightarrow{j_*} \tilde{H}_{q-1}(X) \xrightarrow{p_*} \tilde{H}_{q-1}(X/A) \longrightarrow \cdots$$

called the reduced homology sequence of the pair (X, A).

Replacing (X, A) by (X^+, A^+) does not change the quotient space $X^+/A^+ = X/A$ and gives an unreduced version:

28.6 Corollary Let X and A be the cell spaces underlying a cell complex X and a subcomplex $A \subset X$. Then the maps $A \overset{j}{\hookrightarrow} X \overset{p}{\to} X/A$ induce a natural unreduced exact homology sequence of the pair (X, A):

$$\cdots \longrightarrow H_q(A) \xrightarrow{j_*} H_q(X) \xrightarrow{p_*} \tilde{H}_q(X/A) \xrightarrow{\partial_q}$$

$$\xrightarrow{\partial_q} H_{q-1}(A) \xrightarrow{j_*} H_{q-1}(X) \xrightarrow{p_*} \tilde{H}_{q-1}(X/A) \longrightarrow \cdots$$

Note that the tilde on the quotient term persists. For both versions naturality means, of course, that the homomorphisms in the sequence define natural transformations of functors defined on a category whose objects are topological pairs (X, A).

28.7 Example Assume that the differerence $X \backslash A$ consists of a single n-cell e, so that X in turn is obtained from A by attaching D^n via an attaching map $\varphi: S^{n-1} \longrightarrow A^{n-1}$. Since we know

$$\tilde{H}_q(X/A; \mathsf{k}) \simeq \tilde{H}_q(D^n/S^{n-1}; \mathsf{k}) = \begin{cases} \mathsf{k} & \text{for } q = n \\ 0 & \text{else} \end{cases}$$

the exact sequence of the pair (X, A) decomposes into the isomorphisms

$$0 \longrightarrow H_q(A; \mathsf{k}) \xrightarrow{j_*} H_q(X; \mathsf{k}) \longrightarrow 0$$

for $n-1 \neq q \neq n$, and one longer section:

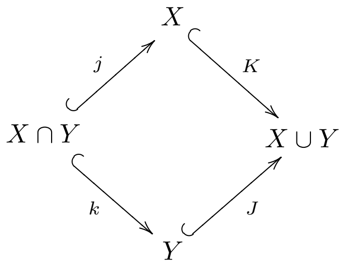
$$0 \longrightarrow H_n(A; \mathsf{k}) \xrightarrow{j_*} H_n(X; \mathsf{k}) \longrightarrow \mathsf{k} \longrightarrow H_{n-1}(A; \mathsf{k}) \xrightarrow{j_*} H_{n-1}(X; \mathsf{k}) \longrightarrow 0$$

Comparing the homology of A and X, we see that the attaching process can only affect homology in degrees n and $n-1$. It may create homology of degree n in the sense that $H_n(j)$ need not be surjective, or it may destroy homology of degree $n-1$ in the sense that $H_{n-1}(j)$ need not be injective.

28.8 Question Illustrate each case explicitly. Can both H_n and H_{n-1} change at the same time? Can both remain unchanged?

The exact sequence as displayed in the example may or may not be sufficient for a particular application. For more precise information the unlabelled arrows must be determined. If need be, this can be done on the level of chains, and you are invited to verify that the homomorphism $H_n(X; \mathsf{k}) \longrightarrow \mathsf{k}$ sends every n-cycle to the coefficient of e in it while $\mathsf{k} \longrightarrow H_{n-1}(A; \mathsf{k})$ may be identified with the composition $\tilde{H}_{n-1}(S^{n-1}; \mathsf{k}) \hookrightarrow H_{n-1}(S^{n-1}; \mathsf{k}) \xrightarrow{\varphi_*} H_{n-1}(A; \mathsf{k})$.

A second interesting type of short exact sequence arises from a so-called *triad*, a cell complex which is given as the union of two subcomplexes X and Y:

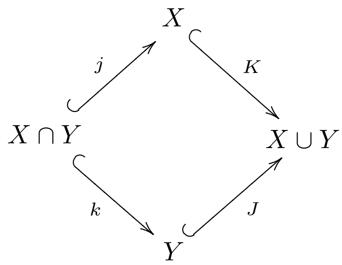


The sequence runs

$$0 \longrightarrow C_\bullet(X\cap Y) \xrightarrow{(j_*,k_*)} C_\bullet(X)\oplus C_\bullet(Y) \xrightarrow{K_*-J_*} C_\bullet(X\cup Y) \longrightarrow 0$$

and its exactness is immediate (note that the minus sign is essential, though in the literature there seems to be no agreement as to where it should go). Once more the application of Theorem 28.4 is straightforward:

28.9 Theorem and **Terminology** Consider the cell spaces X and Y underlying a triad



of cell complexes. There is a natural exact sequence

$$\cdots \longrightarrow H_q(X\cap Y) \xrightarrow{(j_*,k_*)} H_q(X)\oplus H_q(Y) \xrightarrow{K_*-J_*} H_q(X\cup Y) \xrightarrow{\partial_q}$$

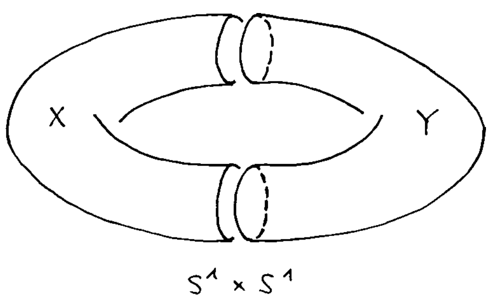
$$\xrightarrow{\partial_q} H_{q-1}(X\cap Y) \xrightarrow{(j_*,k_*)} H_{q-1}(X)\oplus H_{q-1}(Y) \xrightarrow{K_*-J_*} H_{q-1}(X\cup Y) \longrightarrow \cdots$$

called the Mayer-Vietoris sequence of the triad. In case the triad is pointed by a base point in $X\cap Y$ there is a similar sequence with reduced homology throughout.

28.10 Examples (1) We cut the torus $S^1\times S^1$ in two halves, writing

$$S^1\times S^1 = X\cup Y$$

with $X := [0,\pi]\times S^1$ and $Y := [\pi,2\pi]\times S^1$.



The homology modules in the corresponding Mayer-Vietoris sequence

$$0 \longrightarrow H_2(S^1\times S^1) \xrightarrow{\partial_2}$$

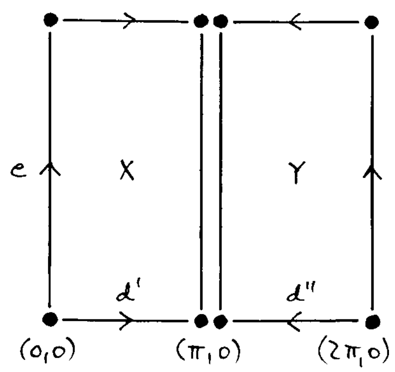
$$\xrightarrow{\partial_2} H_1\big(\{0,\pi\}\times S^1\big) \xrightarrow{(j_*,k_*)} H_1([0,\pi]\times S^1)\oplus H_q([\pi,2\pi]\times S^1) \xrightarrow{K_*-J_*} H_1(S^1\times S^1) \xrightarrow{\partial_1}$$

$$\xrightarrow{\partial_1} H_0\big(\{0,\pi\}\times S^1\big) \xrightarrow{(j_*,k_*)} H_0([0,\pi]\times S^1)\oplus H_0([\pi,2\pi]\times S^1) \xrightarrow{K_*-J_*} H_0(S^1\times S^1) \longrightarrow 0$$

are all free over the base ring k, and bases are singled out if one puts the obvious cell structures on X and Y. Likewise the matrices of the arrows may be determined, going through the definitions step by step. As an example let us compute the 2×2-matrix that describes the boundary homomorphism

$$H_1(S^1\times S^1) \xrightarrow{\partial_1} H_0\big(\{0,\pi\}\times S^1\big).$$

The generators of $H_1(S^1\times S^1)$ are represented by the cycles $d := (0,1)\times\{0\}$ and $e := \{0\}\times(0,1)$:



Let us first deal with d: cccording to the definition of the connecting homomorphisms we must write d as a difference of two chains d' in X and d'' in Y, say

$$d' = (0,\pi)\times\{0\} \quad\text{and}\quad d'' = (2\pi,\pi)\times\{0\}$$

where the peculiar notation $(2\pi,\pi)$ is used to indicate opposite orientation of the interval. The theory then guarantees that $\partial d'$ and $\partial d''$ are one and the same chain, and in fact a cycle in the intersection $\{0,1\}\times S^1$: indeed clearly

$$\partial d' = \{\pi\}\times\{0\} - \{0\}\times\{0\} = \{\pi\}\times\{0\} - \{2\pi\}\times\{0\} = \partial d''.$$

This determines the image of d, and thus the first column of the matrix. With e the situation is different since $e = e - 0$ already is an admissible representation of e as a difference, and so the connecting homomorphism sends the cycle e to its boundary $\partial e = 0$.

In this way the Mayer-Vietoris sequence of our triad can be completely evaluated; an amusing if somewhat lengthy exercise, which results in:

$$0 \longrightarrow \mathsf{k} \xrightarrow{\left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right]}$$

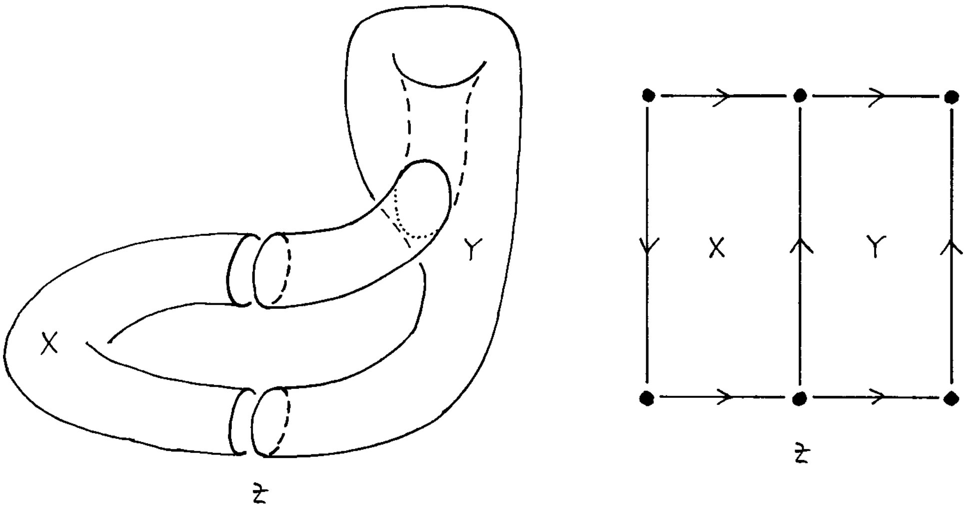
$$\longrightarrow \mathsf{k}\oplus\mathsf{k} \xrightarrow{\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]} \mathsf{k}\oplus\mathsf{k} \xrightarrow{\left[\begin{smallmatrix}0&0\\1&-1\end{smallmatrix}\right]} \mathsf{k}\oplus\mathsf{k} \xrightarrow{\left[\begin{smallmatrix}-1&0\\1&0\end{smallmatrix}\right]}$$

$$\longrightarrow \mathsf{k}\oplus\mathsf{k} \xrightarrow{\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]} \mathsf{k}\oplus\mathsf{k} \xrightarrow{\left[\begin{smallmatrix}1&-1\end{smallmatrix}\right]} \mathsf{k} \longrightarrow 0$$

(2) Let us, now that we have dissected the torus glue the two pieces $X = [0,\pi]\times S^1$ and $Y = [\pi,2\pi]\times S^1$ back together, but with a flip, making the identifications

$$X \ni (\pi, e^{it}) \sim (\pi, e^{it}) \in Y \quad \text{and} \quad X \ni (0, e^{-it}) \sim (2\pi, e^{it}) \in Y.$$

While the resulting space Z, a non-orientable surface called the *Klein bottle*, cannot be embedded in $\mathbb{R}^3$ the figure



gives a good idea of it if you are willing to ignore the artificial self-intersection. It is an easy matter to compute the homology of Z from a cell structure, but instead let us try and see whether the same information can be extracted from the Mayer-Vietoris sequence of the triad defined by the subcomplexes $X, Y \subset Z$. Using the same method as above, and the obvious fact that $H_0(Z;\mathsf{k}) = \mathsf{k}$ the sequence can be determined as far as this:

$$0 \longrightarrow H_2(Z;\mathsf{k}) \longrightarrow$$

$$\longrightarrow \mathsf{k}\oplus\mathsf{k} \xrightarrow{\left[\begin{smallmatrix}-1 & 1\\ 1 & 1\end{smallmatrix}\right]} \mathsf{k}\oplus\mathsf{k} \longrightarrow H_1(Z;\mathsf{k}) \longrightarrow$$

$$\longrightarrow \mathsf{k}\oplus\mathsf{k} \xrightarrow{\left[\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right]} \mathsf{k}\oplus\mathsf{k} \xrightarrow{[\,1\ -1\,]} \mathsf{k} \longrightarrow 0$$

The final result depends on the nature of the base ring, and let us now specialize to the case $\mathsf{k} = \mathbb{Z}$. Then

$$H_2(Z;\mathbb{Z}) = \ker \begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix} = 0$$

while for $H_1(Z)$ we obtain the short exact sequence

$$0 \longrightarrow (\mathbb{Z}\oplus\mathbb{Z})/\left(\mathbb{Z}\begin{bmatrix}-1\\ 1\end{bmatrix} + \mathbb{Z}\begin{bmatrix}1\\ 1\end{bmatrix}\right) \longrightarrow H_1(Z;\mathbb{Z}) \longrightarrow \ker\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} \longrightarrow 0,$$

which is isomorphic to

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow H_1(Z;\mathbb{Z}) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

In general it is not possible to reconstruct the centre term of a short exact sequence from the other two. Nevertheless in this case it is, up to isomorphism of course, since the quotient term is a free module: pick any element $z \in H_1(Z;\mathbb{Z})$ that maps to $1 \in \mathbb{Z}$, then z spans a submodule which projects isomorphically onto $\mathbb{Z}$, and therefore defines an isomorphism $H_1(Z;\mathbb{Z}) \simeq \mathbb{Z}/2 \oplus \mathbb{Z}$.

In the calculus of homology the Mayer-Vietoris sequence is a powerful, and probably the most popular tool. It is closely interrelated with other basic properties of homology that we already know: as an illustration we will now show how some of them can in turn be formally derived from Mayer-Vietoris sequences. Note that it does not matter whether one starts from the reduced or unreduced version of the sequence since Proposition 27.4 allows to pass from one to the other.

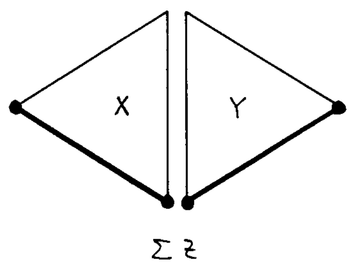
As is immediately clear from the definitions homology commutes with categorical sums: $H_q(X+Y;\mathsf{k})$ is the same as $H_q(X;\mathsf{k}) \oplus H_q(Y;\mathsf{k})$ up to natural equivalence of functors $\mathbf{Top^{cel}} \longrightarrow \mathbf{Lin_k}$. Alternatively this can be proved from the even simpler fact that the empty space has trivial homology, using a Mayer-Vietoris sequence: that of the triad corresponding to the partition $X+Y = X \cup Y$ reduces to a series of isomorphisms

$$H_q(X) \xrightarrow{K_*-J_*} H_q(X+Y)$$

because the intersection term $H_q(X \cap Y) = H_q(\emptyset)$ vanishes for every $q \in \mathbb{Z}$.

A more interesting observation is that the suspension isomorphisms of Theorem 27.5 can be obtained from a Mayer-Vietoris sequence. To this purpose we write the reduced suspension of a pointed cell space Z as the union $\Sigma Z = X \cup Y$ of two subspaces

$$X = \big([0,\tfrac{1}{2}]\times Z\big)/\big(\{0\}\times Z \,\cup\, [0,\tfrac{1}{2}]\times\{\circ\}\big) \quad\text{and}\quad Y = \big([\tfrac{1}{2},1]\times Z\big)/\big(\{1\}\times Z \,\cup\, [\tfrac{1}{2},1]\times\{\circ\}\big).$$



Note that each summand is a homeomorphic copy of the *reduced cone*

$$CZ = \big([0,1]\times Z\big)/\big(\{0\}\times Z \,\cup\, [0,1]\times\{\circ\}\big)$$

of Z, which in turn coincides with the reduced mapping cone[1] $C(\mathrm{id}_Z)$ of the identity map of Z. In particular X and Y are contractible pointed cell spaces, and homotopy invariance therefore implies

$$\tilde{H}_q(X) = \tilde{H}_q\{\circ\} = \tilde{H}_q(\emptyset^+) = H_q(\emptyset) = 0, \text{ and similarly } \tilde{H}_q(Y) = 0.$$

Thus the direct sum terms in the Mayer-Vietoris sequence vanish, and we conclude that the connecting homomorphism

$$\tilde{H}_{q+1}(\Sigma Z) \xrightarrow{\partial_q} \tilde{H}_q(X \cap Y)$$

is an isomorphism for each $q \in \mathbb{Z}$. This proves our claim since $X \cap Y = \{\tfrac{1}{2}\}\times Z$ may be identified with Z.

The resulting equivalence of functors $\partial_q\colon \tilde{H}_{q+1}(\Sigma\,?) \longrightarrow \tilde{H}_q$ turns out to be inverse to the suspension isomorphism σ_q of Theorem 27.4. Indeed, putting a cell structure on Z we can represent every homology class of $\tilde{H}_{q+1}(\Sigma Z)$ in the form

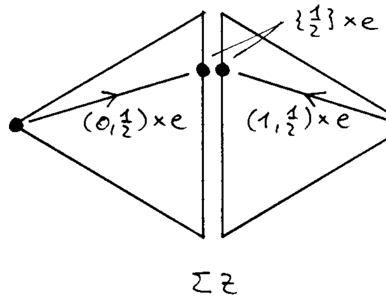
$$\sum_e \lambda_e\cdot(0,1)\times e \in C_{q+1}(Z)$$

for some cycle $\sum_e \lambda_e\cdot e \in C_q(Z)$, and the value of the connecting homomorphism obtained from the splitting

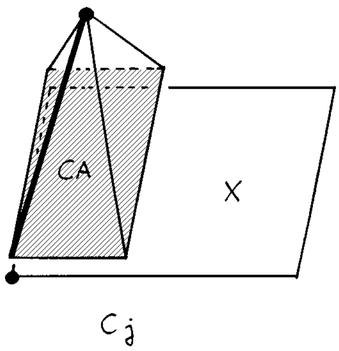
$$\sum_e \lambda_e\cdot(0,1)\times e = \sum_e \lambda_e\cdot\big(0,\tfrac{1}{2}\big)\times e - \textstyle\sum_e \lambda_e\cdot\big(1,\tfrac{1}{2}\big)\times e$$

is the class of $\sum_e \lambda_e\cdot\{\tfrac{1}{2}\}\times e$.

[1] Mapping cones were introduced and studied in Problems 24 and 26



Finally the exact homology sequence associated with an inclusion $j\colon A \hookrightarrow X$ of pointed complexes also may be interpreted as a special Mayer-Vietoris sequence. Recalling that the mapping cone of that inclusion is the quotient of $I{\times}A+X$ with respect to the identifications $I{\times}A \ni (1,x) \sim x \in X$ we write the reduced mapping cone Cj as the union $Cj = X \cup CA$. Then $X \cap CA = A$.



In the associated Mayer-Vietoris sequence

$$\cdots \longrightarrow \tilde{H}_q(A) \longrightarrow \tilde{H}_q(X) \oplus \tilde{H}_q(CA) \longrightarrow \tilde{H}_q(Cj) \xrightarrow{\partial_q}$$

$$\xrightarrow{\partial_q} \tilde{H}_{q-1}(A) \longrightarrow \tilde{H}_{q-1}(X) \oplus \tilde{H}_{q-1}(CA) \longrightarrow \tilde{H}_{q-1}(Cj) \longrightarrow \cdots$$

the homology of the cone CA vanishes, and since the quotient mapping

$$p\colon Cj \longrightarrow Cj/CA = X/A$$

is a pointed homology equivalence we may substitute X/A for Cj:

$$\begin{array}{ccccccc}
\cdots \longrightarrow \tilde{H}_q(X) \oplus 0 & \longrightarrow & \tilde{H}_q(Cj) & \xrightarrow{\partial_q} & \tilde{H}_{q-1}(A) \longrightarrow \cdots \\
& \searrow & \simeq \big\downarrow p_* & \nearrow & \\
& & \tilde{H}_q(X/A) & &
\end{array}$$

The resulting sequence is easily seen to coincide with the exact sequence of the pair (X,A), in the sense of Theorem 28.5.

29 Synopsis of Homology

The purpose of this section is twofold: Firstly I would like to give an overview of further developments that could not be accomodated in this course but could have been treated in the same fashion. Secondly I would like to very briefly describe alternative approaches to homology, and discuss some of their advantages and drawbacks. Some of these approaches are widely used but require a somewhat different setup, and this creates the need for a dictionary between the present course and a terminology which is considered the standard one. Finally we shall have just a glimpse of very substantial generalisations of homology.

29.1 Cohomology There is a contravariant version of homology which is constructed by dualizing chain complexes. If $V_\bullet$ is a chain complex over k then to obtain its *dual complex* $V^\bullet$ the module V_q is replaced by $V^q := \mathrm{Hom}_{\mathsf{k}}(V_q, \mathsf{k})$, and the definition of

$$\cdots \longrightarrow V^{q-1} \xrightarrow{\delta_{q-1}} V^q \xrightarrow{\delta_q} V^{q+1} \longrightarrow \cdots$$

is completed using the transposed linear mappings: $\delta_q := (-1)^q(\partial_{q+1})^{\mathrm{t}}$. Re-labelling q as $-q$ would have made the resulting object a chain complex in the sense of Definition 24.2 but it is preferable to accept it as a so-called *cochain complex* over k. In fact from the algebraic point of view the latter have turned out to be the more natural notion, and they are now often considered as the basic version. The difference is, of course, strictly formal, and any result on complexes implies one for cochain complexes, and vice versa. In particular, every cochain complex $V^\bullet$ gives rise to a sequence of homology modules

$$H^q(V^\bullet) = (\ker \delta_q)/\delta_{q-1}(V^{q-1})$$

(*cocycles* modulo *coboundaries*) now renamed *cohomology*.

We are thus able to assign to every cell complex X the cochain complex $C^\bullet(X;\mathsf{k})$ which the dual of $C_\bullet(X;\mathsf{k})$, and further the cohomology modules

$$H^q(X;\mathsf{k}) := H^q\big(C^\bullet(X;\mathsf{k})\big)$$

for all $q \in \mathbb{Z}$. The reasoning familiar from Section 25 shows how to make them homotopy invariant functors from $\mathbf{Top}^{\mathbf{cel}}$ to $\mathbf{Lin}_{\mathsf{k}}$. While unlike homology these functors are contravariant they turn out to convey essentially the same information, and one would ask whether it is worth the extra trouble to define them at all. The answer is affirmative, and will be explained next.

29.2 Products An obvious subject that we have not been able to discuss is the behavier of homology with respect to cartesian products. Let X and Y be cell spaces: the assignment $(d,e) \longmapsto d\times e$ on the level of cells defines bilinear mappings

$$C_p(X;\mathsf{k}) \times C_q(Y;\mathsf{k}) \longrightarrow C_{p+q}(X\times Y;\mathsf{k}),$$

and these can be shown to induce bilinear maps

$$H_p(X;\mathsf{k}) \times H_q(Y;\mathsf{k}) \longrightarrow H_{p+q}(X\times Y;\mathsf{k}) \quad \text{and}$$
$$H^p(X;\mathsf{k}) \times H^q(Y;\mathsf{k}) \longrightarrow H^{p+q}(X\times Y;\mathsf{k})$$

that constitute the so-called *cross products* in homology and cohomology. While they seem perfect twins it is only the second that allows the construction of a product on the cohomology of a single space X. Indeed, composing with the homomorphism Δ^* induced by the diagonal map $\Delta\colon X \longrightarrow X{\times}X$ we obtain a bilinear mapping

$$H^p(X;\mathsf{k}) \times H^q(X;\mathsf{k}) \xrightarrow{\times} H^{p+q}(X{\times}X;\mathsf{k}) \xrightarrow{\Delta^*} H^{p+q}(X;\mathsf{k})$$

called the *cup product*. In this context it is better to rewrite the graded module $\big(H^q(X;\mathsf{k})\big)_{q\in\mathbb{Z}}$ as the direct sum

$$H^*(X;\mathsf{k}) = \bigoplus_{q=-\infty}^{\infty} H^q(X;\mathsf{k}),$$

which under the cup product becomes a ring, or more precisely a graded k-algebra. This algebra is *graded commutative*: two homogeneous elements $x \in H^p(X;\mathsf{k})$ and $y \in H^q(X;\mathsf{k})$ commute or anticommute according to their degrees:

$$yx = (-1)^{pq}xy$$

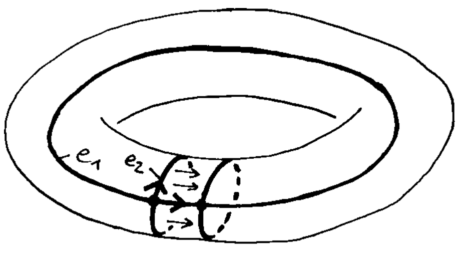
While the cup product is a purely topological notion by definition, its meaning becomes most transparent in the context of differential topology, for if X is a manifold then the cup product on $H^*(X)$ in particular reflects the way submanifolds of X of various dimensions intersect each other. A simple but typical example is the torus $X = S^1{\times}S^1$: its chain complex has vanishing differentials, so that

$$H^0(X;\mathbb{Z}) = H^2(X;\mathbb{Z}) = \mathbb{Z} \quad \text{and} \quad H^1(X;\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z},$$

with respect to bases given by the product cell structure with four cells $\{\circ\}{\times}\{\circ\}$, $e_1 = d{\times}\{\circ\}$, $e_2 = \{\circ\}{\times}d$, and $d{\times}e$. In other words as a graded abelian group $H^*(S^1{\times}S^1;\mathbb{Z})$ may be identified with the algebra

$$\Lambda^*(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \Lambda^2(V)$$

of alternating forms on the lattice $V := \mathbb{Z}e_1{\oplus}\mathbb{Z}e_2$. As it turns out, this identification respects the ring structures too, that is, the cup product corresponds precisely to the wedge product of alternating forms.



$S^1 \times S^1$

The intersections of the generating 1-cycles in $S^1{\times}S^1$ illustrate the commutativity law: e_1 and e_2 in either order intersect in a single point, but switching the factors changes the orientation of the intersection. It is intuitively clear that replacing e_1 and e_2 by other cycles in the same homology class, it is not possible to remove the intersection. By contrast this is clearly the case if, for instance, e_2 is intersected with itself: shift one copy of e_2 along e_1, and it will become disjoint from the other.

Even though cohomology now appears to be superior to homology it does not in general supersede it, and for cell spaces X that are not manifolds both remain important. While homology does not carry a natural internal multiplication there is a so-called *cap product*

$$H^p(X;\mathsf{k}) \times H_{p+q}(X;\mathsf{k}) \longrightarrow H_q(X;\mathsf{k})$$

that makes the homology $H_*(X;\mathsf{k})$ a graded module over $H^*(X;\mathsf{k})$ and thus gives it an equally rich additional structure.

29.3 Axioms Homology (and cohomology) can be characterized by a small set of axioms. Recall that Proposition 27.4 allows to obtain unreduced from reduced homology, and vice versa, in a completely formal way: therefore an axiomatic characterisation of one version will automatically yield one of the other. Nevertheless I will describe axioms for both reduced and unreduced homology, for each has its own merits, and both appear in the literature.

A sequence of functors

$$\tilde{h}_q: \mathbf{Top}^{\mathsf{cel},\circ} \longrightarrow \mathbf{Ab} \quad (q \in \mathbb{Z})$$

and a corresponding sequence of natural transformations

$$\sigma_q: \tilde{h}_q \longrightarrow \tilde{h}_{q+1}(\Sigma\,?)$$

are said to form a *reduced homology theory* if they obey the following axioms:

- each functor $\tilde{h}_q$ is homotopy invariant,
- if $A \subset X$ is a subcomplex with respect to some cell structure then the sequence

  $$\tilde{h}_q(A) \longrightarrow \tilde{h}_q(X) \longrightarrow \tilde{h}_q(X/A)$$

  induced by the inclusion and quotient mappings is exact,
- each σ_q is an equivalence of functors, and
- $\tilde{h}_q(S^0) = 0$ unless $q = 0$.

The unreduced version relies on Mayer-Vietoris sequences rather than suspension isomorphisms: an (unreduced) *homology theory* consists of a sequence of functors

$$h_q: \mathbf{Top}^{\mathsf{cel}} \longrightarrow \mathbf{Ab} \quad (q \in \mathbb{Z})$$

and a corresponding sequence of natural transformations

$$\partial_q(X, Y): h_q(X \cup Y) \longrightarrow h_{q-1}(X \cap Y)$$

which is functorial in cell triads X, Y. The axioms required are:

- each functor h_q is homotopy invariant,
- for every triad X, Y the Mayer-Vietoris sequence

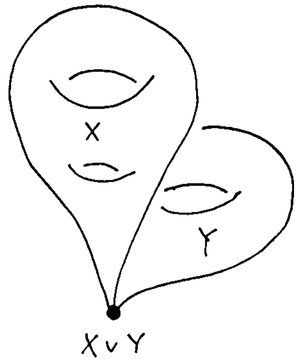
  $$\cdots \longrightarrow h_q(X \cap Y) \longrightarrow h_q(X) \oplus h_q(Y) \longrightarrow h_q(X \cup Y) \xrightarrow{\partial_q(X,Y)} h_{q-1}(X \cup Y) \longrightarrow \cdots$$

  is exact,
- $h_q(\emptyset) = 0$ for all q, and $h_q\{\circ\} = 0$ for all $q \neq 0$.

The abelian group $\mathsf{k} := \tilde{h}_0(S^0)$, repectively $\mathsf{k} := h_0\{\circ\}$ is called the *coefficient* of the homology theory. Our construction of homology in the previous sections may be restated as an existence result: for any given k there exists a homology theory with coefficient k, at least if k has a ring structure. Making this assumption for the sake of simplicity, we also have uniqueness:

Theorem All homology theories $(h_q, \partial_q)_{q\in\mathbb{Z}}$ with given coefficient ring k are functorially equivalent to $(H_q, \partial_q)_{q\in\mathbb{Z}}$ where ∂_q is the connecting homomorphism of 28.9. Likewise all reduced homology theories $(\tilde{h}_q, \sigma_q)$ are functorially equivalent to $(\tilde{H}_q, \sigma_q)_{q\in\mathbb{Z}}$ with σ_q the suspension isomorphism constructed in 27.5.

Let me at least sketch the proof, in the reduced case. The first step is to prove that homology is additive: $\tilde{h}_q(X \vee Y) = \tilde{h}_q(X) \oplus \tilde{h}_q(Y)$, and in particular $\tilde{h}_q\{\circ\} = 0$.



It suffices to observe that the quotient $(X{\vee}Y)/X$ is canonically homeomorphic to Y, and $(X{\vee}Y)/Y$ to X: the first fact supplies an exact sequence

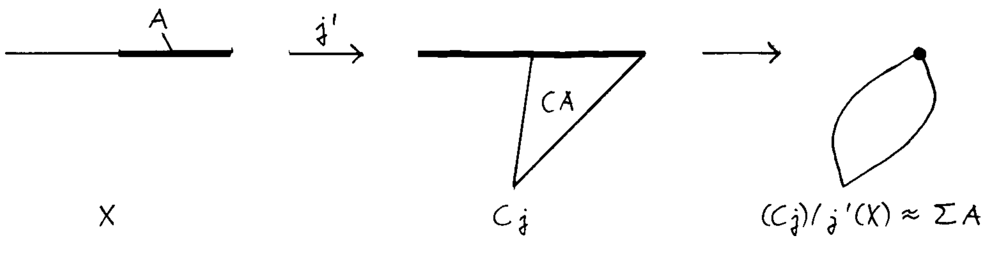
$$\tilde{h}_q(X) \rightleftarrows \tilde{h}_q(X{\vee}Y) \rightleftarrows \tilde{h}_q(Y)$$

and the second the dotted arrows. These show that the sequence is part of a short exact sequence that exhibits $h_q(X{\vee}Y)$ as the direct sum of $h_q(X)$ and $h_q(Y)$.

In the next step the long exact sequence of the pair given by a cell complex X and a subcomplex A is constructed from the exactness axiom. The inclusion $j\colon A \hookrightarrow X$ gives rise to the following diagram involving mapping cones and suspensions[1]:

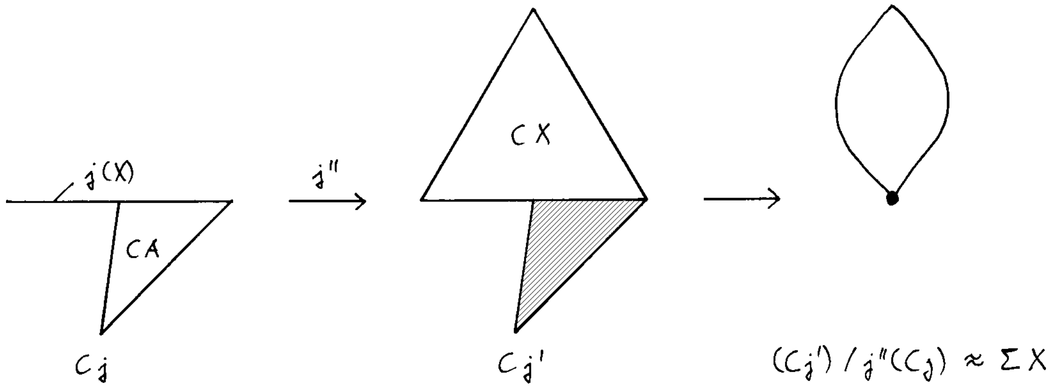
$$\begin{array}{ccccccccc}
\tilde{h}_q(A) & \xrightarrow{j_*} & \tilde{h}_q(X) & \longrightarrow & \tilde{h}_q(X/A) & & & & \\
 & & \| & & \uparrow \simeq & & & & \\
 & & \tilde{h}_q(X) & \xrightarrow{j'_*} & \tilde{h}_q(Cj) & \longrightarrow & \tilde{h}_q(\Sigma A) & & \\
 & & & & \| & & \uparrow \simeq & & \\
 & & & & \tilde{h}_q(Cj) & \xrightarrow{j''_*} & \tilde{h}_q(Cj') & \longrightarrow & \tilde{h}_q(\Sigma X)
\end{array}$$

Recall that the quotient mapping $Cj \longrightarrow X/A$ is a homotopy equivalence. The arrow j' embeds X in Cj sending $x \in X$ to its class in $Cj = (I{\times}A + X)/{\sim}$; then $(Cj)/j'(X)$ is canonically homeomorphic to the reduced suspension ΣA.



Arguing with j' instead of j we similarly obtain the homotopy equivalence $(Cj)/j'(X) \overset{\circ}{\simeq} Cj'$, the embedding $j''\colon Cj \longrightarrow Cj'$, and the homeomorphism $(Cj')/j''(Cj) \approx \Sigma X$.

[1] See Problems 24 and 26 again



The diagram commutes up to sign and has exact rows by a triple application of the exactness axiom. There results an exact sequence

$$\tilde h_q(A) \xrightarrow{j_*} \tilde h_q(X) \longrightarrow \tilde h_q(X/A) \longrightarrow \tilde h_q(\Sigma A) \xrightarrow{(\Sigma j)_*} \tilde h_q(\Sigma X)$$

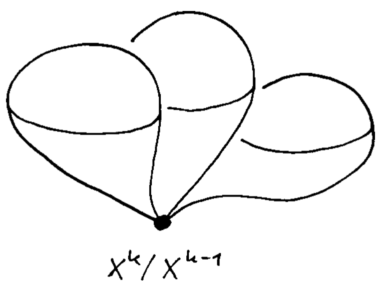
from which the long homology sequence is obtained by desuspension.

Let X be a cell complex, and think of X as being built by successively attaching cells of increasing dimension. We show that for fixed n the group $\tilde h_n(X)$ is only affected by the attachment of n-cells and $(n+1)$-cells:

- $\tilde h_n(X^q) = 0$ for all $q < n$, and
- the inclusion $X^q \hookrightarrow X$ induces an isomorphism $\tilde h_n(X^q) \simeq \tilde h_n(X)$ for all $q > n$.

The proof is by comparing the homology of two consecutive skeletons X^{k-1} and X^k: the quotient X^k/X^{k-1} is a bouquet of k-spheres, and so

$$\tilde h_n(X^k/X^{k-1}) \simeq \bigoplus \tilde h_n(S^k) \simeq \bigoplus \tilde h_n(\Sigma^k S^0) \simeq \bigoplus \tilde h_{n-k}(S^0) = \begin{cases} \bigoplus \mathsf{k} & \text{for } n = k, \\ 0 & \text{for } n \neq k. \end{cases}$$

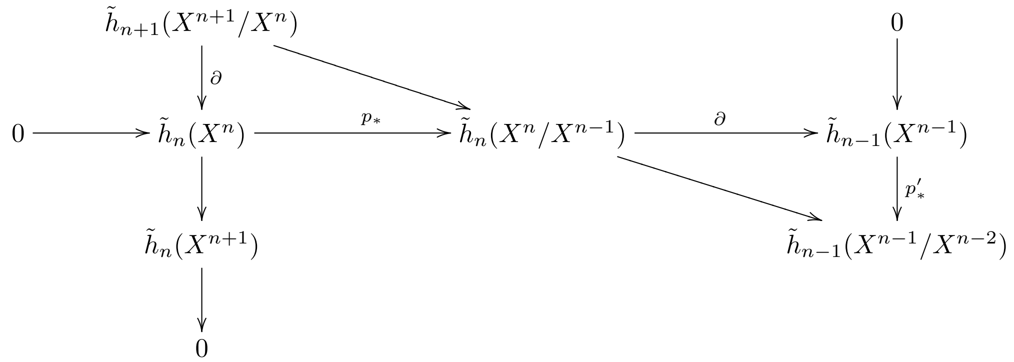


Therefore the exact sequence of $X^{k-1} \subset X^k$ reduces to a series of isomorphisms

$$\tilde h_n(X^{k-1}) \simeq \tilde h_n(X^k) \quad \text{for } n{+}1{<}k, \text{ or } n{>}k.$$

In view of $\tilde h_n(X^0) = 0$ for $n > 0$, and $X^k = X$ for sufficiently large k both claims now follow.

In the final step it is shown that $\tilde h_q(X)$ can be computed from a chain complex in a way which recovers the definition of $\tilde H_q(X; \mathsf{k})$. The relevant groups and homomorphisms appear in the following commutative diagram:



Note that the row and the two columns of this diagram are sections of long exact sequences. We compute

$$\begin{aligned}\tilde h_n(X) &= \tilde h_n(X^{n+1})\\ &\simeq \tilde h_n(X^n) \big/ \partial \tilde h_{n+1}(X^{n+1}/X^n)\\ &\simeq p_*\tilde h_n(X^n) \big/ (p_*\partial)\tilde h_{n+1}(X^{n+1}/X^n)\end{aligned}$$

where the last isomorphism is due to the injectivity of p_*. On the other hand we have

$$p_*\tilde h_n(X^n) = \ker\partial = \ker(p'_*\partial),$$

and therefore $\tilde h_n(X) \simeq \ker(p'_*\partial) \big/ (p_*\partial)\tilde h_{n+1}(X^{n+1}/X^n)$ is canonically isomorphic to the n-th homology group of the complex

$$\cdots \longrightarrow \tilde h_{n+1}(X^{n+1}/X^n) \longrightarrow \tilde h_n(X^n/X^{n-1}) \longrightarrow \tilde h_{n-1}(X^{n-1}/X^{n-2}) \longrightarrow \cdots$$

that can be seen in the diagonal of the diagram. In view of the canonical isomorphism

$$\tilde h_n(X^n/X^{n-1}) \simeq \bigoplus \mathsf{k} = \mathsf{k}^{E_n(X)}$$

the groups of this complex may be identified with those of the cell chain complex $C_\bullet(X;\mathsf{k})$. Finally, analysing the differential of the complex in terms of mapping degrees one will recognise it as the boundary operator of Definition 23.1, and this completes the proof of the theorem. Further details can be found in [Dold] and [tom Dieck].

29.4 Extension The principal limitation of our cell homology is that the cell spaces to which it can be applied are necessarily compact. We will now discuss various ways to extend homology and cohomology to wider categories of topological spaces.

Two approaches stay within the familiar framework. The first is quite straightforward: the reasoning of Section 25 that has shown homology to be independent of a chosen cell structure would likewise allow to define the homology modules of any topological space that is, though not necessarily homeomorphic but at least homotopy equivalent to a cell complex. — Secondly, a wider notion of cell complex may be used, allowing for complexes with infinitely many cells. So-called *CW-complexes* are adequate here, indeed have been specially designed to make the typical arguments used in homotopy theory applicable to that more general class. The definition of cell homology then goes through without change, with but the obvious restrictions on applications: so, a finiteness assumption must be made when dealing with Euler or Lefschetz numbers. As a typical example of an infinite CW-complex you might consider the projective space $\mathbb{R}P^\infty$, which is constructed by iteration of the attaching step that turns $\mathbb{R}P^{n-1}$ into $\mathbb{R}P^n$. The homology of $\mathbb{R}P^\infty$ is determined from its cell chain complex as before and comes out as one would expect, for instance

$$H_q(\mathbb{R}P^\infty;\mathbb{F}_2) \simeq \mathbb{F}_2 \quad \text{for all } q \geq 0.$$

Alternatively the step from our (finite) cell complexes to arbitrary *CW*-complexes may be placed in a categorical framework. The case of $\mathbb{R}P^\infty = \bigcup_{n=0}^\infty \mathbb{R}P^n$ is an example of a *direct limit* in the category **Top**, and quite generally a *CW*-complex may be considered to be the direct limit of its finite subcomplexes. The notion of direct limit likewise applies to the category $\mathbf{Lin}_\mathsf{k}$, and cell homology is compatible with direct limits, so computable in terms of its values on finite complexes: a property that it is useful to add to the list of axioms.

By the forementioned and related methods the domain of homology may be made to comprise most topological spaces of practical interest, among them all differential manifolds, and spaces defined by algebraic or analytic equations and inequalities. Also on this basis it is possible to improve upon the results of Section 20 and treat two classical and quite famous results of topology which we have not had the means (nor the time) to discuss:

Jordan-Brouwer separation theorem If $f: S^{n-1} \longrightarrow S^n$ is an embedding then $S^n \setminus f(S^{n-1})$ has exactly two connected components.

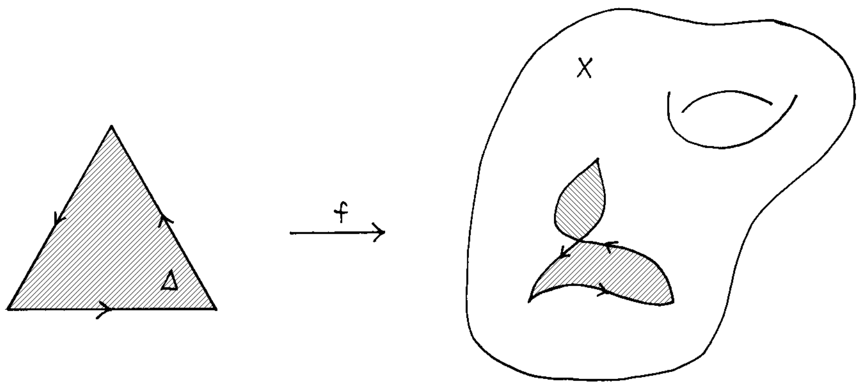
Brouwer's theorem on the invariance of the domain If $f: U^n \longrightarrow \mathbb{R}^n$ is an embedding then $f(U)$ is open in $\mathbb{R}^n$.

Thus homology based on cell structures proves to be perfectly adequate for practical purposes. Nevertheless it would of course be most satisfactory to have homology defined for simply all topological spaces, and the characterisation of homology by axioms makes this possible: now that we know homology can by and large be calculated by skilful use of the axioms we can accept alternative constructions of homology theories that are not computable as such. The best known of such constructions is the so-called *singular homology*.

The idea of singular theories is simple: since there is no hope to find embedded cells or simplices, or other fixed concrete objects in *every* topological space X one replaces them by arbitrary continuous mappings into X. Embedded simplices are thus replaced by maps

$$f: \Delta \longrightarrow X$$

from a closed (standard) simplex into X, called *singular simplices* in X since the image of Δ need, of course, no longer be homeomorphic to Δ.



Given a base ring k one forms the free module which has the set of all singular q-simplices in X as its base. The combinatorial structure of simplices allows to define a differential, thus turning the sequence of these modules into a chain complex over k. The singular homology of the space X is defined as the homology of that complex, and it can be shown to verify the axioms of a homology theory. Note that this definition gives hardly a clue as to how to compute homology: for a typical space X the chain modules are huge, so are the submodules of cycles and boundaries, and it is only the last step — taking homology in the algebraic sense — that cuts everything down to reasonable size.

That singular homology satisfies the axioms of a homology theory on $\mathbf{Top}^{\mathbf{cel}}$ does not mean it satisfies the literal translation of every single axioms to the category **Top**. While the important homotopy axiom poses no problem in this respect the exactness and Mayer-Vietoris axioms do. The reason behind this is that the process of collapsing a subspace, of which we have made extensive use, does not behave well for arbitrary topological spaces. As an example let us look at the following diagram of inclusions:

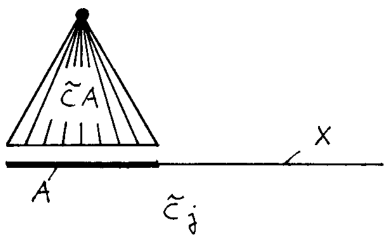
$$\begin{array}{ccc} S^{n-1} & \hookrightarrow & D^n \\ \downarrow & & \| \\ D^n\setminus\{0\} & \overset{j}{\hookrightarrow} & D^n \end{array}$$

If the literal translation of the exactness axiom were true we would have a commutative diagram of homology

$$\begin{array}{ccccccccc} H_q(S^{n-1}) & \longrightarrow & H_q(D^n) & \longrightarrow & \tilde H_q(D^n/S^{n-1}) & \xrightarrow{\partial} & H_{q-1}(S^{n-1}) & \longrightarrow & H_{q-1}(D^n) \\ \downarrow & & \| & & \downarrow & & \downarrow & & \| \\ H_q(D^n\setminus\{0\}) & \longrightarrow & H_q(D^n) & \longrightarrow & \tilde H_q\big(D^n/(D^n\setminus\{0\})\big) & \xrightarrow{\partial} & H_{q-1}(D^n\setminus\{0\}) & \longrightarrow & H_{q-1}(D^n) \end{array}$$

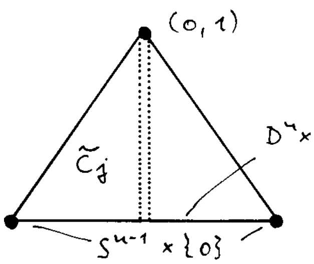
for each q, and its rows would be exact. Since S^{n-1} is a deformation retract of $D^n\setminus\{0\}$ the vertical arrows are isomorphisms, with possible exception of the middle one: then the latter must be isomorphic too, by the five lemma[2]. But this is impossible since the quotient space $D^n/(D^n\setminus\{0\})$ consists of two points, one of which is closed while the other (the base point) is not: unlike D^n/S^{n-1} this quotient is the same for all $n>0$! (You may amuse yourself proving that it is contractible as a pointed space.)

As the example suggests one simply should avoid to collapse non-closed subspaces like $D^n\setminus\{0\}$. Naturally there arises the need for a substitute, and there is one available in form of the mapping cone $\tilde Cj = (I\times A + X)/\sim$ of the inclusion $j\colon A \hookrightarrow X$: attaching the cone $\tilde CA = (I\times A)/(\{0\}\times A)$ to X amounts to making the subspace A contractible.



In our example $\tilde Cj$ can be realised as the subspace

$$\big\{(tx, 1-t) \in \mathbb{R}^n\times\mathbb{R} \,\big|\, t\in I, 0\neq x\in D^n\big\} \;\cup\; D^n\times\{0\}$$



of $\mathbb{R}^{n+1}$, and projection from the point $(0,\frac12)$ defines a deforming retraction of $\tilde Cj$ to the pointed n-sphere

$$\big\{(tx, 1-t) \in \mathbb{R}^n\times\mathbb{R} \,\big|\, t\in I, x\in S^{n-1}\big\} \;\cup\; D^n\times\{0\}.$$

[2] Problem 47

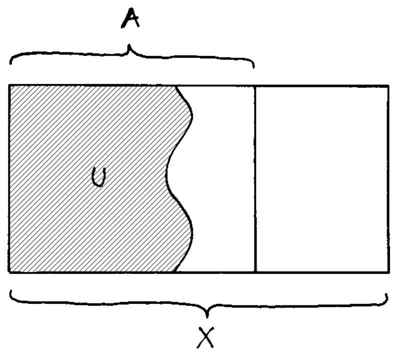
In general singular homology satisfies the exactness axiom if given the inclusion $j: A \hookrightarrow X$ the mapping cone $\tilde{C}j$ is substituted for the quotient X/A. In this context the mapping cone is rarely mentioned though, and the abbreviation

$$H_q(X, A; \mathsf{k}) := \tilde{H}_q(\tilde{C}j; \mathsf{k})$$

is used instead. In fact singular homology allows a direct construction of the so-called *relative homology modules* $H_q(X, A; \mathsf{k})$ using relative singular cycles and boundaries, and by established custom homology is treated as a functor on the category of pairs of topological spaces (X, A) where $A \subset X$ is implicitly understood. Note that by definition $H_q(X, \emptyset) = \tilde{H}_q(X^+) = H_q(X)$, also that $\tilde{C}j$ may be considered as the reduced mapping cone of $j^+: A^+ \hookrightarrow X^+$, and that if j is the inclusion of a cell subcomplex then everything remains as before, in view of the familiar homotopy equivalence $Cj \simeq X/A$.

The Mayer-Vietoris axiom poses a similar problem as it cannot be expected to hold for arbitrary triads: given a topological space Z, every partition $Z = X \cup Y$ of the set Z would otherwise result in isomorphisms $H_q(Z) \simeq H_q(X) \oplus H_q(Y)$ of homology for all q. The axiom can reasonably be imposed only for special triads, which usually are called *excisive*. They include, in particular, the coverings of Z by two open subsets and, as we know, those of a cell complex by two of its subcomplexes. — Most classical texts do not count the Mayer-Vietoris property among the axioms at all but derive it from a so-called *excision axiom* which pertains to the relative homology $H_q(X, A)$: subsets $U \subset A$ which are sufficiently separated from $X \backslash A$ can be excised in the sense that the inclusion of the complement induces isomorphisms

$$H_q(X \backslash U, A \backslash U) \simeq H_q(X, A).$$



29.5 Further realisations of homology The importance of singular homology lies in the fact that it makes sense for all topological spaces. On the other hand alternative constructions of homology theories are often preferred for various reasons. Direct computability may be one, as in the case of cell homology. Another may be that objects considered in a particular context sometimes directly define (co-)homology class if the right realisation of homology is used. As we know, all such realisations — whether defined on the full category **Top** or not — take identical values on finite cell complexes (and on arbitrary *CW*-complexes if they are compatible with limits), but they are likely to differ on more pathological spaces.

One of the simplest constructions of a cohomology theory is based on the *de Rham complex*

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{A}^0(X) \xrightarrow{d} \mathcal{A}^1(X) \xrightarrow{d} \mathcal{A}^2(X) \longrightarrow \cdots$$

of a differential manifold X: this cochain complex links the real vector spaces $\mathcal{A}^q(X)$ of C^∞ differential forms of degree q by the Cartan differential d. The homology of this complex is, by definition, the *de Rham cohomology* of X. It turns out to be a cohomology theory which a priori lives on the category **Dif** of manifolds and C^∞ mappings, but can be extended to a reasonably large category by the familiar technique of homotopic approximation. The coefficient ring of de Rham cohomology clearly is $H^0(X) = \mathbb{R}$, so in particular every closed differential q-form on a manifold X defines a cohomology class in $H^q(X;\mathbb{R})$.

Like singular cohomology, *Čech cohomology* makes sense for arbitrary topological spaces X. It uses open coverings $(X_\lambda)_{\lambda\in\Lambda}$ of X, and the construction of $H^q(X)$ is based on an analysis of the way any $q+1$ of the covering sets X_λ intersect. The a priori dependence on the open covering (X_λ) of X is eliminated by taking a direct limit over all such coverings. Čech cohomology not only allows arbitrary coefficient rings but, much more generally, coefficient *sheaves*: quite naively speaking, this notion allows to make the coefficient ring not a constant but an entity that may vary over the space X. For algebraists Čech cohomology is but one out of many realisations of sheaf cohomology, and emphasis is usually put on its dependence on the coefficient sheaf rather than on the underlying space. In algebraic geometry sheaf cohomology is as basic and important a tool as geometric homology is in topology.

In the literature you will also encounter *homology with closed carriers*, or more frequently its dual *cohomology with compact carriers*, or *compact support*. The reference is not to yet another construction but to a variant of (usually singular) cohomology written H_c^q that is functorial with respect to proper maps. In fact, on the category $\mathbf{Top}^{\mathbf{pr}}$ of locally compact Hausdorff spaces and proper mappings it coincides with reduced cohomology of the Alexandroff compactification:

$$H_c^q(Z) = \tilde{H}^q(\hat{Z})$$

If, in particular $Z = X\backslash A$ is a difference of finite cell complexes then $\hat{Z} = X/A$ and therefore

$$H_c^q(X\backslash A) = \tilde{H}^q(X/A).$$

Cohomology with compact carriers seems to have the wrong kind of variance when applied to the inclusion of one locally compact space in another: review the proof of Theorem 20.4, and you will understand why.

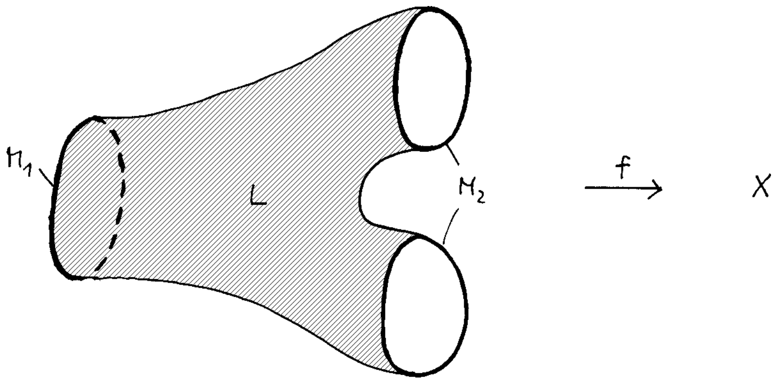
Note Singular *homology* with compact carriers is not a new notion: when occasionally singular homology is said to have compact carriers this is just to emphasize the fact that the homology of an arbitrary space is the direct limit of the homology of its compact subspaces, an axiom satisfied by many natural constructions of homology.

29.6 Generalised homology Apart from describing different realisations of (co-)homology, something which after all essentially is always the same object, one might look for new functors that behave like homology as much as possible but provide additional geometric insight. As is clear from the very construction of cell homology in Sections 23 and 24, homology of a cell complex is based on an analysis of the cell complex by layers, more precisely, of the way the attaching maps of the q-cells interact with the $(q-1)$-cells. On the other hand the discussion in 29.3 has shown that this property is not so much a consequence of that particular construction but one intrinsic to homology as characterised by the axioms.

What we know to be an asset for the purpose of calculation is, on the other hand, a conceptual limitation. In fact there exist interesting *generalized (co-)homology theories* which satisfy all the familiar axioms but the last, the "dimension" axiom that would confine the coefficient of homology, that is the homology of the one-point space, to pure degree zero. While such theories largely behave like the *ordinary homology theories* discussed so far some arguments are bound to fail. In particular it is no longer possible to compute the value of a generalised homology functor h_* on a cell space X from a cell chain complex (only a certain approximation of $h_*(X)$ can thus be determined in general). — Generalised homology and cohomology theories are systematically constructed from so-called *spectra*, by a method that generalises the definition of homotopy groups. Let me here rather present two generalised theories that do not require much construction as they arise naturally, each in quite a different way, and both very beautiful.

Like ordinary singular homology *bordism theories* test a topological space by sending compact spaces into it: not simplices this time but compact differential manifolds without boundary. The vastness of the set of all such "singular" manifolds for a given space X is at once reduced by passing to bordism classes: $M_1 \to X$ is declared *bordant* to $M_2 \to X$ if there exist a compact manifold L with boundary

$\partial L = M_1 + M_2$, and a map $f\colon L \longrightarrow X$ that restricts to the given maps on M_1 and M_2. Thus M_1 and M_2 must necessarily have the same dimension q while L is $(q{+}1)$-dimensional.



The resulting set of equivalence classes becomes an abelian group under the simplest of all addition rules that can be imagined: the sum of the classes $[M_1 \to X]$ and $[M_2 \to X]$ is represented by the sum map $M_1 + M_2 \to X$, and the definition of the cross product

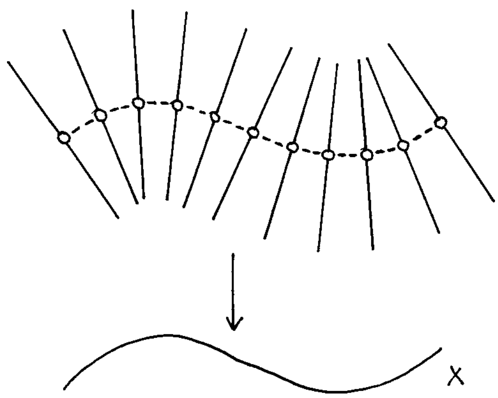
$$[M \to X] \times [N \to Y] := [M \times N \longrightarrow X \times Y]$$

is equally simple. It is a non-trivial fact that the resulting groups of bordism classes do define a generalised homology theory, which is written $N_*(X)$. To compute its coefficient $N_*\{\circ\}$ amounts to the classification of all compact manifolds without boundary up to bordism: a formidable task, which nevertheless has been achieved. $N_q\{\circ\}$ turns out to be non-zero for infinitely many $q \in \mathbb{N}$.

Question Explain why $N_*(X)$ consists of pure 2-torsion: $2x = 0$ for all $x \in N_*(X)$.

Just one of the many pleasing properties of bordism theory can be mentioned here: if X itself is a q-dimensional compact differential manifold without boundary then the identity of X represents a bordism class $[X] \in N_q(X)$, called the *fundamental class* of X. While definitions of fundamental classes exist in other homology theories too none is as simple as this one. — The notion of bordism can be enhanced by considering manifolds $M \to X$ with an additional structure on M, an orientation for example: in that particular case the equation $\partial L = M_1 + M_2$ that establishes the bordism relation would change to $\partial L = (-M_1) + M_2$, and the corresponding oriented bordism theory Ω_* does no longer consist of mere torsion. Oriented bordism has implicitly played a role in this course, as you will gather from another look at the proofs of Theorem 18.5 and Proposition 23.5.

A completely different approach leads to a generalised cohomology theory known as *K-theory*. Its starting point is the notion of *vector bundle* over a topological space X: roughly speaking, a continuous family of, say complex vector spaces parametrized by X (the tangent bundle of a differential manifold is an example).



To vector bundles over a fixed space X the standard constructions of linear algebra apply, in particular the notions of direct sums, tensor products, and duals of vector bundles make sense. The set of all isomorphism classes of vector bundles over X thus becomes a semi-ring under direct sum and tensor product, and by formally adjoining additive inverses there results a ring called $K^0(X)$. Without much difficulty the functor K^0 is seen to be homotopy invariant, and its reduced version to satisfy the exactness axiom at least on a reasonably large subcategory of **Top**. On the other hand no clue as to a grading seems to be in sight. At this point a famous and deep result, the *Bott periodicity theorem* comes in. It supplies a natural isomorphism of modules over $K^0\{\circ\} = \mathbb{Z}$, between $\tilde{K}^0(X)$ and $\tilde{K}^0(\Sigma^2 X)$. A generalised cohomology theory $(K^q)_{q\in\mathbb{Z}}$ can now be built defining

$$\tilde{K}^q(X) := \begin{cases} \tilde{K}^0(X) & \text{for even } q \text{ and} \\ \tilde{K}^0(\Sigma X) & \text{in case of odd } q, \end{cases}$$

since the necessary suspension isomorphism $\tilde{K}^q(X) \simeq \tilde{K}^{q+1}(\Sigma X)$ is provided by the periodicity theorem if q is even, while for odd q it is the identity.

As the definition suggests K-theory is the natural cohomology theory to use whenever one has to do with vector bundles, since every (complex) vector bundle over X, which need be well-defined only up to isomorphism, directly defines an element in $K^0(X)$. Such situations frequently arise in the study of elliptic partial differential equations.

The group $\tilde{K}^1(S^1)$ turns out to be trivial; therefore the coefficient ring of K-theory vanishes in odd degree:

$$K^*\{\circ\} = \bigoplus_{q\equiv 0 \bmod 2} K^q\{\circ\} = \bigoplus_{q\equiv 0 \bmod 2} \mathbb{Z}$$

More appropriately one should formally modify the axioms of homology, allowing the sequence of homology functors to be indexed by an abelian group other than $\mathbb{Z}$: as a $\mathbb{Z}/2$-indexed cohomology the homogeneous decomposition of K-theory simply is

$$K^*(X) = K^0(X) \oplus K^1(X)$$

and its coefficient ring $K^*(X) = \mathbb{Z}$ has pure degree zero. This ironically suggests that K-theory might be but a coarser version of integral cohomology obtained by reducing the degree to a question of even or odd — a guess that turns out to be quite off the mark. Firstly and surprisingly, integral cohomology (including the grading) can, up to torsion, be recovered from K-theory. Secondly, and this is much more interesting, the torsion part of K-theory carries interesting new information and has been successfully applied to a variety of geometric problems that are not, or are not known to be, accessible by other means.

The Hopf invariant mentioned at the end of Section 20 is an integer assigned to maps from S^{4k-1} to S^{2k} for any $k \geq 1$. Using K-theory F. Adams has shown in 1960 that the Hopf invariant cannot be even unless k is equal to 1, 2, or 4. This has a surprising application to finite dimensional division algebras over $\mathbb{R}$ (algebras without zero divisors which need not be commutative nor associative): the dimension of such an algebra as a real vector space must be 1, 2, 4, or 8, that is, one of the dimensions realised by the classical examples $\mathbb{R}$ itself, $\mathbb{C}$, the quaternion algebra $\mathbb{H}$, and the Cayley numbers. While other proofs of this theorem of algebra are known none of them is purely algebraic. — A related question in topology is about vector fields on spheres. Here again a deep result is due to F. Adams: using more K-theory he has determined for each sphere S^n the maximal number of vector fields on it which are linearly independent at every point.

Exercises: Aufgaben

Aufgabe 1 Man konstruiere Homöomorphismen

$$D^n \approx [-1,1]^n \quad \text{und} \quad U^n \approx (-1,1)^n.$$

Aufgabe 2 Man zeige allgemeiner: Ist $U \subset \mathbb{R}^n$ eine offene Menge, die bezüglich eines ihrer Punkte sternförmig ist, so ist U homöomorph zu U^n (das ist einfacher, wenn man U zusätzlich als beschränkt voraussetzt). Ist $D \subset \mathbb{R}^n$ eine kompakte Menge, die bezüglich eines jeden Punktes einer nicht-leeren offenen Kugel $U \subset D$ sternförmig ist, so ist D homöomorph zu D^n. Genügt es dabei, anstelle der Kompaktheit nur die Abgeschlossenheit von D vorauszusetzen?

Aufgabe 3 Sei $h: S^{n-1} \longrightarrow S^{n-1}$ ein Homöomorphismus. Man konstruiere einen Homöomorphismus $H: D^n \longrightarrow D^n$ mit

$$H(x) = h(x) \quad \text{für alle } x \in S^{n-1}.$$

(Warum wäre es nicht korrekt, $H|S^{n-1} = h$ zu schreiben, obwohl das wahrscheinlich nicht mißverstanden würde?)

Aufgabe 4 Beweisen Sie, daß die Abbildung

$$f: \mathbb{R} \longrightarrow S^1 \times S^1; \quad t \mapsto (e^{it}, e^{ict})$$

für irrationales $c \in \mathbb{R}$ stetig und injektiv, aber keine Einbettung ist.

Aufgabe 5 Sei $B \subset \mathbb{R}^n$ eine nicht-leere Teilmenge, und sei $d_B: \mathbb{R}^n \longrightarrow \mathbb{R}$ durch

$$d_B(x) := \inf \big\{ |x-y| \,\big|\, y \in B \big\}$$

definiert. Zeigen Sie, daß d_B stetig ist und daß

$$d_B(x) = 0 \iff x \in \overline{B}$$

gilt.

Aufgabe 6 Welche Abbildungen von einem topologischen Raum X in einen Klumpenraum Y sind stetig? — Begründen Sie: Ist X eine Menge, Y ein topologischer Raum und $f: X \longrightarrow Y$ eine Abbildung, so gibt es eine eindeutig bestimmte gröbste Topologie auf X, die f stetig macht. Erklären Sie, wieso die Unterraumtopologie ein Beispiel dafür ist.

Aufgabe 7 Abstrakter als $\mathbb{R}^n$, aber konkreter als der Begriff des topologischen Raums ist der eines *metrischen Raums* X: hier wird nicht der Umgebungs- oder Offenheitsbegriff, sondern der des Abstandes $d: X\times X \longrightarrow [0,\infty)$ axiomatisiert:

- $d(x,y) = d(y,x)$ immer,
- $d(x,y)=0 \iff x=y$, und
- $d(x,z) \leq d(x,y) + d(y,z)$ (Dreiecksungleichung)

Erklären Sie, in welchem Sinne jeder metrische Raum auch ein topologischer Raum ist. Finden Sie interessante Beispiele, in denen die Elemente von X Abbildungen zwischen fest gegebenen Mengen sind.

Aufgabe 8 Beweisen Sie, daß die Gruppe $SO(n)$ zusammenhängt, daß dagegen $O(n)$ für $n > 0$ zwei Zusammenhangskomponenten hat.

Tip Mindestens zwei Ansätze führen problemlos zum Ziel: entweder kann man mit rein geometrischen Überlegungen die Dimension abbauen und induktiv vorgehen, oder man kann sich auf bekannte Klassifikationssätze der linearen Algebra stützen.

Aufgabe 9 Der in der Literatur übliche Zusammenhangsbegriff nennt einen topologischen Raum X dann zusammenhängend, wenn es außer $\emptyset$ und X keine weiteren Teilmengen von X gibt, die zugleich offen und abgeschlossen sind. Um Mißverständnisse zu vermeiden, wollen wir das in dieser Aufgabe *schwach zusammenhängend* nennen. Zeigen Sie einige der folgenden Dinge:

- Zusammenhang impliziert schwachen Zusammenhang,
- aber nicht umgekehrt.
- Beide Begriffe stimmen bei lokal zusammenhängenden Räumen X überein,
- und auch dann, wenn $X \subset \mathbb{R}$ ist.

Übrigens wird dieser schwache Zusammenhangsbegriff gerade dann gerne verwendet, wenn er zu dem anderen äquivalent ist. Zum Beispiel beweist man den globalen Identitätssatz für analytische Funktionen gerne nach diesem Muster: Statt direkt "für alle x gilt $f(x) = g(x)$" zu beweisen, überlegt man sich, daß $\{x \,|\, T_x f = T_x g\}$ sowohl offen als auch abgeschlossen ist, und schließt dann ...

Aufgabe 10 Sei $f: X \longrightarrow Y$ eine Abbildung zwischen Hausdorff-Räumen. Der Punkt $a \in X$ besitze eine abzählbare Umgebungsbasis. Zeigen Sie, daß unter diesen Voraussetzungen das bekannte Folgenkriterium gilt: f ist bei a stetig, genau wenn für jede Folge $(x_n)_{n\in\mathbb{N}}$ mit $\lim\limits_{n\to\infty} x_n = a$ auch $\lim\limits_{n\to\infty} f(x_n) = f(a)$ gilt.

Aufgabe 11 Normalität topologischer Räume vererbt sich nicht ohne weiteres auf Unterräume; klar ist nur, daß jeder abgeschlossene Teilraum eines normalen Raums selbst normal ist (warum?). Das kann ein Grund sein, sich für einen etwas stärkeren Begriff zu interessieren: X heißt *vollständig normal*, wenn es zu je zwei Mengen $A, B \subset X$ mit $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ Umgebungen U von A und V von B mit $U \cap V = \emptyset$ gibt. Beweisen Sie:

- Jeder metrische Raum ist vollständig normal, und
- vollständige Normalität vererbt sich auf beliebige Teilräume.

Insbesondere ist also nicht nur $\mathbb{R}^n$ selbst, sondern auch jeder Teilraum von $\mathbb{R}^n$ normal.

Aufgabe 12 Sei X ein topologischer Raum und L ein nicht-leerer kompakter Raum; $f: X \times L \longrightarrow \mathbb{R}$ sei eine stetige Funktion. Man zeige: Dann ist

$$F: X \longrightarrow \mathbb{R}; \quad F(x) := \max\{f(x,y) \mid y \in L\}$$

nicht nur wohldefiniert, sondern auch stetig.

Aufgabe 13 Sei X ein lokal kompakter Hausdorff-Raum, und sei als Menge $\hat{X} := \{\infty\} + X$. Beweisen Sie: Wenn man als die offenen Teilmengen von $\hat{X}$

- alle offenen Teilmengen von X und
- alle Mengen der Form $\{\infty\} \cup X \setminus K$ mit kompaktem $K \subset X$

erklärt, dann wird $\hat{X}$ zu einem kompakten Hausdorff-Raum, der $X \subset \hat{X}$ als offenen Teilraum enthält. Man nennt $\hat{X}$ die *Alexandroff-* oder *Ein-Punkt-Kompaktifizierung* von X.

Aufgabe 14 Zeigen Sie, daß die in Aufgabe 13 beschriebene Topologie durch die genannten Eigenschaften ($\hat{X}$ kompakter Hausdorff-Raum, $X \subset \hat{X}$ offener Teilraum) eindeutig bestimmt ist. Präzisieren Sie: Homöomorphe (lokal kompakte Hausdorff-) Räume haben auch homöomorphe Alexandroff-Kompaktifizierungen. Was erhält man für $X = [0,1)$, für $X = (0,1)$ oder weitere Beispiele Ihrer Wahl? Gibt $\hat{X}$ eigentlich auch Sinn, wenn X schon kompakt oder gar leer ist?

Aufgabe 15 Im folgenden werden vier topologische Räume W, X, Y und Z beschrieben, deren zugrundeliegende Menge immer $\mathbb{R}^2$ ist:

- W ist $\mathbb{R}^2$ mit der gewöhnlichen Topologie.
- $\mathbb{R}^2$ wird vermöge der Abbildung

  $$[0,\infty) \times S^1 \ni (t,x) \longmapsto tx \in \mathbb{R}^2$$

  als Quotient von $[0,\infty) \times S^1$ aufgefaßt, und $X = \mathbb{R}^2$ erhält die Quotienttopologie.
- $Y = \mathbb{R}^2$ wird als Quotient des Summenraums $\sum_{S^1} [0,\infty)$ unter

  $$\sum_{x \in S^1} [0,\infty) \ni (x,t) \longmapsto tx \in \mathbb{R}^2$$

  aufgefaßt.
- $Z = \mathbb{R}^2$ als Menge, aber mit der durch die SNCF-Metrik

  $$d(x,y) := \begin{cases} |x-y| & \text{wenn } (x,y) \text{ linear abhängig} \\ |x|+|y| & \text{sonst} \end{cases}$$

  definierten Topologie.

Erklären Sie den Namen *SNCF-Metrik*. Vergleichen Sie einige dieser Topologien auf $\mathbb{R}^2$ oder alle vier miteinander.

Antwort Es ist $W = X$, die Räume von Y und Z sind einander sehr ähnlich (ein Bündel von an den Köpfen zusammenhängenden Stecknadeln), aber doch auf subtile Weise verschieden: die Topologie von Y ist echt feiner als die von Z, und diese wieder (viel) feiner als die von $W = X$.

Aufgabe 16 Sei X ein kompakter und Y ein hausdorffscher Raum. Zeigen Sie: Jede surjektive stetige Abbildung $f\colon X \longrightarrow Y$ ist eine Identifizierungsabbildung.

Aufgabe 17 Eine einfache, aber wichtige Quotientbildung ist das *Zusammenschlagen eines Teilraums zu einem Punkt*, zu Recht so genannt, weil ein eher brutaler Vorgang: Sei X ein topologischer Raum und $A \subset X$ ein nicht-leerer Teilraum. Durch

$$x \sim y \quad :\Longleftrightarrow \quad x = y \text{ oder } \{x, y\} \subset A$$

wird eine Äquivalenzrelation auf X und damit ein Quotientraum bestimmt, den man mit X/A bezeichnet.

Zeigen Sie:

- X/A kann nur dann hausdorffsch sein, wenn $A \subset X$ abgeschlossen ist.
- Ist A abgeschlossen und X ein normaler Raum, so ist X/A hausdorffsch.

Bemerkung Man definiert auch $X/\emptyset$, hält sich dabei aber ausnahmsweise nicht an die formale, sondern die verbale Definition: so wie oben der Teilraum $A \subset X$ zum Punkt $\{A\} \in X/A$ zusammengeschlagen wird, schlägt man hier $\emptyset$ zum Punkt $\{\emptyset\} \in X/\emptyset$ zusammen, indem man $X/\emptyset := \big\{\{\emptyset\}\big\} + X$ vereinbart. Die Inklusion $X \hookrightarrow X/\emptyset$ spielt dann die Rolle der Quotientabbildung, und das nicht mal schlecht (inwiefern?), obwohl sie natürlich nicht surjektiv ist.

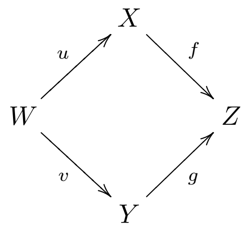
Aufgabe 18 Konstruieren Sie Homöomorphismen $\mathbb{R}^n/D^n \approx \mathbb{R}^n$ und $D^n/S^{n-1} \approx S^n$.

Aufgabe 19 Präzisieren und beweisen Sie: Wenn man die beiden Volltori $D^2 \times S^1$ und $S^1 \times D^2$ auf die naheliegende Weise längs $S^1 \times S^1$ miteinander verklebt, erhält man eine 3-Sphäre.

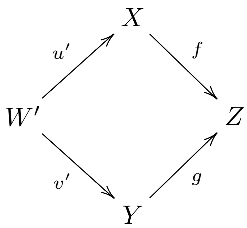
Tip $S^3 \subset \mathbb{C}^2$. Übrigens kann man sich die Sphäre S^3 recht gut vorstellen, etwa als D^3/S^2 oder — hier besonders günstig — als die Alexandroff-Kompaktifizierung von $\mathbb{R}^3$ (siehe Aufgaben 13/14).

Aufgabe 20 Wenn Sie Interesse und etwas Zeit haben, können Sie den Abschnitt über die projektiven Räume als Quotienten gut ergänzen, indem Sie sich allgemeiner mit topologischen Gruppenaktionen vertraut machen (auch Transformationsgruppen genannt). Grundbegriffe finden Sie in [Jänich] 3.4 und 3.5, [Armstrong] 4.3 und 4.4, [tom Dieck] I.5 (schon anspruchsvoller) sowie in den einführenden Abschnitten der Bücher, die diesem Thema ganz gewidmet sind: [Bredon: *Introduction to Compact Transformation Groups* (1972)] und [tom Dieck: *Transformation Groups* (1987)].

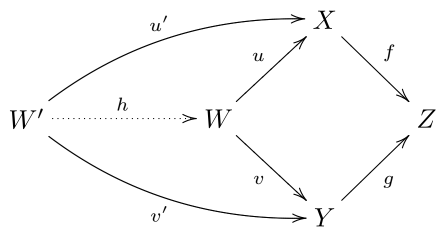
Aufgabe 21 Sei C eine Kategorie. Ein kommutatives Diagramm



in C heißt ein *Pullback-Diagramm* oder *kartesisches Quadrat*, wenn gilt: Sind $u'\colon W' \longrightarrow X$ und $v'\colon W' \longrightarrow Y$ beliebige Morphismen in C, die das Diagramm



kommutieren lassen, so gibt es genau einen Morphismus $h\colon W' \longrightarrow W$, der das gesamte Diagramm



kommutativ macht.

Präzisieren und beweisen Sie die zu Lemma 12.4 analoge Aussage, daß jedes kartesische Quadrat durch die Vorgabe von $X \xrightarrow{f} Z \xleftarrow{g} Y$ im wesentlichen eindeutig bestimmt ist.

Aufgabe 22 Zeigen Sie: Ist $C = \mathbf{Top}$ (oder auch eine von vielen anderen Kategorien, zum Beispiel $C = \mathbf{Lin}_K$), dann läßt sich jedes vorgegebene Diagramm $X \longrightarrow Z \longleftarrow Y$ in C durch geeignete Wahl von $X \longleftarrow W \longrightarrow Y$ zu einem kartesischen Diagramm ergänzen. Was ergibt sich speziell, wenn $Y = \{\circ\}$ nur aus einem Punkt besteht?

Aufgabe 23 Sei X ein kompakter Hausdorff-Raum und $1 \leq k \in \mathbb{N}$. Der Raum Y entstehe aus X durch Anheften einer k-Zelle. Zeigen Sie:

- Wenn X zusammenhängt, dann ist auch Y zusammenhängend.
- Ist $k \geq 2$, so gilt auch die Umkehrung.

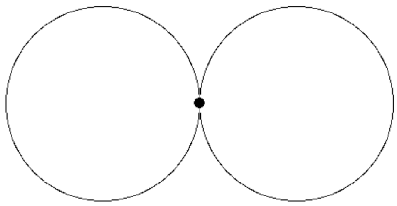
Bemerkung In Wirklichkeit braucht X dafür weder kompakt noch hausdorffsch zu sein; diese Voraussetzungen dienen nur der Bequemlichkeit.

Aufgabe 24 Sei X ein endlicher Zellenkomplex. Überlegen Sie, was man wohl unter einem *Unterkomplex* $S \subset X$ von X versteht. Erklären Sie, in welchem Sinne der Raum X/S (siehe Aufgabe 17) dann wieder ein Zellenkomplex ist.

Exercises: Problems

Problem 1 The *antipodal map* of S^n sends $x \in S^n$ to $-x \in S^n$. Prove that this map is homotopic to the identity provided n is odd.

Problem 2 Let X be the topological space obtained by removing one point p from the torus $S^1 \times S^1$. Show that X is homotopy equivalent to the one point union of two circles:



Problem 3 Let X be a compact Hausdorff space (this for the sake of simplicity), and let $\varphi_0, \varphi_1 \colon S^{n-1} \longrightarrow X$ be continuous maps which are homotopic to each other. Prove that $D^n \cup_{\varphi_0} X$ and $D^n \cup_{\varphi_1} X$ have the same homotopy type.

Problem 4 Prove the claim made about the comb Y of Example 14.13(3): although Y is a contractible space the pointed space $(Y, 0)$ is not contractible in the category $\mathbf{Top}^\circ$.

Problem 5 Let $f \colon D^2 \longrightarrow D^2$ be defined in polar coordinates by

$$f(r, \varphi) = (r, \varphi + 2\pi r).$$

Explain why f is continuous, and hence a morphism in $\mathbf{Top}^{S^1}$. Prove that f is homotopic to the identity relative S^1.

Problem 5 Give a precise definition of the *one point union* of a family $(X_\lambda)_{\lambda \in \Lambda}$ of pointed topological spaces X_λ. It is usually written

$$\bigvee_{\lambda \in \Lambda} X_\lambda \quad (\text{or } X_1 \vee X_2 \vee \cdots \vee X_r \text{ in case of finite } \Lambda = \{1, \ldots, r\})$$

and also called the *bouquet* or just the *wedge* of these spaces. Prove that the bouquet is the sum of the given spaces in the category $\mathbf{Top}^\circ$, in the sense of Proposition 6.8. Is it true that each X_λ is a subspace of $\bigvee_{\lambda \in \Lambda} X_\lambda$?

Problem 7 Consider the compact annulus $X := D_2(0) \setminus U_1(0) \subset \mathbb{R}^2$ and define $f: X \longrightarrow X$ in polar coordinates by

$$f(r, \varphi) = (r, \varphi + 2\pi r).$$

Prove that f is not homotopic to the identity relative the boundary $S_2(0) \cup S_1(0) \subset \mathbb{R}^2$ of X. Use the fact — to be proven in Section 18 — that the identity map of the 1-sphere $I^1/\partial I^1$ represents a non-trivial element of the fundamental group $\pi_1(I^1/\partial I^1, \circ)$.

Problem 8 A *dunce's cap*[1] X is obtained from a triangle, say from

$$T = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ x+y \leq 1\}$$

by making the following identifications on the boundary:

$$(t, 0) \sim (0, t) \sim (1-t, t) \quad \text{for all } t \in I$$

Prove that X is contractible.

Hint This seems hard to see directly but is a consequence of the result stated as Problem 3. For this application there is no need to have solved that problem but, of course, you must understand its statement.

Problem 9 In order to recognise proper maps the following simple rules are often used (all spaces X and Y are assumed to be locally compact Hausdorff spaces).

- If Y is compact then the projection $\mathrm{pr}_1: X \times Y \longrightarrow X$ is proper.
- If $f: X \longrightarrow Y$ is proper, and $F \subset X$ closed then the restriction $f|F: F \longrightarrow Y$ is proper.
- If $f: X \longrightarrow Y$ is proper, and $V \subset Y$ open (or, more generally, locally closed in Y) then the mapping

$$g: f^{-1}(V) \longrightarrow V$$

obtained by restricting f is proper.

Prove these rules.

Problem 10 Show that every proper map $f: X \longrightarrow Y$ must send closed subsets of X to closed subsets of Y. — Prove that every non-constant polynomial $p(X) \in \mathbb{R}[X]$ defines a proper map $p: \mathbb{R} \longrightarrow \mathbb{R}$.

Problem 11 Let $f \in \mathbf{C}(X, Y)$ be a morphism in a category $\mathbf{C}$.

- Show that f induces a natural transformation of functors from $\mathbf{C}$ to Ens

$$\mathbf{C}(f, ?): \mathbf{C}(Y, ?) \longrightarrow \mathbf{C}(X, ?).$$

- Prove that $\mathbf{C}(f, ?)$ is a natural equivalence if and only if f is an isomorphism.
- Prove that every natural transformation from $\mathbf{C}(Y, ?)$ to $\mathbf{C}(X, ?)$ is of the form $\mathbf{C}(f, ?)$, with a uniquely determined morphism $f \in \mathbf{C}(X, Y)$.

[1] dunce's cap: a paper cone formerly put on the head of a dunce [a person slow at learning] at school as a mark of disgrace (quoted from *The Concise Oxford Dictionary*).

Problem 12 Let X be a manifold (or just an open subset of some $\mathbb{R}^n$) and let $a, b, c \in X$ be points. Given two differentiable paths $\alpha: I \longrightarrow X$ from a to b, and $\beta: I \longrightarrow X$ from b to c the homotopy sum $\alpha+\beta$ will not be differentiable in general. Construct therefore a differentiable path $\gamma: I \longrightarrow X$ from a to c that is homotopic to $\alpha+\beta$ relative $\{0,1\}$.

Problem 13 Prove Sard's Theorem for what may be the simplest of all interesting cases: a C^1 function $f: [0,1] \longrightarrow \mathbb{R}$.

Problem 14 Let k and $n \geq 1$ be integers. For $j = 1, \dots, k$ let

$$D_j = D_{r_j}(a_j) \subset U^{n+1}$$

be pairwise disjoint compact subballs of the open unit ball. Denote by $U_j = U_{r_j}(a_j) \subset U^{n+1}$ the interior, and by $S_j = S_{r_j}(a_j) \subset U^{n+1}$ the boundary of D_j. If

$$f: D^{n+1} \setminus \bigcup_{j=1}^{k} U_j \longrightarrow S^n$$

is a map then not only $\deg(f|S^n)$ but also degrees $\deg(f|S_j)$ are defined for all j, using the obvious homeomorphism $S^n \approx S_j$ that sends x to $a_j + r_j x$.

Prove the formula:

$$\deg(f|S^n) = \sum_{j=1}^{k} \deg(f|S_j)$$

Problem 15 In order to compute the mapping degree of a map $g: S^n \longrightarrow S^n$ by counting the points in a regular fibre g need not be differentiable *everywhere*:

- Let $V \subset S^n$ be open such that g is differentiable on $g^{-1}(V)$. Then $\deg g$ may be computed from the fibre $g^{-1}\{c\}$ of a regular value $c \in V$ in the usual way.

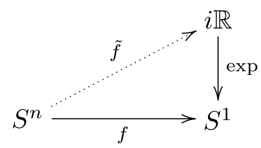
You may wish to prove this in general, adapting the proof of 18.7. Alternatively you may consider the following minimal version, which often is sufficient for g obtained by one point compactification, and admits a more elementary proof.

- If $\circ \in S^n$ is a point with $g^{-1}\{\circ\} = \{\circ\}$ and if g is differentiable on $S^n \setminus \{\circ\}$ then $\deg g$ may be computed from the fibre $g^{-1}\{c\}$ of a regular value $c \in S^n \setminus \{\circ\}$.

Problem 16 Let $p(X) \in \mathbb{R}[X]$ be a non-constant polynomial. Compute the mapping degree of $\hat{p}: \hat{\mathbb{R}} \longrightarrow \hat{\mathbb{R}}$. Which are the mapping degrees that $\hat{f}$ can have for an arbitrary proper map $f: \mathbb{R} \longrightarrow \mathbb{R}$?

Problem 17 Prove that $\pi_n(S^1, 1)$ is the trivial group for all $n > 1$.

Hint Using the homotopy invariance of path integrals over a closed differential form, you can show that every map $f\colon S^n \longrightarrow S^1$ lifts to some $\tilde{f}$ that makes the diagram



commutative. (You may prefer to study $\pi_n(\mathbb{C}^*, 1)$ instead and work with a holomorphic form: it comes down to the same.)

Problem 18 Show that the homotopy group functors commute with cartesian products:

$$\pi_n(X\times Y) = \pi_n(x) \times \pi_n(Y) \quad \text{for every } n \in \mathbb{N} \text{ and all } X, Y \in \mathbf{Top}^\circ$$

It thus follows from the result of the previous problem that the q-dimensional torus $T^q := S^1\times\cdots\times S^1$ has trivial higher homotopy groups $\pi_n(T^q) = \{0\}$ for all $n > 1$. In particular there can be no non-trivial mapping degree for maps $S^n \longrightarrow T^n$.

Problem 19 Let X and Y be pointed spaces, and $n > 1$. Prove that the natural homomorphism

$$\pi_n(X) \oplus \pi_n(Y) \longrightarrow \pi_n(X \vee Y)$$

is injective, and that its image is a direct summand of $\pi_n(X \vee Y)$. (The precise meaning of "$A \subset C$ is a direct summand": there exists a subgroup $B \subset C$ such that the canonical map $A \oplus B \longrightarrow C$ is an isomorphism.)

Problem 20 Using Proposition 20.6 show that no sphere of positive even dimension admits a topological group structure.

Problem 21 Define and compute the mapping degree for an arbitrary complex rational function $f \in \mathbb{C}(Z)$.

Problem 22 Let $j\colon A \hookrightarrow X$ be the inclusion of a closed subspace. Prove that j has the homotopy extension property if and only if

$$\{0\}\times X \;\cup\; I\times A \subset I\times X$$

is a retract of $I\times X$.

Hint First prove that $\{0\}\times X \;\cup\; I\times A$ can be rewritten as the quotient space

$$Z := (X + I\times A)/\!\sim$$

with respect to the identification $X \ni x \sim (0, x) \in I\times A$ for all $x \in A$.

Problem 23 Given be a topological space X the *suspension* of X is defined as the quotient space

$$\tilde{\Sigma}X := (I\times X)/\sim$$

obtained by collapsing $\{0\}\times X$ and $\{1\}\times X$ to one point each (think of a double cone over X). If $(X,\circ)$ is a pointed space then more often the *reduced* suspension

$$\Sigma X := (I\times X)/\big(\{0\}\times X \,\cup\, \{1\}\times X \,\cup\, I\times\{\circ\}\big)$$

is used instead, and itself considered a pointed space. Prove the following facts:

- $\tilde{\Sigma}S^n \approx \Sigma S^n \approx S^{n+1}$ for all $n \in \mathbb{N}$
- If X is a cell complex and $\{\circ\}$ a 0-cell of X then both $\tilde{\Sigma}X$ and ΣX are cell spaces, and homotopy equivalent to each other.

Problem 24 Given be a map $f\colon X \longrightarrow Y$ the *mapping cone* of f is the quotient space

$$\tilde{C}f := (I\times X + Y)/\sim$$

in which $\{0\}\times X$ is collapsed to a point, and $(1,x) \in I\times X$ identified with $f(x) \in Y$ for all $x \in X$. For pointed maps $f\colon (X,\circ) \longrightarrow (Y,\circ)$ a *reduced* version Cf of the mapping cone is also available, and obtained by further collapsing $I\times\{\circ\} \subset I\times X$. Prove the following:

- $\tilde{\Sigma}X$ and ΣX are special cases of mapping cones.
- $\tilde{C}f$ contains a copy of Y as a closed subspace. The same is true of Cf provided $\{\circ\} \subset X$ is closed.
- If X is a cell complex and $f\colon A \hookrightarrow X$ the inclusion of a subcomplex then $\tilde{C}f$ is homotopy equivalent to X/A, and so is Cf in case A is pointed by one of its 0-cells.

Problem 25 Let X be a cell complex. Describe what you feel are the essential features of X, but exclusively in terms of numerical data (discrete numbers). Try to include as much information as you can. To this purpose, consider simple examples like a 2-sphere, a 2-disk, a dunce's cap, the real projective plane, etc.

Remark This problem is meant to be addressed naively — that is, while you are not yet familiar with the material from Section 23 onwards that leads to the notion of homology.

Problem 26 Let X and Y be cell complexes, each pointed by a 0-cell. If $f\colon X \longrightarrow Y$ is a cellular map then the mapping cone Cf also is a cell complex. Let $j\colon Y \hookrightarrow Cf$ denote the embedding mentioned in Problem 24. Prove that there is a canonical pointed homotopy equivalence between Cj and the reduced suspension ΣX.

Problem 27 Prove that $\pi_n(S^n \vee S^n)$ is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$ for each $n \geq 2$.

Hint Realise $S^n \vee S^n$ as a subcomplex of $S^n \times S^n$.

Problem 28 Prove that $\pi_1(S^1 \vee S^1)$ is *not* isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$.

Hint The identity map of S^1 defines two obvious elements x and y in $\pi_1(S^1 \vee S^1)$. Using the technique of Problem 17 you can show that the homotopy sum $x+y-x-y$ is a non-zero element of $\pi_1(S^1 \vee S^1)$: so the latter is not an abelian group.

Problem 29 For given relatively prime integers $p \in \mathbb{N}$ and $q \in \mathbb{Z}$ the *lens space* $L(p,q)$ is defined as the quotient of $S^3 \subset \mathbb{C}^2$ determined by the equivalence relation

$$(w,z) \sim (w',z') \quad :\Longleftrightarrow \quad (w,z) = (\varepsilon\, w', \varepsilon^q z') \text{ for some } p\text{-th root of unity } \varepsilon \in S^1.$$

Prove that $L(p,q)$ is a three-dimensional topological manifold. (It is in fact a differential manifold.)

Remark This and the following two problems deal with three different aspects of one and the same subject, but the solutions are completely independent of each other.

Problem 30 The following description of $L(p,q)$ is copied from the book [Armstrong]. Let P be a regular polygonal region in the plane with centre of gravity at the origin and vertices $a_0, a_1, \ldots, a_{p-1}$, and let X be the solid double pyramid formed from P by joining each of its points by straight lines to the points $b_0 = (0,0,1)$ and $b_q = (0,0,-1)$ of $\mathbb{R}^3$. Identify the triangles with vertices a_i, a_{i+1}, b_0, and a_{i+q}, a_{i+q+1}, b_q for each $i = 0,1,\ldots,p-1$, in such a way that a_i is identified to a_{i+q}, the vertex a_{i+1} to a_{i+q+1}, and b_0 to b_q. (See the book for a picture, or make a cardboard model.)

Prove that the resulting space is indeed homeomorphic to $L(p,q)$.

Problem 31 The description given in Problem 30 suggests a cell structure on $L(p,q)$. Determine the boundary operators of that structure.

Problem 32 Give the n-dimensional torus $X := (S^1)^n$ the product cell structure with 2^n cells. Compute the chain complex $C_\bullet(X)$ and its homology. (Try $S^1 \times S^1$ first.)

Problem 33 Let X be a connected cell complex. Prove that for $q \geq 2$ every q-chain $x \in C_q(X)$ with integral coefficients is of the form $[h]_q$ for some $h\colon D^q \longrightarrow X^q$ that sends S^{q-1} into X^{q-1}.

Problem 34 Compute the induced homomorphisms of chain complexes and homology for some cellular mappings:

- the identity mapping $\mathrm{id}\colon S^n \longrightarrow S^n$ using the structure with two cells on one, and that with $2(n+1)$ cells on the other side,
- the quotient mapping $q\colon S^n \longrightarrow \mathbb{R}P^n$ using suitable cell structures.

Problem 35 Revise the examples of chain complexes we have calculated so far (lecture notes and previous problems). Report if anything interesting happens when the base ring is changed from $\mathbb{Q}$ to $\mathbb{Z}$ (as in the case of $\mathbb{R}P^n$ already discussed).

Problem 36 Let X be a cell complex. Prove that $H_0\big(C_\bullet(X,\mathsf{k})\big)$ always is a free k-module, and that its rank is the number of connected components of X.

Remark There is a quick solution based on Theorem 22.8 and homotopy invariance of the homology modules. But you will find that the direct proof using just the definitions is very instructive.

Problem 37 Let $n \in \mathbb{N}$, and let $V_1, \ldots, V_n$ be finitely generated abelian groups. Prove that there exists a cell complex X with

$$H_q\big(C_\bullet(X, \mathbb{Z})\big) \simeq \begin{cases} \mathbb{Z} & \text{for } q = 0, \\ V_q & \text{for } q = 1, \ldots, n, \quad \text{and} \\ \{0\} & \text{for } q > n. \end{cases}$$

Hint Use what is known about the structure of finitely generated abelian groups.

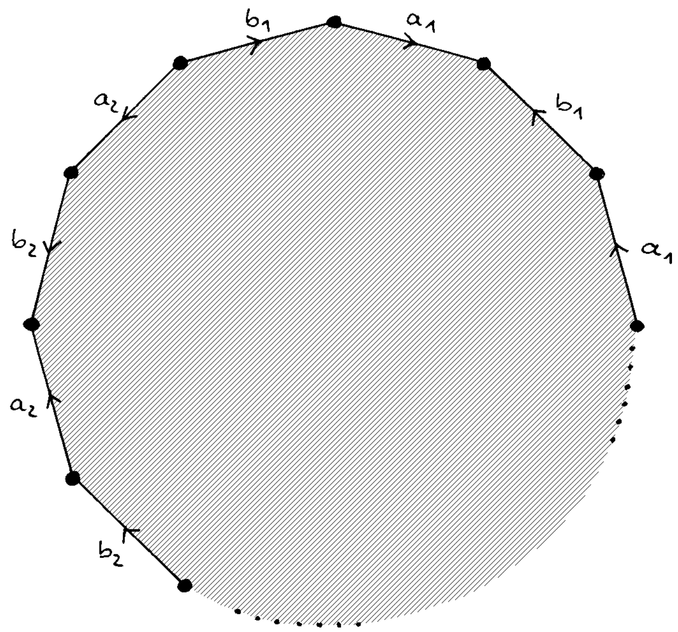
Problem 38 Let X be a cell complex. Describe the chain complex $C_\bullet(S^1 \times X)$ in terms of $C_\bullet(X)$.

Problem 39 Which of the applications of the homotopy groups in Section 20 could have been made using homology instead?

Problem 40 A topological manifold *with boundary* is a Hausdorff space X admitting a countable basis of its topology such that each $x \in X$ has an open neighbourhood U that is homeomorphic to the n-dimensional open ball U^n, or to $\{x \in U^n \mid x_1 \leq 0\}$. In the latter case, if $n > 0$ then x is called a boundary point of X.

Prove that the notion of boundary point is well-defined.

Problem 41 The compact surfaces (topological or differential manifolds of dimension 2) can be completely classified. Those which are orientable (and do not have a boundary) are obtained from a $4g$-gon (for any given $g \in \mathbb{N}$) by making identifications of the edges as indicated (supply a suitable interpretation for the case $g = 0$).



Compute the homology invariants and the Euler number of these surfaces.

Problem 42 Cell complexes of dimension one or less are also called *graphs*. Compute the homology invariants of a graph, and show that up to homotopy equivalence every non-empty connected graph is determined by its Euler number.

You will know that graph theory by itself is a branch of mathematics, and naturally a lot has been written about it. While it considers graphs as combinatorial rather than topological objects the notions are basically equivalent — or should be.

"Many established textbooks of graph theory use a definition that is very simple but fails for graphs that contain loops. Indeed authors who mean to allow loops almost unanimously define blobs: a fact they seem to be quite unaware of, or not to care about."

Scan the literature and find out whether that would be a fair statement to make.

Problem 43 Let X be a topological group whose underlying topological space is a polyhedron. Prove that

- either X is finite (as a set),
- or $e(X) = 0$

(consider the connected case first). So for instance no sphere of positive even dimension can carry a topological group structure.

Problem 44 Let X be a cell complex and $f: X \longrightarrow X$ a self-map of prime order p (so $f^p = \mathrm{id}_X$), and assume that f sends each open cell e of X to a distinct cell $f(e) \neq e$. The equivalence relation

$$x \sim y \quad :\Longleftrightarrow \quad x = f^j(y) \text{ for some } j \in \mathbb{Z}$$

defines a quotient $X \xrightarrow{q} X/f$.

- Prove that X/f carries a cell structure such that q is cellular.

Of course q induces homomorphisms $q_*: H_r(X; \mathsf{k}) \longrightarrow H_r(X/f; \mathsf{k})$ for all r.

- Show that a natural homomorphism in the opposite direction $t: H_r(X/f; \mathsf{k}) \longrightarrow H_r(X; \mathsf{k})$ can be constructed by essentially taking inverse images of cells (t is called the *transfer* homomorphism).
- Show that the image of t is contained in the f-invariant part

$$H_r(X; \mathsf{k})^f := \{x \in H_r(X; \mathsf{k}) \mid f(x) = x\}.$$

Making suitable assumptions on the base ring k, find the relations between t and $q_*|H_r(X; \mathsf{k})^f$.

Problem 45 Prove that the relation between reduced and unreduced homology may also be expressed by the following natural equivalences of functors:

$$H_q(X) \simeq \tilde{H}_q(X^+)$$

for any cell space X, and

$$\tilde{H}_q(X) \simeq \ker \big(H_q(X) \xrightarrow{H_q(\circ)} H_q\{\circ\}\big)$$

for any pointed cell space X. Note that the second equivalence would allow to define reduced homology of any non-empty cell space X, without the need to specify a base point.

Problem 46 Assume that in the commutative diagram of k-modules

$$\begin{array}{ccccccccc} V_2 & \xrightarrow{f_2} & V_1 & \xrightarrow{f_1} & V_0 & \xrightarrow{f_0} & V_{-1} & \xrightarrow{f_{-1}} & V_{-2} \\ \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_{-1}} & & \downarrow{\scriptstyle h_{-2}} \\ W_2 & \xrightarrow{g_2} & W_1 & \xrightarrow{g_1} & W_0 & \xrightarrow{g_0} & W_{-1} & \xrightarrow{g_{-1}} & W_{-2} \end{array}$$

both rows are exact (that is, exact at each point where this notion makes sense). Prove the so-called *five lemma*: if h_i is an isomorphism for each $i \neq 0$ then h_0 is an isomorphism too.

Problem 47 Let X be obtained from a cell space A by attaching an n-cell e via an attaching map $\varphi: S^{n-1} \longrightarrow A^{n-1}$, and let

$$0 \longrightarrow H_n(A;\mathsf{k}) \xrightarrow{j_*} H_n(X;\mathsf{k}) \xrightarrow{\pi} \mathsf{k} \xrightarrow{d} H_{n-1}(A;\mathsf{k}) \xrightarrow{j_*} H_{n-1}(X;\mathsf{k}) \longrightarrow 0$$

be the corresponding exact sequence as defined in Example 28.7. Prove that π sends every n-cycle to the coefficient of e in it while d may be identified with the composition

$$\tilde{H}_{n-1}(S^{n-1};\mathsf{k}) \hookrightarrow H_{n-1}(S^{n-1};\mathsf{k}) \xrightarrow{\varphi_*} H_{n-1}(A;\mathsf{k}).$$

Give an example with $\mathsf{k} = \mathbb{Z}$, and neither $H_n(j)$ nor $H_{n-1}(j)$ an isomorphism.

Problem 48 The spaces

$$Y = \{(x,y) \in S^{n-1}\times\mathbb{R}^n \mid \langle x,y\rangle = 0,\ |y| \leq 1\}$$

and its boundary

$$\partial Y = \{(x,y) \in S^{n-1}\times\mathbb{R}^n \mid \langle x,y\rangle = 0,\ |y| = 1\}$$

are called the *tangent disk*, respectively *tangent sphere bundle* of S^{n-1} since the fibre of the projection of the former to $x \in S^{n-1}$ is just the tangent disk to S^{n-1} at x.

Construct a homeomorphism between $\Sigma(Y/\partial Y)$ and $\Sigma^n\big((S^{n-1})^+\big)$, the n-fold repeated suspension of the pointed space $(S^{n-1})^+$.

Remark Begin with $I\times Y \ni (t,x,y) \longmapsto (x, ?\cdot x + y) \in S^{n-1}\times D^n$.

Problem 49 Use the result of Problem 48 to calculate the homology of $Y/\partial Y$. Try and find out how much information on the integral homology of ∂Y you can extract.

Comment The complete determination of the latter is quite tricky (and not really meant to be a part of this problem). Leaving aside the cases of small n, which are simple but somewhat exceptional, the result is:

$$\tilde{H}_{2n-3}(\partial Y;\mathbb{Z}) \simeq \tilde{H}_{n-1}(\partial Y;\mathbb{Z}) \simeq \tilde{H}_{n-2}(\partial Y;\mathbb{Z}) \simeq \mathbb{Z} \text{ for even } n \geq 4,$$

and

$$\tilde{H}_{2n-3}(\partial Y;\mathbb{Z}) \simeq \mathbb{Z} \text{ and } \tilde{H}_{n-2}(\partial Y;\mathbb{Z}) \simeq \mathbb{Z}/2 \text{ for odd } n \geq 3,$$

all other homology being trivial.