A_∞-bimodules and Serre A_∞-functors

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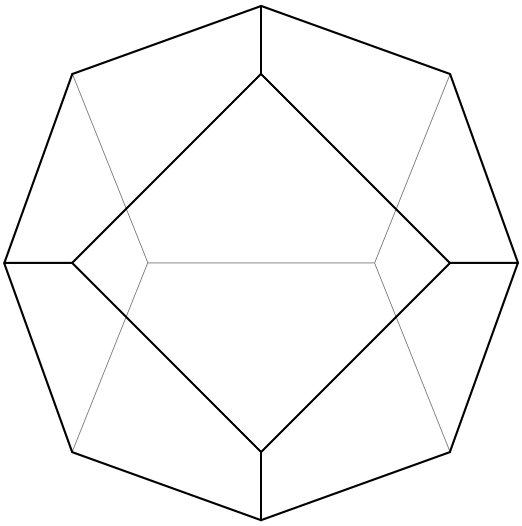

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Abstract

This dissertation is intended to transport the theory of Serre functors into the context of A_∞-categories. We begin with an introduction to multicategories and closed multicategories, which form a framework in which the theory of A_∞-categories is developed. We prove that (unital) A_∞-categories constitute a closed symmetric multicategory. We define the notion of A_∞-bimodule similarly to Tradler and show that it is equivalent to an A_∞-functor of two arguments which takes values in the differential graded category of complexes of $\Bbbk$-modules, where $\Bbbk$ is a commutative ground ring. Serre A_∞-functors are defined via A_∞-bimodules following ideas of Kontsevich and Soibelman. We prove that a unital closed under shifts A_∞-category $\mathcal{A}$ over a field $\Bbbk$ admits a Serre A_∞-functor if and only if its homotopy category $H^0\mathcal{A}$ admits an ordinary Serre functor. The proof uses categories enriched in $\mathcal{K}$, the homotopy category of complexes of $\Bbbk$-modules, and Serre $\mathcal{K}$-functors. Another important ingredient is an A_∞-version of the Yoneda Lemma.

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Contents

CHAPTER 0

Introduction

0.1. Motivation

It is widely accepted that Verdier's notion of triangulated category is not quite satisfactory. It suffers from numerous deficiencies, starting from non-functorial cone up to the failure of descent for the derived category of quasi-coherent sheaves on a scheme, see e.g. [**58**]. As a remedy, Bondal and Kapranov [**5**] introduced the notion of pretriangulated differential graded (dg) category that enhances the notion of triangulated category. In particular, the 0^{th} cohomology (called also the homotopy category) of a pretriangulated dg category admits a natural triangulated structure. Remarkably, triangulated categories arising in algebraic geometry and representation theory are of that kind. It is therefore desirable to develop the relevant homological algebra at the level of pretriangulated dg categories rather than at the level of triangulated categories. Drinfeld gave in [**13**] an explicit construction (implicitly present in Keller's paper [**25**]) of a quotient of dg categories and proved that it is compatible with Verdier's quotient of triangulated categories.

The notion of dg category is a particular case of a more general and more flexible notion of A_∞-category. The notion of pretriangulated dg category generalizes to A_∞-categories. It is being developed independently by Kontsevich and Soibelman [**32**] and by Bespalov, Lyubashenko, and the author [**3**]. Consequently, there are attempts to rewrite homological algebra using A_∞-categories. Lyubashenko and Ovsienko extended in [**42**] Drinfeld's construction of quotients to unital A_∞-categories. In [**39**], the author jointly with Lyubashenko constructed another kind of a quotient and proved that it enjoys a certain universal property. We also proved that both constructions of quotients agree, i.e., produce A_∞-equivalent unital A_∞-categories. In [**3**], it is proven that the homotopy category of a pretriangulated A_∞-category is triangulated. Furthermore, the quotient of a pretriangulated A_∞-category over a pretriangulated A_∞-subcategory is again pretriangulated.

The reasons to work with A_∞-categories rather than with dg categories are the following. On the one hand, it is the mirror symmetry conjecture of Kontsevich. In one of its versions, it asserts that the homotopy category of the Fukaya A_∞-category, constructed from the symplectic structure of a Calabi–Yau manifold, is equivalent to the derived category of coherent sheaves on a dual complex algebraic variety. The construction of the Fukaya A_∞-category is not a settled question yet, it is a subject of current research, see e.g. Seidel [**48**]. However, it is clear that the Fukaya A_∞-category is in general not a dg category. On the other hand, the supply of dg functors between dg categories is not sufficient for the purposes of theory. Instead of extending the class of dg functors to a wider class of (unital) A_∞-functors, some authors prefer to equip the category of dg categories with a suitable model structure and to work in the homotopy category of dg categories. This approach is being developed, e.g., by Tabuada [**54, 53**] and Toën [**57**]. There are evidences that both approaches may be equivalent, at least if the ground ring is a field. However, the precise relation is not yet clear to the author.

The goal of this dissertation is to transport the theory of Serre functors into the context of A_∞-categories.

The notion of Serre functor was introduced by Bondal and Kapranov [**4**], who used it to reformulate Serre-Grothendieck duality for coherent sheaves on a smooth projective variety as follows. Let X be a smooth projective variety of dimension n over a field k. Denote by $D^b(X)$ the bounded derived category of coherent sheaves on X. Then there exists an isomorphism of k-vector spaces

$$\operatorname{Hom}_{D^b(X)}(\mathscr{F}^\bullet, \mathscr{G}^\bullet \otimes \omega_X[n]) \cong \operatorname{Hom}_{D^b(X)}(\mathscr{G}^\bullet, \mathscr{F}^\bullet)^*,$$

natural in $\mathscr{F}^\bullet, \mathscr{G}^\bullet \in D^b(X)$, where ω_X is the canonical sheaf, and $*$ denotes the dual k-vector space. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves concentrated in degrees i and 0 respectively, the above isomorphism specializes to the familiar form of Serre duality:

$$\operatorname{Ext}^{n-i}(\mathscr{F}, \mathscr{G} \otimes \omega_X) \cong \operatorname{Ext}^i(\mathscr{G}, \mathscr{F})^*.$$

In general, a k-linear functor S from a k-linear category $\mathcal{C}$ to itself is a *Serre functor* if there exists an isomorphism of k-vector spaces

$$\mathcal{C}(X, SY) \cong \mathcal{C}(Y, X)^*,$$

natural in $X, Y \in \operatorname{Ob} \mathcal{C}$. A Serre functor, if it exists, is determined uniquely up to natural isomorphism. In the above example, $\mathcal{C} = D^b(X)$ and $S = - \otimes \omega_X[n]$. Being an abstract category theory notion, Serre functors have been discovered in other contexts, for example, in Kapranov's studies of constructible sheaves on stratified spaces [**24**]. The existence of a Serre functor imposes strong restrictions on the category. For example, Reiten and van den Bergh have shown that Serre functors in abelian categories of modules are related to Auslander–Reiten sequences and triangles, and they classified the noetherian hereditary Ext-finite abelian categories with Serre duality [**47**]. Serre functors play an important role in reconstruction of a variety from its derived category of coherent sheaves [**6**]. Another rapidly developing area where Serre functors find applications is non-commutative geometry.

The idea of non-commutative geometry that goes back to Grothendieck is that categories should be thought of as non-commutative counterparts of geometric objects. For example, the derived category of quasi-coherent sheaves on a scheme X reflects a great deal of geometric properties of X. The general philosophy suggests to forget about the scheme itself and to work with its derived category of quasi-coherent sheaves. Thus, instead of defining what a non-commutative scheme is, non-commutative geometry declares an arbitrary (sufficiently nice) triangulated category to be the derived category of quasi-coherent sheaves on a non-commutative scheme. In the spirit of the agenda explained above, Keller in the talk at ICM 2006 and Kontsevich and Soibelman in [**31**] suggested to consider pretriangulated dg categories resp. A_∞-categories as non-commutative schemes. Then, to a (commutative) scheme X its derived dg category $D_{dg}(X)$ is associated. By definition, $D_{dg}(X)$ is Drinfeld's quotient of the dg category of complexes of quasi-coherent sheaves on X modulo the dg subcategory of acyclic complexes. Its homotopy category is the ordinary derived category of X. Geometric properties of the scheme X (smoothness, properness etc.) correspond to certain algebraic properties of its derived dg category. Abstracting these yields a definition of smooth, proper etc. non-commutative schemes. This approach is being actively developed by Toën, see e.g. [**57**] and [**56**].

In non-commutative geometry, triangulated categories admitting a Serre functor (and satisfying some further conditions) are considered as non-commutative analogs of smooth projective varieties. Modern homological algebra insists on working with pretriangulated dg categories or A_∞-categories instead of triangulated categories. Therefore, we need a notion of Serre functor in these settings. The definition of Serre dg functor is straightforward, but apparently useless, for given a pretriangulated dg category $\mathcal{A}$ such that $H^0(\mathcal{A})$ admits a Serre functor, it is not clear whether there exists a Serre dg functor from $\mathcal{A}$ to itself. The definition of Serre A_∞-functor was sketched by Soibelman in [**51**]. We define Serre A_∞-functors using A_∞-bimodules and prove that an A_∞-category $\mathcal{A}$ admits a Serre functor if and only if its homotopy category $H^0(\mathcal{A})$ does, provided that some standard assumptions are satisfied. This result applies in particular to the derived dg category of coherent sheaves on a smooth projective variety X.

The (refined) homological mirror symmetry conjecture of Kontsevich asserts the existence of an A_∞-equivalence between the Fukaya A_∞-category and the derived dg category of coherent sheaves on a dual complex algebraic variety. Both A_∞-categories admit Serre A_∞-functors, cf. [**30**]. Therefore, to approach the conjecture, a theory of A_∞-categories with Serre duality is needed.

A smooth projective variety X is Calabi–Yau if the canonical sheaf is isomorphic to the structure sheaf. In particular, the Serre functor

$$S = -\otimes \omega_X[n] : D^b(X) \to D^b(X)$$

is isomorphic to the shift functor $[n]$. This motivates the following definition: a triangulated category is an n-Calabi–Yau category if the n^{th} shift is a Serre functor. Triangulated 2-Calabi–Yau categories play an important role in the theory of cluster categories, see e.g. Caldero and Keller [**9**]. Correspondingly, there is a notion of n-Calabi–Yau A_∞-category. We believe that the obtained results may have applications to the study of topological conformal field theories due to the work of Costello [**10**].

0.2. Notation and conventions

We tried to avoid questions of ‘size’. Possible set-theoretic difficulties can be solved by using universes [**19**].

The category of sets is denoted by **Set**. The category of (small) categories is denoted by **Cat**. Binary product of sets or of categories is written as $\times$ and arbitrary product as $\prod$. The disjoint union of sets is written as $\sqcup$. The cardinality of a set S is denoted by $|S|$. The set of integers is denoted by $\mathbb{Z}$; the set of non-negative integers is denoted by $\mathbb{N}$. For a finite set I and an element $n = (n_i)_{i\in I} \in \mathbb{N}^I$, denote by $|n|$ the sum $\sum_{i\in I} n_i$.

The symmetric group on n points is denoted by $\mathfrak{S}_n$.

The set of objects of a category $\mathcal{C}$ is denoted by $\operatorname{Ob}\mathcal{C}$, the set of morphisms is denoted by $\operatorname{Mor}\mathcal{C}$. For a pair of objects $X, Y \in \operatorname{Ob}\mathcal{C}$, we prefer the notation $\mathcal{C}(X,Y)$ for the set of morphisms from X to Y to the traditional $\operatorname{Hom}_{\mathcal{C}}(X,Y)$. The composite of morphisms $f : X \to Y$ and $g : Y \to Z$ is denoted by $fg = f \cdot g = g \circ f$. A function $f : X \to Y$ applied to an element $x \in X$ is written $xf = f(x)$ and occasionally fx. The inverse image of an element $y \in Y$ is denoted by $f^{-1}y$. Isomorphism between objects of a category is written $\cong$, whereas the symbol $\simeq$ denotes equivalence of categories. Identity morphisms in a category are denoted by id. The symbol $\mathrm{Id}_{\mathcal{C}}$ denotes the identity functor of a category $\mathcal{C}$.

We assume familiarity with the language of enriched categories. Some of the concepts are recalled in the main text. For a symmetric monoidal category $\mathcal{V}$, a *$\mathcal{V}$-category* $\mathcal{C}$ consists of a set of objects $\operatorname{Ob}\mathcal{C}$, an object of morphisms $\mathcal{C}(X,Y) \in \operatorname{Ob}\mathcal{V}$, for each pair of objects $X, Y \in \operatorname{Ob}\mathcal{C}$, a composition

$$\mu_{\mathcal{C}} : \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z),$$

for each triple $X, Y, Z \in \operatorname{Ob} \mathcal{C}$, and an identity element $1^{\mathcal{C}}_X : \mathbb{1} \to \mathcal{C}(X, X)$, for each object $X \in \operatorname{Ob} \mathcal{C}$. The associativity and identity axioms are expressed by the commutativity of certain diagrams, see [**29**, Section 1.2], or Section 1.3.1 below. Sometimes, when no confusion arises, composition in a $\mathcal{V}$-category is denoted just by comp. The associativity axiom allows to define iterated compositions

$$\mathcal{C}(X_0, X_1) \otimes \dots \otimes \mathcal{C}(X_{n-1}, X_n) \to \mathcal{C}(X_0, X_n),$$

which are denoted by $\mu^{\mathbf{n}}_{\mathcal{C}}$. Identities are often denoted simply by 1_X. The reader is referred to [**29**] for more details on enriched categories.

Throughout, $\Bbbk$ is a commutative ring with 1. The following categories are particularly important for this dissertation: the category **gr** of graded $\Bbbk$-modules; the category $\mathsf{C}_\Bbbk$ of complexes of $\Bbbk$-modules, sometimes also denoted by **dg**; the homotopy category $\mathcal{K}$ of complexes of $\Bbbk$-modules. These categories are discussed in detail in the main text. The category of $\Bbbk$-modules is written as $\Bbbk$-**Mod**.

0.3. Chapter synopsis

Chapter 1. We introduce the definitions of lax Monoidal category and multicategory. We also need 2-dimensional analogs of these concepts, though not in full generality. The appropriate notions are those of lax Monoidal category and multicategory enriched in **Cat**, the category of categories. To get a uniform treatment, we introduce lax Monoidal categories and multicategories enriched in a symmetric Monoidal category $\mathcal{V}$. We discuss thoroughly the relation between lax Monoidal categories and multicategories. More precisely, we introduce appropriate notions of functors and natural transformations in both cases, and observe that lax Monoidal categories as well as multicategories constitute a 2-category, or equivalently a **Cat**-category, a category enriched in **Cat**. We construct a **Cat**-functor from the 2-category of lax Monoidal categories to the 2-category of multicategories. We prove that it is fully faithful (in enriched sense) and describe its essential image. These results hold true in enriched setting. However proofs are provided in the case $\mathcal{V} = \mathbf{Set}$ only, with a few exceptions, where for the sake of being rigorous we included proofs of propositions that are used in the sequel in the case of general $\mathcal{V}$.

We spend some time discussing categories and multicategories enriched in a symmetric multicategory. An important issue is base change. We prove that a symmetric multifunctor $F : \mathsf{V} \to \mathsf{W}$ gives rise to a **Cat**-functor F_*, the base change **Cat**-functor, from the 2-category of (multi)categories enriched in V to the 2-category of (multi)categories enriched in W. The case of multicategories coming from lax Monoidal categories is of particular importance, it is treated at some length. Finally, we introduce the notion of closed multicategory that generalizes the notion of closed Monoidal category. We prove that a closed symmetric multicategory C gives rise to a multicategory $\underline{\mathsf{C}}$ enriched in C. For each symmetric multifunctor $F : \mathsf{C} \to \mathsf{D}$, we introduce its closing transformation, which is a certain naturally arising D-multifunctor $\underline{F} : F_*\underline{\mathsf{C}} \to \underline{\mathsf{D}}$. We spend some time discussing properties of closing transformations.

Chapter 2. We introduce Serre functors for categories enriched in a symmetric Monoidal category $\mathcal{V}$. The examples of interest are $\mathcal{V} = \mathcal{K}$, the homotopy category of complexes of $\Bbbk$-modules, and $\mathcal{V} = \mathbf{gr}$, the category of graded $\Bbbk$-modules. These build a bridge from A_∞-categories to ordinary $\Bbbk$-linear categories. We study behavior of Serre functors under base change and derive some results concerning the existence of Serre functors.

Chapter 3. We begin by explaining our conventions about graded $\Bbbk$-modules and complexes of $\Bbbk$-modules, where $\Bbbk$ is a fixed commutative ground ring, and introduce some notation. Afterwards, we introduce (differential) graded spans and quivers. We recall the definition of augmented coassociative counital coalgebra and introduce the main example of interest, the tensor coalgebra $T\mathcal{A}$ of a quiver $\mathcal{A}$. We spend some time studying morphisms and coderivations from a tensor product of tensor coalgebras to another tensor coalgebra, which is necessary for understanding A_∞-functors and A_∞-transformations.

We define a symmetric multicategory A_∞ of A_∞-categories and A_∞-functors. In particular, we introduce A_∞-functors of several arguments. We prove that the multicategory A_∞ is closed. The disadvantage of our proof is that it is indirect, although short. In particular, it says almost nothing useful about the closed structure. There is a more conceptual approach to A_∞-categories that is developed by Yuri Bespalov, Volodymyr Lyubashenko, and the author in the book in progress [**3**]. It leads to an explicit description of the closed structure which is suitable for computations. It seems impossible to reproduce the contents of the book here, since it would take us far away from the main topic of the dissertation. For the reason of size, only a short summary of results relevant for the further considerations is given. We briefly review the closed structure of A_∞ and describe the multicategory $\underline{\mathsf{A}_\infty}$ enriched in A_∞.

We recall the definition of unital A_∞-category following Lyubashenko [**38**], and generalize the notion of unital A_∞-functor to the case of many arguments. We prove that unital A_∞-categories and unital A_∞-functors constitute a closed symmetric multicategory $\mathsf{A}^{\mathrm{u}}_\infty$. We construct a symmetric multifunctor $\mathsf{k} : \mathsf{A}^{\mathrm{u}}_\infty \to \widehat{\mathcal{K}\text{-}\mathbf{Cat}}$, where $\widehat{\mathcal{K}\text{-}\mathbf{Cat}}$ is the symmetric multicategory arising from the symmetric Monoidal category $\mathcal{K}$-**Cat** of categories enriched in $\mathcal{K}$, the homotopy category of complexes of $\Bbbk$-modules. The category $\mathcal{K}$-**Cat** is enriched in $\Bbbk$-**Cat**, the category of $\Bbbk$-linear categories, therefore so is the multicategory $\widehat{\mathcal{K}\text{-}\mathbf{Cat}}$. Applying results about base change, we conclude that the multicategory $\mathsf{A}^{\mathrm{u}}_\infty$ may be also viewed as a multicategory enriched in $\Bbbk$-**Cat**. The multifunctor k is extended to a $\Bbbk$-**Cat**-multifunctor. We relate the closed multicategory approach to A_∞-categories with the 2-category approach developed in [**39**] and [**40**].

Finally, we briefly consider duality for A_∞-categories. We show that the correspondence that assigns to an A_∞-category its opposite extends to a symmetric multifunctor $\mathrm{op} : \mathsf{A}_\infty \to \mathsf{A}_\infty$. We compute its closing transformation.

Chapter 4. We introduce bimodules over A_∞-categories and prove that, for A_∞-categories $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}$-$\mathcal{B}$-bimodules constitute a differential graded category. It turns out to be isomorphic to the differential category of A_∞-functors $\mathcal{A}^{\mathrm{op}}, \mathcal{B} \to \underline{\mathsf{C}}_\Bbbk$, where $\underline{\mathsf{C}}_\Bbbk$ is the differential graded category of complexes of $\Bbbk$-modules. We introduce also unital A_∞-bimodules. We spend some time discussing operations on A_∞-bimodules.

Chapter 5. As an application of the bimodule technique, we introduce Serre A_∞-functors. We prove a criterion and give certain sufficient conditions for the existence of Serre A_∞-functors. In particular, we prove that if the ground ring $\Bbbk$ is a field, then a Serre A_∞-functor in an A_∞-category $\mathcal{A}$ induces an ordinary Serre functor in the cohomology $H^0\mathcal{A}$. The converse is true if the A_∞-category $\mathcal{A}$ is closed under shifts. Finally, we consider the strict case of a Serre A_∞-functor.

Appendix. As another application of the bimodule technique, we prove the Yoneda Lemma for unital A_∞-categories. As a corollary, we deduce that an arbitrary unital A_∞-category is A_∞-equivalent to a differential graded category. We also obtain a criterion for representability of a unital A_∞-functor $\mathcal{A} \to \underline{\mathsf{C}}_\Bbbk$, which is necessary for our proof of the existence of Serre A_∞-functors.

Author's contribution. More detailed information concerning the author's contribution to the dissertation can be found at the beginning of each chapter. Here we just make general remarks. The results of Chapters 1 and 3 have been obtained jointly with Yuri Bespalov and Volodymyr Lyubashenko, and should appear as part of the book [**3**]. The central notion of closed multicategory as well as the key observation that A_∞-categories form a closed multicategory are due to Yuri Bespalov. Sections 1.1.14 and 3.5 are entirely due to the author. The main results of Chapters 2, 4, 5, as well as the results of Appendix A represent the author's contribution, with the following exceptions: Propositions 2.2.11, 4.4.2, 4.4.4, 5.2.1, and part of Proposition A.7 are due to Prof. Lyubashenko. He should also be credited for suggesting many definitions and formulations of statements.

CHAPTER 1

Tools

The study of A_∞-categories is intertwined with higher-dimensional category theory, in particular, with the study of monoidal categories and multicategories. While monoidal categories are in prevalent use in mathematics, the concept of multicategory, which goes back to Lambek [**33**], has been confined primarily to purposes of categorical logic. In mid-1990s, multicategories enjoyed a resurgence of interest due to applications in higher-dimensional category theory, due to the work, e.g., of Baez and Dolan [**1**], Day and Street [**11**], Leinster [**35**]; in quantum algebra, see e.g. Beilinson and Drinfeld [**2**], Borcherds [**8**], Soibelman [**50**]. We discuss these notions at some length. The emphasis is put on the relation between monoidal categories and multicategories, which is crucial for the subsequent chapters, and on closed multicategories, which are the main tool of the dissertation. Though the notion of closed multicategory is merely a straightforward generalization of that of closed monoidal category, to the best of author's knowledge, it did not appear in the literature before and deserves closer look. The author may have first learned it from Yuri Bespalov.

The chapter is organized as follows. In Section 1.1 we briefly review the definitions of lax Monoidal category, lax Monoidal functor, and Monoidal transformation. We also introduce enriched analogs of these notions. For a symmetric Monoidal category $\mathcal{V}$, the 2-category $\mathcal{V}$-**Cat** of $\mathcal{V}$-categories, $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations provides an example of symmetric Monoidal **Cat**-category. We investigate in Section 1.1.14 the effect of a lax symmetric Monoidal functor $\mathcal{V} \to \mathcal{W}$ on the corresponding enriched categories, functors, and transformations. This issue was intentionally omitted in Kelly's book [**29**]. According to [**7**, Proposition 6.4.3], a lax symmetric Monoidal functor $\mathcal{V} \to \mathcal{W}$ induces a 2-functor $\mathcal{V}$-**Cat** $\to$ $\mathcal{W}$-**Cat**. For our purposes, we need a more precise statement. Namely, we prove that the induced 2-functor $\mathcal{V}$-**Cat** $\to$ $\mathcal{W}$-**Cat** is, in fact, a symmetric Monoidal **Cat**-functor.

Multicategories, multifunctors, and multinatural transformations are introduced in Section 1.2. For the benefit of the reader, we begin with the unenriched picture, since it seems more accessible. The enriched counterparts are discussed afterwards. Section 1.2.16 is devoted to the relation between lax Monoidal categories and multicategories. We explain how a lax Monoidal category gives rise to a multicategory. Furthermore, a lax Monoidal functor induces a multifunctor between the corresponding multicategories, and a Monoidal transformation of lax Monoidal functors induces a multinatural transformation of the corresponding multifunctors. Together, these correspondences constitute a **Cat**-functor from the 2-category of lax Monoidal categories to the 2-category of multicategories. We prove that its if fully faithful and describe its essential image. Though these results may be well-know to the experts, there seem to be no account of these results in the literature (or at least a search turned up nothing).

In Section 1.3.1 we briefly discuss categories and multicategories enriched in a symmetric multicategory V. Closed multicategories are introduced in Section 1.3.10. The fundamental result here is that a closed symmetric multicategory C gives rise to a symmetric multicategory $\underline{\mathsf{C}}$ enriched in C. We also develop some algebra in closed multicategories. Multifunctors between closed multicategories are studied in Section 1.3.27. It turns out that with an arbitrary symmetric multifunctor between closed symmetric multicategories a certain natural closing transformation is associated. We investigate properties of closing transformations.

Much of the material presented here can be found in [**3**], where it is treated in greater generality. Here we develop as much theory as necessary. To keep the exposition self-contained, we included some topics which are not used in the sequel but which supplement the picture.

The results of Section 1.2.16 have been obtained jointly with Volodymyr Lyubashenko, although some proofs given here differ from those provided in [**3**]. These are due to the author. The proofs of Propositions 1.2.26 and 1.2.27 are entirely due to Prof. Lyubashenko. The notion of lax representable multicategory is an invention of Prof. Lyubashenko. The notions of closed multicategory and closing transformation are due to Yuri Bespalov. A thorough proof of Proposition 1.3.14 was given first by Prof. Lyubashenko in [**3**]. Here we provide a lightened version of his proof.

1.1. Lax Monoidal categories

A classical *monoidal category*, as defined by Mac Lane [**43**], is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $(X, Y) \mapsto X \otimes Y$ (tensor product), an object $\mathbb{1}$ (unit object), and isomorphisms

$$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z), \quad \mathbb{1} \otimes X \cong X, \quad X \otimes \mathbb{1} \cong X$$

natural in $X, Y, Z \in \operatorname{Ob} \mathcal{C}$ (associativity and unit constraints, or coherence isomorphisms), such that the pentagon, involving the five ways of bracketing four objects, commutes, and the associativity constraint for $Y = \mathbb{1}$ is compatible with the unit constraints. A monoidal category is called strict if coherence isomorphisms are identities. A version of Mac Lanes's 'coherence theorem' [**43**] asserts that each monoidal category is equivalent to a strict monoidal category.

Four other definitions of monoidal category (and related notions of lax monoidal functor and monoidal transformation) are discussed by Leinster in [**35**, Chapter 3]. These definitions are proven to be equivalent, in a strong sense. More precisely, monoidal categories and lax monoidal functors for a particular definition form a category. Leinster constructs equivalences between these categories for different definitions and remarks that these equivalences can be extended to equivalences of 2-categories (including monoidal transformations). We are not going to explore the subject here.

We adopt a definition of lax Monoidal category which is a weakening of Lyubashenko's notion of Monoidal category [**37**, Definition 1.2.2], which in turn is equivalent to Leinster's notion of unbiased monoidal category [**35**, Definition 3.1.1]. We begin by an informal introduction.

Let $\mathcal{C}$ be a monoidal category. Let $\otimes_1^n, \otimes_2^n : \mathcal{C}^n \to \mathcal{C}$ be arbitrary derived n-ary tensor products, i.e., functors obtained by iterating the functor $\otimes$. For example, for $n = 4$, we may have $\otimes_1^4 = \otimes \circ (\otimes \times \otimes)$ and $\otimes_2^4 = \otimes \circ (1 \times \otimes) \circ (1 \times 1 \times \otimes)$. The meaning of Mac Lane's coherence theorem is that there is a *unique* natural isomorphism $\otimes_1^n \xrightarrow{\sim} \otimes_2^n$ constructed from associativity and unit constraints and their inverses, which means that there is essentially a unique way to extend the operation $\otimes$ to a functor $\otimes^n : \mathcal{C}^n \to \mathcal{C}$. In particular, for each partition $n = n_1 + \cdots + n_k$, there is a unique natural isomorphism

$\otimes^n \xrightarrow{\sim} \otimes^k \circ (\otimes^{n_1} \times \dots \times \otimes^{n_k})$. These isomorphisms satisfy equations

$$\begin{tikzcd}
 & \otimes^k \circ (\otimes^{n_1} \times \dots \times \otimes^{n_k}) \arrow[d, "\cong"] \\
\otimes^n \arrow[ur, bend left, "\cong"] \arrow[dr, bend right, "\cong"'] & \otimes^k \circ (\otimes^{p_1} \times \dots \times \otimes^{p_k}) \circ (\otimes^{n_1^1} \times \dots \times \otimes^{n_1^{p_1}} \times \dots \times \otimes^{n_k^1} \times \dots \times \otimes^{n_k^{p_k}}) \\
 & \otimes^p \circ (\otimes^{n_1^1} \times \dots \times \otimes^{n_1^{p_1}} \times \dots \times \otimes^{n_k^1} \times \dots \times \otimes^{n_k^{p_k}}) \arrow[u, "\cong"]
\end{tikzcd}$$

where $n = n_1 + \dots + n_k$, $p = p_1 + \dots + p_k$, $n_i = n_i^1 + \dots + n_i^{p_i}$, $i = 1, \dots, k$. Indeed, the paths of the diagram represent two natural isomorphisms between the source and the target. These paths must coincide by Mac Lane's coherence theorem.

A *Monoidal category* is defined as a category $\mathcal{C}$ equipped with arbitrary n-ary tensor products $\otimes^n : \mathcal{C}^n \to \mathcal{C}$ and coherence isomorphisms $\otimes^n \xrightarrow{\sim} \otimes^k \circ (\otimes^{n_1} \times \dots \times \otimes^{n_k}) : \mathcal{C}^n \to \mathcal{C}$, for each partition $n = n_1 + \dots + n_k$, satisfying the above equations. The formal definition follows. Since it is no extra work, we define at once lax Monoidal categories in which coherence morphisms $\otimes^n \to \otimes^k \circ (\otimes^{n_1} \times \dots \times \otimes^{n_k}) : \mathcal{C}^n \to \mathcal{C}$ are not necessarily invertible.

Let $\mathcal{O}$ (resp. $\mathcal{S}$) denote the category whose objects are finite linearly ordered sets and whose morphisms are order-preserving (resp. arbitrary) maps. For each integer $n \geqslant 0$, denote by $\mathbf{n}$ the linearly ordered set $\{1 < 2 < \dots < n\}$; in particular, $\mathbf{0}$ is the empty set. For each map $f : I \to J$, the inverse image $f^{-1}j \subset I$ of an element $j \in J$ inherits a linear order from I. Note that there is a bijection between maps $f : \mathbf{m} \to \mathbf{n}$ in $\operatorname{Mor} \mathcal{O}$ and partitions $m = m_1 + \dots + m_n$, given by $m_j = |f^{-1}j|$, $j \in \mathbf{n}$. Similarly, maps $f : \mathbf{m} \to \mathbf{n}$ from $\operatorname{Mor} \mathcal{S}$ correspond to partitions of the set $\mathbf{m}$ into disjoint union of n subsets.

Let $\mathcal{C}$ be a category. For each set $I \in \operatorname{Ob} \mathcal{O}$, denote by $\mathcal{C}^I$ the category of functors from I to $\mathcal{C}$, where I is regarded as a discrete category. Thus an object of $\mathcal{C}^I$ is a function $X : I \to \operatorname{Ob} \mathcal{C}$, $i \mapsto X_i$, i.e., a family $(X_i)_{i \in I}$ of objects of $\mathcal{C}$, and the set of morphisms between $(X_i)_{i \in I}$ and $(Y_i)_{i \in I}$ is given by

$$\mathcal{C}^I((X_i)_{i \in I}, (Y_i)_{i \in I}) = \prod_{i \in I} \mathcal{C}(X_i, Y_i).$$

In particular, $\mathcal{C}^{\mathbf{n}}$ is the ordinary $\mathcal{C}^n$, $\mathcal{C}^{\mathbf{0}}$ is the category with one object and one morphism.

1.1.1. Definition. A *lax Monoidal* (resp. *lax symmetric Monoidal*) *category* $(\mathcal{C}, \otimes^I, \lambda^f)$ consists of the following data.

- A category $\mathcal{C}$.
- A functor $\otimes^I : \mathcal{C}^I \to \mathcal{C}$, $(X_i)_{i \in I} \mapsto \otimes^{i \in I} X_i$, for each $I \in \operatorname{Ob} \mathcal{O}$, such that $\otimes^I = \mathrm{Id}_{\mathcal{C}}$, for each 1-element set I. For each map $f : I \to J$ in $\operatorname{Mor} \mathcal{O}$ (resp. $\operatorname{Mor} \mathcal{S}$), introduce a functor $\otimes^f : \mathcal{C}^I \to \mathcal{C}^J$ that to a function $X : I \to \operatorname{Ob} \mathcal{C}$, $i \mapsto X_i$, assigns the function $J \to \operatorname{Ob} \mathcal{C}$, $j \mapsto \otimes^{i \in f^{-1}j} X_i$. The functor $\otimes^f : \mathcal{C}^I \to \mathcal{C}^J$ acts on morphism via the map
$$\prod_{i \in I} \mathcal{C}(X_i, Y_i) \xrightarrow{\sim} \prod_{j \in J} \prod_{i \in f^{-1}j} \mathcal{C}(X_i, Y_i) \xrightarrow{\prod_{j \in J} \otimes^{f^{-1}j}} \prod_{j \in J} \mathcal{C}(\otimes^{i \in f^{-1}j} X_i, \otimes^{i \in f^{-1}j} Y_i).$$
- A morphism of functors
$$\lambda^f : \otimes^I \to \otimes^J \circ \otimes^f : \mathcal{C}^I \to \mathcal{C}, \qquad \lambda^f : \otimes^{i \in I} X_i \to \otimes^{j \in J} \otimes^{i \in f^{-1}j} X_i,$$
for each map $f : I \to J$ in $\operatorname{Mor} \mathcal{O}$ (resp. $\operatorname{Mor} \mathcal{S}$).

These are required to satisfy the following conditions.

- For each order-preserving bijection $f : I \to J$,
$$\lambda^f = \mathrm{id}\,.$$
For each set $I \in \operatorname{Ob} \mathcal{O}$ and for each 1-element set J,
$$\lambda^{I \to J} = \mathrm{id}\,.$$
- For each pair of composable maps $I \xrightarrow{f} J \xrightarrow{g} K$ from $\operatorname{Mor} \mathcal{O}$ (resp. $\operatorname{Mor} \mathcal{S}$), the diagram
$$\begin{tikzcd} \otimes^{i\in I} X_i \arrow[r, "\lambda^f"] \arrow[d, "\lambda^{fg}"'] & \otimes^{j\in J} \otimes^{i\in f^{-1}j} X_i \arrow[d, "\lambda^g"] \\ \otimes^{k\in K} \otimes^{i\in f^{-1}g^{-1}k} X_i \arrow[r, "\otimes^{k\in K}\lambda^{f_k}"] & \otimes^{k\in K} \otimes^{j\in g^{-1}k} \otimes^{i\in f^{-1}j} X_i \end{tikzcd} \tag{1.1.1}$$
commutes, where $f_k = f|_{f^{-1}g^{-1}k} : f^{-1}g^{-1}k \to g^{-1}k$, $k \in K$.

A *Monoidal* (resp. *symmetric Monoidal*) *category* is a lax Monoidal (resp. symmetric Monoidal) category in which each λ^f is an isomorphism.

1.1.2. Remark. For each $I \in \operatorname{Ob} \mathcal{O}$, there exists a unique order-preserving bijection $f : I \to \mathbf{n}$ for some $n \geqslant 0$. If $I = \{i_1 < i_2 < \cdots < i_n\}$, then f is given by $f(i_k) = k$, $k \in \mathbf{n}$. Since λ^f is the identity, it follows that
$$\otimes^{i\in I} X_i = \otimes^{k\in\mathbf{n}} X_{f^{-1}(k)} = \otimes^{k\in\mathbf{n}} X_{i_k} = \otimes^{\mathbf{n}}(X_{i_1}, X_{i_2}, \dots, X_{i_n}),$$
for each $X_i \in \operatorname{Ob} \mathcal{C}$, $i \in I$. Thus the meaning of the condition that λ^f is the identity, for each order-preserving map f, is that the tensor product of a family of objects depends on the cardinality of the family, but not on the indexing set.

1.1.3. Remark. An arbitrary Monoidal structure on a category $\mathcal{C}$ produces a monoidal structure as follows. The tensor product is given by the functor $\otimes = \otimes^{\mathbf{2}} : \mathcal{C}^2 \to \mathcal{C}$, $(X, Y) \mapsto X \otimes Y \overset{\text{def}}{=} \otimes^{\mathbf{2}}(X, Y)$. The unit object $\mathbb{1}$ is $\otimes^{\mathbf{0}}(*)$, where $*$ is the only object of the category $\mathcal{C}^{\mathbf{0}}$. The associativity constraint is given by the composite
$$a_{X,Y,Z} = \bigl[(X \otimes Y) \otimes Z \xrightarrow{(\lambda^{\mathsf{VI}})^{-1}} \otimes^{\mathbf{3}}(X, Y, Z) \xrightarrow{\lambda^{\mathsf{IV}}} X \otimes (Y \otimes Z)\bigr],$$
where the maps are
$$\mathsf{VI} : \mathbf{3} \to \mathbf{2}, \quad 1 \mapsto 1, \quad 2 \mapsto 1, \quad 3 \mapsto 2,$$
$$\mathsf{IV} : \mathbf{3} \to \mathbf{2}, \quad 1 \mapsto 1, \quad 2 \mapsto 2, \quad 3 \mapsto 2.$$
The unit constraints are
$$\ell_X = \lambda^{\cdot\,\mathsf{I}} : X \to \mathbb{1} \otimes X, \quad r_X = \lambda^{\mathsf{I}\,\cdot} : X \to X \otimes \mathbb{1},$$
where $\mathsf{I}\,. : \mathbf{1} \to \mathbf{2}$, $1 \mapsto 1$, $.\,\mathsf{I} : \mathbf{1} \to \mathbf{2}$, $1 \mapsto 2$. The pentagon axiom and triangle axiom are proven in [**37**, Section 1.2]. If, furthermore, $\mathcal{C}$ is a symmetric Monoidal category, then so defined monoidal structure on $\mathcal{C}$ is symmetric, with the symmetry given by
$$c_{X,Y} = \lambda^{\mathsf{X}} : X \otimes Y \to Y \otimes X,$$
where the map $\mathsf{X} = (12) : \mathbf{2} \to \mathbf{2}$, $1 \mapsto 2$, $2 \mapsto 1$. The axioms for the symmetry are proven in [**37**, Section 1.2]. Conversely, an arbitrary (symmetric) monoidal structure on a category $\mathcal{C}$ extends to a (symmetric) Monoidal structure, cf. [**12**, Proposition 1.5].

1.1.4. Example. Endow the category **Set** of sets with the following symmetric Monoidal structure. For each integer $n \geqslant 0$ and sets $X_1, X_2, \dots, X_n$, choose a product $\prod_{k\in\mathbf{n}} X_k$. In the case $n = 1$, take the product of the family consisting of an object X_1 to be just X_1. Denote by $\mathrm{pr}_p : \prod_{k\in\mathbf{n}} X_k \to X_p$, $p \in \mathbf{n}$, the canonical projections. For an arbitrary linearly ordered set $I = \{i_1 < i_2 < \dots < i_n\}$ and a family of sets $(X_i)_{i\in I}$, define $\prod_{i\in I} X_i$ as $\prod_{k\in\mathbf{n}} X_{i_k}$. The maps $\mathrm{pr}_{i_p} \overset{\text{def}}{=} \mathrm{pr}_p : \prod_{i\in I} X_i = \prod_{k\in\mathbf{n}} X_{i_k} \to X_{i_p}$ turn the set $\prod_{i\in I} X_i$ into a product of the family $(X_i)_{i\in I}$. For a map $f : I \to J$ in $\operatorname{Mor}\mathcal{S}$, the composites

$$\prod_{j\in J}\prod_{i\in f^{-1}j} X_i \xrightarrow{\mathrm{pr}_{f(p)}} \prod_{i\in f^{-1}f(p)} X_i \xrightarrow{\mathrm{pr}_p} X_p, \quad p \in I,$$

turn the iterated product $\prod_{j\in J}\prod_{i\in f^{-1}j} X_i$ into a product of the family $(X_i)_{i\in I}$. By the universal property of product, there exists a unique isomorphism

$$\lambda^f_{\mathbf{Set}} : \prod_{i\in I} X_i \xrightarrow{\sim} \prod_{j\in J}\prod_{i\in f^{-1}j} X_i$$

that makes the diagrams

$$\begin{tikzcd}
\prod_{i\in I} X_i \arrow[r, "\mathrm{pr}_p"] \arrow[d, dashed, "\lambda^f_{\mathbf{Set}}"'] & X_p \\
\prod_{j\in J}\prod_{i\in f^{-1}j} X_i \arrow[r, "\mathrm{pr}_{f(p)}"] & \prod_{i\in f^{-1}f(p)} X_i \arrow[u, "\mathrm{pr}_p"']
\end{tikzcd}$$

commute, for each $p \in I$. Equation (1.1.1) follows from the uniqueness part of the universal property.

1.1.5. Definition. A *lax Monoidal* (resp. *lax symmetric Monoidal*) *functor* between lax Monoidal (resp. lax symmetric Monoidal) categories

$$(F, \phi^I) : (\mathcal{C}, \otimes^I, \lambda^f) \to (\mathcal{D}, \otimes^I, \lambda^f)$$

consists of

- a functor $F : \mathcal{C} \to \mathcal{D}$, and
- a morphism of functors
  $$\phi^I : \otimes^I \circ F^I \to F \circ \otimes^I : \mathcal{C}^I \to \mathcal{D}, \qquad \phi^I : \otimes^{i\in I} FX_i \to F \otimes^{i\in I} X_i,$$
  for each $I \in \operatorname{Ob}\mathcal{O}$,

such that

- $\phi^I = \mathrm{id}_F$, for each 1-element set I, and
- for each map $f : I \to J$ in $\operatorname{Mor}\mathcal{O}$ (resp. $\operatorname{Mor}\mathcal{S}$) and for an arbitrary family $(X_i)_{i\in I}$ of objects of $\mathcal{C}$, the diagram
  $$\begin{tikzcd}
  \otimes^{i\in I} FX_i \arrow[rr, "\phi^I"] \arrow[d, "\lambda^f"'] & & F \otimes^{i\in I} X_i \arrow[d, "F\lambda^f"] \\
  \otimes^{j\in J} \otimes^{i\in f^{-1}j} FX_i \arrow[r, "\otimes^{j\in J}\phi^{f^{-1}j}"] & \otimes^{j\in J} F \otimes^{i\in f^{-1}j} X_i \arrow[r, "\phi^J"] & F \otimes^{j\in J} \otimes^{i\in f^{-1}j} X_i
  \end{tikzcd} \tag{1.1.2}$$
  commutes.

A *Monoidal* (resp. *symmetric Monoidal*) *functor* is a lax Monoidal (resp. lax symmetric Monoidal) functor such that each ϕ^I is an isomorphism.

1.1.6. Definition. A *Monoidal transformation*

$$t : (F, \phi^I) \to (G, \psi^I) : (\mathcal{C}, \otimes^I, \lambda^f) \to (\mathcal{D}, \otimes^I, \lambda^f)$$

is a natural transformation $t : F \to G$ such that for each $I \in \operatorname{Ob} \mathcal{O}$ the diagram

$$\begin{tikzcd}
\otimes^{i\in I} FX_i \arrow[r, "\phi^I"] \arrow[d, "\otimes^{i\in I} t"'] & F \otimes^{i\in I} X_i \arrow[d, "t"] \\
\otimes^{i\in I} GX_i \arrow[r, "\psi^I"] & G \otimes^{i\in I} X_i
\end{tikzcd} \tag{1.1.3}$$

commutes.

A Monoidal category $(\mathcal{C}, \otimes^I, \lambda^f)$ is called *strict* if $\lambda^f : \otimes^I \to \otimes^J \circ \otimes^f$ is the identity transformation, for each order-preserving map $f : I \to J$. A Monoidal (resp. symmetric Monoidal) category is equivalent to a strict Monoidal (resp. strict symmetric Monoidal) category by [**37**, Theorem 1.2.7]. Strict Monoidal (resp. strict symmetric Monoidal) categories are in bijection with strict monoidal (resp. strict symmetric monoidal) categories by [**37**, Propositions 1.2.15, 1.2.17]. These results lead to the following useful observation.

1.1.7. Lemma (Coherence principle)**.** *An equation between isomorphisms of functors constructed from the data of Monoidal (resp. symmetric Monoidal) category holds true if it is satisfied for arbitrary strict Monoidal (resp. strict symmetric Monoidal) categories.*

Proof. A Monoidal category $\mathcal{C}$ is Monoidally equivalent to a strict Monoidal category $\mathcal{A}$ by [**37**, Theorem 1.2.7]. The equation we consider holds in $\mathcal{A}$ by assumption. A Monoidal equivalence $(F, \phi^I) : \mathcal{C} \to \mathcal{A}$ gives rise to a prism (in which edges are isomorphisms) with commutative walls, whose bottom is the considered equation in $\mathcal{A}$. Therefore, its top, which is the required equation in $\mathcal{C}$, also commutes. □

1.1.8. Remark. All isomorphisms of functors which can be constructed for arbitrary symmetric strictly monoidal category data coincide if their source and target coincide. Therefore, the same property holds for isomorphisms of functors which can be constructed for arbitrary symmetric Monoidal category data. Similarly, all isomorphisms of functors which can be constructed from λ^f with monotonic f for arbitrary Monoidal category data coincide if their source and target coincide.

We are going to define lax Monoidal categories enriched in a symmetric Monoidal category $\mathcal{V} = (\mathcal{V}, \otimes^I_{\mathcal{V}}, \lambda^f_{\mathcal{V}})$. The definition of $\mathcal{V}$-category is briefly recalled in Section 0.2. The reader is referred to [**29**] for more details concerning enriched categories.

For $I, J \in \operatorname{Ob} \mathcal{O}$ and a family of objects $I \times J \ni (i, j) \mapsto X_{ij} \in \operatorname{Ob} \mathcal{V}$, define a natural permutation isomorphism

$$\sigma_{(12)} = \bigl[\otimes^{j\in J}_{\mathcal{V}} \otimes^{i\in I}_{\mathcal{V}} X_{ij} \xrightarrow{(\lambda_{\mathcal{V}}^{\mathrm{pr}_2 : I\times J \to J})^{-1}} \otimes^{(i,j)\in I\times J}_{\mathcal{V}} X_{ij} \xrightarrow{\lambda_{\mathcal{V}}^{\mathrm{pr}_1 : I\times J \to I}} \otimes^{i\in I}_{\mathcal{V}} \otimes^{j\in J}_{\mathcal{V}} X_{ij}\bigr]. \tag{1.1.4}$$

For a $\mathcal{V}$-category $\mathcal{C}$ and $I \in \operatorname{Ob} \mathcal{O}$, define $\mathcal{C}^I$ to be the $\mathcal{V}$-category of functions on I with values in $\mathcal{C}$:

$$\operatorname{Ob} \mathcal{C}^I = \{\text{ maps } I \to \operatorname{Ob} \mathcal{C} : i \mapsto X_i\}, \quad \mathcal{C}^I((X_i)_{i\in I}, (Y_i)_{i\in I}) = \otimes^{i\in I}_{\mathcal{V}} \mathcal{C}(X_i, Y_i).$$

Composition in $\mathcal{C}^I$ is given by the morphism in $\mathcal{V}$

$$\mu_{\mathcal{C}^I} = \bigl[(\otimes^{i\in I}_{\mathcal{V}} \mathcal{C}(X_i, Y_i)) \otimes_{\mathcal{V}} (\otimes^{i\in I}_{\mathcal{V}} \mathcal{C}(Y_i, Z_i)) \xrightarrow{\sigma_{(12)}}$$

$$\otimes^{i\in I}_{\mathcal{V}} (\mathcal{C}(X_i, Y_i) \otimes_{\mathcal{V}} \mathcal{C}(Y_i, Z_i)) \xrightarrow{\otimes^{i\in I}_{\mathcal{V}} \mu_{\mathcal{C}}} \otimes^{i\in I}_{\mathcal{V}} \mathcal{C}(X_i, Z_i)\bigr],$$

for each $X_i, Y_i, Z_i \in \operatorname{Ob}\mathcal{C}$, $i \in I$. The identity of an object $(X_i)_{i\in I}$ is given by the composite in $\mathcal{V}$

$$1^{\mathcal{C}^I}_{(X_i)_{i\in I}} = \big[\mathbb{1}_\mathcal{V} \xrightarrow{\lambda^{\varnothing\to I}_\mathcal{V}} \otimes^{i\in I}_\mathcal{V}\mathbb{1}_\mathcal{V} \xrightarrow{\otimes^{i\in I}_\mathcal{V} 1^\mathcal{C}_{X_i}} \otimes^{i\in I}_\mathcal{V}\mathcal{C}(X_i, X_i)\big].$$

In particular, $\mathcal{C}^\varnothing$ is the $\mathcal{V}$-category $\mathbb{1}$ with one object $*$, whose endomorphism object is the unit object $\mathbb{1}_\mathcal{V}$ of $\mathcal{V}$. For each 1-element set I, we identify $\mathcal{C}^I$ with $\mathcal{C}$ via the obvious isomorphism.

1.1.9. Definition. A *lax Monoidal* (resp. *lax symmetric Monoidal*) *$\mathcal{V}$-category* $(\mathcal{C}, \otimes^I, \lambda^f)$ consists of the following data.

- A $\mathcal{V}$-category $\mathcal{C}$.
- A $\mathcal{V}$-functor $\otimes^I : \mathcal{C}^I \to \mathcal{C}$, $(X_i)_{i\in I} \mapsto \otimes^{i\in I} X_i$, for each set $I \in \operatorname{Ob}\mathcal{O}$, such that $\otimes^I = \mathrm{Id}_\mathcal{C}$, for each 1-element set I. For each map $f : I \to J$ in $\operatorname{Mor}\mathcal{O}$ (resp. $\operatorname{Mor}\mathcal{S}$), introduce a $\mathcal{V}$-functor $\otimes^f : \mathcal{C}^I \to \mathcal{C}^J$ that to a function $X : I \to \operatorname{Ob}\mathcal{C}$, $i \mapsto X_i$, assigns the function $J \to \operatorname{Ob}\mathcal{C}$, $j \mapsto \otimes^{i\in f^{-1}j} X_i$. The functor $\otimes^f$ acts on morphisms via the morphism

$$\otimes^{i\in I}_\mathcal{V}\mathcal{C}(X_i, Y_i) \xrightarrow[\sim]{\lambda^f_\mathcal{V}} \otimes^{j\in J}_\mathcal{V} \otimes^{i\in f^{-1}j}_\mathcal{V} \mathcal{C}(X_i, Y_i) \xrightarrow{\otimes^{j\in J}_\mathcal{V}\otimes^{f^{-1}j}} \otimes^{j\in J}_\mathcal{V}\mathcal{C}(\otimes^{i\in f^{-1}j} X_i, \otimes^{i\in f^{-1}j} Y_i).$$

- A morphism of $\mathcal{V}$-functors

$$\begin{tikzcd}
\mathcal{C}^I \arrow[r, "\otimes^f"] \arrow[dr, "\otimes^I"', ""{name=A}] & \mathcal{C}^J \arrow[d, "\otimes^J"] \\
& \mathcal{C}
\arrow[Rightarrow, from=A, to=1-2, "\lambda^f"']
\end{tikzcd}$$

  consisting of morphisms

$$\lambda^f : \mathbb{1}_\mathcal{V} \to \mathcal{C}(\otimes^{i\in I} X_i, \otimes^{j\in J} \otimes^{i\in f^{-1}j} X_i), \quad X_i \in \operatorname{Ob}\mathcal{C}, \quad i \in I,$$

  for each map $f : I \to J$ in $\operatorname{Mor}\mathcal{O}$ (resp. $\operatorname{Mor}\mathcal{S}$).

These data are subject to the following conditions.

(a) For each order-preserving bijection $f : I \to J$,

$$\lambda^f = \mathrm{id}\,.$$

For each set $I \in \operatorname{Ob}\mathcal{O}$ and for each 1-element set J,

$$\lambda^{I\to J} = \mathrm{id}\,.$$

(b) for each pair of composable maps $I \xrightarrow{f} J \xrightarrow{g} K$ from $\operatorname{Mor}\mathcal{O}$ (resp. from $\operatorname{Mor}\mathcal{S}$) the equation

$$\begin{tikzcd}
\mathcal{C}^J \arrow[r, "\otimes^g"] \arrow[dr, "\otimes^J"', ""{name=D}] & \mathcal{C}^K \arrow[d, "\otimes^K"] \\
\mathcal{C}^I \arrow[u, "\otimes^f"] \arrow[r, "\otimes^I"', ""{name=B}] & \mathcal{C}
\arrow[Rightarrow, from=B, to=1-1, "\lambda^f"]
\arrow[Rightarrow, from=D, to=1-2, "\lambda^g"]
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}^J \arrow[r, "\otimes^g"] & \mathcal{C}^K \arrow[d, "\otimes^K"] \\
\mathcal{C}^I \arrow[u, "\otimes^f"] \arrow[ur, "\otimes^{fg}"', ""{name=E}] \arrow[r, "\otimes^I"', ""{name=F}] & \mathcal{C}
\arrow[Rightarrow, from=E, to=1-1, "\otimes^{k\in K}_\mathcal{V}\lambda^{f_k}"]
\arrow[Rightarrow, from=F, to=1-2, "\lambda^{fg}"']
\end{tikzcd}
\tag{1.1.5}$$

holds true, where $f_k = f|_{f^{-1}g^{-1}k} : f^{-1}g^{-1}k \to g^{-1}k$, $k \in K$, and the natural transformation $\otimes_{\mathcal{V}}^{k\in K}\lambda^{f_k}$ is given by the composite

$$\mathbb{1}_{\mathcal{V}} \xrightarrow{\lambda_{\mathcal{V}}^{\varnothing\to K}} \otimes_{\mathcal{V}}^{k\in K}\mathbb{1}_{\mathcal{V}} \xrightarrow{\otimes_{\mathcal{V}}^{k\in K}\lambda^{f_k}} \otimes_{\mathcal{V}}^{k\in K}\mathcal{C}(\otimes^{i\in f^{-1}g^{-1}k}X_i, \otimes^{j\in g^{-1}k}\otimes^{i\in f^{-1}j} X_i).$$

A *Monoidal* (resp. *symmetric Monoidal*) $\mathcal{V}$*-category* is a lax Monoidal (resp. symmetric Monoidal) $\mathcal{V}$-category in which each λ^f is an isomorphism.

A $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$ gives rise to $\mathcal{V}$-functor $F^I : \mathcal{C}^I \to \mathcal{D}^I$, $(X_i)_{i\in I} \mapsto (FX_i)_{i\in I}$, that acts on morphisms via

$$\mathcal{C}^I((X_i)_{i\in I}, (Y_i)_{i\in I}) = \otimes_{\mathcal{V}}^{i\in I}\mathcal{C}(X_i, Y_i) \xrightarrow{\otimes_{\mathcal{V}}^{i\in I}F_{X_i,Y_i}} \otimes_{\mathcal{V}}^{i\in I}\mathcal{D}(FX_i, FY_i) = \mathcal{D}^I((FX_i)_{i\in I}, (FY_i)_{i\in I}),$$

for each $X_i, Y_i \in \operatorname{Ob}\mathcal{C}$, $i \in I$.

1.1.10. Definition. A *lax Monoidal* (resp. *lax symmetric Monoidal*) $\mathcal{V}$*-functor* between lax Monoidal (resp. lax symmetric Monoidal) $\mathcal{V}$-categories

$$(F, \phi^I) : (\mathcal{C}, \otimes_{\mathcal{C}}^I, \lambda_{\mathcal{C}}^f) \to (\mathcal{D}, \otimes_{\mathcal{D}}^I, \lambda_{\mathcal{D}}^f)$$

consists of

- a $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$, and
- a morphism of $\mathcal{V}$-functors

$$\begin{tikzcd}
\mathcal{C}^I \arrow[r, "F^I"] \arrow[d, "\otimes_{\mathcal{C}}^I"'] & \mathcal{D}^I \arrow[d, "\otimes_{\mathcal{D}}^I"] \arrow[dl, Rightarrow, "\phi^I"'] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{D}
\end{tikzcd}$$

for each set $I \in \operatorname{Ob}\mathcal{O}$,

such that $\phi^I = \mathrm{id}_F$, for each 1-element set I, and for each map $f : I \to J$ of $\operatorname{Mor}\mathcal{O}$ (resp. $\operatorname{Mor}\mathcal{S}$) the equation

$$\begin{tikzcd}
 & \mathcal{C}^I \arrow[rr, "F^I"] \arrow[dl, "\otimes_{\mathcal{C}}^f"'] & & \mathcal{D}^I \arrow[dl, "\otimes_{\mathcal{D}}^f"'] \arrow[dd, "\otimes_{\mathcal{D}}^I"] \arrow[dlll, Rightarrow, "\otimes_{\mathcal{V}}^{j\in J}\phi^{f^{-1}j}"'] \\
\mathcal{C}^J \arrow[rr, "F^J"'] \arrow[dr, "\otimes_{\mathcal{C}}^J"'] & & \mathcal{D}^J \arrow[dl, Rightarrow, "\phi^J"'] \arrow[dr, "\otimes_{\mathcal{D}}^J"'] & {} \arrow[l, Rightarrow, "\lambda_{\mathcal{D}}^f"] \\
 & \mathcal{C} \arrow[rr, "F"'] & & \mathcal{D}
\end{tikzcd}
=
\begin{tikzcd}
 & \mathcal{C}^I \arrow[r, "F^I"] \arrow[dl, "\otimes_{\mathcal{C}}^f"'] \arrow[dd, "\otimes_{\mathcal{C}}^I"] & \mathcal{D}^I \arrow[dd, "\otimes_{\mathcal{D}}^I"] \arrow[ddl, Rightarrow, "\phi^I"] \\
\mathcal{C}^J \arrow[dr, "\otimes_{\mathcal{C}}^J"'] & {} \arrow[l, Rightarrow, "\lambda_{\mathcal{C}}^f"'] & \\
 & \mathcal{C} \arrow[r, "F"'] & \mathcal{D}
\end{tikzcd}
\qquad (1.1.6)$$

holds true, where the natural transformation $\otimes_{\mathcal{V}}^{j\in J}\phi^{f^{-1}j}$ is given by the composite

$$\begin{aligned}\mathbb{1}_{\mathcal{V}} \xrightarrow{\lambda_{\mathcal{V}}^{\varnothing\to J}} \otimes_{\mathcal{V}}^{j\in J}\mathbb{1}_{\mathcal{V}} \xrightarrow{\otimes_{\mathcal{V}}^{j\in J}\phi^{f^{-1}j}} \otimes_{\mathcal{V}}^{j\in J}&\mathcal{D}(\otimes^{i\in f^{-1}j}FX_i, F\otimes^{i\in f^{-1}j} X_i) \\ &= \mathcal{D}^J((\otimes^{i\in f^{-1}j}FX_i)_{j\in J}, (F\otimes^{i\in f^{-1}j} X_i)_{j\in J}).\end{aligned}$$

A *Monoidal* (resp. *symmetric Monoidal*) $\mathcal{V}$*-functor* is a lax Monoidal (resp. lax symmetric Monoidal) $\mathcal{V}$-functor such that each ϕ^I is an isomorphism.

An arbitrary $\mathcal{V}$-natural transformation $t : F \to G : \mathcal{C} \to \mathcal{D}$, represented by a family of morphisms $t_X : \mathbb{1} \to \mathcal{D}(FX, GX)$, $X \in \operatorname{Ob} \mathcal{C}$, gives rise to a $\mathcal{V}$-natural transformation $t^I : F^I \to G^I : \mathcal{C}^I \to \mathcal{D}^I$, represented by

$$t^I_{(X_i)_{i\in I}} = \bigl[\mathbb{1}_{\mathcal{V}} \xrightarrow{\lambda^{\varnothing\to I}_{\mathcal{V}}} \otimes^{i\in I}_{\mathcal{V}} \mathbb{1}_{\mathcal{V}} \xrightarrow{\otimes^{i\in I}_{\mathcal{V}} t_{X_i}} \otimes^{i\in I}_{\mathcal{V}} \mathcal{D}(FX_i, GX_i) = \mathcal{D}^I((FX_i)_{i\in I}, (GX_i)_{i\in I})\bigr],$$

for each $X_i \in \operatorname{Ob} \mathcal{C}$, $i \in I$.

1.1.11. Definition. A *Monoidal $\mathcal{V}$-transformation*

$$t : (F, \phi^I) \to (G, \psi^I) : (\mathcal{C}, \otimes^I_{\mathcal{C}}, \lambda^f_{\mathcal{C}}) \to (\mathcal{D}, \otimes^I_{\mathcal{D}}, \lambda^f_{\mathcal{D}})$$

is a $\mathcal{V}$-natural transformation $t : F \to G$ such that for each $I \in \operatorname{Ob} \mathcal{O}$ the following equation holds:

$$\begin{tikzcd}
\mathcal{C}^I \arrow[r, bend left, "F^I", ""{name=U, below}] \arrow[r, bend right, "G^I"', ""{name=L, above}] \arrow[d, "\otimes^I_{\mathcal{C}}"'] & \mathcal{D}^I \arrow[d, "\otimes^I_{\mathcal{D}}"] \arrow[dl, Rightarrow, "\psi^I"] \\
\mathcal{C} \arrow[r, "G"] & \mathcal{D}
\arrow[Rightarrow, from=U, to=L, "t^I"]
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}^I \arrow[r, "F"] \arrow[d, "\otimes^I_{\mathcal{C}}"'] & \mathcal{D}^I \arrow[d, "\otimes^I_{\mathcal{D}}"] \arrow[dl, Rightarrow, "\phi^I"'] \\
\mathcal{C} \arrow[r, bend left, "F", ""{name=U, below}] \arrow[r, bend right, "G"', ""{name=L, above}] & \mathcal{D}
\arrow[Rightarrow, from=U, to=L, "t"]
\end{tikzcd}
\tag{1.1.7}$$

1.1.12. Example. Let $\mathcal{V} = \mathbf{Cat}$ be the symmetric Monoidal category of categories. The tensor product is the Cartesian product of categories. A **Cat**-category is the same thing as a 2-category; **Cat**-functors and **Cat**-natural transformations are particular cases of 2-functors and 2-transformations. A weak 2-functor is required to preserve compositions and identities up to natural isomorphisms [**37**, Definition A.1.2]. On the other hand, a **Cat**-functor is a 2-functor that preserves compositions and identities on the nose. We say that a **Cat**-functor is a strict 2-functor. Similarly, a **Cat**-natural transformation is a strict 2-transformation. Thus, a symmetric Monoidal **Cat**-category $(\mathfrak{C}, \boxtimes^I, \Lambda^f)$ consists of the following data.

- A 2-category $\mathfrak{C}$.
- A strict 2-functor $\boxtimes^I : \mathfrak{C}^I \to \mathfrak{C}$, for each $I \in \operatorname{Ob} \mathcal{S}$, such that $\boxtimes^I = \mathrm{Id}_{\mathfrak{C}}$, for each 1-element set I. For a map $f : I \to J$ in $\operatorname{Mor} \mathcal{S}$, introduce a 2-functor $\boxtimes^f : \mathfrak{C}^I \to \mathfrak{C}^J$ that to a function $X : I \to \operatorname{Ob} \mathfrak{C}$, $i \mapsto X_i$, assigns the function $J \to \operatorname{Ob} \mathfrak{C}$, $j \mapsto \boxtimes^{i\in f^{-1}j} X_i$. The 2-functor $\boxtimes^f$ acts on categories of morphisms via the functor

$$\prod_{i\in I} \mathfrak{C}(X_i, Y_i) \xrightarrow{\sim} \prod_{j\in J} \prod_{i\in f^{-1}j} \mathfrak{C}(X_i, Y_i) \xrightarrow{\prod_{j\in J} \boxtimes^{f^{-1}j}} \prod_{j\in J} \mathfrak{C}(\boxtimes^{i\in f^{-1}j} X_i, \boxtimes^{i\in f^{-1}j} Y_i).$$

- An invertible strict 2-transformation

$$\begin{tikzcd}
\mathfrak{C}^I \arrow[r, "\boxtimes^f"] \arrow[dr, "\boxtimes^I"', ""{name=A}] & \mathfrak{C}^J \arrow[d, "\boxtimes^J"] \\
& \mathfrak{C}
\arrow[Rightarrow, from=A, to=1-2, "\Lambda^f"]
\end{tikzcd}
\tag{1.1.8}$$

consisting of invertible 1-morphisms $\Lambda^f : \boxtimes^{i\in I} X_i \to \boxtimes^{j\in J} \boxtimes^{i\in f^{-1}j} X_i$, $X_i \in \operatorname{Ob} \mathfrak{C}$, for each map $f : I \to J$ in $\operatorname{Mor} \mathcal{S}$.

These data are required to satisfy the following conditions.

(a) For each order-preserving bijection $f : I \to J$,

$$\Lambda^f = \mathrm{id}\,.$$

For each set $I \in \operatorname{Ob} \mathcal{O}$ and for each 1-element set J,

$$\Lambda^{I \to J} = \mathrm{id}\,.$$

(b) For each pair of composable maps $I \xrightarrow{f} J \xrightarrow{g} K$ from $\operatorname{Mor} \mathcal{S}$ the diagram

$$\begin{tikzcd}
\boxtimes^{i\in I} X_i \arrow[r, "\Lambda^f"] \arrow[d, "\Lambda^{fg}"'] & \boxtimes^{j\in J} \boxtimes^{i\in f^{-1}j} X_i \arrow[d, "\Lambda^g"] \\
\boxtimes^{k\in K} \boxtimes^{i\in f^{-1}g^{-1}k} X_i \arrow[r, "\boxtimes^{k\in K}\Lambda^{f_k}"] & \boxtimes^{k\in K} \boxtimes^{j\in g^{-1}k} \boxtimes^{i\in f^{-1}j} X_i
\end{tikzcd} \tag{1.1.9}$$

commutes, where $f_k = f|_{f^{-1}g^{-1}k} : f^{-1}g^{-1}k \to g^{-1}k$, $k \in K$.

A symmetric Monoidal **Cat**-category is a particular case of a symmetric monoidal 2-category, as defined e.g. in [**37**, Definition A.6.1].

1.1.13. Example. Let $\mathcal{V} = (\mathcal{V}, \otimes^I_{\mathcal{V}}, \lambda^f_{\mathcal{V}})$ be a symmetric Monoidal category. The 2-category $\mathfrak{C} = \mathcal{V}$-**Cat** of $\mathcal{V}$-categories, $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations is a symmetric Monoidal **Cat**-category. Indeed, a strict 2-functor $\boxtimes^I : \mathfrak{C}^I \to \mathfrak{C}$ is defined as follows. The tensor product of $\mathcal{V}$-categories $\mathcal{C}_i$, $i \in I$, is a $\mathcal{V}$-category $\mathcal{C} = \boxtimes^{i\in I}\mathcal{C}_i$ whose set of objects is $\operatorname{Ob}\mathcal{C} = \prod_{i\in I}\operatorname{Ob}\mathcal{C}_i$, and whose objects of morphisms are given by $\mathcal{C}((X_i)_{i\in I}, (Y_i)_{i\in I}) = \otimes^{i\in I}_{\mathcal{V}}\mathcal{C}_i(X_i, Y_i)$, for each $X_i, Y_i \in \operatorname{Ob}\mathcal{C}_i$, $i \in I$. Composition in $\mathcal{C}$ is given by the morphism in $\mathcal{V}$

$$\mu_{\mathcal{C}} = \bigl[(\otimes^{i\in I}_{\mathcal{V}}\mathcal{C}_i(X_i, Y_i)) \otimes_{\mathcal{V}} (\otimes^{i\in I}_{\mathcal{V}}\mathcal{C}_i(Y_i, Z_i)) \xrightarrow{\sigma_{(12)}}$$

$$\otimes^{i\in I}_{\mathcal{V}}\left(\mathcal{C}_i(X_i, Y_i) \otimes \mathcal{C}_i(Y_i, Z_i)\right) \xrightarrow{\otimes^{i\in I}_{\mathcal{V}}\mu_{\mathcal{C}_i}} \otimes^{i\in I}_{\mathcal{V}}\mathcal{C}_i(X_i, Z_i)\bigr],$$

for each $X_i, Y_i, Z_i \in \operatorname{Ob}\mathcal{C}_i$, $i \in I$. For each $X_i \in \operatorname{Ob}\mathcal{C}_i$, $i \in I$, the identity of the object $(X_i)_{i\in I} \in \operatorname{Ob}\mathcal{C}$ is given by the composite in $\mathcal{V}$

$$1^{\mathcal{C}}_{(X_i)_{i\in I}} = \bigl[\mathbb{1}_{\mathcal{V}} \xrightarrow{\lambda^{\varnothing\to I}_{\mathcal{V}}} \otimes^{i\in I}\mathbb{1}_{\mathcal{V}} \xrightarrow{\otimes^{i\in I}_{\mathcal{V}}1^{\mathcal{C}_i}_{X_i}} \otimes^{i\in I}_{\mathcal{V}}\mathcal{C}_i(X_i, X_i)\bigr].$$

The tensor product of $\mathcal{V}$-functors $F_i : \mathcal{C}_i \to \mathcal{D}_i$, $i \in I$, is a $\mathcal{V}$-functor $F = \boxtimes^{i\in I}F_i$ such that $\operatorname{Ob}F = \prod_{i\in I}\operatorname{Ob}F_i$ and

$$F = \otimes^{i\in I}_{\mathcal{V}}F_i : \otimes^{i\in I}_{\mathcal{V}}\mathcal{C}_i(X_i, Y_i) \to \otimes^{i\in I}_{\mathcal{V}}\mathcal{D}_i(F_iX_i, F_iY_i),$$

for each $X_i, Y_i \in \operatorname{Ob}\mathcal{C}_i$, $i \in I$. The tensor product of $\mathcal{V}$-natural transformations

$$r_i : F_i \to G_i : \mathcal{C}_i \to \mathcal{D}_i, \quad i \in I,$$

is a $\mathcal{V}$-natural transformation $r = \boxtimes^{i\in I}r_i$ whose components are

$$r = \bigl[\mathbb{1}_{\mathcal{V}} \xrightarrow{\lambda^{\varnothing\to I}_{\mathcal{V}}} \otimes^{i\in I}_{\mathcal{V}}\mathbb{1}_{\mathcal{V}} \xrightarrow{\otimes^{i\in I}_{\mathcal{V}}r_i} \otimes^{i\in I}_{\mathcal{V}}\mathcal{D}_i(F_iX_i, G_iX_i)\bigr],$$

for each $X_i \in \operatorname{Ob}\mathcal{C}_i$, $i \in I$. Showing that $\boxtimes^I$ preserves compositions and identities is a straightforward computation.

For each map $f : I \to J$, 2-transformation (1.1.8) is given by the family of $\mathcal{V}$-functors $\Lambda^f_{\mathfrak{C}} : \boxtimes^{i\in I}\mathcal{C}_i \to \boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}\mathcal{C}_i$, defined by

$$\lambda^f_{\mathbf{Set}} : \prod_{i\in I}\operatorname{Ob}\mathcal{C}_i \to \prod_{j\in J}\prod_{i\in f^{-1}j}\operatorname{Ob}\mathcal{C}_i, \quad (X_i)_{i\in I} \mapsto ((X_i)_{i\in f^{-1}j})_{j\in J},$$

on objects, and by

$$\lambda^f_{\mathcal{V}} : \otimes^{i\in I}_{\mathcal{V}} \mathcal{C}_i(X_i, Y_i) \xrightarrow{\sim} \otimes^{j\in J}_{\mathcal{V}} \otimes^{i\in f^{-1}j}_{\mathcal{V}} \mathcal{C}_i(X_i, Y_i) \tag{1.1.10}$$

on morphisms. It is not difficult to check that $\Lambda^f_{\mathfrak{C}}$ is a strict 2-transformation. Equation (1.1.9) for $\Lambda^f_{\mathfrak{C}}$ follows from similar equation (1.1.5) for $\lambda^f_{\mathcal{V}}$. Therefore, $\mathfrak{C} = \mathcal{V}$-**Cat** is a symmetric Monoidal **Cat**-category.

1.1.14. Base change. Let $(B, \beta^I) : (\mathcal{V}, \otimes^I_{\mathcal{V}}, \lambda^f_{\mathcal{V}}) \to (\mathcal{W}, \otimes^I_{\mathcal{W}}, \lambda^f_{\mathcal{W}})$ be a lax symmetric Monoidal functor. It gives rise to a lax symmetric Monoidal **Cat**-functor

$$(B_*, \beta^I_*) : \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{W}\text{-}\mathbf{Cat},$$

which we are going to describe. For notational simplicity the tensor product in both categories $\mathcal{V}$ and $\mathcal{W}$ is denoted by $\otimes$.

To a $\mathcal{V}$-category $\mathcal{C}$ a $\mathcal{W}$-category $B_*\mathcal{C}$ is assigned, with the same set of objects. For each pair of objects $X, Y \in \operatorname{Ob}\mathcal{C} = \operatorname{Ob} B_*\mathcal{C}$, there is an object $B_*\mathcal{C}(X,Y) = B\mathcal{C}(X,Y)$ of the category $\mathcal{W}$. The identity of an object $X \in \operatorname{Ob}\mathcal{C} = \operatorname{Ob} B_*\mathcal{C}$ is

$$1^{B_*\mathcal{C}}_X = \bigl[\mathbb{1}_{\mathcal{W}} \xrightarrow{\beta^{\varnothing}} B\mathbb{1}_{\mathcal{V}} \xrightarrow{B1^{\mathcal{C}}_X} B\mathcal{C}(X,X)\bigr].$$

Composition in $B_*\mathcal{C}$ is given by

$$\mu_{B_*\mathcal{C}} = \bigl[B\mathcal{C}(X,Y) \otimes B\mathcal{C}(Y,Z) \xrightarrow{\beta^{\mathbf{2}}} B(\mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z)) \xrightarrow{B\mu_{\mathcal{C}}} B\mathcal{C}(X,Z)\bigr],$$

for each $X, Y, Z \in \operatorname{Ob}\mathcal{C}$. The right identity axiom follows from the commutative diagram

$$\begin{tikzcd}
B\mathcal{C}(X,Y) \otimes \mathbb{1}_{\mathcal{W}} \arrow[r, "1\otimes\beta^{\varnothing}"] & B\mathcal{C}(X,Y) \otimes B\mathbb{1}_{\mathcal{V}} \arrow[r, "1\otimes B1^{\mathcal{C}}_X"] \arrow[d, "\beta^{\mathbf{2}}"] & B\mathcal{C}(X,Y) \otimes B\mathcal{C}(Y,Y) \arrow[d, "\beta^{\mathbf{2}}"] \\
 & B(\mathcal{C}(X,Y) \otimes \mathbb{1}_{\mathcal{V}}) \arrow[r, "B(1\otimes 1^{\mathcal{C}}_X)"] & B(\mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Y)) \arrow[d, "B\mu_{\mathcal{C}}"] \\
B\mathcal{C}(X,Y) \arrow[uu, "\lambda^{\mathsf{I}\cdot}_{\mathcal{W}}"] \arrow[ur, "B\lambda^{\mathsf{I}\cdot}_{\mathcal{V}}"] \arrow[rr, equal] & & B\mathcal{C}(X,Y)
\end{tikzcd}$$

where the left trapezoid is equation (1.1.2) written for the map $\mathsf{I}. : \mathbf{1} \to \mathbf{2}$, the square commutes by the naturality of $\beta^{\mathbf{2}}$, and the bottom trapezoid is a consequence of the right identity axiom for $1^{\mathcal{C}}_X$. A similar computation shows that $1^{B_*\mathcal{C}}_X$ is also a left identity. The associativity of $\mu_{B_*\mathcal{C}}$ is expressed by the following equation:

$$\begin{tikzcd}
B_*\mathcal{C}(W,X) \otimes B_*\mathcal{C}(X,Y) \otimes B_*\mathcal{C}(Y,Z) \arrow[r, "\lambda^{\mathsf{IV}}_{\mathcal{W}}"] \arrow[d, "\lambda^{\mathsf{VI}}_{\mathcal{W}}"'] & B_*\mathcal{C}(W,X) \otimes (B_*\mathcal{C}(X,Y) \otimes B_*\mathcal{C}(Y,Z)) \arrow[d, "1\otimes\mu_{B_*\mathcal{C}}"] \\
(B_*\mathcal{C}(W,X) \otimes B_*\mathcal{C}(X,Y)) \otimes B_*\mathcal{C}(Y,Z) \arrow[d, "\mu_{B_*\mathcal{C}}\otimes 1"'] & B_*\mathcal{C}(W,X) \otimes B_*\mathcal{C}(X,Z) \arrow[d, "\mu_{B_*\mathcal{C}}"] \\
B_*\mathcal{C}(W,Y) \otimes B_*\mathcal{C}(Y,Z) \arrow[r, "\mu_{B_*\mathcal{C}}"] & B_*\mathcal{C}(W,Z)
\end{tikzcd}$$

Expanding out the top-right path yields

$$\begin{aligned}
&\big[B\mathcal{C}(W,X)\otimes B\mathcal{C}(X,Y)\otimes B\mathcal{C}(Y,Z) \xrightarrow{\lambda^{\mathsf{IV}}_{\mathcal{W}}} B\mathcal{C}(W,X)\otimes(B\mathcal{C}(X,Y)\otimes B\mathcal{C}(Y,Z))\\
&\quad\xrightarrow{1\otimes\beta^{\mathbf{2}}} B\mathcal{C}(W,X)\otimes B(\mathcal{C}(X,Y)\otimes\mathcal{C}(Y,Z))\\
&\qquad\xrightarrow{1\otimes B\mu_{\mathcal{C}}} B\mathcal{C}(W,X)\otimes B\mathcal{C}(X,Z) \xrightarrow{\beta^{\mathbf{2}}} B(\mathcal{C}(W,X)\otimes\mathcal{C}(X,Z)) \xrightarrow{B\mu_{\mathcal{C}}} B\mathcal{C}(W,Z)\big]\\
=&\big[B\mathcal{C}(W,X)\otimes B\mathcal{C}(X,Y)\otimes B\mathcal{C}(Y,Z) \xrightarrow{\lambda^{\mathsf{IV}}_{\mathcal{W}}} B\mathcal{C}(W,X)\otimes(B\mathcal{C}(X,Y)\otimes B\mathcal{C}(Y,Z))\\
&\quad\xrightarrow{1\otimes\beta^{\mathbf{2}}} B\mathcal{C}(W,X)\otimes B(\mathcal{C}(X,Y)\otimes\mathcal{C}(Y,Z))\\
&\quad\xrightarrow{\beta^{\mathbf{2}}} B(\mathcal{C}(W,X)\otimes(\mathcal{C}(X,Y)\otimes\mathcal{C}(Y,Z)))\\
&\qquad\qquad\xrightarrow{B(1\otimes\mu_{\mathcal{C}})} B(\mathcal{C}(W,X)\otimes\mathcal{C}(X,Z)) \xrightarrow{B\mu_{\mathcal{C}}} B\mathcal{C}(W,Z)\big]\\
=&\big[B\mathcal{C}(W,X)\otimes B\mathcal{C}(X,Y)\otimes B\mathcal{C}(Y,Z) \xrightarrow{\beta^{\mathbf{3}}} B(\mathcal{C}(W,X)\otimes\mathcal{C}(X,Y)\otimes\mathcal{C}(Y,Z))\\
&\quad\xrightarrow{B\lambda^{\mathsf{IV}}_{\mathcal{V}}} B(\mathcal{C}(W,X)\otimes(\mathcal{C}(X,Y)\otimes\mathcal{C}(Y,Z)))\\
&\qquad\qquad\xrightarrow{B(1\otimes\mu_{\mathcal{C}})} B(\mathcal{C}(W,X)\otimes\mathcal{C}(X,Z)) \xrightarrow{B\mu_{\mathcal{C}}} B\mathcal{C}(W,Z)\big]
\end{aligned}$$

by equation (1.1.2) written for the map $\mathsf{IV} : \mathbf{3} \to \mathbf{2}$. Similarly, the left-bottom path becomes

$$\begin{aligned}
&\big[B\mathcal{C}(W,X)\otimes B\mathcal{C}(X,Y)\otimes B\mathcal{C}(Y,Z) \xrightarrow{\beta^{\mathbf{3}}} B(\mathcal{C}(W,X)\otimes\mathcal{C}(X,Y)\otimes\mathcal{C}(Y,Z)) \xrightarrow{B\lambda^{\mathsf{VI}}_{\mathcal{V}}}\\
&\quad B((\mathcal{C}(W,X)\otimes\mathcal{C}(X,Y))\otimes\mathcal{C}(Y,Z)) \xrightarrow{B(\mu_{\mathcal{C}}\otimes 1)} B(\mathcal{C}(W,Y)\otimes\mathcal{C}(Y,Z)) \xrightarrow{B\mu_{\mathcal{C}}} B\mathcal{C}(W,Z)\big].
\end{aligned}$$

Both paths coincide by the associativity of $\mu_{\mathcal{C}}$. Therefore $B_*\mathcal{C}$ is a $\mathcal{W}$-category.

To a $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$ a $\mathcal{W}$-functor $B_*F : B_*\mathcal{C} \to B_*\mathcal{D}$, with $\operatorname{Ob} B_*F = \operatorname{Ob} F$, is assigned. Its action on morphisms is given by

$$(B_*F)_{X,Y} = \big[B\mathcal{C}(X,Y) \xrightarrow{BF_{X,Y}} B\mathcal{D}(FX,FY)\big], \qquad X,Y \in \operatorname{Ob}\mathcal{C}.$$

For each triple $X,Y,Z \in \operatorname{Ob}\mathcal{C}$, consider the diagram

$$\begin{tikzcd}
B\mathcal{C}(X,Y)\otimes B\mathcal{C}(Y,Z) \arrow[r, "BF_{X,Y}\otimes BF_{Y,Z}"] \arrow[d, "\beta^{\mathbf{2}}"'] & B\mathcal{D}(FX,FY)\otimes B\mathcal{D}(FY,FZ) \arrow[d, "\beta^{\mathbf{2}}"] \\
B(\mathcal{C}(X,Y)\otimes\mathcal{C}(Y,Z)) \arrow[r, "B(F_{X,Y}\otimes F_{Y,Z})"] \arrow[d, "B\mu_{\mathcal{C}}"'] & B(\mathcal{D}(FX,FY)\otimes\mathcal{D}(FY,FZ)) \arrow[d, "B\mu_{\mathcal{D}}"] \\
B\mathcal{C}(X,Z) \arrow[r, "BF_{X,Z}"] & B\mathcal{D}(FX,FZ)
\end{tikzcd}$$

The top square commutes by the naturality of $\beta^{\mathbf{2}}$, the bottom square commutes since F preserves composition. The composite in the left (resp. right) column is equal to $\mu_{B_*\mathcal{C}}$ (resp. $\mu_{B_*\mathcal{D}}$). Thus B_*F preserves composition. Compatibility with identities is obvious:

$$\begin{aligned}
1^{B_*\mathcal{C}}_X(B_*F)_{X,X} &= \big[\mathbb{1}_{\mathcal{W}} \xrightarrow{\beta^{\varnothing}} B\mathbb{1}_{\mathcal{V}} \xrightarrow{B1^{\mathcal{C}}_X} B\mathcal{C}(X,X) \xrightarrow{BF_{X,X}} B\mathcal{D}(FX,FX)\big]\\
&= \big[\mathbb{1}_{\mathcal{W}} \xrightarrow{\beta^{\varnothing}} B\mathbb{1}_{\mathcal{V}} \xrightarrow{B1^{\mathcal{D}}_{FX}} B\mathcal{D}(FX,FX)\big] = 1^{B_*\mathcal{D}}_X,
\end{aligned}$$

$$\begin{tikzcd}
B\mathcal{C}(X,Y)\otimes\mathbb{1}_{\mathcal{W}} \arrow[r, "1\otimes\beta^{\varnothing}"] & B\mathcal{C}(X,Y)\otimes B\mathbb{1}_{\mathcal{V}} \arrow[r, "BF_{X,Y}\otimes Bt_Y"] \arrow[d, "\beta^{\mathbf{2}}"] & B\mathcal{D}(FX,FY)\otimes B\mathcal{D}(FY,GY) \arrow[d, "\beta^{\mathbf{2}}"] \\
 & B(\mathcal{C}(X,Y)\otimes\mathbb{1}_{\mathcal{V}}) \arrow[r, "B(F_{X,Y}\otimes t_Y)"] & B(\mathcal{D}(FX,FY)\otimes\mathcal{D}(FY,GY)) \arrow[d, "B\mu^{\mathcal{D}}"] \\
B\mathcal{C}(X,Y) \arrow[uu, "\lambda^{\mathsf{I}\cdot}_{\mathcal{W}}"] \arrow[dd, "\lambda^{\cdot\mathsf{I}}_{\mathcal{W}}"'] \arrow[ur, "B\lambda^{\mathsf{I}\cdot}_{\mathcal{V}}"] \arrow[dr, "B\lambda^{\cdot\mathsf{I}}_{\mathcal{V}}"] & & B\mathcal{D}(FX,GY) \\
 & B(\mathbb{1}_{\mathcal{V}}\otimes\mathcal{C}(X,Y)) \arrow[r, "B(t_X\otimes G_{X,Y})"] & B(\mathcal{D}(FX,GX)\otimes\mathcal{D}(GX,GY)) \arrow[u, "B\mu^{\mathcal{D}}"'] \\
\mathbb{1}_{\mathcal{W}}\otimes B\mathcal{C}(X,Y) \arrow[r, "\beta^{\varnothing}\otimes 1"] & B\mathbb{1}_{\mathcal{V}}\otimes B\mathcal{C}(X,Y) \arrow[r, "Bt_X\otimes BG_{X,Y}"] \arrow[u, "\beta^{\mathbf{2}}"'] & B\mathcal{D}(FX,GX)\otimes B\mathcal{D}(GX,GY) \arrow[u, "\beta^{\mathbf{2}}"']
\end{tikzcd}$$

Diagram 1.1.

so that B_*F is a $\mathcal{W}$-functor.

To a $\mathcal{V}$-natural transformation $t : F \to G : \mathcal{C} \to \mathcal{D}$ a $\mathcal{W}$-natural transformation $B_*t : B_*F \to B_*G : B_*\mathcal{C} \to B_*\mathcal{D}$ is assigned. Its components are given by

$$(B_*t)_X = \bigl[\mathbb{1}_{\mathcal{W}} \xrightarrow{\beta^{\varnothing}} B\mathbb{1}_{\mathcal{V}} \xrightarrow{Bt_X} B\mathcal{D}(FX,GX)\bigr], \qquad X \in \operatorname{Ob}\mathcal{C}.$$

The naturality of B_*t is expressed by the following equation:

$$\begin{tikzcd}
B_*\mathcal{C}(X,Y) \arrow[r, "\lambda^{\mathsf{I}\cdot}_{\mathcal{W}}"] \arrow[d, "\lambda^{\cdot\mathsf{I}}_{\mathcal{W}}"'] & B_*\mathcal{C}(X,Y)\otimes\mathbb{1}_{\mathcal{W}} \arrow[d, "(B_*F)_{X,Y}\otimes(B_*t)_Y"] \\
\mathbb{1}_{\mathcal{W}}\otimes B_*\mathcal{C}(X,Y) \arrow[d, "(B_*t)_X\otimes(B_*G)_{X,Y}"'] & B_*\mathcal{D}(FX,FY)\otimes B_*\mathcal{D}(FY,GY) \arrow[d, "\mu_{B_*\mathcal{D}}"] \\
B_*\mathcal{D}(FX,GX)\otimes B_*\mathcal{D}(GX,GY) \arrow[r, "\mu_{B_*\mathcal{D}}"] & B_*\mathcal{D}(FX,GY)
\end{tikzcd}$$

It coincides with the exterior of Diagram 1.1. The top right square and the bottom right square commute by the naturality of $\beta^{\mathbf{2}}$. The pentagon commutes since t is a $\mathcal{V}$-natural transformation. The remaining quadrilaterals are instances of equation (1.1.2) written for the maps $\mathsf{I}\cdot : \mathbf{1} \to \mathbf{2}$ and $\cdot\mathsf{I} : \mathbf{1} \to \mathbf{2}$. Thus B_*t is a $\mathcal{W}$-natural transformation.

It is obvious that B_* is compatible with composition of 1-morphisms and with action of 1-morphism on 2-morphisms. It is also compatible with vertical composition of 2-morphisms. Indeed, suppose $t : F \to G : \mathcal{C} \to \mathcal{D}$ and $u : G \to H : \mathcal{C} \to \mathcal{D}$ are $\mathcal{V}$-natural transformations. The composite $tu : F \to H : \mathcal{C} \to \mathcal{D}$ has components

$$(tu)_X = \bigl[\mathbb{1}_{\mathcal{V}} \xrightarrow{\lambda^{\varnothing\to\mathbf{2}}_{\mathcal{V}}} \mathbb{1}_{\mathcal{V}}\otimes\mathbb{1}_{\mathcal{V}} \xrightarrow{t_X\otimes u_X} \mathcal{D}(FX,GX)\otimes\mathcal{D}(GX,HX) \xrightarrow{\mu_{\mathcal{D}}} \mathcal{D}(FX,FH)\bigr],$$

for each $X \in \operatorname{Ob}\mathcal{C}$. Consider the diagram

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{W}} \arrow[r, "\lambda_{\mathcal{W}}^{\varnothing\to\mathbf{2}}"] \arrow[d, "\beta^{\varnothing}"'] & \mathbb{1}_{\mathcal{W}} \otimes \mathbb{1}_{\mathcal{W}} \arrow[r, "\beta^{\varnothing}\otimes\beta^{\varnothing}"] & B\mathbb{1}_{\mathcal{V}} \otimes B\mathbb{1}_{\mathcal{V}} \arrow[r, "Bt_X\otimes Bu_X"] \arrow[d, "\beta^{\mathbf{2}}"'] & B\mathcal{D}(FX,GX) \otimes B\mathcal{D}(GX,HX) \arrow[d, "\beta^{\mathbf{2}}"] \\
B\mathbb{1}_{\mathcal{V}} \arrow[rr, "B\lambda_{\mathcal{V}}^{\varnothing\to\mathbf{2}}"] & & B(\mathbb{1}_{\mathcal{V}} \otimes \mathbb{1}_{\mathcal{V}}) \arrow[r, "B(t_X\otimes u_X)"] & B(\mathcal{D}(FX,GX) \otimes \mathcal{D}(GX,HX))
\end{tikzcd}$$

The pentagon is equation (1.1.2) written for the map $\varnothing \to \mathbf{2}$. The square commutes by the naturality of $\beta^{\mathbf{2}}$. The top-right composite followed by $B\mu_{\mathcal{D}}$ yields the component $(B_*(tu))_X$, whereas the left-bottom composite followed by $B\mu_{\mathcal{D}}$ is equal to $((B_*t)(B_*u))_X$. We conclude that $B_*(tu) = (B_*t)(B_*u)$.

The above considerations can be summarized by saying that $B_* : \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{W}\text{-}\mathbf{Cat}$ is a **Cat**-functor. In order to turn it into a lax symmetric Monoidal **Cat**-functor we need a natural transformation of **Cat**-functors

$$\begin{tikzcd}
(\mathcal{V}\text{-}\mathbf{Cat})^I \arrow[r, "B_*^I"] \arrow[d, "\boxtimes^I"'] & (\mathcal{W}\text{-}\mathbf{Cat})^I \arrow[d, "\boxtimes^I"] \arrow[dl, Rightarrow, "\beta_*^I"'] \\
\mathcal{V}\text{-}\mathbf{Cat} \arrow[r, "B_*"] & \mathcal{W}\text{-}\mathbf{Cat}
\end{tikzcd}$$

for each set $I \in \operatorname{Ob}\mathcal{O}$. For an arbitrary family $(\mathcal{C}_i)_{i\in I}$ of $\mathcal{V}$-categories, there is a $\mathcal{W}$-functor $\beta_*^I : \boxtimes^{i\in I} B_*\mathcal{C}_i \to B_* \boxtimes^{i\in I} \mathcal{C}_i$, identity on objects, whose action on morphisms is given by

$$(\beta_*^I)_{(X_i)_{i\in I},(Y_i)_{i\in I}} = \bigl[\otimes^{i\in I} B\mathcal{C}_i(X_i,Y_i) \xrightarrow{\beta^I} B \otimes^{i\in I} \mathcal{C}_i(X_i,Y_i)\bigr].$$

The proof that β_*^I is a $\mathcal{W}$-functor is based on the following technical lemma.

1.1.15. Lemma. *Let $I, J \in \operatorname{Ob}\mathcal{O}$. For each family of objects $I \times J \ni (i,j) \mapsto X_{ij} \in \operatorname{Ob}\mathcal{V}$, the diagram*

$$\begin{tikzcd}
\otimes^{j\in J} \otimes^{i\in I} BX_{ij} \arrow[r, "\otimes^{j\in J}\beta^I"] \arrow[d, "\sigma_{(12)}"'] & \otimes^{j\in J} B \otimes^{i\in I} X_{ij} \arrow[r, "\beta^J"] & B \otimes^{j\in J} \otimes^{i\in I} X_{ij} \arrow[d, "B\sigma_{(12)}"] \\
\otimes^{i\in I} \otimes^{j\in J} BX_{ij} \arrow[r, "\otimes^{i\in I}\beta^J"] & \otimes^{i\in I} B \otimes^{j\in J} X_{ij} \arrow[r, "\beta^I"] & B \otimes^{i\in I} \otimes^{j\in J} X_{ij}
\end{tikzcd}$$

commutes.

Proof. The diagram in question coincides with the exterior of the following diagram:

$$\begin{tikzcd}
\otimes^{j\in J} \otimes^{i\in I} BX_{ij} \arrow[r, "\otimes^{j\in J}\beta^I"] \arrow[d, "(\lambda_{\mathcal{W}}^{\mathrm{pr}_2:I\times J\to J})^{-1}"'] & \otimes^{j\in J} B \otimes^{i\in I} X_{ij} \arrow[r, "\beta^J"] & B \otimes^{j\in J} \otimes^{i\in I} X_{ij} \arrow[d, "B(\lambda_{\mathcal{V}}^{\mathrm{pr}_2:I\times J\to J})^{-1}"] \\
\otimes^{(i,j)\in I\times J} BX_{ij} \arrow[rr, "\beta^{I\times J}"] \arrow[d, "\lambda_{\mathcal{W}}^{\mathrm{pr}_1:I\times J\to I}"'] & & B \otimes^{(i,j)\in I\times J} X_{ij} \arrow[d, "B\lambda_{\mathcal{V}}^{\mathrm{pr}_1:I\times J\to I}"] \\
\otimes^{i\in I} \otimes^{j\in J} BX_{ij} \arrow[r, "\otimes^{i\in I}\beta^J"] & \otimes^{i\in I} B \otimes^{j\in J} X_{ij} \arrow[r, "\beta^I"] & B \otimes^{i\in I} \otimes^{j\in J} X_{ij}
\end{tikzcd}$$

The pentagons are instances of equation (1.1.2). □

That β^I_* preserves composition follows from the commutative diagram

$$\begin{tikzcd}
(\otimes^{i\in I}B\mathcal{C}_i(X_i,Y_i))\otimes(\otimes^{i\in I}B\mathcal{C}_i(Y_i,Z_i)) \arrow[r, "\beta^I\otimes\beta^I"] \arrow[d, "\sigma_{(12)}"'] & (B\otimes^{i\in I}\mathcal{C}_i(X_i,Y_i))\otimes(B\otimes^{i\in I}\mathcal{C}_i(Y_i,Z_i)) \arrow[d, "\beta^{\mathbf{2}}"] \\
\otimes^{i\in I}(B\mathcal{C}_i(X_i,Y_i)\otimes B\mathcal{C}_i(Y_i,Z_i)) \arrow[d, "\otimes^{i\in I}\beta^{\mathbf{2}}"'] & B\big((\otimes^{i\in I}\mathcal{C}_i(X_i,Y_i))\otimes(\otimes^{i\in I}\mathcal{C}_i(Y_i,Z_i))\big) \arrow[d, "B\sigma_{(12)}"] \\
\otimes^{i\in I}B(\mathcal{C}_i(X_i,Y_i)\otimes\mathcal{C}_i(Y_i,Z_i)) \arrow[r, "\beta^I"] \arrow[d, "\otimes^{i\in I}B\mu_{\mathcal{C}_i}"'] & B\otimes^{i\in I}(\mathcal{C}_i(X_i,Y_i)\otimes\mathcal{C}_i(Y_i,Z_i)) \arrow[d, "B\otimes^{i\in I}\mu_{\mathcal{C}_i}"] \\
\otimes^{i\in I}B\mathcal{C}_i(X_i,Z_i) \arrow[r, "\beta^I"] & B\otimes^{i\in I}\mathcal{C}_i(X_i,Z_i)
\end{tikzcd}$$

where the hexagon is a particular case of Lemma 1.1.15, and the square commutes by the naturality of β^I. The composite in the left (resp. right) column is equal to $\mu_{\boxtimes^{i\in I}B_*\mathcal{C}_i}$ (resp. $\mu_{B_*\boxtimes^{i\in I}\mathcal{C}_i}$). Compatibility with identities follows from the commutative diagram

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{W}} \arrow[r, "\lambda^{\varnothing\to I}_{\mathcal{W}}"] \arrow[d, "\beta^{\varnothing}"'] & \otimes^I\mathbb{1}_{\mathcal{W}} \arrow[r, "\otimes^I\beta^{\varnothing}"] & \otimes^I B\mathbb{1}_{\mathcal{V}} \arrow[r, "\otimes^{i\in I}B1^{\mathcal{C}_i}_{X_i}"] \arrow[d, "\beta^I"'] & \otimes^{i\in I}B\mathcal{C}_i(X_i,X_i) \arrow[d, "\beta^I"] \\
B\mathbb{1}_{\mathcal{V}} \arrow[rr, "B\lambda^{\varnothing\to I}_{\mathcal{V}}"] & & B\otimes^I\mathbb{1}_{\mathcal{V}} \arrow[r, "B\otimes^{i\in I}1^{\mathcal{C}_i}_{X_i}"] & B\otimes^{i\in I}\mathcal{C}_i(X_i,X_i)
\end{tikzcd}$$

where the pentagon is equation (1.1.2) written for the map $\varnothing\to I$, and the square commutes by the naturality of β^I. The top composite is equal to $1^{\boxtimes^{i\in I}B_*\mathcal{C}_i}_{(X_i)_{i\in I}}$, the left-bottom composite is equal to $1^{B_*\boxtimes^{i\in I}\mathcal{C}_i}_{(X_i)_{i\in I}}$. Therefore $\beta_*:\boxtimes^{i\in I}B_*\mathcal{C}_i\to B_*\boxtimes^{i\in I}\mathcal{C}_i$ is a $\mathcal{W}$-functor.

The **Cat**-naturality of β^I_* means that it is compatible with $\mathcal{V}$-functors and with $\mathcal{V}$-natural transformations. The former condition is expressed by the equation

$$\begin{tikzcd}
\boxtimes^{i\in I}B_*\mathcal{C}_i \arrow[r, "\beta^I_*"] \arrow[d, "\boxtimes^{i\in I}B_*F_i"'] \arrow[dr, phantom, "="] & B_*\boxtimes^{i\in I}\mathcal{C}_i \arrow[d, "B_*\boxtimes^{i\in I}F_i"] \\
\boxtimes^{i\in I}B_*\mathcal{D}_i \arrow[r, "\beta^I_*"] & B_*\boxtimes^{i\in I}\mathcal{D}_i
\end{tikzcd}$$

for an arbitrary family of $\mathcal{V}$-functors $F_i:\mathcal{C}_i\to\mathcal{D}_i$, $i\in I$. It holds true by the naturality of β^I. The latter condition is expressed by the equation

$$\begin{tikzcd}
\boxtimes^{i\in I}B_*\mathcal{C}_i \arrow[r, "\beta^I_*"] \arrow[d, shift right=8, "\boxtimes^{i\in I}B_*F_i"'] \arrow[d, shift left=8, "\boxtimes^{i\in I}B_*G_i"] \arrow[d, phantom, "\overset{\boxtimes^{i\in I}B_*t_i}{\Longrightarrow}"] & B_*\boxtimes^{i\in I}\mathcal{C}_i \arrow[d, "B_*\boxtimes^{i\in I}G_i"] \\
\boxtimes^{i\in I}B_*\mathcal{D}_i \arrow[r, "\beta^I_*"] & B_*\boxtimes^{i\in I}\mathcal{D}_i
\end{tikzcd}
=
\begin{tikzcd}
\boxtimes^{i\in I}B_*\mathcal{C}_i \arrow[r, "\beta^I_*"] \arrow[d, "\boxtimes^{i\in I}B_*F_i"'] & B_*\boxtimes^{i\in I}\mathcal{C}_i \arrow[d, shift right=8, "B_*\boxtimes^{i\in I}F_i"'] \arrow[d, shift left=8, "B_*\boxtimes^{i\in I}G_i"] \arrow[d, phantom, "\overset{B_*\boxtimes^{i\in I}t_i}{\Longrightarrow}"] \\
\boxtimes^{i\in I}B_*\mathcal{D}_i \arrow[r, "\beta^I_*"] & B_*\boxtimes^{i\in I}\mathcal{D}_i
\end{tikzcd}$$

for arbitrary $\mathcal{V}$-natural transformations $t_i:F_i\to G_i:\mathcal{C}_i\to\mathcal{D}_i$, $i\in I$. Comparing components of the $\mathcal{V}$-natural transformations in both sides of the equation, we obtain an

$$\begin{tikzcd}
\prod_{i\in I}\mathcal{V}(\mathbb{1}_{\mathcal{V}},X_i) \arrow[r, "\otimes^I_{\mathcal{V}}"] \arrow[ddd, "\lambda^f_{\mathbf{Set}}"'] & \mathcal{V}(\otimes^I\mathbb{1}_{\mathcal{V}},\otimes^{i\in I}X_i) \arrow[r, "\mathcal{V}(\lambda^{\varnothing\to I}_{\mathcal{V}},1)"] \arrow[d, "\mathcal{V}(1,\lambda^f_{\mathcal{V}})"] & \mathcal{V}(\mathbb{1}_{\mathcal{V}},\otimes^{i\in I}X_i) \arrow[d, "\mathcal{V}(1,\lambda^f_{\mathcal{V}})"] \\
 & \mathcal{V}(\otimes^I\mathbb{1}_{\mathcal{V}},\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i) \arrow[r, "\mathcal{V}(\lambda^{\varnothing\to I}_{\mathcal{V}},1)"] & \mathcal{V}(\mathbb{1}_{\mathcal{V}},\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i) \\
 & \mathcal{V}(\otimes^{j\in J}\otimes^{i\in f^{-1}j}\mathbb{1}_{\mathcal{V}},\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i) \arrow[r, "\mathcal{V}(\otimes^{j\in J}\lambda^{\varnothing\to f^{-1}j}_{\mathcal{V}},1)"] \arrow[u, "\mathcal{V}(\lambda^f_{\mathcal{V}},1)"] & \mathcal{V}(\otimes^J\mathbb{1}_{\mathcal{V}},\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i) \arrow[u, "\mathcal{V}(\lambda^{\varnothing\to J}_{\mathcal{V}},1)"] \\
\prod_{j\in J}\prod_{i\in f^{-1}j}\mathcal{V}(\mathbb{1}_{\mathcal{V}},X_i) \arrow[r, "\prod_{j\in J}\otimes^{f^{-1}j}_{\mathcal{V}}"] & \prod_{j\in J}\mathcal{V}(\otimes^{f^{-1}j}\mathbb{1}_{\mathcal{V}},\otimes^{i\in f^{-1}j}X_i) \arrow[r, "\prod_{j\in J}\mathcal{V}(\lambda^{\varnothing\to f^{-1}j}_{\mathcal{V}},1)"] \arrow[u, "\otimes^J_{\mathcal{V}}"] & \prod_{j\in J}\mathcal{V}(\mathbb{1}_{\mathcal{V}},\otimes^{i\in f^{-1}j}X_i) \arrow[u, "\otimes^J_{\mathcal{V}}"]
\end{tikzcd}$$

Diagram 1.2.

equation expressed by the exterior of the diagram

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{W}} \arrow[r, "\lambda^{\varnothing\to I}_{\mathcal{W}}"] \arrow[d, "\beta^{\varnothing}"'] & \otimes^I\mathbb{1}_{\mathcal{W}} \arrow[r, "\otimes^I\beta^{\varnothing}"] & \otimes^I B\mathbb{1}_{\mathcal{V}} \arrow[r, "\otimes^{i\in I}B(t_i)_{X_i}"] \arrow[d, "\beta^I"'] & \otimes^{i\in I}B\mathcal{D}_i(F_iX_i,G_iX_i) \arrow[d, "\beta^I"] \\
B\mathbb{1}_{\mathcal{V}} \arrow[rr, "B\lambda^{\varnothing\to I}_{\mathcal{V}}"] & & B\otimes^I\mathbb{1}_{\mathcal{V}} \arrow[r, "B\otimes^{i\in I}(t_i)_{X_i}"] & B\otimes^{i\in I}\mathcal{D}_i(F_iX_i,G_iX_i)
\end{tikzcd}$$

where $X_i \in \operatorname{Ob}\mathcal{C}_i$, $i \in I$. The square commutes by the naturality of β^I, the pentagon is equation (1.1.2) written for the map $\varnothing \to I$. Finally, equation (1.1.6) for β^I_* follows from analogous equation (1.1.2) for β^I.

1.1.16. Example. Let $\mathcal{V} = (\mathcal{V}, \otimes^I_{\mathcal{V}}, \lambda^f_{\mathcal{V}})$ be a symmetric Monoidal category. There is a lax symmetric Monoidal functor $(F, \phi^I) : \mathcal{V} \to \mathbf{Set}$, $F : X \mapsto \mathcal{V}(\mathbb{1}_{\mathcal{V}}, X)$,

$$\phi^I = \bigl[\textstyle\prod_{i\in I}\mathcal{V}(\mathbb{1}_{\mathcal{V}},X_i) \xrightarrow{\otimes^I_{\mathcal{V}}} \mathcal{V}(\otimes^I\mathbb{1}_{\mathcal{V}},\otimes^{i\in I}X_i) \xrightarrow{\mathcal{V}(\lambda^{\varnothing\to I}_{\mathcal{V}},1)} \mathcal{V}(\mathbb{1}_{\mathcal{V}},\otimes^{i\in I}X_i)\bigr].$$

Clearly, $\phi^I = \mathrm{id}$ for each 1-element set I. Equation (1.1.2) coincides with the exterior of Diagram 1.2. The hexagon expresses the naturality of $\lambda^f_{\mathcal{V}}$. The bottom square commutes since $\otimes^J_{\mathcal{V}}$ is a functor. The square in the middle is a consequence of equation (1.1.1) written for the par of maps $\varnothing \to I \xrightarrow{f} J$. The functor (F, ϕ^I) gives rise to a lax symmetric Monoidal **Cat**-functor $(F_*, \phi^I_*) : \mathcal{V}$-**Cat** $\to$ **Cat**. For a $\mathcal{V}$-category $\mathcal{C}$, denote by $\overline{\mathcal{C}}$ the ordinary category $F_*\mathcal{C}$. Sometimes it is called the *underlying category* of the $\mathcal{V}$-category $\mathcal{C}$. For example, if $\mathcal{V} = \mathbf{Cat}$, and $\mathcal{C}$ is a **Cat**-category (i.e., a 2-category), then $\overline{\mathcal{C}}$ is the ordinary category obtained from $\mathcal{C}$ by forgetting 2-morphisms.

1.2. Multicategories

1.2.1. Multicategories. We refer the reader to [**35**, Chapter 2] for a modern introduction to multicategories.

The notion of multicategory (known also as colored operad or pseudo-tensor category) is a many-object version of the notion of operad. If morphisms in a category are considered as analogous to functions, morphisms in a multicategory are analogous to functions in several variables. The most familiar example of multicategory is the multicategory of vector spaces and multilinear maps. An arrow in a multicategory looks like $X_1, X_2, \dots, X_n \to Y$, with a finite sequence of objects as the source and one object as the target, and composition turns a tree of arrows into a single arrow.

We begin by giving formal definitions of multicategory, multifunctor, and multinatural transformation.

1.2.2. Definition. A *multiquiver* C consists of a set of objects $\operatorname{Ob}\mathsf{C}$ and a set of morphisms $\mathsf{C}((X_i)_{i\in I};Y)$, for each $I \in \operatorname{Ob}\mathcal{O}$ and $X_i, Y \in \operatorname{Ob}\mathsf{C}$, $i \in I$, such that

$$\mathsf{C}((X_i)_{i\in I};Y) = \mathsf{C}((X_{\sigma^{-1}(j)})_{j\in J};Y),$$

for each order-preserving bijection $\sigma : I \to J$. A *morphism of multiquivers* $F : \mathsf{C} \to \mathsf{D}$ consists of a function $\operatorname{Ob}F : \operatorname{Ob}\mathsf{C} \to \operatorname{Ob}\mathsf{D}$, $X \mapsto FX$, and functions

$$F = F_{(X_i)_{i\in I};Y} : \mathsf{C}((X_i)_{i\in I};Y) \to \mathsf{D}((FX_i)_{i\in I};FY), \quad f \mapsto Ff,$$

for each $I \in \operatorname{Ob}\mathcal{O}$ and $X_i, Y \in \operatorname{Ob}\mathsf{C}$, $i \in I$, such that $F_{(X_i)_{i\in I};Y} = F_{(X_{\sigma^{-1}(j)})_{j\in J};Y}$, for each order-preserving bijection $\sigma : I \to J$.

1.2.3. Remark. If $I = \{i_1 < i_2 < \dots < i_n\}$ is a linearly ordered set and $I \to \mathbf{n}$, $i_k \mapsto k$, is the unique order-preserving bijection, then it follows that

$$\mathsf{C}((X_i)_{i\in I};Y) = \mathsf{C}((X_{i_k})_{k\in\mathbf{n}};Y) = \mathsf{C}(X_{i_1}, X_{i_2}, \dots, X_{i_n};Y).$$

Elements of $\mathsf{C}((X_i)_{i\in I};Y)$ are depicted as arrows $(X_i)_{i\in I} \to Y$, or as

$$X_{i_1}, X_{i_2}, \dots, X_{i_n} \to Y.$$

If $I = \varnothing$, elements of $\mathsf{C}(;Y)$ are depicted as $() \to Y$.

The following definition of (symmetric) multicategory is very close to Leinster's definition of 'fat symmetric multicategory' [**35**, Definition A.2.1].

1.2.4. Definition. A (*symmetric*) *multicategory* C consists of the following data.

- A multiquiver C.
- For each $\phi : I \to J$ in $\operatorname{Mor}\mathcal{O}$ (resp. $\operatorname{Mor}\mathcal{S}$) and $X_i, Y_j, Z \in \operatorname{Ob}\mathsf{C}$, $i \in I$, $j \in J$, a function

  $$\mu_\phi = \mu_\phi^{\mathsf{C}} : \prod_{j\in J} \mathsf{C}((X_i)_{i\in\phi^{-1}j};Y_j) \times \mathsf{C}((Y_j)_{j\in J};Z) \to \mathsf{C}((X_i)_{i\in I};Z), \tag{1.2.1}$$

  called *composition* and written

  $$((f_j)_{j\in J}, g) \mapsto (f_j)_{j\in J} \cdot_\phi g.$$

  It is required to be compatible with the multiquiver structure. Namely, suppose we are given a commutative diagram in $\mathcal{O}$ (resp. $\mathcal{S}$ in the symmetric case)

  $$\begin{tikzcd} I \arrow[r, "\phi"] \arrow[d, "\sigma"'] & J \arrow[d, "\tau"] \\ K \arrow[r, "\psi"] & L \end{tikzcd}$$

  where the vertical arrows are order-preserving bijections. Suppose further that $X_i, Y_j, U_k, V_l \in \operatorname{Ob}\mathsf{C}$, $i \in I$, $j \in J$, $k \in K$, $l \in L$, are objects of C such that $X_i = U_{\sigma(i)}$ and $Y_j = V_{\tau(j)}$, for each $i \in I$, $j \in J$. Then the mapping

  $$\mu_\psi : \prod_{l\in L} \mathsf{C}((U_k)_{k\in\psi^{-1}l};V_l) \times \mathsf{C}((V_l)_{l\in L};Z) \to \mathsf{C}((U_k)_{k\in K};Z)$$

must coincide with (1.2.1).

- For each $X \in \operatorname{Ob}\mathsf{C}$, an element $1_X \in \mathsf{C}(X;X)$, called the *identity* of X.

These data are required to satisfy the following axioms.

- Associativity:
$$(f_j)_{j\in J}\cdot_\phi \bigl((g_k)_{k\in K}\cdot_\psi h\bigr) = \bigl((f_j)_{j\in\psi^{-1}k}\cdot_{\phi_k} g_k\bigr)\cdot_{\phi\psi} h,$$
for each pair of composable maps $I \xrightarrow{\phi} J \xrightarrow{\psi} K$ in $\operatorname{Mor}\mathcal{O}$ (resp. $\operatorname{Mor}\mathcal{S}$) and for arbitrary morphisms $f_j : (X_i)_{i\in\phi^{-1}j} \to Y_j$, $j \in J$, $g_k : (Y_j)_{j\in\psi^{-1}k} \to Z_k$, $k \in K$, $h : (Z_k)_{k\in K} \to W$. Here $\phi_k = \phi|_{\phi^{-1}\psi^{-1}k} : \phi^{-1}\psi^{-1}k \to \psi^{-1}k$, $k \in K$.
- Identity:
$$(1_{X_i})_{i\in I}\cdot_{\mathrm{id}_I} f = f = f\cdot_{\triangleright} 1_Y,$$
for each morphism $f : (X_i)_{i\in I} \to Y$, where $\triangleright : I \to \mathbf{1}$ is the only map.

Restricting to morphisms with one input object as the source, we find that an arbitrary multicategory (symmetric or not) has an underlying category.

Suppose C is a symmetric multicategory, $\phi : I \to J$ is an arbitrary bijection, and $X_i, Y_j \in \operatorname{Ob}\mathsf{C}$, $i \in I$, $j \in J$, are objects of C such that $X_i = Y_{\phi(i)}$, for each $i \in I$. For each object $W \in \operatorname{Ob}\mathsf{C}$, define a mapping

$$\mathsf{C}(\phi;W) : \mathsf{C}((Y_j)_{j\in J};W) \to \mathsf{C}((X_i)_{i\in I};W), \quad f \mapsto (1_{Y_j})_{j\in J}\cdot_\phi f, \tag{1.2.2}$$

where $1_{Y_j} \in \mathsf{C}(Y_j;Y_j) = \mathsf{C}((X_i)_{i\in\phi^{-1}j};Y_j)$, $j \in J$. The identity axiom and the compatibility of composition with the multiquiver structure imply that

$$\mathsf{C}(\phi;W) = \mathrm{id} : \mathsf{C}((Y_j)_{j\in J};W) \to \mathsf{C}((X_i)_{i\in I};W),$$

for each order-preserving bijection $\phi : I \to J$. Furthermore, suppose $I \xrightarrow{\phi} J \xrightarrow{\psi} K$ are bijections, and $X_i, Y_j, Z_k \in \operatorname{Ob}\mathsf{C}$, $i \in I$, $j \in J$, $k \in K$, are objects of C such that $X_i = Y_{\phi(i)}$, $Y_j = Z_{\psi(j)}$, for each $i \in I$, $j \in J$. Then the equation

$$\mathsf{C}(\phi\psi;W) = \bigl[\mathsf{C}((Z_k)_{k\in K};W) \xrightarrow{\mathsf{C}(\psi;W)} \mathsf{C}((Y_j)_{j\in J};W) \xrightarrow{\mathsf{C}(\phi;W)} \mathsf{C}((X_i)_{i\in I};W)\bigr]$$

holds true. Indeed, for an arbitrary $f : (Z_k)_{k\in K} \to W$, we have

$$\mathsf{C}(\phi;W)\mathsf{C}(\psi;W)(f) = (1_{Y_j})_{j\in J}\cdot_\phi \bigl((1_{Z_k})_{k\in K}\cdot_\psi f\bigr) = \bigl((1_{Y_j})_{j\in\psi^{-1}k}\cdot_{\triangleright} 1_{Z_k}\bigr)_{k\in K}\cdot_{\phi\psi} f$$

by the associativity axiom. By the identity axiom, $(1_{Y_j})_{j\in\psi^{-1}k}\cdot_{\triangleright} 1_{Z_k} = 1_{Z_k}$, $k \in K$, and the equation follows. In particular, $\mathsf{C}(\phi;W)$ is a bijection with the inverse $\mathsf{C}(\phi^{-1};W)$.

1.2.5. Proposition. *Suppose we are given a commutative diagram*

$$\begin{tikzcd}
I \arrow[r, "\phi"] \arrow[d, "\sigma"'] & J \arrow[d, "\tau"] \\
K \arrow[r, "\psi"] & L
\end{tikzcd}$$

where the vertical arrows are bijections. Suppose further that $X_i, Y_j, U_k, V_l \in \operatorname{Ob}\mathsf{C}$, $i \in I$, $j \in J$, $k \in K$, $l \in L$, are objects of C such that $X_i = U_{\sigma(i)}$ and $Y_j = V_{\tau(j)}$, for each $i \in I$,

$j \in J$. Then the diagram

$$\begin{tikzcd}
\prod_{l\in L} \mathsf{C}((U_k)_{k\in\psi^{-1}l}; V_l) \times \mathsf{C}((V_l)_{l\in L}; Z) \arrow[rr, "\mu_\psi"] \arrow[d, "\prod_{l\in L} \mathsf{C}(\sigma_l;V_l)\times\mathsf{C}(\tau;Z)"'] & & \mathsf{C}((U_k)_{k\in K}; Z) \arrow[dd, "\mathsf{C}(\sigma;Z)"] \\
\prod_{l\in L} \mathsf{C}((X_i)_{i\in\phi^{-1}\tau^{-1}(l)}; Y_{\tau^{-1}(l)}) \times \mathsf{C}((Y_j)_{j\in J}; Z) \arrow[d, "\lambda^{\tau^{-1}}_{\mathbf{Set}}\times 1"'] & & \\
\prod_{j\in J} \mathsf{C}((X_i)_{i\in\phi^{-1}j}; Y_j) \times \mathsf{C}((Y_j)_{j\in J}; Z) \arrow[rr, "\mu_\phi"] & & \mathsf{C}((X_i)_{i\in I}; Z)
\end{tikzcd}$$

commutes, where $\sigma_l = \sigma|_{\sigma^{-1}\psi^{-1}l} : \sigma^{-1}\psi^{-1}l \to \psi^{-1}l$, $l \in L$, are the induced bijections.

Proof. Suppose $f_l : (U_k)_{k\in\psi^{-1}l} \to V_l$, $l \in L$, $g : (V_l)_{l\in L} \to Z$ are morphisms in C. Evaluating their image along the left-bottom composite yields

$$\begin{aligned}
\big(\mathsf{C}(\sigma_{\tau(j)}; Y_j)(f_{\tau(j)})\big)_{j\in J} \cdot_\phi \mathsf{C}(\tau; Z)(g) &= \big((1_{U_k})_{k\in\psi^{-1}\tau(j)} \cdot_{\sigma_{\tau(j)}} f_{\tau(j)}\big)_{j\in J} \cdot_\phi \big((1_{V_l})_{l\in L} \cdot_\tau g\big) \\
&= \Big(\big((1_{U_k})_{k\in\psi^{-1}l} \cdot_{\sigma_l} f_l\big) \cdot_\triangleright 1_{V_l}\Big)_{l\in L} \cdot_{\phi\tau} g,
\end{aligned}$$

by the associativity axiom for the pair of maps $I \xrightarrow{\phi} J \xrightarrow{\tau} L$. By the identity axiom,

$$\big((1_{U_k})_{k\in\psi^{-1}l} \cdot_{\sigma_l} f_l\big) \cdot_\triangleright 1_{V_l} = (1_{U_k})_{k\in\psi^{-1}l} \cdot_{\sigma_l} f_l.$$

Since $\phi\tau = \sigma\psi$, it follows that

$$\begin{aligned}
\big(\mathsf{C}(\sigma_{\tau(j)}; Y_j)(f_{\tau(j)})\big)_{j\in J} \cdot_\phi \mathsf{C}(\tau; Z)(g) &= \big((1_{U_k})_{k\in\psi^{-1}l} \cdot_{\sigma_l} f_l\big)_{l\in L} \cdot_{\sigma\psi} g \\
&= (1_{U_k})_{k\in K} \cdot_\sigma \big((f_l)_{l\in L} \cdot_\psi g\big) \\
&= \mathsf{C}(\sigma; Z)((f_l)_{l\in L} \cdot_\psi g),
\end{aligned}$$

by the associativity axiom for the pair of maps $I \xrightarrow{\sigma} K \xrightarrow{\psi} L$. The proposition is proven. □

Proposition 1.2.5 implies, in particular, that a symmetric multicategory can be defined as a multicategory C equipped with a family of bijections (1.2.2), satisfying axioms. We prefer Definition 1.2.4 since it allows a uniform treatment of both non-symmetric and symmetric cases.

1.2.6. Definition. Suppose C and D are (symmetric) multicategories. A (*symmetric*) *multifunctor* $F : \mathsf{C} \to \mathsf{D}$ is a morphism of multiquivers such that composition and identities are preserved.

The following proposition yields a useful criterion for a multifunctor to be symmetric.

1.2.7. Proposition. *Suppose C and D are symmetric multicategories, $F : \mathsf{C} \to \mathsf{D}$ is a multifunctor. It is a symmetric multifunctor if and only if for each bijection $\sigma : I \to K$ and objects $X_i, U_k, Z \in \operatorname{Ob}\mathsf{C}$, $i \in I$, $k \in K$, such that $X_i = U_{\sigma(i)}$, for each $i \in I$, the diagram*

$$\begin{tikzcd}
\mathsf{C}((X_i)_{i\in I}; Z) \arrow[r, "F_{(X_i)_{i\in I};Z}"] \arrow[d, "\mathsf{C}(\sigma;Z)"'] & \mathsf{D}((FX_i)_{i\in I}; FZ) \arrow[d, "\mathsf{D}(\sigma;FZ)"] \\
\mathsf{C}((U_k)_{k\in K}; Z) \arrow[r, "F_{(U_k)_{k\in K};Z}"] & \mathsf{D}((FU_k)_{k\in K}; FZ)
\end{tikzcd}$$

commutes.

Proof. Obviously, the formulated condition is necessary. Let us prove it is also sufficient. Note that, for each map $\phi : I \to J$ in $\operatorname{Mor}\mathcal{S}$, there is a bijection $\sigma : I \to K$ in $\operatorname{Mor}\mathcal{S}$ such that the composite $K \xrightarrow{\sigma^{-1}} I \xrightarrow{\phi} J$ is an order-preserving map. Indeed, take $K = I$ as sets, and introduce a linear ordering $<$ on K by the rule:

$$k < k' \quad \overset{\text{def}}{\Leftrightarrow} \quad \phi(k) < \phi(k') \text{ or } (\phi(k) = \phi(k') \text{ and } k < k').$$

Then the identity map $I \to K$ satisfies the required property. Suppose $X_i, Y_j, Z \in \operatorname{Ob}\mathsf{C}$, $i \in I$, $j \in J$, are objects of C. Denote the composite $\sigma^{-1} \cdot \phi : K \to J$ by ψ and take $U_k = X_{\sigma^{-1}(k)}$, $k \in K$. By Proposition 1.2.5, the diagram

$$\begin{tikzcd}
\prod_{j\in J} \mathsf{C}((U_k)_{k\in\psi^{-1}j}; Y_j) \times \mathsf{C}((Y_j)_{j\in J}; Z) \arrow[r, "\mu^{\mathsf{C}}_{\psi}"] \arrow[d, "\prod_{j\in J}\mathsf{C}(\sigma_j;Y_j)\times 1"'] & \mathsf{C}((U_k)_{k\in K}; Z) \arrow[d, "\mathsf{C}(\sigma;Z)"] \\
\prod_{j\in J} \mathsf{C}((X_i)_{i\in\phi^{-1}j}; Y_j) \times \mathsf{C}((Y_j)_{j\in J}; Z) \arrow[r, "\mu^{\mathsf{C}}_{\phi}"] & \mathsf{C}((X_i)_{i\in I}; Z)
\end{tikzcd}$$

commutes, where $\sigma_j = \sigma|_{\phi^{-1}j} : \phi^{-1}j \to \psi^{-1}j$, $j \in J$. A similar diagram for D and for the objects FX_i, FU_k, FY_j, FZ, $i \in I$, $j \in J$, $k \in K$, commutes by the same proposition. These diagrams are the ceiling and the floor of a prism whose edges are components of the multifunctor F. Three out of four side faces of the prism commute by assumptions. Since $\prod_{j\in J}\mathsf{C}(\sigma_j; Y_j) \times 1$ is a bijection, the remaining face

$$\begin{tikzcd}
\prod_{j\in J} \mathsf{C}((X_i)_{i\in\phi^{-1}j}; Y_j) \times \mathsf{C}((Y_j)_{j\in J}; Z) \arrow[r, "\mu^{\mathsf{C}}_{\phi}"] \arrow[d, "\prod_{j\in J} F_{(X_i)_{i\in\phi^{-1}j};Y_j} \times F_{(Y_j)_{j\in J};Z}"'] & \mathsf{C}((X_i)_{i\in I}; Z) \arrow[d, "F_{(X_i)_{i\in I};Z}"] \\
\prod_{j\in J} \mathsf{D}((FX_i)_{i\in\phi^{-1}j}; FY_j) \times \mathsf{C}((FY_j)_{j\in J}; FZ) \arrow[r, "\mu^{\mathsf{D}}_{\phi}"] & \mathsf{C}((FX_i)_{i\in I}; FZ)
\end{tikzcd}$$

commutes as well, hence F preserves an arbitrary composition. ☐

1.2.8. Definition. Suppose $F, G : \mathsf{C} \to \mathsf{D}$ are multifunctors. A *multinatural transformation* $r : F \to G : \mathsf{C} \to \mathsf{D}$ is a family of elements $r_X \in \mathsf{D}(FX; GX)$, $X \in \operatorname{Ob}\mathsf{C}$, such that

$$Ff \cdot_{\triangleright} r_Y = (r_{X_i})_{i\in I} \cdot_{\mathrm{id}_I} Gf,$$

for each morphism $f : (X_i)_{i\in I} \to Y$, where $\triangleright : I \to \mathbf{1}$ is the only map.

1.2.9. Example. Consider the symmetric multicategory $\widehat{\Bbbk\text{-}\mathbf{Mod}}$. Its objects are $\Bbbk$-modules. A morphism $f \in \widehat{\Bbbk\text{-}\mathbf{Mod}}((X_i)_{i\in I}; Y)$ is a $\Bbbk$-multilinear map $f : \prod_{i\in I} X_i \to Y$. For a map $\phi : I \to J$ in $\operatorname{Mor}\mathcal{S}$, the composite of $\Bbbk$-multilinear maps $f_j : \prod_{i\in\phi^{-1}j} X_i \to Y_j$, $j \in J$, and $g : \prod_{j\in J} Y_j \to Z$ is the $\Bbbk$-multilinear map

$$\prod_{i\in I} X_i \cong \prod_{j\in J} \prod_{i\in\phi^{-1}j} X_i \xrightarrow{\prod_{j\in J} f_j} \prod_{j\in J} Y_j \xrightarrow{g} Z.$$

1.2.10. Example. Consider the symmetric multicategory $\widehat{\mathbf{gr}}$. Its objects are $\mathbb{Z}$-graded $\Bbbk$-modules, i.e., functions $X : \mathbb{Z} \to \operatorname{Ob}\Bbbk\text{-}\mathbf{Mod}$, $n \mapsto X^n$. A morphism $f \in \widehat{\mathbf{gr}}((X_i)_{i\in I}; Y)$

is a family of $\Bbbk$-multilinear maps

$$\bigl(f^{(n_i)_{i\in I}}:\prod_{i\in I}X_i^{n_i}\to Y^{\sum_{i\in I}n_i}\bigr)_{(n_i)\in\mathbb{Z}^I}$$

of degree 0. For a map $\phi:I\to J$, the composite of morphisms

$$\begin{aligned} f_j&=\bigl(f_j^{(n_i)_{i\in\phi^{-1}j}}:\prod_{i\in\phi^{-1}j}X_i^{n_i}\to Y_j^{\sum_{i\in\phi^{-1}j}n_i}\bigr)_{(n_i)\in\mathbb{Z}^{\phi^{-1}j}}, \qquad j\in J,\\ g&=\bigl(g^{(m_j)_{j\in J}}:\prod_{j\in J}Y_j^{m_j}\to Z^{\sum_{j\in J}m_j}\bigr)_{(m_j)\in\mathbb{Z}^J} \end{aligned}$$

is the $\Bbbk$-multilinear map

$$\prod_{i\in I}X_i^{n_i}\cong\prod_{j\in J}\prod_{i\in\phi^{-1}j}X_i^{n_i}\xrightarrow{\prod_{j\in J}f_j^{(n_i)}}\prod_{j\in J}Y_j^{\sum_{i\in\phi^{-1}j}n_i}$$
$$\xrightarrow{g^{(\sum_{i\in\phi^{-1}j}n_i)}}Z^{\sum_{j\in J}\sum_{i\in\phi^{-1}j}n_i}\xrightarrow{(-1)^\sigma}Z^{\sum_{i\in I}n_i},$$

where the sign

$$\sigma=\sum_{i<p,\,\phi(i)>\phi(p)}^{i,p\in I}n_in_p \tag{1.2.3}$$

is prescribed by the Koszul sign rule.

1.2.11. Example. Consider the symmetric multicategory $\widehat{\mathbf{dg}}$. Its objects are $\mathbb{Z}$-graded $\Bbbk$-modules X equipped with a differential, a family of $\Bbbk$-linear maps $(d:X^n\to X^{n+1})_{n\in\mathbb{Z}}$ such that $d^2=0$. A morphism $f\in\widehat{\mathbf{dg}}((X_i)_{i\in I};Y)$ is a chain multimap, i.e., an element $f\in\widehat{\mathbf{gr}}((X_i)_{i\in I};Y)$ such that

$$\Bigl[\prod_{i\in I}X_i^{n_i}\xrightarrow{f^{(n_i)_{i\in I}}}Y^{\sum_{i\in I}n_i}\xrightarrow{d}Y^{1+\sum_{i\in I}n_i}\Bigr]$$
$$=\sum_{q\in I}(-1)^{\sum_{i>q}n_i}\Bigl[\prod_{i\in I}X_i^{n_i}\xrightarrow{\prod_{i\in I}[(1)_{i<q},d,(1)_{i>q}]}\prod_{i\in I}X_i^{n_i+\delta_{iq}}\xrightarrow{f^{(n_i+\delta_{iq})_{i\in I}}}Y^{1+\sum_{i\in I}n_i}\Bigr]$$

for all $(n_i)_{i\in I}\in\mathbb{Z}^I$. Here $\delta_{iq}=1$ if $i=q$ and $\delta_{iq}=0$ otherwise. Composition in $\widehat{\mathbf{gr}}$ of chain multimaps is again a chain multimap. Therefore, there is a faithful multifunctor $\widehat{\mathbf{dg}}\to\widehat{\mathbf{gr}}$ that forgets the differential.

Let $\mathcal{V}$ be a symmetric Monoidal category. We are going to define $\mathcal{V}$-multicategories, $\mathcal{V}$-multifunctors, and $\mathcal{V}$-multinatural transformations. When $\mathcal{V}=\mathbf{Set}$, we recover the above definitions. The reason we are introducing also multicategories enriched in a symmetric Monoidal category is because the symmetric multicategory of unital A_∞-categories may be considered, as we will see, as a multicategory enriched in the category of $\Bbbk$-linear categories.

1.2.12. Definition. A *$\mathcal{V}$-multiquiver* C consists of a set of objects $\operatorname{Ob}\mathsf{C}$ and an object of morphisms $\mathsf{C}((X_i)_{i\in I};Y)\in\operatorname{Ob}\mathcal{V}$, for each $I\in\operatorname{Ob}\mathcal{O}$ and $X_i,Y\in\operatorname{Ob}\mathsf{C}$, $i\in I$, such that

$$\mathsf{C}((X_i)_{i\in I};Y)=\mathsf{C}((X_{\sigma^{-1}(j)})_{j\in J};Y),$$

for each order-preserving bijection $\sigma : I \to J$. A *morphism of $\mathcal{V}$-multiquivers* $F : \mathsf{C} \to \mathsf{D}$ consists of a function $\operatorname{Ob} F : \operatorname{Ob}\mathsf{C} \to \operatorname{Ob}\mathsf{D}$, $X \mapsto FX$, and morphisms in $\mathcal{V}$

$$F = F_{(X_i)_{i\in I};Y} : \mathsf{C}((X_i)_{i\in I};Y) \to \mathsf{D}((FX_i)_{i\in I};FY),$$

for each $I \in \operatorname{Ob}\mathcal{O}$ and $X_i, Y \in \operatorname{Ob}\mathsf{C}$, $i \in I$, such that $F_{(X_i)_{i\in I};Y} = F_{(X_{\sigma^{-1}(j)})_{j\in J};Y}$, for each order-preserving bijection $\sigma : I \to J$.

A map $\phi : I \to J$ gives rise to a map $\overline{\phi} : I \sqcup J \sqcup \mathbf{1} \to J \sqcup \mathbf{1}$ given by

$$\overline{\phi}|_{J\sqcup\mathbf{1}} = \mathrm{id} : J \sqcup \mathbf{1} \to J \sqcup \mathbf{1}, \quad \overline{\phi}|_I = \bigl(I \xrightarrow{\phi} J \hookrightarrow J \sqcup \mathbf{1}\bigr). \tag{1.2.4}$$

1.2.13. Definition. A (*symmetric*) *$\mathcal{V}$-multicategory* C consists of the following data.

- A $\mathcal{V}$-multiquiver C.
- For each $\phi : I \to J$ in $\operatorname{Mor}\mathcal{O}$ (resp. $\operatorname{Mor}\mathcal{S}$) and $X_i, Y_j, Z \in \operatorname{Ob}\mathsf{C}$, $i \in I$, $j \in J$, a morphism in $\mathcal{V}$
$$\mu_\phi = \mu_\phi^{\mathsf{C}} : \otimes^{J\sqcup\mathbf{1}}\bigl((\mathsf{C}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J}, \mathsf{C}((Y_j)_{j\in J};Z)\bigr) \to \mathsf{C}((X_i)_{i\in I};Z), \tag{1.2.5}$$
called *composition.* It is required to be compatible with the $\mathcal{V}$-multiquiver structure. Namely, suppose we are given a commutative diagram in $\mathcal{O}$ (resp. $\mathcal{S}$)
$$\begin{tikzcd} I \arrow[r, "\phi"] \arrow[d, "\sigma"'] & J \arrow[d, "\tau"] \\ K \arrow[r, "\psi"] & L \end{tikzcd}$$
where the vertical arrows are order-preserving bijections. Suppose further that $X_i, Y_j, U_k, V_l \in \operatorname{Ob}\mathsf{C}$, $i \in I$, $j \in J$, $k \in K$, $l \in L$, are objects of C such that $X_i = U_{\sigma(i)}$ and $Y_j = V_{\tau(j)}$, for each $i \in I$, $j \in J$. Then the morphism
$$\mu_\psi : \otimes^{L\sqcup\mathbf{1}}\bigl((\mathsf{C}((U_k)_{k\in\psi^{-1}l};V_l))_{l\in L}, \mathsf{C}((V_l)_{l\in L};Z)\bigr) \to \mathsf{C}((U_k)_{k\in K};Z)$$
must coincide with (1.2.5).
- For each $X \in \operatorname{Ob}\mathsf{C}$, a morphism $1_X = 1_X^{\mathsf{C}} : \mathbb{1}_{\mathcal{V}} \to \mathsf{C}(X;X)$ in $\mathcal{V}$, called the *identity* of X.

These data are required to satisfy the following axioms.

- Associativity: for each pair of composable maps $I \xrightarrow{\phi} J \xrightarrow{\psi} K$ in $\operatorname{Mor}\mathcal{O}$ (resp. $\operatorname{Mor}\mathcal{S}$) and objects $X_i, Y_j, Z_k, W \in \operatorname{Ob}\mathsf{C}$, $i \in I$, $j \in J$, $k \in K$, Diagram 1.3 commutes. The map ϕ_k is the restriction $\phi|_{\phi^{-1}\psi^{-1}k} : \phi^{-1}\psi^{-1}k \to \psi^{-1}k$, $k \in K$. The map $\mathrm{id}_J \sqcup \triangleright : J \sqcup K \sqcup \mathbf{1} \to J \sqcup \mathbf{1}$ preserves the order, whereas the map $\overline{\psi} : J \sqcup K \sqcup \mathbf{1} \to K \sqcup \mathbf{1}$ is not necessarily order-preserving; it is given by (1.2.4).
- Identity:

$$\bigl[\mathsf{C}((X_i)_{i\in I};Z) \xrightarrow{\lambda^{\mathsf{I}\cdot}} \mathsf{C}((X_i)_{i\in I};Z) \otimes \mathbb{1}_{\mathcal{V}} \xrightarrow{1\otimes 1_Z} \mathsf{C}((X_i)_{i\in I};Z) \otimes \mathsf{C}(Z;Z) \xrightarrow{\mu_{\triangleright:I\to\mathbf{1}}} \mathsf{C}((X_i)_{i\in I};Z)\bigr] = \mathrm{id}, \tag{1.2.6}$$

$$\bigl[\mathsf{C}((X_i)_{i\in I};Z) \xrightarrow{\lambda^{\mathbf{1}\hookrightarrow I\sqcup\mathbf{1}}} \otimes^{I\sqcup\mathbf{1}}\bigl((\mathbb{1}_{\mathcal{V}})_{i\in I}, \mathsf{C}((X_i)_{i\in I};Z)\bigr) \xrightarrow{\otimes^{I\sqcup\mathbf{1}}((1_{X_i})_{i\in I},1)} \otimes^{I\sqcup\mathbf{1}}\bigl[\bigl(\mathsf{C}(X_i;X_i)\bigr)_{i\in I}, \mathsf{C}((X_i)_{i\in I};Z)\bigr] \xrightarrow{\mu_{\mathrm{id}_I}} \mathsf{C}((X_i)_{i\in I};Z)\bigr] = \mathrm{id}, \tag{1.2.7}$$

for each $I \in \operatorname{Ob}\mathcal{O}$ and $X_i, Z \in \operatorname{Ob}\mathsf{C}$, $i \in I$. Here $\triangleright : I \to \mathbf{1}$ is the only map.

$$\begin{tikzcd}
 & \bigotimes^{J\sqcup\mathbf{1}}\Big[\big(\mathsf{C}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J},\bigotimes^{K\sqcup\mathbf{1}}\Big(\big(\mathsf{C}((Y_j)_{j\in\psi^{-1}k};Z_k)\big)_{k\in K},\mathsf{C}((Z_k)_{k\in K};W)\Big)\Big] \arrow[dd, "\bigotimes^{J\sqcup\mathbf{1}}((1)_{j\in J},\mu_\psi)"] \\
\bigotimes^{J\sqcup K\sqcup\mathbf{1}}\Big[\big(\mathsf{C}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J},\big(\mathsf{C}((Y_j)_{j\in\psi^{-1}k};Z_k)\big)_{k\in K},\mathsf{C}((Z_k)_{k\in K};W)\Big] \arrow[ur, "\lambda_{\mathcal{V}}^{\mathrm{id}_J\sqcup\triangleright}"] \arrow[dd, "\lambda_{\mathcal{V}}^{\overline{\psi}}"'] & \\
 & \bigotimes^{J\sqcup\mathbf{1}}\Big[\big(\mathsf{C}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J},\mathsf{C}((Y_j)_{j\in J};W)\Big] \arrow[dd, "\mu_\phi"] \\
\bigotimes^{K\sqcup\mathbf{1}}\Big[\Big(\bigotimes^{\psi^{-1}k\sqcup\mathbf{1}}\big[\big(\mathsf{C}((X_i)_{i\in\phi_k^{-1}j};Y_j)\big)_{j\in\psi^{-1}k},\mathsf{C}((Y_j)_{j\in\psi^{-1}k};Z_k)\big]\Big)_{k\in K},\mathsf{C}((Z_k)_{k\in K};W)\Big] \arrow[dd, "\bigotimes^{K\sqcup\mathbf{1}}((\mu_{\phi_k})_{k\in K},1)"'] & \\
 & \mathsf{C}((X_i)_{i\in I};W) \\
\bigotimes^{K\sqcup\mathbf{1}}\big[\big(\mathsf{C}((X_i)_{i\in(\phi\psi)^{-1}(k)};Z_k)\big)_{k\in K},\mathsf{C}((Z_k)_{k\in K};W)\big] \arrow[ur, "\mu_{\phi\psi}"] &
\end{tikzcd}$$

DIAGRAM 1.3.

1.2.14. Definition. Suppose C and D are (symmetric) multicategories. A (*symmetric*) $\mathcal{V}$-*multifunctor* $F : \mathsf{C} \to \mathsf{D}$ is a morphism of multiquivers such that composition and identities are preserved. The former means that the diagram

$$\begin{tikzcd}
\otimes^{J\sqcup\mathbf{1}}\big[\big(\mathsf{C}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J},\mathsf{C}((Y_j)_{j\in J};Z)\big] \arrow[r, "\mu^{\mathsf{C}}_{\phi}"] \arrow[d, "\otimes^{J\sqcup\mathbf{1}}[(F_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J},F_{(Y_j)_{j\in J};Z}]"'] & \mathsf{C}((X_i)_{i\in I};Z) \arrow[d, "F_{(X_i)_{i\in I};Z}"] \\
\otimes^{J\sqcup\mathbf{1}}\big[\big(\mathsf{D}((FX_i)_{i\in\phi^{-1}j};FY_j)\big)_{j\in J},\mathsf{D}((FY_j)_{j\in J};FZ)\big] \arrow[r, "\mu^{\mathsf{D}}_{\phi}"] & \mathsf{D}((FX_i)_{i\in I};FZ)
\end{tikzcd}$$

commutes, for each map $\phi : I \to J$ in $\operatorname{Mor}\mathcal{O}$ (resp. $\operatorname{Mor}\mathcal{S}$) and $X_i, Y_j, Z \in \operatorname{Ob}\mathsf{C}$, $i \in I$, $j \in J$. Preservation of identities is expressed by the equation

$$\big[\mathbb{1}_{\mathcal{V}} \xrightarrow{1^{\mathsf{C}}_X} \mathsf{C}(X;X) \xrightarrow{F_{X;X}} \mathsf{D}(FX;FX)\big] = 1^{\mathsf{D}}_{FX},$$

for each $X \in \operatorname{Ob}\mathsf{C}$.

1.2.15. Definition. A $\mathcal{V}$-*multinatural transformation* $r : F \to G : \mathsf{C} \to \mathsf{D}$ is a family of morphisms $r_X : \mathbb{1}_{\mathcal{V}} \to \mathsf{D}(FX;GX)$, $X \in \operatorname{Ob}\mathsf{C}$, such that the diagram

$$\begin{tikzcd}
\mathsf{C}((X_i)_{i\in I};Y) \arrow[r, "\lambda^{\mathsf{I}\cdot}_{\mathcal{V}}"] \arrow[d, "\lambda^{\mathbf{1}\hookrightarrow I\sqcup\mathbf{1}}_{\mathcal{V}}"'] & \mathsf{C}((X_i)_{i\in I};Y)\otimes\mathbb{1}_{\mathcal{V}} \arrow[d, "F_{(X_i)_{i\in I};Y}\otimes r_Y"] \\
\otimes^{I\sqcup\mathbf{1}}[(\mathbb{1}_{\mathcal{V}})_{i\in I},\mathsf{C}((X_i)_{i\in I};Y)] \arrow[d, "\otimes^{I\sqcup\mathbf{1}}[(r_{X_i})_{i\in I},G_{(X_i)_{i\in I};Y}]"'] & \mathsf{D}((FX_i)_{i\in I};FY)\otimes\mathsf{D}(FY;GY) \arrow[d, "\mu^{\mathsf{D}}_{\triangleright}"] \\
\otimes^{I\sqcup\mathbf{1}}[(\mathsf{D}(FX_i,GX_i))_{i\in I},\mathsf{D}((GX_i)_{i\in I};GY)] \arrow[r, "\mu^{\mathsf{D}}_{\mathrm{id}_I}"] & \mathsf{D}((FX_i)_{i\in I};GY)
\end{tikzcd}$$

commutes, for each $I \in \operatorname{Ob}\mathcal{O}$ and $X_i, Y \in \operatorname{Ob}\mathsf{C}$, $i \in I$, where $\triangleright : I \to \mathbf{1}$ is the only map.

1.2.16. From lax Monoidal categories to multicategories and back. The aim of the present section is to justify the point of view that multicategories are just as good as lax Monoidal categories. In the passage below, we briefly explain the ideas. In order to simplify the way of speaking, we speak about non-symmetric case. Similar results hold true in symmetric case as well.

We will shortly see that a lax Monoidal category $\mathcal{C} = (\mathcal{C}, \otimes^I, \lambda^f)$ gives rise to a multicategory $\widehat{\mathcal{C}}$ whose objects are those of $\mathcal{C}$, and a morphism

$$X_1, X_2, \dots, X_n \to Y$$

in $\widehat{\mathcal{C}}$ is a morphism

$$\otimes^{\mathbf{n}}(X_1, X_2, \dots, X_n) \to Y$$

in $\mathcal{C}$. Composition in $\widehat{\mathcal{C}}$ is derived from composition and tensor product in $\mathcal{C}$. Furthermore, a lax Monoidal functor $(F, \phi^I) : \mathcal{C} \to \mathcal{D}$ defines a multifunctor $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$, and a Monoidal transformation $t : (F, \phi^I) \to (G, \psi^I) : \mathcal{C} \to \mathcal{D}$ gives rise to a multinatural transformation $\widehat{t} : \widehat{F} \to \widehat{G} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$. Conversely, an arbitrary multifunctor $\widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ is of the form $\widehat{F}$ for a unique F, and an arbitrary multinatural transformation $\widehat{F} \to \widehat{G} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ is of the form $\widehat{t}$ for a unique t.

Here is a fancy way of seeing these statements. Note that lax Monoidal categories, lax Monoidal functors, and Monoidal transformations form a 2-category **LaxMonCat**. Similarly, multicategories, multifunctors, and multinatural transformations form a 2-category **Multicat**. A 2-category is the same thing as a **Cat**-category in the sense of enriched categories. The correspondences $\mathcal{C} \mapsto \widehat{\mathcal{C}}$, $F \mapsto \widehat{F}$, $t \mapsto \widehat{t}$ define a **Cat**-functor

$$\mathbf{LaxMonCat} \to \mathbf{Multicat},$$

which is a special case of a 2-functor, the difference being that a **Cat**-functor preserves composition of 1-morphisms on the nose, not only up to 2-isomorphisms. The **Cat**-functor **LaxMonCat** → **Multicat** is fully faithful (in enriched sense). In particular, it induces a **Cat**-equivalence between **LaxMonCat** and its essential image, i.e., the full **Cat**-subcategory of **Multicat** consisting of multicategories isomorphic to $\widehat{\mathcal{C}}$ for some $\mathcal{C}$. These can be described by a simple axiom, which leads to the notion of lax representable multicategory. The essence of the axiom is the existence, for each finite sequence $(X_i)_{i\in I}$ of objects, of an object X and a morphisms $(X_i)_{i\in I} \to X$ enjoying a universal property resembling that of tensor product of modules. The notion of lax representable multicategory seems to be more satisfactory than that of lax Monoidal category. It formalizes a notion of monoidal category in which tensor product is only defined up to canonical isomorphism. For example, as in the case of modules, it is not important to remember the construction of tensor product; it is only the universal property of tensor product that matters. Another advantage is that there is no need to bother with morphisms λ^f.

The relation between lax Monoidal categories and multicategories is apparently part of common consciousness of category-theorists, see e.g. [**33, 11, 35**]. We discuss it in details since it is crucial for what follows.

The results of this section are true in more general, enriched setting. We did not aim to exhaust the subject and the reader. Thus we have decided to provide proofs in the case $\mathcal{V} = \mathbf{Set}$ only, with a few exceptions, where for the sake of being rigorous we included proofs of propositions that are used in the sequel in the case of general $\mathcal{V}$. The reader is advised to skip these proofs on the first reading. In particular, since we will not need representable $\mathcal{V}$-multicategories, we are not going to discuss them. The curious reader is referred to [**3**].

1.2.17. Proposition. *A (symmetric) lax Monoidal category* $(\mathcal{C}, \otimes^I, \lambda^f)$ *gives rise to a (symmetric) multicategory* $\widehat{\mathcal{C}}$ *defined by the following prescriptions:* $\operatorname{Ob}\widehat{\mathcal{C}} = \operatorname{Ob}\mathcal{C}$, *and* $\widehat{\mathcal{C}}((X_i)_{i\in I}; Y) = \mathcal{C}(\otimes^{i\in I}X_i, Y)$, *for each* $I \in \operatorname{Ob}\mathcal{O}$ *and* $X_i, Y \in \operatorname{Ob}\mathcal{C}$, $i \in I$. *Thus a morphism* $(X_i)_{i\in I} \to Y$ *in* $\widehat{\mathcal{C}}$ *is a morphism* $\otimes^{i\in I}X_i \to Y$ *in* $\mathcal{C}$. *For each map* $\phi : I \to J$ *in* $\operatorname{Mor}\mathcal{O}$ *(resp.* $\operatorname{Mor}\mathcal{S}$*), the composition of morphisms* $f_j : (X_i)_{i\in\phi^{-1}j} \to Y_j$, $j \in J$, *and* $g : (Y_j)_{j\in J} \to Z$ *is given by the composite*

$$(f_j)_{j\in J} \cdot_\phi g = \bigl[\otimes^{i\in I}X_i \xrightarrow{\lambda^\phi} \otimes^{j\in J} \otimes^{i\in\phi^{-1}j} X_i \xrightarrow{\otimes^{j\in J}f_j} \otimes^{j\in J}Y_j \xrightarrow{g} Z\bigr]. \tag{1.2.8}$$

Identities of $\widehat{\mathcal{C}}$ *are those of* $\mathcal{C}$.

Proof. The condition

$$\widehat{\mathcal{C}}((X_i)_{i\in I}; Y) = \mathcal{C}(\otimes^{i\in I}X_i, Y) = \mathcal{C}(\otimes^{j\in J}X_{\sigma^{-1}(j)}, Y) = \widehat{\mathcal{C}}((X_{\sigma^{-1}(j)})_{j\in J}; Y),$$

for each order-preserving bijection $\sigma : I \to J$, is satisfied since λ^σ is the identity morphism, in particular $\otimes^{i\in I}X_i = \otimes^{j\in J}X_{\sigma^{-1}(j)}$. Thus, $\widehat{\mathcal{C}}$ is a multiquiver. For the same reasons composition in $\widehat{\mathcal{C}}$ is compatible with the multiquiver structure.

Clearly, for each morphism $f : (X_i)_{i\in I} \to Y$ in $\widehat{\mathcal{C}}$,

$$f \cdot_{\triangleright} 1_Y = \bigl[\otimes^{i\in I}X_i \xrightarrow{\lambda^{\triangleright}} \otimes^{i\in I}X_i \xrightarrow{f} Y \xrightarrow{1_Y} Y\bigr] = f,$$

since $\lambda^{\triangleright} = \mathrm{id}$. Similarly,

$$(1_{X_i})_{i\in I} \cdot_{\mathrm{id}_I} f = \bigl[\otimes^{i\in I}X_i \xrightarrow{\lambda^{\mathrm{id}_I}} \otimes^{i\in I}X_i \xrightarrow{\otimes^{i\in I}1_{X_i}} \otimes^{i\in I}X_i \xrightarrow{f} Y\bigr] = f,$$

since $\lambda^{\mathrm{id}_I} = \mathrm{id}$, and $\otimes^I$ is a functor, in particular, $\otimes^{i\in I}1_{X_i} = 1_{\otimes^{i\in I}X_i}$.

Let us check that composition is associative. Suppose $I \xrightarrow{\phi} J \xrightarrow{\psi} K$ is a pair of composable maps in $\operatorname{Mor}\mathcal{O}$ (resp. $\operatorname{Mor}\mathcal{S}$). Consider morphisms $f_j : (X_i)_{i\in\phi^{-1}j} \to Y_j$, $j \in J$, $g_k : (Y_j)_{j\in\psi^{-1}k} \to Z_k$, $k \in K$, $h : (Z_k)_{k\in K} \to W$ in $\widehat{\mathcal{C}}$. Then

$$(g_k)_{k\in K} \cdot_{\psi} h = \bigl[\otimes^{j\in J}Y_j \xrightarrow{\lambda^{\psi}} \otimes^{k\in K} \otimes^{j\in\psi^{-1}k} Y_j \xrightarrow{\otimes^{k\in K}g_k} \otimes^{k\in K}Z_k \xrightarrow{h} W\bigr],$$

therefore

$$\begin{aligned}(f_j)_{j\in J} \cdot_{\phi} \bigl((g_k)_{k\in K} \cdot_{\psi} h\bigr) &= \bigl[\otimes^{i\in I}X_i \xrightarrow{\lambda^{\phi}} \otimes^{j\in J} \otimes^{i\in\phi^{-1}j} X_i \xrightarrow{\otimes^{j\in J}f_j} \otimes^{j\in J}Y_j \\ &\qquad \xrightarrow{\lambda^{\psi}} \otimes^{k\in K} \otimes^{j\in\psi^{-1}k} Y_j \xrightarrow{\otimes^{k\in K}g_k} \otimes^{k\in K}Z_k \xrightarrow{h} W\bigr] \\ &= \bigl[\otimes^{i\in I}X_i \xrightarrow{\lambda^{\phi}} \otimes^{j\in J} \otimes^{i\in\phi^{-1}j} X_i \xrightarrow{\lambda^{\psi}} \otimes^{k\in K} \otimes^{j\in\psi^{-1}k} \otimes^{i\in\phi^{-1}j}X_i \\ &\qquad \xrightarrow{\otimes^{k\in K}\otimes^{j\in\psi^{-1}k}f_j} \otimes^{j\in\psi^{-1}k}Y_j \xrightarrow{\otimes^{k\in K}g_k} \otimes^{k\in K}Z_k \xrightarrow{h} W\bigr],\end{aligned}$$

by naturality of λ^{ψ}. Similarly, $\bigl((f_j)_{j\in\psi^{-1}k} \cdot_{\phi_k} g_k\bigr) \cdot_{\phi\psi} h$ is given by the composite

$$\begin{aligned}&\bigl[\otimes^{i\in I}X_i \xrightarrow{\lambda^{\phi\psi}} \otimes^{k\in K} \otimes^{i\in\phi^{-1}\psi^{-1}k} X_i \xrightarrow{\otimes^{k\in K}\lambda^{\phi_k}} \otimes^{k\in K} \otimes^{j\in\psi^{-1}k} \otimes^{i\in\phi^{-1}j}X_i \\ &\qquad \xrightarrow{\otimes^{k\in k}\otimes^{j\in\psi^{-1}k}f_j} \otimes^{k\in K} \otimes^{j\in\psi^{-1}k} Y_j \xrightarrow{\otimes^{k\in K}g_k} \otimes^{k\in K}Z_k \xrightarrow{h} W\bigr].\end{aligned}$$

The equation $(f_j)_{j\in J} \cdot_{\phi} \bigl((g_k)_{k\in K} \cdot_{\psi} h\bigr) = \bigl((f_j)_{j\in\psi^{-1}k} \cdot_{\phi_k} g_k\bigr) \cdot_{\phi\psi} h$ is a consequence of equation (1.1.1). □

1.2.18. Proposition. *A lax (symmetric) Monoidal functor*

$$(F, \phi^I) : (\mathcal{C}, \otimes^I, \lambda^f) \to (\mathcal{D}, \otimes^I, \lambda^f)$$

between lax (symmetric) Monoidal categories gives rise to a (symmetric) multifunctor $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ *defined as follows. The function* $\operatorname{Ob}\widehat{F} : \operatorname{Ob}\widehat{\mathcal{C}} \to \operatorname{Ob}\widehat{\mathcal{D}}$ *is equal to* $\operatorname{Ob}F$. *The action on morphisms is given by*

$$\widehat{F} : \widehat{\mathcal{C}}((X_i)_{i\in I}; Y) = \mathcal{C}(\otimes^{i\in I}X_i, Y) \to \mathcal{D}(\otimes^{i\in I}FX_i, FY) = \widehat{\mathcal{D}}((FX_i)_{i\in I}; FY), \quad f \mapsto \widehat{F}f,$$

where $\widehat{F}f$ *is given by the composite*

$$\widehat{F}f = \bigl[\otimes^{i\in I}FX_i \xrightarrow{\phi^I} F \otimes^{i\in I} X_i \xrightarrow{Ff} FY\bigr].$$

Proof. Equation (1.1.2) implies that

$$\phi^I = \phi^J : \otimes^{i\in I}FX_i = \otimes^{j\in J}FX_{\sigma^{-1}(j)} \to F \otimes^{i\in I} X_i = F \otimes^{j\in J} X_{\sigma^{-1}(j)},$$

for each order-preserving bijection $\sigma : I \to J$. Therefore, $\widehat{F}_{(X_i)_{i\in I};Y} = \widehat{F}_{(X_{\sigma^{-1}(j)})_{j\in J};Y}$, hence $\widehat{F}$ is a morphism of multiquivers. To show that $\widehat{F}$ is a multifunctor, we must show that it preserves identities and composition. The former is obvious, since

$$\widehat{F}1_X = \bigl[FX \xrightarrow{\phi^{\mathbf{1}}} FX \xrightarrow{F1_X} FX\bigr] = 1_{FX},$$

as $\phi^{\mathbf{1}}$ is the identity transformation of the functor F. To show $\widehat{F}$ preserves composition, consider a map $\psi : I \to J$ in $\operatorname{Mor}\mathcal{O}$ (resp. $\operatorname{Mor}\mathcal{S}$) and morphisms $f_j : (X_i)_{i\in\psi^{-1}j} \to Y_j$, $j \in J$, $g : (Y_j)_{j\in J} \to Z$ in $\widehat{\mathcal{C}}$. Recalling the definition of composition in $\widehat{\mathcal{C}}$, we find that

$$\begin{aligned}\widehat{F}\bigl((f_j)_{j\in J} \cdot_\psi g\bigr) = \bigl[\otimes^{i\in I}FX_i \xrightarrow{\phi^I} F\otimes^{i\in I} X_i \xrightarrow{F\lambda^\psi} \\ F\otimes^{j\in J}\otimes^{i\in\psi^{-1}j}X_i \xrightarrow{F\otimes^{j\in J}f_j} F\otimes^{j\in J} Y_j \xrightarrow{Fg} FZ\bigr].\end{aligned}$$

On the other hand,

$$\begin{aligned}(\widehat{F}f_j)_{j\in J} \cdot_\psi (\widehat{F}g) &= \bigl[\otimes^{i\in I}FX_i \xrightarrow{\lambda^\psi} \otimes^{j\in J}\otimes^{i\in\phi^{-1}j} FX_i \xrightarrow{\otimes^{j\in J}\phi^{\psi^{-1}j}} \otimes^{j\in J}F\otimes^{i\in\psi^{-1}j} X_i \\ &\qquad \xrightarrow{\otimes^{j\in J}Ff_j} \otimes^{j\in J}FY_j \xrightarrow{\phi^J} F\otimes^{j\in J} Y_j \xrightarrow{Fg} FZ\bigr] \\ &= \bigl[\otimes^{i\in I}FX_i \xrightarrow{\lambda^\psi} \otimes^{j\in J}\otimes^{i\in\phi^{-1}j} FX_i \xrightarrow{\otimes^{j\in J}\phi^{\psi^{-1}j}} \otimes^{j\in J}F\otimes^{i\in\psi^{-1}j} X_i \\ &\qquad \xrightarrow{\phi^J} F\otimes^{j\in J}\otimes^{i\in\psi^{-1}j}X_i \xrightarrow{F\otimes^{j\in J}f_j} F\otimes^{j\in J} Y_j \xrightarrow{Fg} FZ\bigr]\end{aligned}$$

by the naturality of ϕ^J. Equation (1.1.2) implies that

$$\widehat{F}\bigl((f_j)_{j\in J} \cdot_\psi g\bigr) = (\widehat{F}f_j)_{j\in J} \cdot_\psi (\widehat{F}g),$$

hence $\widehat{F}$ is a multifunctor. □

1.2.19. Proposition. *A Monoidal transformation $r : (F,\phi^I) \to (G,\psi^I) : \mathcal{C} \to \mathcal{D}$ gives rise to a multinatural transformation $\widehat{r} : \widehat{F} \to \widehat{G} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$, determined by the morphisms*

$$\widehat{r}_X = r_X \in \mathcal{D}(FX, GX), \qquad X \in \operatorname{Ob}\mathcal{C}.$$

Proof. To show $\widehat{r}$ is a multinatural transformation, let $f : (X_i)_{i\in I} \to Y$ be a morphism in $\widehat{\mathcal{C}}$, i.e., a morphism $f : \otimes^{i\in I}X_i \to Y$ in $\mathcal{C}$. We must show

$$\widehat{F}f \cdot_\triangleright r_Y = (r_{X_i})_{i\in I} \cdot_{\mathrm{id}_I} \widehat{G}f.$$

Expanding out the left hand side we obtain

$$\widehat{F}f \cdot_\triangleright r_Y = \bigl[\otimes^{i\in I}FX_i \xrightarrow{\phi^I} F\otimes^{i\in I} X_i \xrightarrow{Ff} FY \xrightarrow{r_Y} GY\bigr],$$

while the right hand side yields

$$(r_{X_i})_{i\in I} \cdot_{\mathrm{id}_I} \widehat{G}f = \bigl[\otimes^{i\in I}FX_i \xrightarrow{\otimes^{i\in I}r_{X_i}} \otimes^{i\in I}GX_i \xrightarrow{\psi^I} G\otimes^{i\in I} X_i \xrightarrow{Gf} GY\bigr].$$

The equation in question follows from the following commutative diagram:

$$\begin{tikzcd}
\otimes^{i\in I}FX_i \arrow[r, "\phi^I"] \arrow[d, "\otimes^{i\in I}r_{X_i}"'] & F\otimes^{i\in I} X_i \arrow[r, "Ff"] \arrow[d, "r_{\otimes^{i\in I}X_i}"] & FY \arrow[d, "r_Y"] \\
\otimes^{i\in I}GX_i \arrow[r, "\psi^I"] & G\otimes^{i\in I} X_i \arrow[r, "Gf"] & GY
\end{tikzcd}$$

The right square commutes by the naturality of r. The left square is a particular case of (1.1.3). □

1.2.20. Definition. A (symmetric) multicategory C is *lax representable* if the functors $\mathsf{C}((X_i)_{i\in I};-) : \mathsf{C} \to \mathbf{Set}$ are representable for all families $(X_i)_{i\in I}$ of objects of C. Equivalently, for each $X_i \in \operatorname{Ob}\mathsf{C}$, $i \in I$, there exists an object $X \in \operatorname{Ob}\mathsf{C}$ and a morphism $\tau_{(X_i)_{i\in I}} : (X_i)_{i\in I} \to X$ enjoying the following universal property: for an arbitrary morphism $f : (X_i)_{i\in I} \to Y$, there exists a unique morphism $\overline{f} : X \to Y$ such that $\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \overline{f} = f$:

$$\begin{tikzcd}
 & X \arrow[dd, dashed, "\exists!\overline{f}"] \\
(X_i)_{i\in I} \arrow[ur, "\tau_{(X_i)_{i\in I}}"] \arrow[dr, "f"'] & \\
 & Y
\end{tikzcd}$$

We may assume that, for each 1-element set I, the chosen X coincides with X_i for the only $i \in I$, and $\tau_X = 1_X \in \mathsf{C}(X;X)$ is chosen to be the identity; then $\overline{f} = f$, for each $f : X \to Y$. Furthermore, since the functors $\mathsf{C}((X_i)_{i\in I};-)$ and $\mathsf{C}((X_{\sigma^{-1}(j)})_{j\in J};-)$ coincide, for each order-preserving bijection $\sigma : I \to J$, we may choose the same representing object for these functors. In the sequel, we assume that the choices of X and $\tau_{(X_i)_{i\in I}}$ satisfy the above conditions.

A strong notion of representability is given by Hermida [**21**, Definition 8.3]. The above condition of lax representability is taken from Definition 8.1(1) of [**21**]. It is due to Volodymyr Lyubashenko. Day and Street make in [**11**] remarks similar to the following proposition.

1.2.21. Proposition. *A (symmetric) multicategory C is lax representable if and only if it is isomorphic to $\widehat{\mathcal{C}}$ for some lax (symmetric) Monoidal category $\mathcal{C}$.*

Proof. The "if" part is rather straightforward. For the proof of the "only if" part, suppose C is a lax representable (symmetric) multicategory. Let $\mathcal{C}$ denote its underlying category. It has the same objects as C, and $\mathcal{C}(X,Y) = \mathsf{C}(X;Y)$, for each pair of objects $X, Y \in \operatorname{Ob}\mathsf{C}$. Composition in $\mathcal{C}$ is induced by composition in C:

$$\mathsf{C}(X;Y) \times \mathsf{C}(Y;Z) \to \mathsf{C}(X;Z), \quad (f,g) \mapsto f \cdot g \overset{\text{def}}{=} f \cdot_{\triangleright} g,$$

where $\triangleright : \mathbf{1} \to \mathbf{1}$ is the only map. Identities of $\mathcal{C}$ are those of C.

Claim. The category $\mathcal{C}$ admits the following structure of a lax (symmetric) Monoidal category. For each $I \in \operatorname{Ob}\mathcal{O}$, the functor $\otimes^I : \mathcal{C}^I \to \mathcal{C}$ takes $(X_i)_{i\in I}$ to an object $\otimes^{i\in I}X_i$ representing the functor $\mathsf{C}((X_i)_{i\in I};-)$. For each $X_i \in \operatorname{Ob}\mathsf{C}$, $i \in I$, choose a universal morphism $\tau_{(X_i)_{i\in I}} : (X_i)_{i\in I} \to \otimes^{i\in I}X_i$. The action of the functor $\otimes^I$ on morphisms is given by

$$\otimes^I : \prod_{i\in I}\mathsf{C}(X_i;Y_i) \to \mathsf{C}(\otimes^{i\in I}X_i;\otimes^{i\in I}Y_i), \quad (f_i)_{i\in I} \mapsto \otimes^{i\in I}f_i,$$

where $\otimes^{i\in I}f_i$ is the unique morphism satisfying $\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \otimes^{i\in I}f_i = (f_i)_{i\in I} \cdot_{\mathrm{id}_I} \tau_{(Y_i)_{i\in I}}$:

$$\begin{tikzcd}
(X_i)_{i\in I} \arrow[r, "\tau_{(X_i)_{i\in I}}"] \arrow[d, "(f_i)_{i\in I}"'] & \otimes^{i\in I}X_i \arrow[d, dashed, "\otimes^{i\in I}f_i"] \\
(Y_i)_{i\in I} \arrow[r, "\tau_{(Y_i)_{i\in I}}"] & \otimes^{i\in I}Y_i
\end{tikzcd}$$

For each map $\phi : I \to J$ in $\operatorname{Mor} \mathcal{O}$ (resp. $\operatorname{Mor} \mathcal{S}$), the morphism

$$\lambda^{\phi} \in \mathsf{C}(\otimes^{i\in I} X_i; \otimes^{j\in J} \otimes^{i\in\phi^{-1}j} X_i)$$

is defined as the unique morphism such that

$$\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \lambda^{\phi} = \big(\tau_{(X_i)_{i\in\phi^{-1}j}}\big)_{j\in J} \cdot_{\phi} \tau_{(\otimes^{i\in\phi^{-1}j} X_i)_{j\in J}},$$

or diagrammatically

$$\begin{tikzcd}
(X_i)_{i\in I} \arrow[r, "\tau_{(X_i)_{i\in I}}"] \arrow[d, "(\tau_{(X_i)_{i\in\phi^{-1}j}})_{j\in J}"'] & \otimes^{i\in I} X_i \arrow[d, dashed, "\lambda^{\phi}"] \\
(\otimes^{i\in\phi^{-1}j} X_i)_{j\in J} \arrow[r, "\tau_{(\otimes^{i\in\phi^{-1}j} X_i)_{j\in J}}"] & \otimes^{j\in J} \otimes^{i\in\phi^{-1}j} X_i
\end{tikzcd}$$

Let us prove the claim. To show that $\otimes^I$ is a functor, we must check that it preserves composition and identities. We first show that $\otimes^{i\in I} 1_{X_i} = 1_{\otimes^{i\in I} X_i}$, for each $X_i \in \operatorname{Ob} \mathsf{C}$, $i \in I$. Indeed, the morphism $\otimes^{i\in I} 1_{X_i}$ is uniquely determined by the property

$$\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \otimes^{i\in I} 1_{X_i} = (1_{X_i})_{i\in I} \cdot_{\mathrm{id}_I} \tau_{(X_i)_{i\in I}} = \tau_{(X_i)_{i\in I}} \text{ (identity)},$$

and $1_{\otimes^{i\in I} X_i}$ clearly satisfies it. To show that $\otimes^I$ preserves composition, consider morphisms $(f_i)_{i\in I} \in \prod_{i\in I} \mathsf{C}(X_i; Y_i)$, $(g_i)_{i\in I} \in \prod_{i\in I} \mathsf{C}(Y_i; Z_i)$. We must show

$$\otimes^{i\in I} (f_i \cdot g_i) = \otimes^{i\in I} f_i \cdot \otimes^{i\in I} g_i.$$

Composing both sides with $\tau_{(X_i)_{i\in I}}$ yields an equivalent equation:

$$\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \big(\otimes^{i\in I}(f_i \cdot g_i)\big) = \tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \big(\otimes^{i\in I} f_i \cdot \otimes^{i\in I} g_i\big).$$

The left hand side equals

$$\begin{aligned}
\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \big(\otimes^{i\in I}(f_i \cdot g_i)\big) &= (f_i \cdot g_i)_{i\in I} \cdot_{\mathrm{id}_I} \tau_{(Z_i)_{i\in I}} && \text{(definition of } \otimes^I) \\
&= (f_i)_{i\in I} \cdot_{\mathrm{id}_I} \big((g_i)_{i\in I} \cdot_{\mathrm{id}_I} \tau_{(Z_i)_{i\in I}}\big) && \text{(associativity)},
\end{aligned}$$

while the right hand side equals

$$\begin{aligned}
\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \big(\otimes^{i\in I} f_i \cdot \otimes^{i\in I} g_i\big) &= \big(\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \otimes^{i\in I} f_i\big) \cdot_{\triangleright} \otimes^{i\in I} g_i && \text{(associativity)} \\
&= \big((f_i)_{i\in I} \cdot_{\mathrm{id}_I} \tau_{(Y_i)_{i\in I}}\big) \cdot_{\triangleright} \otimes^{i\in I} g_i && \text{(definition of } \otimes^I) \\
&= (f_i)_{i\in I} \cdot_{\mathrm{id}_I} \big(\tau_{(Y_i)_{i\in I}} \cdot_{\triangleright} \otimes^{i\in I} g_i\big) && \text{(associativity)} \\
&= (f_i)_{i\in I} \cdot_{\mathrm{id}_I} \big((g_i)_{i\in I} \cdot_{\mathrm{id}_I} \tau_{(Z_i)_{i\in I}}\big) && \text{(definition of } \otimes^I),
\end{aligned}$$

hence the assertion. Therefore, $\otimes^I$ is a functor. The property $\otimes^I = \mathrm{Id}_{\mathcal{C}}$, for a 1-element set I, follows by the choice of representing objects $\otimes^{i\in I} X_i$ and universal morphisms $\tau_{(X_i)_{i\in I}}$.

To show that λ^{ϕ} is a natural transformation, let $(f_i)_{i\in I} \in \prod_{i\in I} \mathsf{C}(X_i; Y_i)$ be a morphism. We must show

$$\otimes^{i\in I} f_i \cdot \lambda^{\phi} = \lambda^{\phi} \cdot \otimes^{j\in J} \otimes^{i\in\phi^{-1}j} f_i.$$

By the universal property of $\tau_{(X_i)_{i\in I}}$, the above equation is equivalent to

$$\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \big(\otimes^{i\in I} f_i \cdot \lambda^{\phi}\big) = \tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \big(\lambda^{\phi} \cdot \otimes^{j\in J} \otimes^{i\in\phi^{-1}j} f_i\big).$$

Expanding out the left hand side we obtain

$$\begin{aligned}
\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \big(\otimes^{i\in I} f_i \cdot \lambda^{\phi}\big) &= \big(\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \otimes^{i\in I} f_i\big) \cdot_{\triangleright} \lambda^{\phi} && \text{(associativity)} \\
&= \big((f_i)_{i\in I} \cdot_{\mathrm{id}_I} \tau_{(Y_i)_{i\in I}}\big) \cdot_{\triangleright} \lambda^{\phi} && \text{(definition of } \otimes^I\text{)} \\
&= (f_i)_{i\in I} \cdot_{\mathrm{id}_I} \big(\tau_{(Y_i)_{i\in I}} \cdot_{\triangleright} \lambda^{\phi}\big) && \text{(associativity)} \\
&= (f_i)_{i\in I} \cdot_{\mathrm{id}_I} \big((\tau_{(Y_i)_{i\in\phi^{-1}j}})_{j\in J} \cdot_{\phi} \tau_{(\otimes^{i\in\phi^{-1}j} Y_i)_{j\in J}}\big) && \text{(definition of } \lambda^{\phi}\text{),}
\end{aligned}$$

while the right hand side becomes

$$\begin{array}{cl}
\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \big(\lambda^{\phi} \cdot \otimes^{j\in J} \otimes^{i\in\phi^{-1}j} f_i\big) & \\
\Vert & \text{(associativity)} \\
\big(\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \lambda^{\phi}\big) \cdot_{\triangleright} \otimes^{j\in J} \otimes^{i\in\phi^{-1}j} f_i & \\
\Vert & \text{(definition of } \lambda^{\phi}\text{)} \\
\big((\tau_{(X_i)_{i\in\phi^{-1}j}})_{j\in J} \cdot_{\phi} \tau_{(\otimes^{i\in\phi^{-1}j} X_i)_{j\in J}}\big) \cdot_{\triangleright} \otimes^{j\in J} \otimes^{i\in\phi^{-1}j} f_i & \\
\Vert & \text{(associativity)} \\
(\tau_{(X_i)_{i\in\phi^{-1}j}})_{j\in J} \cdot_{\phi} \big(\tau_{(\otimes^{i\in\phi^{-1}j} X_i)_{j\in J}} \cdot_{\triangleright} \otimes^{j\in J} \otimes^{i\in\phi^{-1}j} f_i\big) & \\
\Vert & \text{(definition of } \otimes^J\text{)} \\
(\tau_{(X_i)_{i\in\phi^{-1}j}})_{j\in J} \cdot_{\phi} \big((\otimes^{i\in\phi^{-1}j} f_i)_{j\in J} \cdot_{\mathrm{id}_J} \tau_{(\otimes^{i\in\phi^{-1}j} Y_i)_{j\in J}}\big) & \\
\Vert & \text{(associativity)} \\
\big(\tau_{(X_i)_{i\in\phi^{-1}j}} \cdot_{\triangleright} \otimes^{i\in\phi^{-1}j} f_i\big)_{j\in J} \cdot_{\phi} \tau_{(\otimes^{i\in\phi^{-1}j} Y_i)_{j\in J}} & \\
\Vert & \text{(definition of } \otimes^{\phi^{-1}j}\text{)} \\
\big((f_i)_{i\in\phi^{-1}j} \cdot_{\mathrm{id}_{\phi^{-1}j}} \tau_{(Y_i)_{i\in\phi^{-1}j}}\big) \cdot_{\phi} \tau_{(\otimes^{i\in\phi^{-1}j} Y_i)_{j\in J}}. &
\end{array}$$

The obtained expressions coincide by the associativity axiom, so that λ^{ϕ} is a natural transformation. The properties $\lambda^{\triangleright} = \mathrm{id}$ and $\lambda^{\mathrm{id}_I} = \mathrm{id}$ follow by the choice of the representing objects $\otimes^{i\in I} X_i$ and the universal morphisms $\tau_{(X_i)_{i\in I}}$. For example, $\lambda^{\mathrm{id}_I} : \otimes^{i\in I} X_i \to \otimes^{i\in I} X_i$ is defined as the unique morphism that satisfies the property

$$\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \lambda^{\mathrm{id}_I} = (\tau_{X_i})_{i\in I} \cdot_{\mathrm{id}_I} \tau_{(X_i)_{i\in I}}.$$

However, the right hand side equals $\tau_{(X_i)_{i\in I}}$ by the identity axiom, since $\tau_{X_i} = 1_{X_i}$, $i \in I$. Clearly, the identity morphism satisfies the required property, thus by uniqueness $\lambda^{\mathrm{id}_I} = \mathrm{id}$. Furthermore, for each order-preserving bijection $\sigma : I \to J$ holds $\lambda^{\sigma} = \mathrm{id}$. Indeed, λ^{σ} is uniquely determined by the equation

$$\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \lambda^{\sigma} = (\tau_{X_{\sigma^{-1}(j)}})_{j\in J} \cdot_{\triangleright} \tau_{(X_{\sigma^{-1}(j)})_{j\in J}}.$$

The right hand side equals $\tau_{(X_{\sigma^{-1}(j)})_{j\in J}}$ by the identity axiom, since $\tau_{X_{\sigma^{-1}(j)}} = 1_{X_{\sigma^{-1}(j)}}$, $j \in J$. Now, $\otimes^{i\in I} X_i = \otimes^{j\in J} X_{\sigma^{-1}(j)}$ and $\tau_{(X_i)_{i\in I}} = \tau_{(X_{\sigma^{-1}(j)})_{j\in J}}$ by assumption. Clearly, the identity morphism satisfies the required equation, therefore by uniqueness $\lambda^{\sigma} = \mathrm{id}$.

Finally, let us prove equation (1.1.1). Suppose $I \xrightarrow{\phi} J \xrightarrow{\psi} K$ is a pair of composable maps in $\operatorname{Mor} \mathcal{O}$ (resp. $\operatorname{Mor} \mathcal{S}$). We must show

$$\lambda^{\phi} \cdot \lambda^{\psi} = \lambda^{\phi\psi} \cdot \otimes^{k\in K} \lambda^{\phi_k},$$

where $\phi_k = \phi|_{\phi^{-1}\psi^{-1}k} : \phi^{-1}\psi^{-1}k \to \psi^{-1}k$, $k \in K$. An equivalent equation is obtained by composing with $\tau_{(X_i)_{i\in I}}$:

$$\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \big(\lambda^{\phi} \cdot \lambda^{\psi}\big) = \tau_{(X_i)_{i\in I}} \cdot \big(\lambda^{\phi\psi} \cdot \otimes^{k\in K} \lambda^{\phi_k}\big).$$

Transforming the left hand side we obtain

$$\begin{array}{cr}
\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \big(\lambda^{\phi} \cdot \lambda^{\psi}\big) & \\
\| & \text{(associativity)} \\
\big(\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \lambda^{\phi}\big) \cdot_{\triangleright} \lambda^{\psi} & \\
\| & \text{(definition of } \lambda^{\phi}\text{)} \\
\big((\tau_{(X_i)_{i\in\phi^{-1}j}})_{j\in J} \cdot_{\phi} \tau_{(\otimes^{i\in\phi^{-1}j}X_i)_{j\in J}}\big) \cdot_{\triangleright} \lambda^{\psi} & \\
\| & \text{(associativity)} \\
(\tau_{(X_i)_{i\in\phi^{-1}j}})_{j\in J} \cdot_{\phi} \big(\tau_{(\otimes^{i\in\phi^{-1}j}X_i)_{j\in J}} \cdot_{\triangleright} \lambda^{\psi}\big) & \\
\| & \text{(definition of } \lambda^{\psi}\text{)} \\
(\tau_{(X_i)_{i\in\phi^{-1}j}})_{j\in J} \cdot_{\phi} \big((\tau_{(\otimes^{i\in\phi^{-1}j}X_i)_{j\in\psi^{-1}k}})_{k\in K} \cdot_{\psi} \tau_{(\otimes^{j\in\psi^{-1}k}\otimes^{i\in\phi^{-1}j}X_i)_{k\in K}}\big). &
\end{array}$$

The right hand side equals

$$\begin{array}{cr}
\tau_{(X_i)_{i\in I}} \cdot \big(\lambda^{\phi\psi} \cdot \otimes^{k\in K}\lambda^{\phi_k}\big) & \\
\| & \text{(associativity)} \\
\big(\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \lambda^{\phi\psi}\big) \cdot_{\triangleright} \otimes^{k\in K}\lambda^{\phi_k} & \\
\| & \text{(definition of } \lambda^{\phi\psi}\text{)} \\
\big((\tau_{(X_i)_{i\in\phi^{-1}\psi^{-1}k}})_{k\in K} \cdot_{\phi\psi} \tau_{(\otimes^{i\in\phi^{-1}\psi^{-1}k}X_i)_{k\in K}}\big) \cdot_{\triangleright} \otimes^{k\in K}\lambda^{\phi_k} & \\
\| & \text{(associativity)} \\
(\tau_{(X_i)_{i\in\phi^{-1}\psi^{-1}k}})_{k\in K} \cdot_{\phi\psi} \big(\tau_{(\otimes^{i\in\phi^{-1}\psi^{-1}k}X_i)_{k\in K}} \cdot_{\triangleright} \otimes^{k\in K}\lambda^{\phi_k}\big) & \\
\| & \text{(definition of } \otimes^{K}\text{)} \\
(\tau_{(X_i)_{i\in\phi^{-1}\psi^{-1}k}})_{k\in K} \cdot_{\phi\psi} \big((\lambda^{\phi_k})_{k\in K} \cdot_{\mathrm{id}_K} \tau_{(\otimes^{j\in\psi^{-1}k}\otimes^{i\in\phi^{-1}j}X_i)_{k\in K}}\big) & \\
\| & \text{(associativity)} \\
\big(\tau_{(X_i)_{i\in\phi^{-1}\psi^{-1}k}} \cdot_{\triangleright} \lambda^{\phi_k}\big)_{k\in K} \cdot_{\phi\psi} \tau_{(\otimes^{j\in\psi^{-1}k}\otimes^{i\in\phi^{-1}j}X_i)_{k\in K}} & \\
\| & \text{(definition of } \lambda^{\phi_k}\text{)} \\
\big((\tau_{(X_i)_{i\in\phi^{-1}j}})_{j\in\psi^{-1}k} \cdot_{\phi_k} \tau_{(\otimes^{i\in\phi^{-1}j}X_i)_{j\in\psi^{-1}k}}\big)_{k\in K} \cdot_{\phi\psi} \tau_{(\otimes^{j\in\psi^{-1}k}\otimes^{i\in\phi^{-1}j}X_i)_{k\in K}}. &
\end{array}$$

The resulting expressions are equal by the associativity axiom, thus equation (1.1.1) is satisfied, and $(\mathcal{C}, \otimes^{I}, \lambda^{\phi})$ is a lax Monoidal category.

Claim. The multicategory C is isomorphic to $\widehat{\mathcal{C}}$.

We begin by constructing a multifunctor $\rho : \mathsf{C} \to \widehat{\mathcal{C}}$, identity on objects. The action on morphisms is given by

$$\mathsf{C}((X_i)_{i\in I}; Y) \to \widehat{\mathcal{C}}((X_i)_{i\in I}; Y) = \mathsf{C}(\otimes^{i\in I}X_i; Y), \quad f \mapsto \overline{f},$$

where $\overline{f} : \otimes^{i\in I}X_i \to Y$ is the unique morphism satisfying $\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \overline{f} = f$. Clearly, ρ preserves identities ($\overline{1_X} = 1_X$, due to the choice of τ_X, $X \in \operatorname{Ob}\mathsf{C}$). To show ρ preserves composition, consider a map $\phi : I \to J$ in $\operatorname{Mor}\mathcal{O}$ (resp. $\operatorname{Mor}\mathcal{S}$) and morphisms $f_j : (X_i)_{i\in\phi^{-1}j} \to Y_j$, $j \in J$, $g : (Y_j)_{j\in J} \to Z$ in C. We must show

$$\overline{(f_j)_{j\in J} \cdot_{\phi} g} = (\overline{f_j})_{j\in J} \cdot_{\phi} \overline{g}.$$

By the universal property of $\tau_{(X_i)_{i\in I}}$, the above equation is equivalent to

$$\tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \overline{(f_j)_{j\in J} \cdot_{\phi} g} = \tau_{(X_i)_{i\in I}} \cdot_{\triangleright} \big((\overline{f_j})_{j\in J} \cdot_{\phi} \overline{g}\big).$$

The left hand side is equal to $(f_j)_{j\in J}\cdot_\phi g$ by definition. Expanding out the right hand side we find

$$\begin{array}{cl}
\tau_{(X_i)_{i\in I}}\cdot_{\triangleright}\big((\overline{f_j})_{j\in J}\cdot_\phi\overline{g}\big) & \\
\| & \text{(definition of } \widehat{\mathcal{C}}\text{)} \\
\tau_{(X_i)_{i\in I}}\cdot_{\triangleright}\big(\lambda^\phi\cdot\otimes^{j\in J}\overline{f_j}\cdot\overline{g}\big) & \\
\| & \text{(associativity)} \\
\big(\tau_{(X_i)_{i\in I}}\cdot_{\triangleright}\lambda^\phi\big)\cdot_{\triangleright}\big(\otimes^{j\in J}\overline{f_j}\cdot\overline{g}\big) & \\
\| & \text{(definition of } \lambda^\phi\text{)} \\
\big((\tau_{(X_i)_{i\in\phi^{-1}j}})_{j\in J}\cdot_\phi\tau_{(\otimes^{i\in\phi^{-1}j}X_i)_{j\in J}}\big)\cdot_{\triangleright}\big(\otimes^{j\in J}\overline{f_j}\cdot\overline{g}\big) & \\
\| & \text{(associativity)} \\
\big(\tau_{(X_i)_{i\in\phi^{-1}j}}\big)_{j\in J}\cdot_\phi\big(\big(\tau_{(\otimes^{i\in\phi^{-1}j}X_i)_{j\in J}}\cdot_{\triangleright}\otimes^{j\in J}\overline{f_j}\big)\cdot_{\triangleright}\overline{g}\big) & \\
\| & \text{(definition of } \otimes^J\text{)} \\
\big(\tau_{(X_i)_{i\in\phi^{-1}j}}\big)_{j\in J}\cdot_\phi\big(\big((\overline{f_j})_{j\in J}\cdot_{\mathrm{id}_J}\tau_{(Y_j)_{j\in J}}\big)\cdot_{\triangleright}\overline{g}\big) & \\
\| & \text{(associativity)} \\
\big(\tau_{(X_i)_{i\in\phi^{-1}j}}\big)_{j\in J}\cdot_\phi\big((\overline{f_j})_{j\in J}\cdot_{\mathrm{id}_J}\big(\tau_{(Y_j)_{j\in J}}\cdot_{\triangleright}\overline{g}\big)\big) & \\
\| & \text{(definition of } \overline{g}\text{)} \\
\big(\tau_{(X_i)_{i\in\phi^{-1}j}}\big)_{j\in J}\cdot_\phi\big((\overline{f_j})_{j\in J}\cdot_{\mathrm{id}_J}g\big) & \\
\| & \text{(associativity)} \\
\big(\tau_{(X_i)_{i\in\phi^{-1}j}}\cdot_{\triangleright}\overline{f_j}\big)_{j\in J}\cdot_\phi g & \\
\| & \text{(definition of } \overline{f_j}\text{)} \\
(f_j)_{j\in J}\cdot_\phi g, &
\end{array}$$

so that ρ preserves composition. The induced maps on the sets of morphisms are bijective by the universal property of τ. Hence, ρ is an isomorphism. □

1.2.22. Example. The symmetric multicategory $\widehat{\Bbbk\text{-}\mathbf{Mod}}$ from Example 1.2.9 is representable. Thus, it comes from some symmetric Monoidal structure of the category $\Bbbk$-**Mod**. For instance, we define $\otimes_{\Bbbk}^{i\in I}X_i$ to be the free $\Bbbk$-module, generated by the set $\prod_{i\in I}X_i$, divided by $\Bbbk$-multilinearity relations. The tautological map $\tau:\prod_{i\in I}X_i\to\otimes_{\Bbbk}^{i\in I}X_i$ determines isomorphisms $\lambda_{\Bbbk}^f$.

1.2.23. Example. The symmetric multicategory $\widehat{\mathbf{gr}}$ from Example 1.2.10 is representable. Thus, it comes from some symmetric Monoidal structure of the category $\mathbf{gr}=\mathbf{gr}(\Bbbk\text{-}\mathbf{Mod})$. We define $(\otimes_{\mathbf{gr}}^{i\in I}X_i)^n=\bigoplus_{\sum_i n_i=n}\otimes_{\Bbbk}^{i\in I}X_i^{n_i}$. The isomorphism $\lambda_{\mathbf{gr}}^f$ is $\lambda_{\Bbbk}^f$, extended additively to direct sums, multiplied with the sign $(-1)^\sigma$, where σ is given by Koszul sign rule (1.2.3).

1.2.24. Example. The symmetric multicategory $\widehat{\mathbf{dg}}$ from Example 1.2.11 is representable. Thus, it comes from some symmetric Monoidal structure of the category $\mathbf{dg}=\mathbf{dg}(\Bbbk\text{-}\mathbf{Mod})$. We define $\otimes_{\mathbf{dg}}^{i\in I}X_i$ as the graded $\Bbbk$-module $\otimes_{\mathbf{gr}}^{i\in I}X_i$ equipped with the differential d whose matrix elements are given by

$$\sum_{j\in I}(-1)^{\sum_{i>j}n_i}1\otimes\cdots\otimes d\otimes\cdots\otimes 1:\otimes_{\Bbbk}^{i\in I}X_i^{n_i}\to\otimes_{\Bbbk}^{i\in I}X_i^{n_i+\delta_{ij}}.$$

The isomorphism $\lambda_{\mathbf{dg}}^f$ coincides with $\lambda_{\mathbf{gr}}^f$.

1.2.25. Proposition. *Let $\mathcal{C}$, $\mathcal{D}$ be lax (symmetric) Monoidal categories. Then the maps*

$$\mathbf{LaxMonCat}(\mathcal{C},\mathcal{D})\to\mathbf{Multicat}(\widehat{\mathcal{C}},\widehat{\mathcal{D}}),\quad F\mapsto\widehat{F},\quad r\mapsto\widehat{r},$$

constructed in Propositions 1.2.18, 1.2.19 are bijective. The same is true for lax symmetric Monoidal categories.

Proof. We begin by constructing an inverse map to the map $F \mapsto \widehat{F}$. Let $G : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ be a (symmetric) multifunctor. Define a functor $F : \mathcal{C} \to \mathcal{D}$ by $\operatorname{Ob} F = \operatorname{Ob} G$,

$$F_{X,Y} = G_{X;Y} : \mathcal{C}(X,Y) = \widehat{\mathcal{C}}(X;Y) \to \widehat{\mathcal{D}}(GX;GY) = \mathcal{D}(GX,GY).$$

For each $X_i \in \operatorname{Ob} \mathcal{C}$, $i \in I$, the set $\widehat{\mathcal{C}}((X_i)_{i\in I}; \otimes^{i\in I} X_i) = \mathcal{C}(\otimes^{i\in I} X_i, \otimes^{i\in I} X_i)$ contains a distinguished element $1_{\otimes^{i\in I} X_i}$. Applying to it $G_{(X_i)_{i\in I};\otimes^{i\in I} X_i}$ we obtain an element

$$\phi^I = G_{(X_i)_{i\in I};\otimes^{i\in I} X_i}(1_{\otimes^{i\in I} X_i}) \in \widehat{\mathcal{D}}((GX_i)_{i\in I}; G \otimes^{i\in I} X_i) = \mathcal{D}(\otimes^{i\in I} GX_i, G \otimes^{i\in I} X_i).$$

We claim the the morphisms $\phi^I : \otimes^{i\in I} FX_i \to F \otimes^{i\in I} X_i$ constitute a natural transformation $\phi^I : \otimes^I \circ F \to F \circ \otimes^I$. Indeed, suppose $f_i : X_i \to Y_i$, $i \in I$, is a family of morphisms in $\mathcal{C}$. We must show

$$\bigl[\otimes^{i\in I} FX_i \xrightarrow{\phi^I} F \otimes^{i\in I} X_i \xrightarrow{F\otimes^{i\in I} f_i} F \otimes^{i\in I} Y_i\bigr] = \bigl[\otimes^{i\in I} FX_i \xrightarrow{\otimes^{i\in I} Ff_i} \otimes^{i\in I} FY_i \xrightarrow{\phi^I} F \otimes^{i\in I} Y_i\bigr].$$

By the definition of composition in the multicategory $\widehat{\mathcal{D}}$, the left hand side admits the presentation

$$\phi^I \cdot F \otimes^{i\in I} f_i = G_{(X_i)_{i\in I};\otimes^{i\in I} X_i}(1_{\otimes^{i\in I} X_i}) \cdot_\triangleright G_{\otimes^{i\in I} X_i;\otimes^{i\in I} Y_i}(\otimes^{i\in I} f_i),$$

where $\otimes^{i\in I} f_i$ is regarded as a morphism $\otimes^{i\in I} X_i \to \otimes^{i\in I} Y_i$ in $\widehat{\mathcal{C}}$, and the composition in the right hand side is performed in the sense of $\widehat{\mathcal{D}}$. Since G is a multifunctor, it follows that

$$\begin{aligned} G_{(X_i)_{i\in I};\otimes^{i\in I} X_i}(1_{\otimes^{i\in I} X_i}) \cdot_\triangleright G_{\otimes^{i\in I} X_i;\otimes^{i\in I} Y_i}(\otimes^{i\in I} f_i) &= G_{(X_i)_{i\in I};\otimes^{i\in I} Y_i}(1_{\otimes^{i\in I} X_i} \cdot_\triangleright \otimes^{i\in I} f_i) \\ &= G_{(X_i)_{i\in I};\otimes^{i\in I} Y_i}(\otimes^{i\in I} f_i). \end{aligned}$$

The last equality holds by the definition of composition in $\widehat{\mathcal{C}}$. Similarly, the right hand side can be written as

$$\begin{aligned} \otimes^{i\in I} G_{X_i;Y_i}(f_i) \cdot G_{(Y_i)_{i\in I};\otimes^{i\in I} Y_i}(1_{\otimes^{i\in I} Y_i}) &= (G_{X_i;Y_i}(f_i))_{i\in I} \cdot_{\mathrm{id}_I} G_{(Y_i)_{i\in I};\otimes^{i\in I} Y_i}(1_{\otimes^{i\in I} Y_i}) \\ &= G_{(X_i)_{i\in I};\otimes^{i\in I} Y_i}((f_i)_{i\in I} \cdot_{\mathrm{id}_I} 1_{\otimes^{i\in I} Y_i}) \\ &= G_{(X_i)_{i\in I};\otimes^{i\in I} Y_i}(\otimes^{i\in I} f_i). \end{aligned}$$

The first and the third equalities follow from the definition of composition in $\widehat{\mathcal{D}}$ resp. $\widehat{\mathcal{C}}$. The second equality is implied by the fact that G is a multifunctor. Thus ϕ^I is a natural transformation.

Let us prove that $(F, \phi^I) : \mathcal{C} \to \mathcal{D}$ is a lax (symmetric) Monoidal functor. If I is a 1-element set, then $\phi^I = G_{X;X}(1_X) = 1_{GX}$ define the identity transformation of the functor F. It remains to prove that diagram (1.1.2) commutes, for each map $f : I \to J$ in $\operatorname{Mor} \mathcal{O}$ (resp. $\operatorname{Mor} \mathcal{S}$). By the definition of composition in $\widehat{\mathcal{D}}$, the left-bottom path of the diagram can be written as $(\phi^{f^{-1}j})_{j\in J} \cdot_f \phi^J$, or, substituting the expressions for the transformations, as

$$\bigl(G_{(X_i)_{i\in f^{-1}j};\otimes^{i\in f^{-1}j} X_i}(1_{\otimes^{i\in f^{-1}j} X_i})\bigr)_{j\in J} \cdot_f G_{(\otimes^{i\in f^{-1}j} X_i)_{j\in J};\otimes^{j\in J}\otimes^{i\in f^{-1}j} X_i}(1_{\otimes^{j\in J}\otimes^{i\in f^{-1}j} X_i}).$$

Since G is a multifunctor, the above expression equals

$$G_{(X_i)_{i\in I};\otimes^{j\in J}\otimes^{i\in f^{-1}j} X_i}\bigl((1_{\otimes^{i\in f^{-1}j} X_i})_{j\in J} \cdot_f 1_{\otimes^{j\in J}\otimes^{i\in f^{-1}j} X_i}\bigr).$$

The composite in the brackets is equal to $\lambda^f : \otimes^{i\in I} X_i \to \otimes^{j\in J} \otimes^{i\in f^{-1}j} X_i$ viewed as a morphism $(X_i)_{i\in I} \to \otimes^{j\in J} \otimes^{i\in f^{-1}j} X_i$ in $\widehat{\mathcal{C}}$. The top-right path of diagram (1.1.2) can be written as

$$G_{(X_i)_{i\in I};\otimes^{i\in I}X_i}(1_{\otimes^{i\in I}X_i}) \cdot_{\triangleright} G_{\otimes^{i\in I}X_i;\otimes^{j\in J}\otimes^{i\in \phi^{-1}j}X_i}(\lambda^f).$$

Since G is a multifunctor, the above composite is equal to

$$G_{(X_i)_{i\in I};\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i}\bigl(1_{\otimes^{i\in I}X_i} \cdot_{\triangleright} \lambda^f\bigr).$$

The composite in the brackets equals λ^f. We conclude that both paths of the diagram compose to $G_{(X_i)_{i\in I};\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i}(\lambda^f)$, hence diagram (1.1.2) is commutative, and $(F, \phi^I) : \mathcal{C} \to \mathcal{D}$ is a lax (symmetric) Monoidal functor.

Now we are going to prove that the two constructed maps are inverse to each other. Given a multifunctor G, we have produced a lax Monoidal functor (F, ϕ^I) out of it. Let us prove that $\widehat{F} = G$. Indeed, both multifunctors give $\operatorname{Ob} F = \operatorname{Ob} G$ on objects. To show that $\widehat{F}$ and G coincide on morphisms, let $f : (X_i)_{i\in I} \to Y$ be a morphism in $\widehat{\mathcal{C}}$, i.e., a morphism $f : \otimes^{i\in I} X_i \to Y$ in $\mathcal{C}$. Recall the definition of $\widehat{F}$:

$$\widehat{F}f = \bigl[\otimes^{i\in I} FX_i \xrightarrow{\phi^I} F \otimes^{i\in I} X_i \xrightarrow{Ff} FY\bigr].$$

The above composite can be written as the composite

$$G_{(X_i)_{i\in I};\otimes^{i\in I}X_i}(1_{\otimes^{i\in I}X_i}) \cdot_{\triangleright} G_{\otimes^{i\in I}X_i;Y}(f)$$

in $\widehat{\mathcal{D}}$. Since G is a multifunctor, the above composite equals

$$G_{(X_i)_{i\in I};Y}(1_{\otimes^{i\in I}X_i} \cdot_{\triangleright} f) = G_{(X_i)_{i\in I};Y}(f).$$

Therefore, $\widehat{F} = G$.

Given a lax (symmetric) Monoidal functor $(F, \phi^I) : \mathcal{C} \to \mathcal{D}$, we make a (symmetric) multifunctor $G = \widehat{F}$ out of it via the recipe of Proposition 1.2.27. The multifunctor G gives rise to a lax (symmetric) Monoidal functor $(H, \psi^I) : \mathcal{C} \to \mathcal{D}$. Let us prove that $(H, \psi^I) = (F, \phi^I)$. Indeed, both functors give $\operatorname{Ob} F = \operatorname{Ob} G = \operatorname{Ob} H$ on objects. Both coincide on morphisms, $H_{X,Y} = G_{X;Y} = F_{X,Y}$. Furthermore, for each $X_i \in \operatorname{Ob}\mathcal{C}$, $i \in I$,

$$\psi^I = \widehat{F}_{(X_i)_{i\in I};\otimes^{i\in I}X_i}(1_{\otimes^{i\in I}X_i}) = \phi^I \cdot F_{\otimes^{i\in I}X_i,\otimes^{i\in I}X_i}(1_{\otimes^{i\in I}X_i}) = \phi^I \cdot 1_{F\otimes^{i\in I}X_i} = \phi^I.$$

Thus $(H, \psi^I) = (F, \phi^I)$, and bijectivity on (multi)functors is proven.

Bijectivity on transformations is clear. Thus Proposition 1.2.25 is proven. □

Finally, as announced at the beginning of the section, let us formulate and prove the analogs of Propositions 1.2.17, 1.2.18 for enriched multicategories. We will not need the analog of Proposition 1.2.19, so it is omitted.

Let us introduce some notation. Suppose $\mathcal{C}$ is a $\mathcal{V}$-category, $f : \mathbb{1}_{\mathcal{V}} \to \mathcal{C}(X, Y)$ is a morphism. It gives rise to morphisms

$$f \cdot - = \mathcal{C}(f, 1) = \bigl[\mathcal{C}(Y, Z) \xrightarrow{\lambda_{\mathcal{V}}^{\cdot\,\mathsf{I}}} \mathbb{1}_{\mathcal{V}} \otimes \mathcal{C}(Y, Z) \xrightarrow{f\otimes 1} \mathcal{C}(X, Y) \otimes \mathcal{C}(Y, Z) \xrightarrow{\text{comp}} \mathcal{C}(X, Z)\bigr],$$

where comp means composition in $\mathcal{C}$. It follows from the associativity of composition in $\mathcal{C}$ that $\mathcal{C}(g, 1)\mathcal{C}(f, 1) = \mathcal{C}(fg, 1)$ whenever makes sense. The identity axiom implies $\mathcal{C}(1_X, 1) = \mathrm{id}$. Similarly, there are morphism

$$- \cdot f = \mathcal{C}(1, f) = \bigl[\mathcal{C}(V, X) \xrightarrow{\lambda^{\cdot\,\mathsf{I}}} \mathcal{C}(V, X) \otimes \mathbb{1}_{\mathcal{V}} \xrightarrow{1\otimes f} \mathcal{C}(V, X) \otimes \mathcal{C}(X, Y) \xrightarrow{\text{comp}} \mathcal{C}(V, Y)\bigr].$$

enjoying similar properties. Moreover, $\mathcal{C}(f, 1)\mathcal{C}(1, g) = \mathcal{C}(1, g)\mathcal{C}(f, 1)$, whenever makes sense.

1.2.26. Proposition. *A lax (symmetric) Monoidal $\mathcal{V}$-category $\mathcal{C} = (\mathcal{C}, \otimes^I, \lambda^f)$ gives rise to a (symmetric) $\mathcal{V}$-multicategory defined as follows. The set of objects* $\operatorname{Ob}\widehat{\mathcal{C}}$ *equals* $\operatorname{Ob}\mathcal{C}$. *For each $I \in$* Ob *and $X_i, Y \in \operatorname{Ob}\mathcal{C}$, $i \in I$, the object of morphisms $\widehat{\mathcal{C}}((X_i)_{i\in I}; Y)$ is $\mathcal{C}(\otimes^{i\in I}X_i, Y)$. Identities of $\widehat{\mathcal{C}}$ are those of $\mathcal{C}$. For each map $\phi : I \to J$ in* $\operatorname{Mor}\mathcal{O}$ *(resp.* $\operatorname{Mor}\mathcal{S}$*) and objects $X_i, Y_j, Z \in \operatorname{Ob}\mathcal{C}$, $i \in I$, $j \in J$, the composition morphism*

$$\mu_\phi^{\widehat{\mathcal{C}}} : \otimes^{J\sqcup\mathbf{1}}\big[\big(\widehat{\mathcal{C}}((X_i)_{i\in\phi^{-1}j}; Y_j)\big)_{j\in J}, \widehat{\mathcal{C}}((Y_j)_{j\in J}; Z)\big] \to \widehat{\mathcal{C}}((X_i)_{i\in I}; Z)$$

is given by the composite in $\mathcal{V}$

$$\begin{tikzcd}
\otimes^{J\sqcup\mathbf{1}}[(\mathcal{C}(\otimes^{i\in\phi^{-1}j}X_i, Y_j))_{j\in J}, \mathcal{C}(\otimes^{j\in J}Y_j, Z)] \arrow[d, "\lambda_{\mathcal{V}}^{\gamma}"] \\
\big(\otimes^{j\in J}\mathcal{C}(\otimes^{i\in\phi^{-1}j}X_i, Y_j)\big) \otimes \mathcal{C}(\otimes^{j\in J}Y_j, Z) \arrow[d, "\otimes^J\otimes 1"] \\
\mathcal{C}(\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}X_i, \otimes^{j\in J}Y_j) \otimes \mathcal{C}(\otimes^{j\in J}Y_j, Z) \arrow[d, "\lambda^{\phi\cdot - \cdot -}"] \\
\mathcal{C}(\otimes^{i\in I}X_i, Z),
\end{tikzcd}$$

where $\gamma : J \sqcup \mathbf{1} \to \mathbf{2}$, $\gamma(j) = 1$, $j \in J$, $\gamma(1) = 2$, $1 \in \mathbf{1}$.

Proof. Clearly, $\widehat{\mathcal{C}}$ is a $\mathcal{V}$-multiquiver and composition respects the $\mathcal{V}$-multiquiver structure. The left identity axiom follows from the commutative diagram

$$\begin{tikzcd}
\otimes^{I\sqcup\mathbf{1}}[(\mathbb{1}_{\mathcal{V}})_{i\in I}, \mathcal{C}(\otimes^{i\in I}X_i, Y)] \arrow[rr, "\otimes^{I\sqcup\mathbf{1}}((1_{X_i})_{i\in I},1)"] \arrow[dr, "\lambda_{\mathcal{V}}^{\gamma}"] & & \otimes^{I\sqcup\mathbf{1}}[(\mathcal{C}(X_i, X_i))_{i\in I}, \mathcal{C}(\otimes^{i\in I}X_i, Y)] \arrow[d, "\lambda_{\mathcal{V}}^{\gamma}"] \\
& (\otimes^{i\in I}\mathbb{1}_{\mathcal{V}}) \otimes \mathcal{C}(\otimes^{i\in I}X_i, Y) \arrow[r, "\otimes^{i\in I}1_{X_i}\otimes 1"] & (\otimes^{i\in I}\mathcal{C}(X_i, X_i)) \otimes \mathcal{C}(\otimes^{i\in I}X_i, Y) \arrow[d, "\otimes^I\otimes 1"] \\
& \mathbb{1}_{\mathcal{V}} \otimes \mathcal{C}(\otimes^{i\in I}X_i, Y) \arrow[u, "\lambda_{\mathcal{V}}^{\varnothing\to I}\otimes 1"] \arrow[r, "1_{\otimes^{i\in I}X_i}\otimes 1"] & \mathcal{C}(\otimes^{i\in I}X_i, \otimes^{i\in I}X_i) \otimes \mathcal{C}(\otimes^{i\in I}X_i, Y) \arrow[d, "\mathrm{comp}"] \\
\mathcal{C}(\otimes^{i\in I}X_i, Y) \arrow[uuu, "\lambda_{\mathcal{V}}^{\mathbf{1}\hookrightarrow I\sqcup\mathbf{1}}"] \arrow[ur, "\lambda_{\mathcal{V}}^{;\,|}"] \arrow[rr, equal] & & \mathcal{C}(\otimes^{i\in I}X_i, Y)
\end{tikzcd}$$

The top trapezoid commutes by the naturality of $\lambda_{\mathcal{V}}^{\gamma}$. The left quadrilateral is a particular case of equation (1.1.1) written for the pair of maps $\mathbf{1} \hookrightarrow I \sqcup \mathbf{1} \xrightarrow{\gamma} \mathbf{2}$. The square in the middle commutes since $\otimes^I$ is a $\mathcal{V}$-functor. Finally, the bottom trapezoid commutes since $1_{\otimes^{i\in I}X_i}$ satisfies the identity axiom in $\mathcal{C}$. The right identity axiom is obvious.

To prove the associativity equation for a composable pair $I \xrightarrow{f} J \xrightarrow{g} K$, consider objects X_i, Y_j, Z_k, W of $\mathcal{C}$ and substitute the definition of composition in $\widehat{\mathcal{C}}$. The last factor $\mathcal{C}(\otimes^{k\in K}Z_k, W)$ splits out and the equation takes the form of the exterior of Diagram 1.4. Here f_k denotes the map $f|_{f^{-1}g^{-1}k} : f^{-1}g^{-1}k \to g^{-1}k$, $k \in K$. Pentagon 1 commutes due to $\otimes^K$ being a $\mathcal{V}$-functor. Square 2 commutes by equation (1.1.5). Polygon 3 in Diagram 1.4 can be rewritten as Diagram 1.5. Here pentagon 4 commutes due to $\otimes^K$ being a $\mathcal{V}$-functor. Square 5 is the definition of $\otimes_{\mathcal{C}}^{g}$. Quadrilateral 6 follows from the associativity

$$\begin{tikzcd}
\bigotimes_{j\in J}\mathcal{C}(\bigotimes_{i\in f^{-1}j}X_i, Y_j)\otimes\bigotimes_{k\in K}\mathcal{C}(\bigotimes_{j\in g^{-1}k}Y_j, Z_k) \arrow[rrrr, "\otimes^J\otimes\otimes^K"] \arrow[d, "\wr"'] & & & & \mathcal{C}(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j}X_i, \bigotimes_{j\in J}Y_j)\otimes\mathcal{C}(\bigotimes_{k\in K}\bigotimes_{j\in g^{-1}k}Y_j, \bigotimes_{k\in K}Z_k) \arrow[d, "(-\cdot\lambda^g)\otimes1"] \\
\bigotimes_{k\in K}[\bigotimes_{j\in g^{-1}k}\mathcal{C}(\bigotimes_{i\in f^{-1}j}X_i, Y_j)\otimes\mathcal{C}(\bigotimes_{j\in g^{-1}k}Y_j, Z_k)] \arrow[rrrr, phantom, "\boxed{3}"] \arrow[dd, "\bigotimes_{k\in K}[\otimes^{g^{-1}k}\otimes1]"'] & & & & \mathcal{C}(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j}X_i, \bigotimes_{k\in K}\bigotimes_{j\in g^{-1}k}Y_j)\otimes\mathcal{C}(\bigotimes_{k\in K}\bigotimes_{j\in g^{-1}k}Y_j, \bigotimes_{k\in K}Z_k) \arrow[ddd, "\mathrm{comp}"] \\
 & & \bigotimes_{k\in K}\mathcal{C}(\bigotimes_{j\in g^{-1}k}\bigotimes_{i\in f^{-1}j}X_i, Z_k) \arrow[dd, "\otimes^K"] & & \\
\bigotimes_{k\in K}[\mathcal{C}(\bigotimes_{j\in g^{-1}k}\bigotimes_{i\in f^{-1}j}X_i, \bigotimes_{j\in g^{-1}k}Y_j)\otimes\mathcal{C}(\bigotimes_{j\in g^{-1}k}Y_j, Z_k)] \arrow[urr, "\bigotimes_{k\in K}\mathrm{comp}"] \arrow[dd, "\bigotimes_{k\in K}[\lambda^{f_k}\cdot-\cdot-]"'] & & & & \\
 & \boxed{1} & \mathcal{C}(\bigotimes_{k\in K}\bigotimes_{j\in g^{-1}k}\bigotimes_{i\in f^{-1}j}X_i, \bigotimes_{k\in K}Z_k) \arrow[rr, "\lambda^g\cdot-"] \arrow[d, "(\bigotimes_{k\in K}\lambda^{f_k})\cdot-"] \arrow[drr, phantom, "\boxed{2}"] & & \mathcal{C}(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j}X_i, \bigotimes_{k\in K}Z_k) \arrow[d, "\lambda^f\cdot-"] \\
\bigotimes_{k\in K}\mathcal{C}(\bigotimes_{i\in f^{-1}g^{-1}k}X_i, Z_k) \arrow[rr, "\otimes^K"] & & \mathcal{C}(\bigotimes_{k\in K}\bigotimes_{i\in f^{-1}g^{-1}k}X_i, \bigotimes_{k\in K}Z_k) \arrow[rr, "\lambda^{fg}\cdot-"] & & \mathcal{C}(\bigotimes_{i\in I}X_i, \bigotimes_{k\in K}Z_k)
\end{tikzcd}$$

DIAGRAM 1.4.

$$\begin{tikzcd}
\bigotimes\limits_{j\in J}\mathcal{C}\Big(\bigotimes\limits_{i\in f^{-1}j}X_i, Y_j\Big)\otimes\bigotimes\limits_{k\in K}\mathcal{C}\Big(\bigotimes\limits_{j\in g^{-1}k}Y_j, Z_k\Big) \arrow[rrrr, "\otimes^J\otimes\otimes^K"] \arrow[drr, "\sim"] \arrow[dd, "\wr"'] & & & & \mathcal{C}\Big(\bigotimes\limits_{j\in J}\bigotimes\limits_{i\in f^{-1}j}X_i, \bigotimes\limits_{j\in J}Y_j\Big)\otimes\mathcal{C}\Big(\bigotimes\limits_{k\in K}\bigotimes\limits_{j\in g^{-1}k}Y_j, \bigotimes\limits_{k\in K}Z_k\Big) \arrow[dddd, "(-\cdot\lambda^g)\otimes 1"] \\
& & \bigotimes\limits_{k\in K}\bigotimes\limits_{j\in g^{-1}k}\mathcal{C}\Big(\bigotimes\limits_{i\in f^{-1}j}X_i, Y_j\Big)\otimes\bigotimes\limits_{k\in K}\mathcal{C}\Big(\bigotimes\limits_{j\in g^{-1}k}Y_j, Z_k\Big) \arrow[dd, "\otimes^g_{\mathcal{C}}\otimes 1"] & & \\
\bigotimes\limits_{k\in K}\Big[\bigotimes\limits_{j\in g^{-1}k}\mathcal{C}\Big(\bigotimes\limits_{i\in f^{-1}j}X_i, Y_j\Big)\otimes\mathcal{C}\Big(\bigotimes\limits_{j\in g^{-1}k}Y_j, Z_k\Big)\Big] \arrow[urr, "\sim"] \arrow[dd, "\bigotimes\limits_{k\in K}{[\otimes^{g^{-1}k}\otimes 1]}"'] & & & \boxed{7} & \\
& \boxed{5} & \bigotimes\limits_{k\in K}\mathcal{C}\Big(\bigotimes\limits_{j\in g^{-1}k}\bigotimes\limits_{i\in f^{-1}j}X_i, \bigotimes\limits_{j\in g^{-1}k}Y_j\Big)\otimes\bigotimes\limits_{k\in K}\mathcal{C}\Big(\bigotimes\limits_{j\in g^{-1}k}Y_j, Z_k\Big) \arrow[dd, "\otimes^K\otimes\otimes^K"'] & & \\
\bigotimes\limits_{k\in K}\Big[\mathcal{C}\Big(\bigotimes\limits_{j\in g^{-1}k}\bigotimes\limits_{i\in f^{-1}j}X_i, \bigotimes\limits_{j\in g^{-1}k}Y_j\Big)\otimes\mathcal{C}\Big(\bigotimes\limits_{j\in g^{-1}k}Y_j, Z_k\Big)\Big] \arrow[urr, "\sim"] \arrow[dd, "\bigotimes\limits_{k\in K}\mathrm{comp}"'] & & & & \mathcal{C}\Big(\bigotimes\limits_{j\in J}\bigotimes\limits_{i\in f^{-1}j}X_i, \bigotimes\limits_{k\in K}\bigotimes\limits_{j\in g^{-1}k}Y_j\Big)\otimes\mathcal{C}\Big(\bigotimes\limits_{k\in K}\bigotimes\limits_{j\in g^{-1}k}Y_j, \bigotimes\limits_{k\in K}Z_k\Big) \arrow[dd, "\mathrm{comp}"] \\
& \boxed{4} & \mathcal{C}\Big(\bigotimes\limits_{k\in K}\bigotimes\limits_{j\in g^{-1}k}\bigotimes\limits_{i\in f^{-1}j}X_i, \bigotimes\limits_{k\in K}\bigotimes\limits_{j\in g^{-1}k}Y_j\Big)\otimes\mathcal{C}\Big(\bigotimes\limits_{k\in K}\bigotimes\limits_{j\in g^{-1}k}Y_j, \bigotimes\limits_{k\in K}Z_k\Big) \arrow[urr, "(\lambda^g\cdot -)\otimes 1"] \arrow[d, "\mathrm{comp}"] & \boxed{6} & \\
\bigotimes\limits_{k\in K}\mathcal{C}\Big(\bigotimes\limits_{j\in g^{-1}k}\bigotimes\limits_{i\in f^{-1}j}X_i, Z_k\Big) \arrow[rr, "\otimes^K"] & & \mathcal{C}\Big(\bigotimes\limits_{k\in K}\bigotimes\limits_{j\in g^{-1}k}\bigotimes\limits_{i\in f^{-1}j}X_i, \bigotimes\limits_{k\in K}Z_k\Big) \arrow[rr, "\lambda^g\cdot -"] & & \mathcal{C}\Big(\bigotimes\limits_{j\in J}\bigotimes\limits_{i\in f^{-1}j}X_i, \bigotimes\limits_{k\in K}Z_k\Big)
\end{tikzcd}$$

Diagram 1.5.

of composition in $\mathcal{C}$. Polygon 7 expresses the $\mathcal{V}$-naturality of the transformation λ^g. Therefore, the associativity of composition in $\widehat{\mathcal{C}}$ is proven. $\square$

1.2.27. Proposition. *A lax (symmetric) Monoidal $\mathcal{V}$-functor*

$$(F, \phi^I) : (\mathcal{C}, \otimes^I, \lambda^f_{\mathcal{C}}) \to (\mathcal{D}, \otimes^I, \lambda^f_{\mathcal{D}})$$

between lax (symmetric) Monoidal $\mathcal{V}$-categories gives rise to a (symmetric) $\mathcal{V}$-multifunctor $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ defined as follows. The mapping of objects is $\operatorname{Ob}\widehat{F} = \operatorname{Ob}F$. *The action on objects of morphisms is given by*

$$\widehat{F}_{(X_i);Y} = \bigl[\widehat{\mathcal{C}}((X_i)_{i\in I}; Y) = \mathcal{C}(\otimes^{i\in I}X_i, Y) \xrightarrow{F_{\otimes^{i\in I}X_i,Y}} \mathcal{D}(F\otimes^{i\in I}X_i, FY) \\ \xrightarrow{\phi^I\cdot -} \mathcal{D}(\otimes^{i\in I}FX_i, FY) = \widehat{\mathcal{D}}((FX_i)_{i\in I}; FY)\bigr]. \qquad (1.2.9)$$

Proof. Clearly, $\widehat{F}$ is a morphism of $\mathcal{V}$-multiquivers. Since $\phi^{\mathbf{1}} = \mathrm{id}$, it follows that

$$\widehat{F}_{X;X} = F_{X,X} : \mathcal{C}(X, X) \to \mathcal{D}(FX, FX)$$

preserves identities.

Compatibility of $\widehat{F}$ with composition that corresponds to a map $f : I \to J$ is expressed by the equation

$$\begin{tikzcd}
\otimes^{J\sqcup\mathbf{1}}\bigl[\bigl(\widehat{\mathcal{C}}((X_i)_{i\in f^{-1}j}; Y_j)\bigr)_{j\in J}, \widehat{\mathcal{C}}((Y_j)_{j\in J}; Z)\bigr] \arrow[r, "\mu^{\widehat{\mathcal{C}}}_f"] \arrow[d, "\otimes^{J\sqcup\mathbf{1}}[(\widehat{F}_{(X_i)_{i\in f^{-1}j};Y_j})_{j\in J}, \widehat{F}_{(Y_j)_{j\in J};Z}]"'] & \widehat{\mathcal{C}}((X_i)_{i\in I}; Z) \arrow[d, "\widehat{F}_{(X_i)_{i\in I};Z}"] \\
\otimes^{J\sqcup\mathbf{1}}\bigl[\bigl(\widehat{\mathcal{D}}((FX_i)_{i\in f^{-1}j}; FY_j)\bigr)_{j\in J}, \widehat{\mathcal{D}}((FY_j)_{j\in J}; FZ)\bigr] \arrow[r, "\mu^{\widehat{\mathcal{D}}}_f"] & \widehat{\mathcal{D}}((FX_i)_{i\in I}; FZ)
\end{tikzcd}$$

It coincides with the exterior of Diagram 1.6. Here quadrilateral 1 commutes due to $\otimes^J$ being a $\mathcal{V}$-functor. Quadrilateral 2 follows from the associativity of composition in $\mathcal{D}$. Quadrilateral 3 commutes due to F being a $\mathcal{V}$-functor. The remaining polygon 4 is the exterior of Diagram 1.7. In this diagram, quadrilateral 5 is due to F being a $\mathcal{V}$-functor. Triangle 6 commutes, as equation (1.1.6) shows. Hexagon 7 follows from the $\mathcal{V}$-naturality of transformation ϕ^J. Therefore, the whole diagram commutes, and $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ is a $\mathcal{V}$-multifunctor. $\square$

1.2.28. Remark. The change of symmetric Monoidal base category considered in Section 1.1.14 can be expressed in particularly convenient form using multifunctors. Let $(B, \beta^I) : (\mathcal{V}, \otimes^I_{\mathcal{V}}, \lambda^f_{\mathcal{V}}) \to (\mathcal{W}, \otimes^I_{\mathcal{W}}, \lambda^f_{\mathcal{W}})$ be a lax symmetric Monoidal functor. By Proposition 1.2.27, it gives rise to a symmetric multifunctor $\widehat{B} : \widehat{\mathcal{V}} \to \widehat{\mathcal{W}}$. The induced lax symmetric **Cat**-functor $(B_*, \beta^I_*) : \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{W}\text{-}\mathbf{Cat}$ constructed in Section 1.1.14 yields a symmetric **Cat**-multifunctor $\widehat{B_*} : \widehat{\mathcal{V}\text{-}\mathbf{Cat}} \to \widehat{\mathcal{W}\text{-}\mathbf{Cat}}$. There is a simple relation between the multifunctor $\widehat{B}$ and the **Cat**-multifunctor $\widehat{B_*}$. For a $\mathcal{V}$-category $\mathcal{C}$, $(\widehat{B_*}\mathcal{C})(X, Y) = B(\mathcal{C}(X, Y))$, $\mu_{\widehat{B_*}\mathcal{C}} = \widehat{B}(\mu_{\mathcal{C}}) \in \widehat{\mathcal{V}}(B\mathcal{C}(X, Y), B\mathcal{C}(Y, Z); B\mathcal{C}(X, Z))$, and $\eta^{\widehat{B_*}\mathcal{C}}_X = \widehat{B}(\eta^{\mathcal{C}}_X) \in \widehat{\mathcal{V}}(; B\mathcal{C}(X, X))$. For a $\mathcal{V}$-functor $F \in \widehat{\mathcal{V}\text{-}\mathbf{Cat}}((\mathcal{C}_i)_{i\in I}; \mathcal{D})$, $\operatorname{Ob}\widehat{B_*}F = \operatorname{Ob}F$, and

$$(\widehat{B_*}F)_{(X_i)_{i\in I},(Y_i)_{i\in I}} = \widehat{B}(F_{(X_i)_{i\in I},(Y_i)_{i\in I}}) \in \widehat{\mathcal{W}}\bigl((B\mathcal{C}_i(X_i, Y_i))_{i\in I}, B\mathcal{D}(F(X_i)_{i\in I}, F(Y_i)_{i\in I})\bigr).$$

For a natural transformation $t : F \to G : (\mathcal{C}_i)_{i\in I} \to \mathcal{D}$ of $\mathcal{V}$-functors,

$$(\widehat{B_*}t)_{(X_i)_{i\in I}} = \widehat{B}(t_{(X_i)_{i\in I}}) \in \widehat{\mathcal{W}}\bigl(; B\mathcal{D}(F(X_i)_{i\in I}, G(X_i)_{i\in I})\bigr).$$

Diagram 1.6.

$$\begin{tikzcd}
\bigotimes_{j\in J}\mathcal{C}(\bigotimes_{i\in f^{-1}j}X_i,Y_j)\otimes\mathcal{C}(\bigotimes_{j\in J}Y_j,Z) \arrow[r, "\otimes^J\otimes 1"] \arrow[ddddd, "\bigotimes_{j\in J}F_{\bigotimes_{i\in f^{-1}j}X_i,Y_j}\otimes F_{\bigotimes_{j\in J}Y_j,Z}"'] & \mathcal{C}(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j}X_i,\bigotimes_{j\in J}Y_j)\otimes\mathcal{C}(\bigotimes_{j\in J}Y_j,Z) \arrow[rr, "\lambda^f_{\mathcal{C}}\cdot-\cdot-"] \arrow[d, "\mathrm{comp}"'] & & \mathcal{C}(\bigotimes_{i\in I}X_i,Z) \arrow[dd, "F_{\bigotimes_{i\in I}X_i,Z}"] \\
& \mathcal{C}(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j}X_i,Z) \arrow[urr, "\lambda^f_{\mathcal{C}}\cdot-"'] \arrow[d, "F_{\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j}X_i,Z}"'] & \boxed{3} & \\
& \mathcal{D}(F(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j}X_i),FZ) \arrow[rr, "(F\lambda^f_{\mathcal{C}})\cdot-"] \arrow[drr, "(\phi^I\cdot F\lambda^f_{\mathcal{C}})\cdot-"] & & \mathcal{D}(F(\bigotimes_{i\in I}X_i),FZ) \arrow[d, "\phi^I\cdot-"] \\
& \boxed{4} & & \mathcal{D}(\bigotimes_{i\in I}FX_i,FZ) \\
& \mathcal{D}(\bigotimes_{j\in J}(F(\bigotimes_{i\in f^{-1}j}X_i)),F(\bigotimes_{j\in J}Y_j))\otimes\mathcal{D}(F(\bigotimes_{j\in J}Y_j),FZ) \arrow[urr, "(\lambda^f_{\mathcal{D}}\cdot\bigotimes_{j\in J}\phi^{f^{-1}j})\cdot-\cdot-"] & & \\
\bigotimes_{j\in J}\mathcal{D}(F(\bigotimes_{i\in f^{-1}j}X_i),FY_j)\otimes\mathcal{D}(F(\bigotimes_{j\in J}Y_j),FZ) \arrow[r, "\otimes^J\otimes 1"] \arrow[d, "\bigotimes_{j\in J}(\phi^{f^{-1}j}\cdot-)\otimes(\phi^J\cdot-)"'] & \mathcal{D}(\bigotimes_{j\in J}(F(\bigotimes_{i\in f^{-1}j}X_i)),\bigotimes_{j\in J}FY_j)\otimes\mathcal{D}(F(\bigotimes_{j\in J}Y_j),FZ) \arrow[u, "(-\cdot\phi^J)\otimes 1"] \arrow[drr, "{[(\bigotimes_{j\in J}\phi^{f^{-1}j})\cdot-]\otimes(\phi^J\cdot-)}"] & \boxed{2} & \\
\bigotimes_{j\in J}\mathcal{D}(\bigotimes_{i\in f^{-1}j}FX_i,FY_j)\otimes\mathcal{D}(\bigotimes_{j\in J}FY_j,FZ) \arrow[rrr, "\otimes^J\otimes 1"] & \boxed{1} & & \mathcal{D}(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j}FX_i,\bigotimes_{j\in J}FY_j)\otimes\mathcal{D}(\bigotimes_{j\in J}FY_j,FZ) \arrow[uuu, "\lambda^f_{\mathcal{D}}\cdot-\cdot-"']
\end{tikzcd}$$

$$\begin{tikzcd}
\bigotimes_{j\in J}\mathcal{C}(\bigotimes_{i\in f^{-1}j} X_i, Y_j)\otimes\mathcal{C}(\bigotimes_{j\in J} Y_j, Z) \arrow[r, "\otimes^J\otimes 1"] \arrow[dd, "\bigotimes_{j\in J} F_{\bigotimes_{i\in f^{-1}j} X_i, Y_j}\otimes F_{\bigotimes_{j\in J} Y_j, Z}"'] & \mathcal{C}(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j} X_i, \bigotimes_{j\in J} Y_j)\otimes\mathcal{C}(\bigotimes_{j\in J} Y_j, Z) \arrow[r, "\mathrm{comp}"] \arrow[d, "F_{\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j} X_i, \bigotimes_{j\in J} Y_j}\otimes F_{\bigotimes_{j\in J} Y_j, Z}"] & \mathcal{C}(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j} X_i, Z) \arrow[dd, "F_{\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j} X_i, Z}"] \\
 & \mathcal{D}(F(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j} X_i), F(\bigotimes_{j\in J} Y_j))\otimes\mathcal{D}(F(\bigotimes_{j\in J} Y_j), FZ) \arrow[dr, "\mathrm{comp}"] \arrow[ddr, "(\phi^I\cdot F\lambda^f_{\mathcal{C}})\cdot-\cdot-"'] \arrow[ddd, "(\phi^J\cdot-)\otimes 1"'] & \boxed{5} \\
\bigotimes_{j\in J}\mathcal{D}(F(\bigotimes_{i\in f^{-1}j} X_i), FY_j)\otimes\mathcal{D}(F(\bigotimes_{j\in J} Y_j), FZ) \arrow[d, "\otimes^J\otimes 1"'] & \boxed{7} & \mathcal{D}(F(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j} X_i), FZ) \arrow[d, "(\phi^I\cdot F\lambda^f_{\mathcal{C}})\cdot-"] \\
\mathcal{D}(\bigotimes_{j\in J}(F(\bigotimes_{i\in f^{-1}j} X_i)), \bigotimes_{j\in J} FY_j)\otimes\mathcal{D}(F(\bigotimes_{j\in J} Y_j), FZ) \arrow[dr, "(-\cdot\phi^J)\otimes 1"] & \boxed{6} & \mathcal{D}(\bigotimes_{i\in I} FX_i, FZ) \\
 & \mathcal{D}(\bigotimes_{j\in J}(F(\bigotimes_{i\in f^{-1}j} X_i)), F(\bigotimes_{j\in J} Y_j))\otimes\mathcal{D}(F(\bigotimes_{j\in J} Y_j), FZ) \arrow[ur, "(\lambda^f_{\mathcal{D}}\cdot\bigotimes_{j\in J}\phi^{f^{-1}j})\cdot-\cdot-"] &
\end{tikzcd}$$

Diagram 1.7.

In practice it is often more convenient to work with the **Cat**-multifunctor $\widehat{B_*}$ rather than with the lax symmetric Monoidal **Cat**-functor (B_*, β^I_*).

1.3. Closed multicategories

1.3.1. Categories and multicategories enriched in symmetric multicategories. According to the classical picture, categories can be enriched in Monoidal categories and lax (symmetric) Monoidal categories can be enriched in symmetric Monoidal categories. As we have seen in Section 1.2.16, multicategories generalize lax Monoidal categories. It is therefore not surprising that categories can be enriched in multicategories and multicategories can be enriched in symmetric multicategories.

Throughout the section, V is a symmetric multicategory. We denote composition in V simply by the dot $\cdot$, whenever no ambiguity is likely.

A V-*category* $\mathcal{C}$ consists of a set of objects $\operatorname{Ob}\mathcal{C}$, an object of morphisms $\mathcal{C}(X,Y) \in \operatorname{Ob}\mathsf{V}$, for each pair $X, Y \in \operatorname{Ob}\mathcal{C}$, a composition morphism

$$\mu_{\mathcal{C}} : \mathcal{C}(X,Y), \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z),$$

for each triple $X, Y, Z \in \operatorname{Ob}\mathcal{C}$, and an identity morphism $1^{\mathcal{C}}_X : () \to \mathcal{C}(X,X)$, for each $X \in \operatorname{Ob}\mathcal{C}$. The associativity of composition is expressed by the commutativity of

$$\begin{tikzcd}
\mathcal{C}(W,X), \mathcal{C}(X,Y), \mathcal{C}(Y,Z) \arrow[r, "{1,\mu_{\mathcal{C}}}"] \arrow[d, "{\mu_{\mathcal{C}},1}"'] & \mathcal{C}(W,X), \mathcal{C}(X,Z) \arrow[d, "\mu_{\mathcal{C}}"] \\
\mathcal{C}(W,Y), \mathcal{C}(Y,Z) \arrow[r, "\mu_{\mathcal{C}}"] & \mathcal{C}(W,Z)
\end{tikzcd}$$

for each quadruple $W, X, Y, Z \in \operatorname{Ob}\mathcal{C}$. The identity axiom is expressed by the equations

$$\big[\mathcal{C}(X,Y) \xrightarrow{1^{\mathsf{V}}_{\mathcal{C}(X,Y)}, 1^{\mathcal{C}}_Y} \mathcal{C}(X,Y), \mathcal{C}(Y,Y) \xrightarrow{\mu_{\mathcal{C}}} \mathcal{C}(X,Y)\big] = 1^{\mathsf{V}}_{\mathcal{C}(X,Y)},$$

$$\big[\mathcal{C}(X,Y) \xrightarrow{1^{\mathcal{C}}_X, 1^{\mathsf{V}}_{\mathcal{C}(X,Y)}} \mathcal{C}(X,X), \mathcal{C}(X,Y) \xrightarrow{\mu_{\mathcal{C}}} \mathcal{C}(X,Y)\big] = 1^{\mathsf{V}}_{\mathcal{C}(X,Y)},$$

for each pair $X, Y \in \operatorname{Ob}\mathcal{C}$.

Each V-category $\mathcal{C}$ gives rise to a V-category $\mathcal{C}^{\mathrm{op}}$, the *opposite* V-*category*, with the same set of objects. For each pair $X, Y \in \operatorname{Ob}\mathcal{C}$, the object of morphisms $\mathcal{C}^{\mathrm{op}}(X,Y)$ is $\mathcal{C}(Y,X)$. Composition in $\mathcal{C}^{\mathrm{op}}$ is given by

$$\mu_{\mathcal{C}^{\mathrm{op}}} = \mathsf{V}(\mathsf{X}; \mathcal{C}(Z,X))(\mu_{\mathcal{C}}) = (1^{\mathsf{V}}_{\mathcal{C}(Y,X)}, 1^{\mathsf{V}}_{\mathcal{C}(Z,Y)}) \cdot_{\mathsf{X}} \mu_{\mathcal{C}} : \mathcal{C}(Y,X), \mathcal{C}(Z,Y) \to \mathcal{C}(Z,X),$$

where $\mathsf{X} = (12) : \mathbf{2} \to \mathbf{2}$, $1 \mapsto 2$, $2 \mapsto 1$. Identities of $\mathcal{C}^{\mathrm{op}}$ are those of $\mathcal{C}$. The associativity of composition is established as follows. The top-right composite in the associativity diagram equals

$$\begin{aligned}
(1, \mu_{\mathcal{C}^{\mathrm{op}}}) \cdot_{\mathsf{IV}} \mu_{\mathcal{C}^{\mathrm{op}}} &= \big(1^{\mathsf{V}}_{\mathcal{C}(X,W)}, (1^{\mathsf{V}}_{\mathcal{C}(Y,X)}, 1^{\mathsf{V}}_{\mathcal{C}(Z,Y)}) \cdot_{\mathsf{X}} \mu_{\mathcal{C}}\big) \cdot_{\mathsf{IV}} \big((1^{\mathsf{V}}_{\mathcal{C}(X,W)}, 1^{\mathsf{V}}_{\mathcal{C}(Z,X)}) \cdot_{\mathsf{X}} \mu_{\mathcal{C}}\big) \\
&= \big(1^{\mathsf{V}}_{\mathcal{C}(X,W)} \cdot 1^{\mathsf{V}}_{\mathcal{C}(X,W)}, ((1^{\mathsf{V}}_{\mathcal{C}(Y,X)}, 1^{\mathsf{V}}_{\mathcal{C}(Z,Y)}) \cdot_{\mathsf{X}} \mu_{\mathcal{C}}) \cdot 1^{\mathsf{V}}_{\mathcal{C}(Z,X)}\big) \cdot_{\mathsf{VI}\cdot\mathsf{X}} \mu_{\mathcal{C}} \\
&= \big(1^{\mathsf{V}}_{\mathcal{C}(X,W)}, (1^{\mathsf{V}}_{\mathcal{C}(Y,X)}, 1^{\mathsf{V}}_{\mathcal{C}(Z,Y)}) \cdot_{\mathsf{X}} \mu_{\mathcal{C}}\big) \cdot_{\mathsf{VI}\cdot\mathsf{X}} \mu_{\mathcal{C}} \\
&= (1^{\mathsf{V}}_{\mathcal{C}(X,W)}, 1^{\mathsf{V}}_{\mathcal{C}(Y,X)}, 1^{\mathsf{V}}_{\mathcal{C}(Z,Y)}) \cdot_{(13)} \big((\mu_{\mathcal{C}}, 1^{\mathsf{V}}_{\mathcal{C}(X,W)}) \cdot_{\mathsf{VI}} \mu_{\mathcal{C}}\big),
\end{aligned}$$

by the associativity of composition in V and by the identity axiom. Here $(13) : \mathbf{3} \to \mathbf{3}$ is the transposition that interchanges 1 and 3. Similarly, the left-bottom composite equals

$$(\mu_{\mathcal{C}^{\mathrm{op}}}, 1) \cdot_{\mathsf{VI}} \mu_{\mathcal{C}^{\mathrm{op}}} = (1^{\mathsf{V}}_{\mathcal{C}(X,W)}, 1^{\mathsf{V}}_{\mathcal{C}(Y,X)}, 1^{\mathsf{V}}_{\mathcal{C}(Z,Y)}) \cdot_{(13)} \big((1^{\mathsf{V}}_{\mathcal{C}(Z,Y)}, \mu_{\mathcal{C}}) \cdot_{\mathsf{VI}} \mu_{\mathcal{C}}\big).$$

The associativity of $\mu_{\mathcal{C}}$ implies the associativity of $\mu_{\mathcal{C}^{\mathrm{op}}}$. The identity axiom is obvious.

For V-categories $\mathcal{C}$, $\mathcal{D}$, a V-*functor* $F : \mathcal{C} \to \mathcal{D}$ consists of a function

$$\operatorname{Ob} F : \operatorname{Ob} \mathcal{C} \to \operatorname{Ob} \mathcal{D}, \quad X \mapsto FX,$$

and morphisms $F = F_{X,Y} : \mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$, $X, Y \in \operatorname{Ob} \mathcal{C}$, such that composition and identities are preserved. The compatibility with composition means that the diagram in V

$$\begin{tikzcd}
\mathcal{C}(X,Y), \mathcal{C}(Y,Z) \arrow[r, "\mu_{\mathcal{C}}"] \arrow[d, "F,F"'] & \mathcal{C}(X,Z) \arrow[d, "F"] \\
\mathcal{D}(FX,FY), \mathcal{D}(FY,FZ) \arrow[r, "\mu_{\mathcal{D}}"] & \mathcal{D}(FX,FZ)
\end{tikzcd}$$

commutes, for each $X, Y, Z \in \operatorname{Ob} \mathcal{C}$. The compatibility with identities is expressed by the equation in V

$$\big[() \xrightarrow{1^{\mathcal{C}}_X} \mathcal{C}(X, X) \xrightarrow{F} \mathcal{D}(FX, FX)\big] = 1^{\mathcal{D}}_{FX},$$

for each $X \in \operatorname{Ob} \mathcal{C}$.

Given V-functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, define their composite $FG : \mathcal{C} \to \mathcal{E}$ by $\operatorname{Ob} FG = \operatorname{Ob} F \cdot \operatorname{Ob} G$ and $(FG)_{X,Y} = F_{X,Y} \cdot G_{FX,FY}$, $X, Y \in \operatorname{Ob} \mathcal{C}$. Showing that FG is a V-functor is a straightforward computation.

For V-functors $F, G : \mathcal{C} \to \mathcal{D}$, a V-*natural transformation* $t : F \to G : \mathcal{C} \to \mathcal{D}$ is a family of morphisms $t_X : () \to \mathcal{D}(XF, XG)$, $X \in \operatorname{Ob} \mathcal{C}$, such that the diagram in V

$$\begin{tikzcd}
\mathcal{C}(X,Y) \arrow[r, "F{,}t_Y"] \arrow[d, "t_X{,}G"'] & \mathcal{D}(XF,YF), \mathcal{D}(YF,YG) \arrow[d, "\mu_{\mathcal{D}}"] \\
\mathcal{D}(XF,XG), \mathcal{D}(XG,YG) \arrow[r, "\mu_{\mathcal{D}}"] & \mathcal{D}(XF,YG)
\end{tikzcd}$$

commutes, for each $X, Y \in \operatorname{Ob} \mathcal{C}$.

The vertical composite of V-natural transformations $t : F \to G : \mathcal{C} \to \mathcal{D}$ and $u : G \to H : \mathcal{C} \to \mathcal{D}$ has components

$$(tu)_X = \big[() \xrightarrow{t_X, u_X} \mathcal{D}(FX, GX), \mathcal{D}(GX, HX) \xrightarrow{\mu_{\mathcal{D}}} \mathcal{D}(FX, HX)\big], \quad X \in \operatorname{Ob} \mathcal{C}.$$

The composite tK of t and a V-functor $K : \mathcal{D} \to \mathcal{E}$ has for its component $(tK)_X$ the composite

$$() \xrightarrow{t_X} \mathcal{D}(FX, GX) \xrightarrow{K} \mathcal{E}(KFX, KGX).$$

The composite Et of t and a V-functor $E : \mathcal{B} \to \mathcal{C}$ has for its component $(Et)_X$ simply t_{EX}.

It is routine to check that with so defined compositions V-categories, V-functors, and V-natural transformations form a 2-category V-**Cat**.

1.3.2. Example. Let $\mathcal{V} = (\mathcal{V}, \otimes^I, \lambda^f)$ be a symmetric Monoidal category. The usual notions of $\mathcal{V}$-category, $\mathcal{V}$-functor, and $\mathcal{V}$-natural transformation are recovered by considering $\mathsf{V} = \widehat{\mathcal{V}}$.

1.3.3. Remark. Let $\mathcal{V} = (\mathcal{V}, \otimes^I, \lambda^f)$ be a symmetric Monoidal category. As we have seen in Example 1.1.13, $\mathcal{V}$-categories, $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations form not merely a 2-category, but a symmetric Monoidal **Cat**-category. We could define V-functors of several variables and V-natural transformations between these V-functors, and would end up with a symmetric **Cat**-multicategory of V-categories. Such generality, however, seems superfluous for the purposes of the dissertation.

1.3.4. Definition. A V-*multiquiver* C consists of a set of objects $\operatorname{Ob}\mathsf{C}$ and an object of morphisms $\mathsf{C}((X_i)_{i\in I};Y)\in\operatorname{Ob}\mathsf{V}$, for each $I\in\operatorname{Ob}\mathcal{O}$ and $X_i,Y\in\operatorname{Ob}\mathsf{C}$, $i\in I$, such that

$$\mathsf{C}((X_i)_{i\in I};Y)=\mathsf{C}((X_{\sigma^{-1}(j)})_{j\in J};Y),$$

for each order-preserving bijection $\sigma:I\to J$. A *morphism* $F:\mathsf{C}\to\mathsf{D}$ *of* V-*multiquivers* consists of a function $\operatorname{Ob}F:\operatorname{Ob}\mathsf{C}\to\operatorname{Ob}\mathsf{D}$, $X\mapsto FX$, and a morphism

$$F=F_{(X_i)_{i\in I};Y}:\mathsf{C}((X_i)_{i\in I};Y)\to\mathsf{D}((FX_i)_{i\in I};FY)$$

in V, for each $I\in\operatorname{Ob}\mathcal{O}$ and $X_i,Y\in\operatorname{Ob}\mathsf{C}$, $i\in I$, such that $F_{(X_i)_{i\in I};Y}=F_{(X_{\sigma^{-1}(j)})_{j\in J};Y}$, for each order-preserving bijection $\sigma:I\to J$.

1.3.5. Definition. A *(symmetric)* V-*multicategory*, consists of the following data.

- A V-multiquiver C.
- For each map $\phi:I\to J$ in $\operatorname{Mor}\mathcal{O}$ (resp. $\operatorname{Mor}\mathcal{S}$) and $X_i,Y_j,Z\in\operatorname{Ob}\mathsf{C}$, $i\in I$, $j\in J$, a morphism

$$\mu^{\mathsf{C}}_\phi:\big(\mathsf{C}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J},\mathsf{C}((Y_j)_{j\in J};Z)\to\mathsf{C}((X_i)_{i\in I};Z)\tag{1.3.1}$$

  in V, called *composition.* It is required to be compatible with the V-multiquiver structure. Namely, suppose we are given a commutative diagram in $\mathcal{O}$ (resp. $\mathcal{S}$)

$$\begin{tikzcd}
I \arrow[r, "\phi"] \arrow[d, "\sigma"'] & J \arrow[d, "\tau"] \\
K \arrow[r, "\psi"] & L
\end{tikzcd}$$

  where the vertical arrows are order-preserving bijections. Suppose further that $X_i,Y_j,U_k,V_l\in\operatorname{Ob}\mathsf{C}$, $i\in I$, $j\in J$, $k\in K$, $l\in L$, are objects of C such that $X_i=U_{\sigma(i)}$ and $Y_j=V_{\tau(j)}$, for each $i\in I$, $j\in J$. Then the morphism

$$\mu^{\mathsf{C}}_\psi:(\mathsf{C}((U_k)_{k\in\psi^{-1}l};V_l))_{l\in L},\mathsf{C}((V_l)_{l\in L};Z)\to\mathsf{C}((U_k)_{k\in K};Z)$$

  must coincide with (1.3.1).
- For each $X\in\operatorname{Ob}\mathsf{C}$, a morphism $1^{\mathsf{C}}_X:()\to\mathsf{C}(X;X)$ in V, called the *identity* of X.

These data are subject to the following axioms.

- Associativity: the diagram

$$\begin{tikzcd}
\big(\mathsf{C}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J},\big(\mathsf{C}((Y_j)_{j\in\psi^{-1}k};Z_k)\big)_{k\in K},\mathsf{C}((Z_k)_{k\in K};W) \arrow[dr, "(1)_{j\in J}{,}\mu^{\mathsf{C}}_\psi"] \arrow[ddd, "(\mu^{\mathsf{C}}_{\phi_k})_{k\in K}{,}1"'] & \\
& \big(\mathsf{C}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J},\mathsf{C}((Y_j)_{j\in J};W) \arrow[d, "\mu^{\mathsf{C}}_\phi"] \\
& \mathsf{C}((X_i)_{i\in I};W) \\
\big(\mathsf{C}((X_i)_{i\in(\phi\psi)^{-1}k};Z_k)\big)_{k\in K},\mathsf{C}((Z_k)_{k\in K};W) \arrow[ur, "\mu^{\mathsf{C}}_{\phi\psi}"'] &
\end{tikzcd}$$

commutes; more precisely,

$$\big((1^{\mathsf{V}}_{\mathsf{C}((X_i)_{i\in\phi^{-1}j};Y_j)})_{j\in J},\mu^{\mathsf{C}}_{\psi}\big)\cdot_{\mathrm{id}_J\sqcup\rhd}\mu^{\mathsf{C}}_{\phi}=\big((\mu^{\mathsf{C}}_{\phi_k})_{k\in K},1^{\mathsf{V}}_{\mathsf{C}((Z_k)_{k\in K};W)}\big)\cdot_{\overline{\psi}}\mu^{\mathsf{C}}_{\phi\psi}, \tag{1.3.2}$$

for each pair of composable maps $I\xrightarrow{\phi}J\xrightarrow{\psi}K$ in $\operatorname{Mor}\mathcal{O}$ (resp. $\operatorname{Mor}\mathcal{S}$) and objects $X_i,Y_j,Z_k,W\in\operatorname{Ob}\mathsf{C}$, $i\in I$, $j\in J$, $k\in K$. The map ϕ_k is the restriction $\phi|_{\phi^{-1}\psi^{-1}k}:\phi^{-1}\psi^{-1}k\to\psi^{-1}k$, $k\in K$. The map $\mathrm{id}_J\sqcup\rhd:J\sqcup K\sqcup\mathbf{1}\to J\sqcup\mathbf{1}$ preserves the order, while the map $\overline{\psi}:J\sqcup K\sqcup\mathbf{1}\to K\sqcup\mathbf{1}$ is not necessarily order-preserving; it is given by (1.2.4).

- Identity: the diagram in V

$$\begin{tikzcd}
\mathsf{C}((X_i)_{i\in I};Y) \arrow[r, "1^{\mathsf{V}}_{\mathsf{C}((X_i)_{i\in I};Y)},1^{\mathsf{C}}_{Y}"] \arrow[d, "(1^{\mathsf{C}}_{X_i})_{i\in I},1^{\mathsf{V}}_{\mathsf{C}((X_i)_{i\in I};Y)}"'] \arrow[dr, "1^{\mathsf{V}}_{\mathsf{C}((X_i)_{i\in I};Y)}"] & \mathsf{C}((X_i)_{i\in I};Y),\mathsf{C}(Y;Y) \arrow[d, "\mu^{\mathsf{C}}_{I\to\mathbf{1}}"] \\
\big(\mathsf{C}(X_i;X_i)\big)_{i\in I},\mathsf{C}((X_i)_{i\in I};Y) \arrow[r, "\mu^{\mathsf{C}}_{\mathrm{id}_I}"] & \mathsf{C}((X_i)_{i\in I};Y)
\end{tikzcd}$$

commutes, for each $I\in\operatorname{Ob}\mathcal{O}$ and $X_i,Y\in\operatorname{Ob}\mathsf{C}$, $i\in I$.

An arbitrary V-multicategory has an underlying V-category. We will not distinguish the two notationally; this should cause only minimal confusion.

1.3.6. Definition. Let C, D be (symmetric) V-multicategories. A (*symmetric*) V-*multifunctor* $F:\mathsf{C}\to\mathsf{D}$ is a morphism of V-multiquivers such that

- F preserves identities:

$$\big[()\xrightarrow{1^{\mathsf{C}}_X}\mathsf{C}(X;X)\xrightarrow{F_{X;X}}\mathsf{D}(FX;FX)\big]=1^{\mathsf{D}}_{FX},$$

for each $X\in\operatorname{Ob}\mathsf{C}$, and

- F preserves composition: the diagram in V

$$\begin{tikzcd}
\big(\mathsf{C}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J},\mathsf{C}((Y_j)_{j\in J};Z) \arrow[r, "\mu^{\mathsf{C}}_{\phi}"] \arrow[d, "(F_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J},F_{(Y_j)_{j\in J};Z}"'] & \mathsf{C}((X_i)_{i\in I};Z) \arrow[d, "F_{(X_i)_{i\in I};Z}"] \\
\big(\mathsf{D}((FX_i)_{i\in\phi^{-1}j};FY_j)\big)_{j\in J},\mathsf{D}((FY_j)_{j\in J};FZ) \arrow[r, "\mu^{\mathsf{D}}_{\phi}"] & \mathsf{D}((FX_i)_{i\in I};FZ)
\end{tikzcd} \tag{1.3.3}$$

commutes, for each map $\phi:I\to J$ in $\operatorname{Mor}\mathcal{O}$ (resp. $\operatorname{Mor}\mathcal{S}$) and $X_i,Y_j,Z\in\operatorname{Ob}\mathsf{C}$, $i\in I$, $j\in J$.

Given V-multifunctors $F:\mathsf{C}\to\mathsf{D}$, $G:\mathsf{D}\to\mathsf{E}$, define the composite $FG:\mathsf{C}\to\mathsf{E}$ by $\operatorname{Ob}(FG)=\operatorname{Ob}F\cdot\operatorname{Ob}G$ and $(FG)_{(X_i)_{i\in I};Y}=F_{(X_i)_{i\in I};Y}\cdot G_{(FX_i)_{i\in I};FY}$, for each $I\in\operatorname{Ob}\mathcal{O}$ and $X_i,Y\in\operatorname{Ob}\mathsf{C}$, $i\in I$. Clearly, FG is a V-multifunctor.

1.3.7. Definition. For V-multifunctors $F,G:\mathsf{C}\to\mathsf{D}$, a V-*multinatural transformation* $r:F\to G:\mathsf{C}\to\mathsf{D}$ is a family of morphisms in V

$$r_X:()\to\mathsf{D}(FX;GX),\qquad X\in\operatorname{Ob}\mathsf{C},$$

such that the diagram in V

$$\begin{tikzcd}
\mathsf{C}((X_i)_{i\in I};Y) \arrow[r, "F_{(X_i)_{i\in I};Y},r_Y"] \arrow[d, "(r_{X_i})_{i\in I},G_{(X_i)_{i\in I};Y}"'] \arrow[dr, phantom, "="] & \mathsf{D}((FX_i)_{i\in I};FY),\mathsf{D}(FY;GY) \arrow[d, "\mu^{\mathsf{D}}_{I\to\mathbf{1}}"] \\
\big(\mathsf{D}(FX_i;GX_i)\big)_{i\in I},\mathsf{D}((GX_i)_{i\in I};GY) \arrow[r, "\mu^{\mathsf{D}}_{\mathrm{id}_I}"] & \mathsf{D}((FX_i)_{i\in I};GY)
\end{tikzcd} \tag{1.3.4}$$

commutes, for each $I\in\operatorname{Ob}\mathcal{O}$ and $X_i,Y\in\operatorname{Ob}\mathsf{C}$, $i\in I$.

The 'vertical' composite $t\cdot u$ of V-multinatural transformations $t:F\to G:\mathsf{C}\to\mathsf{D}$ and $u:G\to H:\mathsf{C}\to\mathsf{D}$ has the component $(t\cdot u)_X$ given by

$$()\xrightarrow{t_X,u_X}\mathsf{D}(FX;GX),\mathsf{D}(GX;HX)\xrightarrow{\mu^{\mathsf{D}}_{\mathbf{1}\to\mathbf{1}}}\mathsf{D}(FX;HX).$$

The composite of $t:F\to G:\mathsf{C}\to\mathsf{D}$ as above with a V-multifunctor $H:\mathsf{D}\to\mathsf{E}$ has for its component $(t\cdot H)_X$ the composite

$$()\xrightarrow{t_X}\mathsf{D}(FX;GX)\xrightarrow{H_{FX;GX}}\mathsf{E}(HFX;HGX);$$

while the composite of t with $E:\mathsf{B}\to\mathsf{C}$ has for its component $(E\cdot t)_X$ simply t_{EX}.

It is a routine computation to check that (symmetric) V-multicategories, (symmetric) V-multifunctors, and V-multinatural transformations constitute a 2-category.

1.3.8. Example. In the case of $\mathsf{V}=\widehat{\mathcal{V}}$, where $\mathcal{V}=(\mathcal{V},\otimes^I,\lambda^f)$ is a symmetric Monoidal category, we recover the definitions of $\mathcal{V}$-multicategories, $\mathcal{V}$-multifunctors, and $\mathcal{V}$-multinatural transformations from the preceding section.

1.3.9. Base change. A symmetric multifunctor $B:\mathsf{V}\to\mathsf{W}$ gives rise to a **Cat**-functor B_* from the 2-category of (symmetric) V-multicategories to the 2-category of (symmetric) W-multicategories. A V-multicategory C gives rise to a W-multicategory $B_*\mathsf{C}$ with the same set of objects and with objects of morphisms $(B_*\mathsf{C})((X_i)_{i\in I};Y)=B\mathsf{C}((X_i)_{i\in I};Y)$. Composition in $B_*\mathsf{C}$ is given by

$$\mu_\phi^{B_*\mathsf{C}}=B(\mu_\phi^{\mathsf{C}}):\big(B\mathsf{C}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J},B\mathsf{C}((Y_j)_{j\in J};Z)\to B\mathsf{C}((X_i)_{i\in I};Z).$$

The identity of an object $X\in\operatorname{Ob}\mathsf{C}$ is $1_X^{B_*\mathsf{C}}=B(1_X^{\mathsf{C}}):()\to B\mathsf{C}(X;X)$. A V-multifunctor $F:\mathsf{C}\to\mathsf{D}$ induces a W-multifunctor $B_*F:B_*\mathsf{C}\to B_*\mathsf{D}$ given by $\operatorname{Ob}B_*F=\operatorname{Ob}F$ and

$$(B_*F)_{(X_i)_{i\in I};Y}=B(F_{(X_i)_{i\in I};Y}):B\mathsf{C}((X_i)_{i\in I};Y)\to B\mathsf{D}((FX_i)_{i\in I};FY).$$

A V-multinatural transformation $r:F\to G:\mathsf{C}\to\mathsf{D}$ gives rise to a W-natural transformation $B_*r:B_*F\to B_*G:B_*\mathsf{C}\to B_*\mathsf{D}$ whose components are given by

$$(B_*r)_X=B(r_X):()\to B\mathsf{D}(FX;GX),\quad X\in\operatorname{Ob}\mathsf{C}.$$

The compatibility conditions (associativity axiom (1.3.2), the identity axiom, preservation of composition and identities, and multinaturality condition (1.3.4)) hold true since B preserves composition and identities. By the same reason, B_* is a **Cat**-functor. Note that the assumption that B is a *symmetric* multifunctor is essential. Indeed, the associativity axiom for a V-multicategory (resp. W-multicategory) involves composition in V (resp. W) with respect to not necessarily order-preserving maps. Quite similarly, the multifunctor B induces a **Cat**-functor $B_*:\mathsf{V}$-**Cat** $\to\mathsf{W}$-**Cat**. If $\mathsf{V}=\widehat{\mathcal{V}}$ and $\mathsf{W}=\widehat{\mathcal{W}}$, where $\mathcal{V}$ and $\mathcal{W}$ are lax symmetric Monoidal categories, then by Proposition 1.2.25 the multifunctor B comes from a certain lax symmetric Monoidal functor, and we recover the base change **Cat**-functor from Section 1.1.14 and Remark 1.2.28.

1.3.10. Closed multicategories. First, we recall that a symmetric Monoidal category $\mathcal{V} = (\mathcal{V}, \otimes^I, \lambda)$ is *closed* if the functor $X \otimes -$ admits a right adjoint $\underline{\mathcal{V}}(X, -)$, for each $X \in \operatorname{Ob}\mathcal{V}$, cf. [**44**, Section VII.7]. In particular, we have an adjunction

$$\mathcal{V}(X \otimes Y, Z) \cong \mathcal{V}(Y, \underline{\mathcal{V}}(X, Z)).$$

The unit $\operatorname{coev}^{\mathcal{V}} : Y \to \underline{\mathcal{V}}(X, X \otimes Y)$ and the counit $\operatorname{ev}^{\mathcal{V}} : X \otimes \underline{\mathcal{V}}(X, Y) \to Y$ of this adjunction are called *coevaluation* and *evaluation* respectively. These satisfy so called triangular identities, see e.g. [**44**, Theorem IV.1.1]. The mutually inverse adjunction isomorphisms are given explicitly as follows:

$$\begin{aligned} \mathcal{V}(Y, \underline{\mathcal{V}}(X, Z)) &\to \mathcal{V}(X \otimes Y, Z), & \mathcal{V}(X \otimes Y, Z) &\to \mathcal{V}(Y, \underline{\mathcal{V}}(X, Z)), \\ f &\mapsto (1_X \otimes f) \operatorname{ev}_{X,Z}, & g &\mapsto \operatorname{coev}_{X,Y} \underline{\mathcal{V}}(X, g). \end{aligned} \tag{1.3.5}$$

For each pair of objects $X, Y \in \operatorname{Ob}\mathcal{V}$, the object $\underline{\mathcal{V}}(X, Y) \in \operatorname{Ob}\mathcal{V}$ is called an *internal* Hom-*object*. There is a $\mathcal{V}$-category $\underline{\mathcal{V}}$ whose objects are those of $\mathcal{V}$, and for each pair of objects X and Y, the object $\underline{\mathcal{V}}(X, Y) \in \operatorname{Ob}\mathcal{V}$ is the internal Hom-object of $\mathcal{V}$. The composition is found from the following equation:

$$\begin{tikzcd}
X \otimes \underline{\mathcal{V}}(X,Y) \otimes \underline{\mathcal{V}}(Y,Z) \arrow[r, "\lambda^{\mathsf{IV}}"] \arrow[d, "\lambda^{\mathsf{VI}}"'] & X \otimes \bigl(\mathcal{V}(X,Y) \otimes \mathcal{V}(Y,Z)\bigr) \arrow[r, "1\otimes\mu_{\underline{\mathcal{V}}}"] \arrow[d, phantom, "="] & X \otimes \underline{\mathcal{V}}(X,Z) \arrow[d, "\operatorname{ev}^{\mathcal{V}}"] \\
X \otimes \underline{\mathcal{V}}(X,Y) \otimes \underline{\mathcal{V}}(Y,Z) \arrow[r, "\operatorname{ev}^{\mathcal{V}} \otimes 1"] & \bigl(X \otimes \underline{\mathcal{V}}(X,Y)\bigr) \otimes \underline{\mathcal{V}}(Y,Z) \arrow[r, "\operatorname{ev}^{\mathcal{V}}"] & Z
\end{tikzcd} \tag{1.3.6}$$

The identity morphism $1^{\mathcal{V}}_X : \mathbb{1}_{\mathcal{V}} \to \underline{\mathcal{V}}(X, X)$ is found from the following equation:

$$\bigl[X \xrightarrow[\sim]{\lambda^{\mathsf{I}\cdot}} X \otimes \mathbb{1}_{\mathcal{V}} \xrightarrow{1\otimes 1^{\mathcal{V}}_X} X \otimes \underline{\mathcal{V}}(X, X) \xrightarrow{\operatorname{ev}^{\mathcal{V}}} X\bigr] = \mathrm{id}_X .$$

The definition of closedness transfers without significant changes from Monoidal categories to multicategories.

1.3.11. Definition. A (symmetric) multicategory C is *closed* if for each $I \in \operatorname{Ob}\mathcal{O}$ and $X_i, Z \in \operatorname{Ob}\mathsf{C}$, $i \in I$, there exist an object $\underline{\mathsf{C}}((X_i)_{i\in I}; Z)$ of C, called *internal* Hom-*object*, and an *evaluation* morphism

$$\operatorname{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z} : (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \to Z$$

such that the function

$$\varphi_{(Y_j)_{j\in J};(X_i)_{i\in I};Z} : \mathsf{C}\bigl((Y_j)_{j\in J}; \underline{\mathsf{C}}((X_i)_{i\in I}; Z)\bigr) \to \mathsf{C}\bigl((X_i)_{i\in I}, (Y_j)_{j\in J}; Z\bigr)$$

that takes a morphism $f : (Y_j)_{j\in J} \to \underline{\mathsf{C}}((X_i)_{i\in I}; Z)$ to the composite

$$\bigl[(X_i)_{i\in I}, (Y_j)_{j\in J} \xrightarrow{(1^{\mathsf{C}}_{X_i})_{i\in I}, f} (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \xrightarrow{\operatorname{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}} Z\bigr] \tag{1.3.7}$$

is a bijection, for an arbitrary sequence $(Y_j)_{j\in J}$, $J \in \operatorname{Ob}\mathcal{O}$, of objects of C. Here concatenation of sequences indexed by I and J is indexed by the disjoint union $I \sqcup J$, where $i < j$ for all $i \in I$, $j \in J$.

Notice that for $I = \varnothing$ an object $\underline{\mathsf{C}}(;Z)$ and an element $\operatorname{ev}^{\mathsf{C}}_{;Z}$ with the required property always exist. Namely, we may always take $\underline{\mathsf{C}}(;Z) = Z$ and $\operatorname{ev}^{\mathsf{C}}_{;Z} = 1^{\mathsf{C}}_Z : Z \to Z$. With this choice $\varphi_{(Y_j)_{j\in J};;Z} : \mathsf{C}\bigl((Y_j)_{j\in J}; Z\bigr) \to \mathsf{C}\bigl((Y_j)_{j\in J}; Z\bigr)$ is the identity map. Furthermore, we may assume that $\underline{\mathsf{C}}((X_i)_{i\in I}; Z) = \underline{\mathsf{C}}((X_{\sigma^{-1}(j)})_{j\in J}; Z)$ and $\operatorname{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z} = \operatorname{ev}^{\mathsf{C}}_{(X_{\sigma^{-1}(j)})_{j\in J};Z}$, for each order-preserving bijection $\sigma : I \to J$.

The condition that $\varphi_{(Y_j)_{j\in J};(X_i)_{i\in I};Z}$ is a bijection translates into the following universal property: for each morphism $g : (X_i)_{i\in I}, (Y_j)_{j\in J} \to Z$, there exists a unique morphism $f : (Y_j)_{j\in J} \to \underline{\mathsf{C}}((X_i)_{i\in I}; Z)$ such that $\bigl((1^{\mathsf{C}}_{X_i})_{i\in I}, f\bigr) \cdot \mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z} = g$:

$$\begin{tikzcd}
& (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[dd, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}"] \\
(X_i)_{i\in I}, (Y_j)_{j\in J} \arrow[ur, "(1^{\mathsf{C}}_{X_i})_{i\in I}{,}f"] \arrow[dr, "g"'] & \\
& Z
\end{tikzcd}$$

An example of closed symmetric multicategory is provided by a closed symmetric Monoidal category.

1.3.12. Proposition. *Let* $(\mathcal{C}, \otimes^I, \lambda^f)$ *be a closed symmetric Monoidal category. Then the multicategory* $\widehat{\mathcal{C}}$ *is closed. For each* $I \in \operatorname{Ob} \mathcal{O}$ *and* $X_i, Y \in \operatorname{Ob} \mathcal{C}$, $i \in I$, *an internal* Hom-*object* $\underline{\widehat{\mathcal{C}}}((X_i)_{i\in I}; Z)$ *is* $\underline{\mathcal{C}}(\otimes^{i\in I} X_i, Z)$, *and evaluation morphisms are represented by the composites in* $\mathcal{C}$:

$$\mathrm{ev}^{\widehat{\mathcal{C}}} = \bigl[\otimes^{I\sqcup\mathbf{1}}((X_i)_{i\in I}, \underline{\mathcal{C}}(\otimes^{i\in I} X_i, Z)) \xrightarrow{\lambda^{\gamma}} (\otimes^{i\in I} X_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in I} X_i, Z) \xrightarrow{\mathrm{ev}^{\mathcal{C}}} Z\bigr],$$

where $\gamma : I \sqcup \mathbf{1} \to \mathbf{2}$ *is given by* $I \ni i \mapsto 1$, $\mathbf{1} \ni 1 \mapsto 2$.

Proof. Since $\mathcal{C}$ is a closed symmetric Monoidal category, the map

$$\varphi^{\mathcal{C}} : \mathcal{C}(Y, \underline{\mathcal{C}}(X, Z)) \to \mathcal{C}(X \otimes Y, Z), \quad g \mapsto (1^{\mathcal{C}}_X \otimes g) \cdot \mathrm{ev}^{\mathcal{C}}_{X,Z},$$

is bijective. To compute the map

$$\varphi^{\widehat{\mathcal{C}}} : \widehat{\mathcal{C}}((Y_j)_{j\in J}; \underline{\widehat{\mathcal{C}}}((X_i)_{i\in I}; Z)) \to \widehat{\mathcal{C}}((X_i)_{i\in I}, (Y_j)_{j\in J}; Z),$$

consider an arbitrary morphism $g : (Y_j)_{j\in J} \to \underline{\widehat{\mathcal{C}}}((X_i)_{i\in I}; Z)$ in $\widehat{\mathcal{C}}$, that is, a morphism $g : \otimes^{j\in J} Y_j \to \underline{\mathcal{C}}(\otimes^{i\in I} X_i, Z)$ in $\mathcal{C}$. Then

$$\varphi^{\widehat{\mathcal{C}}}(g) = \bigl((1^{\widehat{\mathcal{C}}}_{X_i})_{i\in I}, g\bigr) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\widehat{\mathcal{C}}}, \tag{1.3.8}$$

where $\mathrm{id}_I \sqcup \triangleright : I \sqcup J \to I \sqcup \mathbf{1}$. Expanding out the right hand side of (1.3.8) using formula (1.2.8) for composition in $\widehat{\mathcal{C}}$, we obtain that $\varphi^{\widehat{\mathcal{C}}}(g)$ is given by the composite

$$\begin{aligned}
&\bigl[\otimes^{I\sqcup J}\bigl((X_i)_{i\in I}, (Y_j)_{j\in J}\bigr) \xrightarrow{\lambda^{\mathrm{id}_I \sqcup \triangleright : I\sqcup J \to I \sqcup \mathbf{1}}} \otimes^{I\sqcup\mathbf{1}}\bigl((X_i)_{i\in I}, \otimes^{j\in J} Y_j\bigr) \xrightarrow{\otimes^{I\sqcup\mathbf{1}}((1^{\mathcal{C}}_{X_i})_{i\in I}, g)} \\
&\qquad \otimes^{I\sqcup\mathbf{1}} \bigl((X_i)_{i\in I}, \underline{\mathcal{C}}(\otimes^{i\in I}, Z)\bigr) \xrightarrow{\lambda^{\gamma: I\sqcup\mathbf{1}\to\mathbf{2}}} \bigl(\otimes^{i\in I} X_i\bigr) \otimes \underline{\mathcal{C}}(\otimes^{i\in I} X_i, Z) \xrightarrow{\mathrm{ev}^{\mathcal{C}}} Z\bigr].
\end{aligned}$$

It can be written as

$$\begin{aligned}
&\bigl[\otimes^{I\sqcup J}\bigl((X_i)_{i\in I}, (Y_j)_{j\in J}\bigr) \xrightarrow{\lambda^{\mathrm{id}_I \sqcup \triangleright : I\sqcup J \to I \sqcup \mathbf{1}}} \otimes^{I\sqcup\mathbf{1}}\bigl((X_i)_{i\in I}, \otimes^{j\in J} Y_j\bigr) \xrightarrow{\lambda^{\gamma: I\sqcup\mathbf{1}\to\mathbf{2}}} \\
&\qquad \bigl(\otimes^{i\in I} X_i\bigr) \otimes \bigl(\otimes^{j\in J} Y_j\bigr) \xrightarrow{(\otimes^{i\in I} 1^{\mathcal{C}}_{X_i}) \otimes g} \bigl(\otimes^{i\in I} X_i\bigr) \otimes \underline{\mathcal{C}}(\otimes^{i\in I} X_i, Z) \xrightarrow{\mathrm{ev}^{\mathcal{C}}} Z\bigr],
\end{aligned}$$

by the naturality of $\lambda^{\gamma: I\sqcup\mathbf{1}\to\mathbf{2}}$. Since the functor $\otimes^I$ preserves identities, it follows that $\otimes^{i\in I} 1^{\mathcal{C}}_{X_i} = 1^{\mathcal{C}}_{\otimes^{i\in I} X_i}$. Furthermore, by equation (1.1.1),

$$\lambda^{\mathrm{id}_I \sqcup \triangleright : I\sqcup J \to I\sqcup\mathbf{1}} \cdot \lambda^{\gamma: I\sqcup\mathbf{1}\to\mathbf{2}} = \lambda^{\pi: I\sqcup J\to\mathbf{2}} : \otimes^{I\sqcup J}\bigl((X_i)_{i\in I}, (Y_j)_{j\in J}\bigr) \to \bigl(\otimes^{i\in I} X_i\bigr) \otimes \bigl(\otimes^{j\in J} Y_j\bigr),$$

where $\pi : I \sqcup J \to \mathbf{2}$ is given by $I \ni i \mapsto 1$, $J \ni j \mapsto 2$. Therefore

$$\varphi^{\widehat{\mathcal{C}}}(g) = \Big[\otimes^{I \sqcup J}\big((X_i)_{i\in I}, (Y_j)_{j\in J}\big) \xrightarrow{\lambda^{\pi:I\sqcup J\to\mathbf{2}}} \big(\otimes^{i\in I} X_i\big) \otimes \big(\otimes^{j\in J} Y_j\big) \\ \xrightarrow{1^{\mathcal{C}}_{\otimes^{i\in I}X_i}\otimes g} \big(\otimes^{i\in I} X_i\big) \otimes \underline{\mathcal{C}}(\otimes^{i\in I} X_i, Z) \xrightarrow{\mathrm{ev}^{\mathcal{C}}} Z\Big],$$

in other words,

$$\varphi^{\widehat{\mathcal{C}}} = \Big[\mathcal{C}(\otimes^{j\in J} Y_j, \underline{\mathcal{C}}(\otimes^{i\in I} X_i, Z)) \underset{\sim}{\xrightarrow{\varphi^{\mathcal{C}}}} \mathcal{C}\big((\otimes^{i\in I} X_i) \otimes (\otimes^{j\in J} Y_j), Z\big) \\ \underset{\sim}{\xrightarrow{\mathcal{C}(\lambda^{\pi:I\sqcup J\to\mathbf{2}},1)}} \mathcal{C}\big(\otimes^{I\sqcup J}\big((X_i)_{i\in I}, (Y_j)_{j\in J}\big), Z\big)\Big].$$

It follows that $\varphi^{\widehat{\mathcal{C}}}$ is bijective, hence $\widehat{\mathcal{C}}$ is closed. □

The following observation simplifies checking that a particular multicategory is closed. However, the choice of internal Hom-objects and evaluations that it suggests is not always convenient.

1.3.13. Proposition. *Suppose that, for each pair of objects $X, Z \in \operatorname{Ob} \mathsf{C}$, there exist an object $\underline{\mathsf{C}}(X; Z)$ and a morphism $\mathrm{ev}^{\mathsf{C}}_{X;Z} : X, \underline{\mathsf{C}}(X; Z) \to Z$ of C such that the function $\varphi_{(Y_j)_{j\in J};X;Z} : \mathsf{C}((Y_j)_{j\in J}; \underline{\mathsf{C}}(X; Z)) \to \mathsf{C}(X, (Y_j)_{j\in J}; Z)$ given by* (1.3.7) *is a bijection, for each finite sequence $(Y_j)_{j\in J}$ of objects of C. Then C is a closed multicategory.*

Proof. Define internal Hom-objects $\underline{\mathsf{C}}(X_1, \dots, X_n; Z)$ and evaluations

$$\mathrm{ev}^{\mathsf{C}}_{X_1,\dots,X_n;Z} : X_1, \dots, X_n, \underline{\mathsf{C}}(X_1, \dots, X_n; Z) \to Z$$

by induction on n. For $n = 0$, choose $\underline{\mathsf{C}}(;Z) = Z$ and $\mathrm{ev}^{\mathsf{C}}_{;Z} = 1_Z : Z \to Z$, as explained above. For $n = 1$, we are already given $\underline{\mathsf{C}}(X; Z)$ and $\mathrm{ev}^{\mathsf{C}}_{X;Z}$. Assume that we have defined $\underline{\mathsf{C}}(X_1, \dots, X_k; Z)$ and $\mathrm{ev}^{\mathsf{C}}_{X_1,\dots,X_k;Z}$ for each $k < n$, and that the function

$$\varphi_{(Y_j)_{j\in J},X_1,\dots,X_k;Z} : \mathsf{C}((Y_j)_{j\in J}; \underline{\mathsf{C}}(X_1, \dots, X_k; Z)) \to \mathsf{C}(X_1, \dots, X_k, (Y_j)_{j\in J}; Z)$$

is a bijection, for each $k < n$ and for each finite sequence $(Y_j)_{j\in J}$ of objects of C. For $X_1, \dots, X_n, Z \in \operatorname{Ob} \mathsf{C}$, define

$$\underline{\mathsf{C}}(X_1, \dots, X_n; Z) \overset{\text{def}}{=} \underline{\mathsf{C}}(X_n; \underline{\mathsf{C}}(X_1, \dots, X_{n-1}; Z)).$$

An evaluation morphism $\mathrm{ev}^{\mathsf{C}}_{X_1,\dots,X_n;Z}$ is given by the composite

$$\begin{tikzcd}
X_1, \dots, X_n, \underline{\mathsf{C}}(X_n; \underline{\mathsf{C}}(X_1, \dots, X_{n-1}; Z)) \arrow[d, "(1)_{\mathbf{n-1}}{,}\mathrm{ev}^{\mathsf{C}}_{X_n;\underline{\mathsf{C}}(X_1,\dots,X_{n-1};Z)}"] \\
X_1, \dots, X_{n-1}, \underline{\mathsf{C}}(X_1, \dots, X_{n-1}; Z) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{X_1,\dots,X_{n-1};Z}"] \\
Z.
\end{tikzcd}$$

It is easy to see that with these choices the function $\varphi_{(Y_j)_{j\in J};X_1,\dots,X_n;Z}$ decomposes as

$$\begin{tikzcd}
\mathsf{C}((Y_j)_{j\in J};\underline{\mathsf{C}}(X_1,\dots,X_n;Z)) \arrow[d, "\varphi_{(Y_j)_{j\in J};X_n;\underline{\mathsf{C}}(X_1,\dots,X_{n-1};Z)}", "\wr"'] \\
\mathsf{C}(X_n,(Y_j)_{j\in J};\underline{\mathsf{C}}(X_1,\dots,X_{n-1};Z)) \arrow[d, "\varphi_{X_n,(Y_j)_{j\in J};X_1,\dots,X_{n-1};Z}", "\wr"'] \\
\underline{\mathsf{C}}(X_1,\dots,X_n,(Y_j)_{j\in J};Z),
\end{tikzcd}$$

hence it is a bijection, and the induction goes through. □

1.3.14. Proposition. *A closed symmetric multicategory* C *gives rise to a symmetric multicategory* $\underline{\mathsf{C}}$ *enriched in* C.

Proof. For each map $\phi : I \to J$ in $\operatorname{Mor}\mathcal{S}$ and $X_i, Y_j, Z \in \operatorname{Ob}\mathsf{C}$, $i \in I$, $j \in J$, there exists a unique morphism

$$\mu_\phi^{\underline{\mathsf{C}}} : \bigl(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\bigr)_{j\in J}, \underline{\mathsf{C}}((Y_j)_{j\in J};Z) \to \underline{\mathsf{C}}((X_i)_{i\in I};Z)$$

that makes the diagram

$$\begin{tikzcd}
(X_i)_{i\in I}, \bigl(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\bigr)_{j\in J}, \underline{\mathsf{C}}((Y_j)_{j\in J};Z) \arrow[r, "(1^{\mathsf{C}}_{X_i})_{i\in I}{,}\mu^{\underline{\mathsf{C}}}_\phi"] \arrow[d, "(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J}{,}1^{\mathsf{C}}_{\underline{\mathsf{C}}((Y_j)_{j\in J};Z)}"'] & (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I};Z) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}"] \\
(Y_j)_{j\in J}, \underline{\mathsf{C}}((Y_j)_{j\in J};Z) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(Y_j)_{j\in J};Z}"] & Z
\end{tikzcd}$$

commute. More precisely, the commutativity in the above diagram means that the equation

$$\bigl((1^{\mathsf{C}}_{X_i})_{i\in I}, \mu^{\underline{\mathsf{C}}}_\phi\bigr) \cdot_{\mathrm{id}_I\sqcup\triangleright} \mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z} = \bigl((\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J}, 1^{\mathsf{C}}_{\underline{\mathsf{C}}((Y_j)_{j\in J};Z)}\bigr) \cdot_{\overline{\phi}} \mathrm{ev}^{\mathsf{C}}_{(Y_j)_{j\in J};Z}$$

holds true, where $\mathrm{id}_I\sqcup\triangleright : I \sqcup J \sqcup \mathbf{1} \to I \sqcup \mathbf{1}$, and $\overline{\phi} : I \sqcup J \sqcup \mathbf{1} \to J \sqcup \mathbf{1}$ is given by (1.2.4). Furthermore, for each $X \in \operatorname{Ob}\mathsf{C}$, there is a morphism

$$1^{\underline{\mathsf{C}}}_X \overset{\text{def}}{=} \varphi^{-1}_{;X;X}(1^{\mathsf{C}}_X) : () \to \underline{\mathsf{C}}(X;X).$$

It is a unique solution to the equation

$$\bigl[X \xrightarrow{1^{\mathsf{C}}_X, 1^{\underline{\mathsf{C}}}_X} X, \underline{\mathsf{C}}(X;X) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{X;X}} X\bigr] = 1^{\mathsf{C}}_X.$$

Let us check the conditions of Definition 1.3.5. Equation (1.3.2) reads

$$\bigl((1^{\mathsf{C}}_{\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)})_{j\in J}, \mu^{\underline{\mathsf{C}}}_\psi\bigr) \cdot_{\mathrm{id}_J\sqcup\triangleright} \mu^{\underline{\mathsf{C}}}_\phi = \bigl((\mu^{\underline{\mathsf{C}}}_{\phi_k})_{k\in K}, 1^{\mathsf{C}}_{\underline{\mathsf{C}}((Z_k)_{k\in K};W)}\bigr) \cdot_{\overline{\psi}} \mu^{\underline{\mathsf{C}}}_{\phi\psi},$$

where $\mathrm{id}_J\sqcup\triangleright : J \sqcup K \sqcup \mathbf{1} \to J \sqcup \mathbf{1}$, and $\overline{\psi} : J \sqcup K \sqcup \mathbf{1} \to K \sqcup \mathbf{1}$ is given by (1.2.4). Applying the transformation $\varphi_{(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J},\underline{\mathsf{C}}((Y_j)_{j\in J};Z);(X_i)_{i\in I};Z}$ yields an equivalent equation

$$\Bigl((1^{\mathsf{C}}_{X_i})_{i\in I}, \bigl[\bigl((1^{\mathsf{C}}_{\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)})_{j\in J}, \mu^{\underline{\mathsf{C}}}_\psi\bigr) \cdot_{\mathrm{id}_J\sqcup\triangleright} \mu^{\underline{\mathsf{C}}}_\phi\bigr]\Bigr) \cdot_{\mathrm{id}_I\sqcup\triangleright} \mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}$$
$$= \Bigl((1^{\mathsf{C}}_{X_i})_{i\in I}, \bigl[\bigl((\mu^{\underline{\mathsf{C}}}_{\phi_k})_{k\in K}, 1^{\mathsf{C}}_{\underline{\mathsf{C}}((Z_k)_{k\in K};W)}\bigr) \cdot_{\overline{\psi}} \mu^{\underline{\mathsf{C}}}_{\phi\psi}\bigr]\Bigr) \cdot_{\mathrm{id}_I\sqcup\triangleright} \mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z},$$

where $\mathrm{id}_I\sqcup\triangleright : I \sqcup J \sqcup \mathbf{1} \to I \sqcup \mathbf{1}$. It is proven as follows. To shorten the notation, we drop the subscripts of identities and of evaluation morphisms. The detailed form can be

$$\begin{tikzcd}
\begin{array}{l} (X_i)_{i\in I}, \\ \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J} \\ \big(\underline{\mathsf{C}}((Y_j)_{j\in\psi^{-1}k};Z_k)\big)_{k\in K}, \\ \underline{\mathsf{C}}((Z_k)_{k\in K};W) \end{array} \arrow[r, "(1^{\mathsf{C}}_{X_i})_{i\in I},(1^{\mathsf{C}}_{\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)})_{j\in J},\mu^{\mathsf{C}}_{\psi}"] \arrow[dd, "\begin{array}{l}(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J}, \\ (1^{\mathsf{C}}_{\underline{\mathsf{C}}((Y_j)_{j\in\psi^{-1}k};Z_k)})_{k\in K}, \\ 1^{\mathsf{C}}_{\underline{\mathsf{C}}((Z_k)_{k\in K};W)}\end{array}"'] & \begin{array}{l} (X_i)_{i\in I}, \\ \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J} \\ \underline{\mathsf{C}}((Y_j)_{j\in J};W) \end{array} \arrow[r, "(1^{\mathsf{C}}_{X_i})_{i\in I},\mu^{\mathsf{C}}_{\phi}"] \arrow[d, "(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J},1^{\mathsf{C}}_{\underline{\mathsf{C}}((Y_j)_{j\in J};W)}"] & (X_i)_{i\in I},\underline{\mathsf{C}}((X_i)_{i\in I};W) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};W}"] \\
 & (Y_j)_{j\in J},\underline{\mathsf{C}}((Y_j)_{j\in J};W) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(Y_j)_{j\in J};W}"] & W \\
\begin{array}{c} (Y_j)_{j\in J},\big(\underline{\mathsf{C}}((Y_j)_{j\in\psi^{-1}k};Z_k)\big)_{k\in K} \\ \underline{\mathsf{C}}((Z_k)_{k\in K};W) \end{array} \arrow[ur, "(1^{\mathsf{C}}_{Y_j})_{j\in J},\mu^{\mathsf{C}}_{\psi}"] \arrow[rr, "(\mathrm{ev}^{\mathsf{C}}_{(Y_j)_{j\in\psi^{-1}k};Z_k})_{k\in K},1^{\mathsf{C}}_{\underline{\mathsf{C}}((Z_k)_{k\in K};W)}"] & & (Z_k)_{k\in K},\underline{\mathsf{C}}((Z_k)_{k\in K};W) \arrow[u, "\mathrm{ev}^{\mathsf{C}}_{(Z_k)_{k\in K};W}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{array}{l} (X_i)_{i\in I}, \\ \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J} \\ \big(\underline{\mathsf{C}}((Y_j)_{j\in\psi^{-1}k};Z_k)\big)_{k\in K}, \\ \underline{\mathsf{C}}((Z_k)_{k\in K};W) \end{array} \arrow[r, "(1^{\mathsf{C}}_{X_i})_{i\in I},(\mu^{\mathsf{C}}_{\phi_k})_{k\in K},1^{\mathsf{C}}_{\underline{\mathsf{C}}((Z_k)_{k\in K};W)}"] \arrow[d, "\begin{array}{l}(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J}, \\ (1^{\mathsf{C}}_{\underline{\mathsf{C}}((Y_j)_{j\in\psi^{-1}k};Z_k)})_{k\in K}, \\ 1^{\mathsf{C}}_{\underline{\mathsf{C}}((Z_k)_{k\in K};W)}\end{array}"'] & \begin{array}{l} (X_i)_{i\in I}, \\ \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}\psi^{-1}k};Z_k)\big)_{k\in K}, \\ \underline{\mathsf{C}}((Z_k)_{k\in K};W) \end{array} \arrow[r, "(1^{\mathsf{C}}_{X_i})_{i\in I},\mu^{\mathsf{C}}_{\phi\psi}"] \arrow[d, "(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}\psi^{-1}k};Z_k})_{k\in K},1^{\mathsf{C}}_{\underline{\mathsf{C}}((Z_k)_{k\in K};W)}"] & (X_i)_{i\in I},\underline{\mathsf{C}}((X_i)_{i\in I};W) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};W}"] \\
\begin{array}{l} (Y_j)_{j\in J}, \\ \big(\underline{\mathsf{C}}((Y_j)_{j\in\psi^{-1}k};Z_k)\big)_{k\in K}, \\ \underline{\mathsf{C}}((Z_k)_{k\in K};W) \end{array} \arrow[r, "(\mathrm{ev}^{\mathsf{C}}_{(Y_j)_{j\in\psi^{-1}k};Z_k})_{k\in K},1^{\mathsf{C}}_{\underline{\mathsf{C}}((Z_k)_{k\in K};W)}"] & (Z_k)_{k\in K},\underline{\mathsf{C}}((Z_k)_{k\in K};W) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(Z_k)_{k\in K};W}"] & W
\end{tikzcd}$$

DIAGRAM 1.8.

read from Diagram 1.8. Since identities in C are idempotent, the associativity axiom for the pair of maps

$$I \sqcup J \sqcup K \sqcup \mathbf{1} \xrightarrow{\mathrm{id}_I \sqcup \mathrm{id}_J \sqcup \triangleright} I \sqcup J \sqcup \mathbf{1} \xrightarrow{\mathrm{id}_I \sqcup \triangleright} I \sqcup \mathbf{1}$$

yields

$$\Bigl((1)_{i\in I}, \bigl[\bigl((1)_{j\in J}, \mu^{\mathsf{C}}_{\psi}\bigr) \cdot_{\mathrm{id}_J \sqcup \triangleright} \mu^{\mathsf{C}}_{\phi}\bigr]\Bigr) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}} \\ = \Bigl((1)_{i\in I}, (1)_{j\in J}, \mu^{\mathsf{C}}_{\psi}\Bigr) \cdot_{\mathrm{id}_I \sqcup \mathrm{id}_J \sqcup \triangleright} \Bigl(\bigl((1)_{i\in I}, \mu^{\mathsf{C}}_{\phi}\bigr) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}}\Bigr).$$

By the definition of μ^{C}_{ϕ},

$$\bigl((1)_{i\in I}, \mu^{\mathsf{C}}_{\phi}\bigr) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}} = \bigl((\mathrm{ev}^{\mathsf{C}})_{j\in J}, 1\bigr) \cdot_{\overline{\phi}} \mathrm{ev}^{\mathsf{C}}.$$

The associativity axiom for the pair of maps

$$I \sqcup J \sqcup K \sqcup \mathbf{1} \xrightarrow{\mathrm{id}_I \sqcup \mathrm{id}_J \sqcup \triangleright} I \sqcup J \sqcup \mathbf{1} \xrightarrow{\overline{\phi}} J \sqcup \mathbf{1}$$

implies

$$\Bigl((1)_{i\in I}, \bigl[\bigl((1)_{j\in J}, \mu^{\mathsf{C}}_{\psi}\bigr) \cdot_{\mathrm{id}_J \sqcup \triangleright} \mu^{\mathsf{C}}_{\phi}\bigr]\Bigr) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}} \\ = \Bigl(\bigl(((1)_{i\in\phi^{-1}j}, 1) \cdot_{\mathrm{id}_{\phi^{-1}j\sqcup\mathbf{1}}} \mathrm{ev}^{\mathsf{C}}\bigr)_{j\in J}, \mu^{\mathsf{C}}_{\psi} \cdot_{K\sqcup\mathbf{1}\to\mathbf{1}} 1\Bigr) \cdot_{(\mathrm{id}_I \sqcup \mathrm{id}_J \sqcup \triangleright)\cdot\overline{\psi}} \mathrm{ev}^{\mathsf{C}}.$$

By the identity axiom,

$$((1)_{i\in\phi^{-1}j}, 1) \cdot_{\mathrm{id}_{\phi^{-1}j\sqcup\mathbf{1}}} \mathrm{ev}^{\mathsf{C}} = \mathrm{ev}^{\mathsf{C}} \cdot_{\phi^{-1}j\sqcup\mathbf{1}\to\mathbf{1}} 1, \quad \mu^{\mathsf{C}}_{\psi} \cdot_{K\sqcup\mathbf{1}\to\mathbf{1}} 1 = \bigl((1)_{k\in K}, 1\bigr) \cdot_{\mathrm{id}_{K\sqcup\mathbf{1}}} \mu^{\mathsf{C}}_{\psi}.$$

Therefore, by the associativity axiom for the pair of maps

$$I \sqcup J \sqcup K \sqcup \mathbf{1} \xrightarrow{\pi} J \sqcup K \sqcup \mathbf{1} \xrightarrow{\mathrm{id}_J \sqcup \triangleright} J \sqcup \mathbf{1},$$

where π is given by

$$\pi|_{J\sqcup K\sqcup\mathbf{1}} = \mathrm{id} : J \sqcup K \sqcup \mathbf{1} \to J \sqcup K \sqcup \mathbf{1}, \quad \pi|_I = \bigl(I \xrightarrow{\phi} J \hookrightarrow J \sqcup K \sqcup \mathbf{1}\bigr),$$

it follows that

$$\Bigl((1)_{i\in I}, \bigl[\bigl((1)_{j\in J}, \mu^{\mathsf{C}}_{\psi}\bigr) \cdot_{\mathrm{id}_J \sqcup \triangleright} \mu^{\mathsf{C}}_{\phi}\bigr]\Bigr) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}} \\ = \Bigl((\mathrm{ev}^{\mathsf{C}})_{j\in J}, (1)_{k\in K}, 1\Bigr) \cdot_{\pi} \Bigl(\bigl((1)_{j\in J}, \mu^{\mathsf{C}}_{\psi}\bigr) \cdot_{\mathrm{id}_J \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}}\Bigr).$$

By the definition of μ^{C}_{ψ},

$$\bigl((1)_{j\in J}, \mu^{\mathsf{C}}_{\psi}\bigr) \cdot_{\mathrm{id}_J \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}} = \bigl((\mathrm{ev}^{\mathsf{C}})_{k\in K}, 1\bigr) \cdot_{\overline{\psi}} \mathrm{ev}^{\mathsf{C}},$$

therefore

$$\Bigl((1)_{i\in I}, \bigl[\bigl((1)_{j\in J}, \mu^{\mathsf{C}}_{\psi}\bigr) \cdot_{\mathrm{id}_J \sqcup \triangleright} \mu^{\mathsf{C}}_{\phi}\bigr]\Bigr) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}} \\ = \Bigl((\mathrm{ev}^{\mathsf{C}})_{j\in J}, (1)_{k\in K}, 1\Bigr) \cdot_{\pi} \Bigl(\bigl((\mathrm{ev}^{\mathsf{C}})_{k\in K}, 1\bigr) \cdot_{\overline{\psi}} \mathrm{ev}^{\mathsf{C}}\Bigr).$$

Similarly, by the associativity axiom for the pair of maps

$$I \sqcup J \sqcup K \sqcup \mathbf{1} \xrightarrow{\mathrm{id}_I \sqcup \overline{\psi}} I \sqcup K \sqcup \mathbf{1} \xrightarrow{\mathrm{id}_I \sqcup \triangleright} I \sqcup \mathbf{1}$$

and the identity axiom,

$$\Big((1)_{i\in I}, \big[\big((\mu^{\underline{\mathsf{C}}}_{\phi_k})_{k\in K}, 1\big) \cdot_{\overline{\psi}} \mu^{\underline{\mathsf{C}}}_{\phi\psi}\big]\Big) \cdot_{\mathrm{id}_I \sqcup \rhd} \mathrm{ev}^{\mathsf{C}} \\ = \Big((1)_{i\in I}, (\mu^{\underline{\mathsf{C}}}_{\phi_k})_{k\in K}, 1\Big) \cdot_{\mathrm{id}_I \sqcup \overline{\psi}} \Big(\big((1)_{i\in I}, \mu^{\underline{\mathsf{C}}}_{\phi\psi}\big) \cdot_{\mathrm{id}_I \sqcup \rhd} \mathrm{ev}^{\mathsf{C}}\Big).$$

By the definition of $\mu^{\underline{\mathsf{C}}}_{\phi\psi}$,

$$\big((1)_{i\in I}, \mu^{\underline{\mathsf{C}}}_{\phi\psi}\big) \cdot_{\mathrm{id}_I \sqcup \rhd} \mathrm{ev}^{\mathsf{C}} = \big((\mathrm{ev}^{\mathsf{C}})_{k\in K}, 1\big) \cdot_{\overline{\phi\psi}} \mathrm{ev}^{\mathsf{C}} .$$

Applying the associativity axiom for the pair of maps

$$I \sqcup J \sqcup K \sqcup \mathbf{1} \xrightarrow{\mathrm{id}_I \sqcup \overline{\psi}} I \sqcup K \sqcup \mathbf{1} \xrightarrow{\overline{\phi\psi}} K \sqcup \mathbf{1}$$

leads to

$$\Big((1)_{i\in I}, (\mu^{\underline{\mathsf{C}}}_{\phi_k})_{k\in K}, 1\Big) \cdot_{\mathrm{id}_I \sqcup \overline{\psi}} \Big(\big((1)_{i\in I}, \mu^{\underline{\mathsf{C}}}_{\phi\psi}\big) \cdot_{\mathrm{id}_I \sqcup \rhd} \mathrm{ev}^{\mathsf{C}}\Big) \\ = \Big(\Big(\big((1)_{i\in \phi^{-1}\psi^{-1}k}, \mu^{\underline{\mathsf{C}}}_{\phi_k}\big) \cdot_{\mathrm{id}_{\phi^{-1}\psi^{-1}k} \sqcup \rhd} \mathrm{ev}^{\mathsf{C}}\Big)_{k\in K}, 1\Big) \cdot_{\overline{\psi}} \mathrm{ev}^{\mathsf{C}},$$

where $\mathrm{id}_{\phi^{-1}\psi^{-1}k} \sqcup \rhd : \phi^{-1}\psi^{-1}k \sqcup \psi^{-1}k \sqcup \mathbf{1} \to \phi^{-1}\psi^{-1}k \sqcup \mathbf{1}$. Finally, since by the definition of $\mu^{\underline{\mathsf{C}}}_{\phi_k}$,

$$\big((1)_{i\in \phi^{-1}\psi^{-1}k}, \mu^{\underline{\mathsf{C}}}_{\phi_k}\big) \cdot_{\mathrm{id}_{\phi^{-1}\psi^{-1}k} \sqcup \rhd} \mathrm{ev}^{\mathsf{C}} = \big((\mathrm{ev}^{\mathsf{C}})_{i\in \phi^{-1}\psi^{-1}k}, 1\big) \cdot_{\overline{\phi_k}} \mathrm{ev}^{\mathsf{C}},$$

it follows that

$$\Big((1)_{i\in I}, \big[\big((\mu^{\underline{\mathsf{C}}}_{\phi_k})_{k\in K}, 1\big) \cdot_{\overline{\psi}} \mu^{\underline{\mathsf{C}}}_{\phi\psi}\big]\Big) \cdot_{\mathrm{id}_I \sqcup \rhd} \mathrm{ev}^{\mathsf{C}} \\ = \Big(\Big(\big((\mathrm{ev}^{\mathsf{C}})_{i\in \phi^{-1}\psi^{-1}k}, 1\big) \cdot_{\overline{\phi_k}} \mathrm{ev}^{\mathsf{C}}\Big)_{k\in K}, 1\Big) \cdot_{(\mathrm{id}_I \sqcup \overline{\psi})\cdot\overline{\phi\psi}} \mathrm{ev}^{\mathsf{C}} .$$

The equation in question is a consequence of the associativity of composition in C, written for the pair of maps

$$I \sqcup J \sqcup K \sqcup \mathbf{1} \xrightarrow{\pi} J \sqcup K \sqcup \mathbf{1} \xrightarrow{\overline{\psi}} K \sqcup \mathbf{1}$$

and morphisms

$$\begin{aligned} \mathrm{ev}^{\mathsf{C}} &: (X_i)_{i\in\phi^{-1}j}, \underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j) \to Y_j, & j \in J, \\ \mathrm{ev}^{\mathsf{C}} &: (Y_j)_{j\in\psi^{-1}k}, \underline{\mathsf{C}}((Y_j)_{j\in\psi^{-1}k}; Z_k) \to Z_k, & k \in K, \\ \mathrm{ev}^{\mathsf{C}} &: (Z_k)_{k\in K}, \underline{\mathsf{C}}((Z_k)_{k\in K}; W) \to W. \end{aligned}$$

Note that $\pi \cdot \overline{\psi} = (\mathrm{id}_I \sqcup \overline{\psi}) \cdot \overline{\phi\psi}$, and the map

$$\pi_k = \pi|_{\pi^{-1}\overline{\psi}^{-1}k} : \pi^{-1}\overline{\psi}^{-1}k = \phi^{-1}\psi^{-1}k \sqcup \psi^{-1}k \sqcup \mathbf{1} \to \psi^{-1}k \sqcup \mathbf{1} = \overline{\psi}^{-1}k$$

coincides with $\overline{\phi_k}$, $k \in K$. The verification of the identity axiom is left to the reader. □

The proof of Proposition 1.3.14 is quite detailed. In the sequel, we are not going to explain each instance of the associativity axiom and identity axiom, letting a pedantic reader fill in details by herself.

Suppose C is a (symmetric) multicategory, $\phi : I \to J$ is a map in $\mathrm{Mor}\,\mathcal{O}$ (resp. $\mathrm{Mor}\,\mathcal{S}$), and $f_j : (X_i)_{i\in\phi^{-1}j} \to Y_j$, $j \in J$, is a family of morphisms in C. Then there is a map

$$\mathsf{C}((f_j)_{j\in J}; Z) : \mathsf{C}((Y_j)_{j\in J}; Z) \to \mathsf{C}((X_i)_{i\in I}; Z), \quad g \mapsto (f_j)_{j\in J} \cdot_\phi g.$$

These maps are compatible with composition: if $I \xrightarrow{\phi} J \xrightarrow{\psi} K$ is a pair of composable maps in $\operatorname{Mor}\mathcal{O}$ (resp. $\operatorname{Mor}\mathcal{S}$), $f_j : (X_i)_{i\in\phi^{-1}j} \to Y_j$, $j \in J$, $g_k : (Y_j)_{j\in\psi^{-1}k} \to Z_k$, $k \in K$, are morphisms in C, then

$$\mathsf{C}((g_k)_{k\in K}; W) \cdot \mathsf{C}((f_j)_{j\in J}; W) = \mathsf{C}(((f_j)_{j\in\psi^{-1}k} \cdot_{\phi_k} g_k)_{k\in K}; W)$$

as maps $\mathsf{C}((Z_k)_{k\in K}; W) \to \mathsf{C}((X_i)_{i\in I}; W)$. This is a consequence of the associativity of composition in C.

1.3.15. Proposition. *For each closed (symmetric) multicategory C, the bijection*

$$\begin{aligned}\varphi_{(Y_j)_{j\in J};(X_i)_{i\in I};Z} : \mathsf{C}\big((Y_j)_{j\in J}; \underline{\mathsf{C}}((X_i)_{i\in I}; Z)\big) &\to \mathsf{C}\big((X_i)_{i\in I}, (Y_j)_{j\in J}; Z\big),\\ g &\mapsto \big((1^{\mathsf{C}}_{X_i})_{i\in I}, g\big) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z},\end{aligned}$$

is natural in $(Y_j)_{j\in J}$. That is, for an arbitrary map $\phi : K \to J$ in $\operatorname{Mor}\mathcal{O}$ *(resp.* $\operatorname{Mor}\mathcal{S}$*) and morphisms $f_j : (W_k)_{k\in\phi^{-1}j} \to Y_j$, $j \in J$, the diagram*

$$\begin{tikzcd}
\mathsf{C}\big((Y_j)_{j\in J}; \underline{\mathsf{C}}((X_i)_{i\in I}; Z)\big) \arrow[r, "\varphi_{(Y_j)_{j\in J};(X_i)_{i\in I};Z}"] \arrow[d, "\mathsf{C}((f_j)_{j\in J};\underline{\mathsf{C}}((X_i)_{i\in I};Z))"'] & \mathsf{C}\big((X_i)_{i\in I}, (Y_j)_{j\in J}; Z\big) \arrow[d, "\mathsf{C}((1^{\mathsf{C}}_{X_i})_{i\in I},(f_j)_{j\in J};Z)"] \\
\mathsf{C}\big((W_k)_{k\in K}; \underline{\mathsf{C}}((X_i)_{i\in I}; Z)\big) \arrow[r, "\varphi_{(W_k)_{k\in K};(X_i)_{i\in I};Z}"] & \mathsf{C}\big((X_i)_{i\in I}, (W_k)_{k\in K}; Z\big)
\end{tikzcd}$$

commutes.

The naturality in the remaining arguments can also be proven. However, since we are not going to make use of the fact, it is not explored in details.

Proof. Indeed, taking an arbitrary morphism $g : (Y_j)_{j\in J} \to \underline{\mathsf{C}}((X_i)_{i\in I}; Z)$ and pushing it along the top-right path produces

$$\Big((1^{\mathsf{C}}_{X_i})_{i\in I}, (f_j)_{j\in J}\Big) \cdot_{\mathrm{id}_I \sqcup \phi} \Big(\big((1^{\mathsf{C}}_{X_i})_{i\in I}, g\big) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}\Big),$$

while pushing g along the left-bottom path yields

$$\Big((1^{\mathsf{C}}_{X_i})_{i\in I}, \big((f_j)_{j\in J} \cdot_\phi g\big)\Big) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}\,.$$

The former expression is equal to

$$\Big((1^{\mathsf{C}}_{X_i} \cdot 1^{\mathsf{C}}_{X_i})_{i\in I}, \big((f_j)_{j\in J} \cdot_\phi g\big)\Big) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z} = \Big((1^{\mathsf{C}}_{X_i})_{i\in I}, \big((f_j)_{j\in J} \cdot_\phi g\big)\Big) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}$$

by the associativity axiom for the pair of maps $I \sqcup K \xrightarrow{\mathrm{id}_I \sqcup \phi} I \sqcup J \xrightarrow{\mathrm{id}_I \sqcup \triangleright} I \sqcup \mathbf{1}$, and since $1^{\mathsf{C}}_{X_i} \cdot 1^{\mathsf{C}}_{X_i} = 1^{\mathsf{C}}_{X_i}$, $i \in I$. The proposition is proven. □

1.3.16. Proposition. *The choice of evaluations $\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}$ for a closed multicategory C determines a unique isomorphism*

$$\underline{\varphi}_{(Y_j)_{j\in J};(X_i)_{i\in I};Z} : \underline{\mathsf{C}}\big((Y_j)_{j\in J}; \underline{\mathsf{C}}((X_i)_{i\in I}; Z)\big) \to \underline{\mathsf{C}}\big((X_i)_{i\in I}, (Y_j)_{j\in J}; Z\big).$$

Proof. Let $\psi_{(W_k)_{k\in K}}$ denote the bijection

$$\begin{tikzcd}
\mathsf{C}\big((W_k)_{k\in K};\underline{\mathsf{C}}\big((Y_j)_{j\in J};\underline{\mathsf{C}}((X_i)_{i\in I};Z)\big)\big) \arrow[d, "\varphi_{(W_k)_{k\in K};(Y_j)_{j\in J};\underline{\mathsf{C}}((X_i)_{i\in I};Z)}"] \\
\mathsf{C}\big((Y_j)_{j\in J},(W_k)_{k\in K};\underline{\mathsf{C}}((X_i)_{i\in I};Z)\big) \arrow[d, "\varphi_{(Y_j)_{j\in J},(W_k)_{k\in K};(X_i)_{i\in I};Z}"] \\
\mathsf{C}\big((X_i)_{i\in I},(Y_j)_{j\in J},(W_k)_{k\in K};Z\big) \arrow[d, "\varphi^{-1}_{(W_k)_{k\in K};(X_i)_{i\in I},(Y_j)_{j\in J};Z}"] \\
\mathsf{C}\big((W_k)_{k\in K};\underline{\mathsf{C}}((X_i)_{i\in I},(Y_j)_{j\in J};Z)\big).
\end{tikzcd}$$

It is natural in $(W_k)_{k\in K}$ by Proposition 1.3.15. Denote $A = \underline{\mathsf{C}}\big((Y_j)_{j\in J};\underline{\mathsf{C}}((X_i)_{i\in I};Z)\big)$ and $B = \underline{\mathsf{C}}((X_i)_{i\in I},(Y_j)_{j\in J};Z)$. Considering $K = \mathbf{1}$ we get an isomorphism between the functors $W \mapsto \mathsf{C}(W;A)$ and $W \mapsto \mathsf{C}(W;B)$, which by the ordinary Yoneda Lemma gives an isomorphism $\underline{\varphi}_{(Y_j)_{j\in J};(X_i)_{i\in I};Z} : A \to B$ in C, the image of 1^{C}_A under the map ψ_A.

Notice that, for an arbitrary family $(W_k)_{k\in K}$, the isomorphism $\psi_{(W_k)_{k\in K}}$ is obtained by composition with $\underline{\varphi}_{(Y_j)_{j\in J};(X_i)_{i\in I};Z}$. Indeed, the naturality of $\psi_{(W_k)_{k\in K}}$ is expressed by the commutative diagram

$$\begin{tikzcd}
\mathsf{C}\big((U_l)_{l\in L};A\big) \arrow[r, "\psi_{(U_l)_{l\in L}}"] \arrow[d, "\mathsf{C}((f_l)_{l\in L};A)"'] & \mathsf{C}\big((U_l)_{l\in L};B\big) \arrow[d, "\mathsf{C}((f_l)_{l\in L};B)"] \\
\mathsf{C}\big((W_k)_{k\in K};A\big) \arrow[r, "\psi_{(W_k)_{k\in K}}"] & \mathsf{C}\big((W_k)_{k\in K};B\big)
\end{tikzcd}$$

for each $\phi : K \to L$ and morphisms $f_l : (W_k)_{k\in\phi^{-1}l} \to U_l$, $l \in L$. Consider $\phi = \triangleright : K \to \mathbf{1}$, $U_1 = A$, and an arbitrary morphism $f : (W_k)_{k\in K} \to A$. Pushing the identity $1^{\mathsf{C}}_A \in \mathsf{C}(A;A)$ along the top-right path produces $f \cdot \psi_A(1^{\mathsf{C}}_A) = f \cdot \underline{\varphi}_{(Y_j)_{j\in J};(X_i)_{i\in I};Z}$, while pushing it along the left-bottom path gives $\psi_{(W_k)_{k\in K}}(f)$, hence the assertion. □

1.3.17. Corollary. *The isomorphism* $\underline{\varphi}_{(Y_j)_{j\in J};(X_i)_{i\in I};Z}$ *makes the diagram*

$$\begin{tikzcd}
\mathsf{C}(;\underline{\mathsf{C}}((Y_j)_{j\in J};\underline{\mathsf{C}}((X_i)_{i\in I};Z))) \arrow[r, "\mathsf{C}(;\underline{\varphi}^{\mathsf{C}}_{(Y_j)_{j\in J};(X_i)_{i\in I};Z})"] \arrow[d, "\varphi^{\mathsf{C}}_{;(Y_j)_{j\in J};\underline{\mathsf{C}}((X_i)_{i\in I};Z)}"'] & \mathsf{C}(;\underline{\mathsf{C}}((X_i)_{i\in I},(Y_j)_{j\in J};Z)) \arrow[d, "\varphi^{\mathsf{C}}_{;(X_i)_{i\in I},(Y_j)_{j\in J};Z}"] \\
\mathsf{C}((Y_j)_{j\in J};\underline{\mathsf{C}}((X_i)_{i\in I};Z)) \arrow[r, "\varphi^{\mathsf{C}}_{(Y_j)_{j\in J};(X_i)_{i\in I};Z}"] & \mathsf{C}((X_i)_{i\in I},(Y_j)_{j\in J};Z)
\end{tikzcd}$$

commute.

Proof. The proof follows from the definition of $\psi_{(W_k)_{k\in K}}$ for $K = \varnothing$. □

1.3.18. Example. Let C be a closed symmetric multicategory. Let X be an object of C. It gives rise to a C-functor $\underline{\mathsf{C}}(X;-) : \underline{\mathsf{C}} \to \underline{\mathsf{C}}$, $Y \mapsto \underline{\mathsf{C}}(X;Y)$, where $\underline{\mathsf{C}}$ denotes the C-category underlying the symmetric C-multicategory $\underline{\mathsf{C}}$. The action on objects of morphisms is found from the following equation in C:

$$\big[\underline{\mathsf{C}}(X;Y),\underline{\mathsf{C}}(Y;Z) \xrightarrow{1,\underline{\mathsf{C}}(X;-)} \underline{\mathsf{C}}(X;Y),\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;Z)) \xrightarrow{\mathrm{ev}^{\mathsf{C}}} \underline{\mathsf{C}}(X;Z)\big] = \mu^{\underline{\mathsf{C}}}.$$

The existence and uniqueness of the solution are guaranteed by the closedness of C. That so defined $\underline{\mathsf{C}}(X;-)$ preserves identities is a consequence of the identity axiom. The compatibility with composition is established as follows. Consider the diagram

$$\begin{tikzcd}
\begin{array}{l}\underline{\mathsf{C}}(X;Y),\\ \underline{\mathsf{C}}(Y;Z),\\ \underline{\mathsf{C}}(Z;W)\end{array} \arrow[r, "{1,\underline{\mathsf{C}}(X;-),\underline{\mathsf{C}}(X;-)}"] \arrow[d, "{1,\mu^{\underline{\mathsf{C}}}}"'] & \begin{array}{l}\underline{\mathsf{C}}(X;Y),\\ \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;Z)),\\ \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Z);\underline{\mathsf{C}}(X;W))\end{array} \arrow[r, "{\mathrm{ev}^{\mathsf{C}},1}"] \arrow[d, "{1,\mu^{\underline{\mathsf{C}}}}"] & \begin{array}{l}\underline{\mathsf{C}}(X;Z),\\ \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Z);\underline{\mathsf{C}}(X;W))\end{array} \arrow[d, "\mathrm{ev}^{\mathsf{C}}"] \\
\begin{array}{l}\underline{\mathsf{C}}(X;Y),\\ \underline{\mathsf{C}}(Y;W)\end{array} \arrow[r, "{1,\underline{\mathsf{C}}(X;-)}"] & \begin{array}{l}\underline{\mathsf{C}}(X;Y),\\ \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;W))\end{array} \arrow[r, "\mathrm{ev}^{\mathsf{C}}"] & \underline{\mathsf{C}}(X;W)
\end{tikzcd}$$

By the above definition of $\underline{\mathsf{C}}(X;-)$, the exterior expresses the associativity of $\mu^{\underline{\mathsf{C}}}$. The right square is the definition of $\mu^{\underline{\mathsf{C}}}$. By the closedness of C, the square

$$\begin{tikzcd}
\underline{\mathsf{C}}(Y;Z),\underline{\mathsf{C}}(Z;W) \arrow[r, "{\underline{\mathsf{C}}(X;-),\underline{\mathsf{C}}(X;-)}"] \arrow[d, "\mu^{\underline{\mathsf{C}}}"'] & \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;Z)),\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Z);\underline{\mathsf{C}}(X;W)) \arrow[d, "\mu^{\underline{\mathsf{C}}}"] \\
\underline{\mathsf{C}}(Y;W) \arrow[r, "\underline{\mathsf{C}}(X;-)"] & \underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Y);\underline{\mathsf{C}}(X;W))
\end{tikzcd}$$

commutes, hence the assertion.

1.3.19. Example. Similarly, an arbitrary object $Z \in \operatorname{Ob} \mathsf{C}$ gives rise to a C-functor $\underline{\mathsf{C}}(-;Z) : \underline{\mathsf{C}}^{\mathrm{op}} \to \underline{\mathsf{C}}$, $X \mapsto \underline{\mathsf{C}}(X;Z)$. Its action on objects of morphisms is found from the following equation in C:

$$\big[\underline{\mathsf{C}}(X;Z),\underline{\mathsf{C}}(Y;X) \xrightarrow{1,\underline{\mathsf{C}}(-;Z)} \underline{\mathsf{C}}(X;Z),\underline{\mathsf{C}}(\underline{\mathsf{C}}(X;Z);\underline{\mathsf{C}}(Y;Z)) \xrightarrow{\mathrm{ev}^{\mathsf{C}}} \underline{\mathsf{C}}(Y;Z)\big] = \mu_{\underline{\mathsf{C}}^{\mathrm{op}}}.$$

The existence and uniqueness of the solution follow from the closedness of C.

Let $g : (Y_j)_{j\in J} \to Z$ be a morphism in C, $X_i \in \operatorname{Ob} \mathsf{C}$, $i \in I$, a family of objects, and let $\phi : I \to J$ be a map in $\operatorname{Mor} \mathcal{S}$. The morphism g gives rise to a morphism

$$\underline{\mathsf{C}}(\phi;g) : (\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J} \to \underline{\mathsf{C}}((X_i)_{i\in I};Z)$$

in C determined in a unique way via the diagram

$$\begin{tikzcd}
(X_i)_{i\in I},(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J} \arrow[r, "{(1)_I,\underline{\mathsf{C}}(\phi;g)}"] \arrow[d, "{(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J}}"'] & (X_i)_{i\in I},\underline{\mathsf{C}}((X_i)_{i\in I};Z) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}"] \\
(Y_j)_{j\in J} \arrow[r, "g"] & Z
\end{tikzcd} \tag{1.3.9}$$

The existence and uniqueness of $\underline{\mathsf{C}}(\phi;g)$ follow from the closedness of C.

Let $\psi : K \to I$ be a map in $\operatorname{Mor} \mathcal{S}$. Let $f_i : (W_k)_{k\in\psi^{-1}i} \to X_i$, $i \in I$, be morphisms in C. A morphism $\underline{\mathsf{C}}((f_i)_{i\in I};1) : \underline{\mathsf{C}}((X_i)_{i\in I};Z) \to \underline{\mathsf{C}}((W_k)_{k\in K};Z)$ is defined as the only morphism that makes the following diagram commutative:

$$\begin{tikzcd}
(W_k)_{k\in K},\underline{\mathsf{C}}((X_i)_{i\in I};Z) \arrow[r, "{(1)_K,\underline{\mathsf{C}}((f_i)_{i\in I};1)}"] \arrow[d, "{(f_i)_{i\in I},1}"'] & (W_k)_{k\in K},\underline{\mathsf{C}}((W_k)_{k\in K};Z) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{(W_k)_{k\in K};Z}"] \\
(X_i)_{i\in I},\underline{\mathsf{C}}((X_i)_{i\in I};Z) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}"] & Z
\end{tikzcd} \tag{1.3.10}$$

The following statements are proven in the same manner: instead of proving an equation $f = g$ between morphisms $f, g : (Y_j)_{j\in J} \to \underline{\mathsf{C}}((X_i)_{i\in I}; Z)$, we prove an equivalent equation $\varphi(f) = \varphi(g)$ between the morphisms $\varphi(f), \varphi(g) : (X_i)_{i\in I}, (Y_j)_{j\in J} \to Z$. Usually, it suffices to draw a commutative diagram whose exterior coincides with the required equation, and to explain its commutativity. When the diagram does not fit into a page, as it happens in the proof of the following lemma, we provide an inline proof.

1.3.20. Lemma. *In the above assumptions, the introduced morphisms satisfy the commutativity relation:*

$$\begin{tikzcd}
\big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j)\big)_{j\in J} \arrow[r, "\underline{\mathsf{C}}(\phi;g)"] \arrow[d, "(\underline{\mathsf{C}}((f_i)_{i\in\phi^{-1}j};1))_{j\in J}"'] \arrow[dr, phantom, "="] & \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[d, "\underline{\mathsf{C}}((f_i)_{i\in I};1)"] \\
\big(\underline{\mathsf{C}}((W_k)_{k\in\psi^{-1}\phi^{-1}j}; Y_j)\big)_{j\in J} \arrow[r, "\underline{\mathsf{C}}(\psi\phi;g)"] & \underline{\mathsf{C}}((W_k)_{k\in K}; Z)
\end{tikzcd}$$

Proof. We may rewrite the required equation in an equivalent form:

$$\begin{aligned}
&\big[(W_k)_{k\in K}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j)\big)_{j\in J} \xrightarrow{(1)_K,\underline{\mathsf{C}}(\phi;g)} (W_k)_{k\in K}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \\
&\qquad \xrightarrow{(1)_K,\underline{\mathsf{C}}((f_i)_{i\in I};1)} (W_k)_{k\in K}, \underline{\mathsf{C}}((W_k)_{k\in K}; Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}} Z\big] \\
&= \big[(W_k)_{k\in K}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j)\big)_{j\in J} \xrightarrow{(1)_K,(\underline{\mathsf{C}}((f_i)_{i\in\phi^{-1}j};1))_{j\in J}} \\
&\qquad (W_k)_{k\in K}, \big(\underline{\mathsf{C}}((W_k)_{k\in\psi^{-1}\phi^{-1}j}; Y_j)\big)_{j\in J} \xrightarrow{(1)_K,\underline{\mathsf{C}}(\psi\cdot\phi;g)} (W_k)_{k\in K}, \underline{\mathsf{C}}((W_k)_{k\in K}; Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}} Z\big].
\end{aligned}$$

The left hand side can be transformed as follows:

$$\begin{aligned}
&\big[(W_k)_{k\in K}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j)\big)_{j\in J} \xrightarrow{(1)_K,\underline{\mathsf{C}}(\phi;g)} (W_k)_{k\in K}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \\
&\qquad \xrightarrow{(f_i)_{i\in I},1} (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}} Z\big] \\
&= \big[(W_k)_{k\in K}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j)\big)_{j\in J} \xrightarrow{(f_i)_{i\in I},(1)_J} (X_i)_{i\in I}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j)\big)_{j\in J} \\
&\qquad \xrightarrow{(1)_I,\underline{\mathsf{C}}(\phi;g)} (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}} Z\big] \\
&= \big[(W_k)_{k\in K}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j)\big)_{j\in J} \xrightarrow{(f_i)_{i\in I},(1)_J} (X_i)_{i\in I}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j)\big)_{j\in J} \\
&\qquad \xrightarrow{(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J}} (Y_j)_{j\in J} \xrightarrow{g} Z\big].
\end{aligned} \tag{1.3.11}$$

The right hand side can be transformed to

$$\begin{aligned}
&\big[(W_k)_{k\in K}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j)\big)_{j\in J} \xrightarrow{(1)_K,(\underline{\mathsf{C}}((f_i)_{i\in\phi^{-1}j};1))_{j\in J}} \\
&\qquad (W_k)_{k\in K}, \big(\underline{\mathsf{C}}((W_k)_{k\in\psi^{-1}\phi^{-1}j}; Y_j)\big)_{j\in J} \xrightarrow{(\mathrm{ev}^{\mathsf{C}}_{(W_k)_{k\in\psi^{-1}\phi^{-1}j};Y_j})_{j\in J}} (Y_j)_{j\in J} \xrightarrow{g} Z\big],
\end{aligned}$$

which coincides with the last expression of (1.3.11). □

1.3.21. Lemma. *Let* $I \xrightarrow{\phi} J \xrightarrow{\psi} K$ *be maps in* $\operatorname{Mor} \mathcal{S}$, *and let* $f_j : (X_i)_{i\in\phi^{-1}j} \to Y_j$, $j \in J$, $g_k : (Y_j)_{j\in\psi^{-1}k} \to Z_k$, $k \in K$, *be morphisms in a closed symmetric multicategory* C. *Then*

$$\begin{aligned}&\underline{\mathsf{C}}\big(((f_j)_{j\in\psi^{-1}k} \cdot g_k)_{k\in K};1\big)\\ &\qquad = \big[\underline{\mathsf{C}}((Z_k)_{k\in K};W) \xrightarrow{\underline{\mathsf{C}}((g_k)_{k\in K};1)} \underline{\mathsf{C}}((Y_j)_{j\in J};W) \xrightarrow{\underline{\mathsf{C}}((f_j)_{j\in J};1)} \underline{\mathsf{C}}((X_i)_{i\in I};W)\big].\end{aligned}$$

Proof. The commutative diagram

$$\begin{tikzcd}
(X_i)_{i\in I}, \underline{\mathsf{C}}((Z_k)_{k\in K};W) \arrow[r, "{(1)_I,\underline{\mathsf{C}}((g_k)_{k\in K};1)}"] \arrow[d, "{(f_j)_{j\in J},1}"'] & (X_i)_{i\in I}, \underline{\mathsf{C}}((Y_j)_{j\in J};W) \arrow[d, "{(f_j)_{j\in J},1}"'] \arrow[dr, "{(1)_I,\underline{\mathsf{C}}((f_j)_{j\in J};1)}"] & \\
(Y_j)_{j\in J}, \underline{\mathsf{C}}((Z_k)_{k\in K};W) \arrow[r, "{(1)_J,\underline{\mathsf{C}}((g_k)_{k\in K};1)}"] \arrow[d, "{(g_k)_{k\in K},1}"'] & (Y_j)_{j\in J}, \underline{\mathsf{C}}((Y_j)_J;W) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{(Y_j)_{j\in J};W}"'] & (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_I;W) \arrow[dl, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};W}"] \\
(Z_k)_{k\in K}, \underline{\mathsf{C}}((Z_k)_{k\in K};W) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(Z_k)_{k\in K};W}"] & W &
\end{tikzcd}$$

implies the lemma. The lower square is the definition of $\underline{\mathsf{C}}((g_k)_{k\in K};1)$. The left quadrilateral is the definition of $\underline{\mathsf{C}}((f_j)_{j\in J};1)$. □

1.3.22. Lemma. *Let* $I \xrightarrow{\phi} J \xrightarrow{\psi} K$ *be maps in* $\operatorname{Mor} \mathcal{S}$, *and let* $f_k : (Y_j)_{j\in\psi^{-1}k} \to Z_k$, $k \in K$, $g : (Z_k)_{k\in K} \to W$ *be morphisms in a closed symmetric multicategory* C. *Denote by* ϕ_k *the restriction* $\phi|_{\phi^{-1}\psi^{-1}k} : \phi^{-1}\psi^{-1}k \to \psi^{-1}k$, $k \in K$. *Then*

$$\begin{aligned}&\big[\big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J} \xrightarrow{\underline{\mathsf{C}}(\phi;(f_k)_{k\in K}\cdot_\psi g)} \underline{\mathsf{C}}((X_i)_{i\in I};W)\big]\\ &= \big[\big((\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in\psi^{-1}k}\big)_{k\in K} \xrightarrow{(\underline{\mathsf{C}}(\phi_k;f_k))_{k\in K}} \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}\psi^{-1}k};Z_k)\big)_{k\in K}\\ &\hspace{20em} \xrightarrow{\underline{\mathsf{C}}(\phi\psi;g)} \underline{\mathsf{C}}((X_i)_{i\in I};W)\big].\end{aligned}$$

Proof. The claim follows from the commutative diagram

$$\begin{tikzcd}
(X_i)_{i\in I}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J} \arrow[r, "(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J}"] \arrow[d, "{(1)_I,(\underline{\mathsf{C}}(\phi_k;f_k))_{k\in K}}"'] & (Y_j)_{j\in J} \arrow[d, "(f_k)_{k\in K}"] \\
(X_i)_{i\in I}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}\psi^{-1}k};Z_k)\big)_{k\in K} \arrow[r, "(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}\psi^{-1}k};Z_k})_{k\in K}"] \arrow[d, "{(1)_I,\underline{\mathsf{C}}(\phi\psi;g)}"'] & (Z_k)_{k\in K} \arrow[d, "g"] \\
(X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I};W) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};W}"] & W
\end{tikzcd}$$

where the lower square is the definition of $\underline{\mathsf{C}}(\phi\psi;g)$, and the commutativity of the upper square is a consequence of the definition of $\underline{\mathsf{C}}(\phi_k;f_k)$, $k \in K$. □

Notation. Let $g : (Y_j)_{j\in J} \to Z$ be a morphism in a closed symmetric multicategory C. Denote by $\dot{g} : () \to \underline{\mathsf{C}}((Y_j)_{j\in J};Z)$ the morphism $\varphi^{-1}_{;(Y_j)_{j\in J};Z}(g) \in \mathsf{C}(;\underline{\mathsf{C}}((Y_j)_{j\in J};Z))$.

1.3.23. Lemma. *Let the above assumptions hold. Let* $(X_i)_{i\in I}$ *be a family of objects of* C*, and let* $\phi : I \to J$ *be a map in* $\operatorname{Mor}\mathcal{S}$*. Then* $\underline{\mathsf{C}}(\phi; g)$ *coincides with the composite*

$$\bigl[(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J} \xrightarrow{(1)_J, \dot{g}} (\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J}, \underline{\mathsf{C}}((Y_j)_{j\in J}; Z) \xrightarrow{\mu_\phi^{\underline{\mathsf{C}}}} \underline{\mathsf{C}}((X_i)_{i\in I}; Z)\bigr].$$

Proof. Plugging the right hand side into defining equation (1.3.9) for $\underline{\mathsf{C}}(\phi; g)$ we obtain the diagram in C

$$\begin{tikzcd}
(X_i)_{i\in I}, (\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J} \arrow[r, "(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J}"] \arrow[d, "(1)_I, (1)_J, \dot{g}"'] & (Y_j)_{j\in J} \arrow[d, "(1)_J, \dot{g}"'] \arrow[dd, bend left, "g"] \\
(X_i)_{i\in I}, (\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J}, \underline{\mathsf{C}}((Y_j)_{j\in J}; Z) \arrow[r, "(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J}, 1"] \arrow[d, "(1)_I, \mu_\phi^{\underline{\mathsf{C}}}"'] & (Y_j)_{j\in J}, \underline{\mathsf{C}}((Y_j)_{j\in J}; Z) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{(Y_j)_{j\in J};Z}"'] \\
(X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}"] & Z
\end{tikzcd}$$

The lower square is the definition of $\mu_\phi^{\underline{\mathsf{C}}}$. The right triangle expresses the equation $\varphi_{;(Y_j)_{j\in J};Z}(\dot{g}) = g$. The commutativity of the exterior of the diagram implies the statement of the lemma. □

1.3.24. Remark. Suppose $g : Y \to Z$ is a morphism in C. It follows that

$$\bigl(\underline{\mathsf{C}}(\triangleright; g)\bigr)^{\cdot} = \bigl[() \xrightarrow{\dot{g}} \underline{\mathsf{C}}(Y; Z) \xrightarrow{\underline{\mathsf{C}}(X;-)} \underline{\mathsf{C}}(\underline{\mathsf{C}}(X; Y); \underline{\mathsf{C}}(X; Z))\bigr],$$

for each object $X \in \operatorname{Ob}\mathsf{C}$.

1.3.25. Lemma. *Let* $\psi : K \to I$ *be a map in* $\operatorname{Mor}\mathcal{S}$*. Let* $f_i : (W_k)_{k\in\psi^{-1}i} \to X_i$*,* $i \in I$*, be morphisms in* C*. Then* $\underline{\mathsf{C}}((f_i)_{i\in I}; 1)$ *coincides with the composite*

$$\bigl[\underline{\mathsf{C}}((X_i)_{i\in I}; Z) \xrightarrow{(\dot{f}_i)_{i\in I}, 1} (\underline{\mathsf{C}}((W_k)_{k\in\psi^{-1}i}; X_i))_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \xrightarrow{\mu_\psi^{\underline{\mathsf{C}}}} \underline{\mathsf{C}}((W_k)_{k\in K}; Z)\bigr].$$

Proof. Plug the right hand side into defining equation (1.3.10) for $\underline{\mathsf{C}}((f_i)_{i\in I}; 1)$. We obtain the diagram in C

$$\begin{tikzcd}
(W_k)_{k\in K}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[d, "(1)_K, (\dot{f}_i)_{i\in I}, 1"'] \arrow[dr, bend left, "(f_i)_{i\in I}, 1"] & \\
(W_k)_{k\in K}, (\underline{\mathsf{C}}((W_k)_{k\in\psi^{-1}i}; X_i))_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[r, "(\mathrm{ev}^{\mathsf{C}}_{(W_k)_{k\in\psi^{-1}i};X_i})_{i\in I}, 1"] \arrow[d, "(1)_K, \mu_\psi^{\underline{\mathsf{C}}}"'] & (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}"] \\
(W_k)_{k\in K}, \underline{\mathsf{C}}((W_k)_{k\in K}; Z) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(W_k)_{k\in K};Z}"] & Z
\end{tikzcd}$$

The square is the definition of $\mu_\psi^{\underline{\mathsf{C}}}$. The triangle follows from the equations

$$\varphi_{;(W_k)_{k\in\psi^{-1}i};X_i}(\dot{f}_i) = f_i, \quad i \in I.$$

The commutativity of the exterior of the diagram implies the statement of the lemma. □

1.3.26. Remark. Suppose $f : W \to X$ is a morphism in C. It follows that

$$\bigl(\underline{\mathsf{C}}(f; 1)\bigr)^{\cdot} = \bigl[() \xrightarrow{\dot{f}} \underline{\mathsf{C}}(W; X) \xrightarrow{\underline{\mathsf{C}}(-;Z)} \underline{\mathsf{C}}(\underline{\mathsf{C}}(X; Z); \underline{\mathsf{C}}(W; Z))\bigr].$$

1.3.27. Closing transformations. Let C, D be closed symmetric multicategories. Let $F : \mathsf{C} \to \mathsf{D}$ be a (symmetric) multifunctor. Define a morphism in D

$$\underline{F}_{(X_i)_{i\in I};Y} : F\underline{\mathsf{C}}((X_i)_{i\in I};Y) \to \underline{\mathsf{D}}((FX_i);FY)$$

as the only morphism that makes the diagram

$$\begin{tikzcd}
 & (FX_i)_{i\in I}, \underline{\mathsf{D}}((FX_i)_{i\in I};FY) \arrow[d, "\mathrm{ev}^{\mathsf{D}}_{(FX_i);FY}"] \\
(FX_i)_{i\in I}, F\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[ru, "(1)_I{,}\underline{F}_{(X_i);Y}"] \arrow[rd, "F\,\mathrm{ev}^{\mathsf{C}}_{(X_i);Y}"'] & \\
 & FY
\end{tikzcd} \tag{1.3.12}$$

commute. It is called the *closing transformation* of the multifunctor F.

1.3.28. Lemma. *The diagram*

$$\begin{tikzcd}
\mathsf{C}\big((Y_j)_{j\in J};\underline{\mathsf{C}}((X_i)_{i\in I};Z)\big) \arrow[r, "F"] \arrow[dd, "\varphi_{(Y_j)_{j\in J};(X_i)_{i\in I};Z}"'] & \mathsf{D}\big((FY_j)_{j\in J};F\underline{\mathsf{C}}((X_i)_{i\in I};Z)\big) \arrow[d, "\mathsf{D}(1;\underline{F}_{(X_i);Z})"] \\
 & \mathsf{D}\big((FY_j)_{j\in J};\underline{\mathsf{D}}((FX_i)_{i\in I};FZ)\big) \arrow[d, "\varphi_{(FY_j)_{j\in J};(FX_i)_{i\in I};FZ}"] \\
\mathsf{C}\big((X_i)_{i\in I},(Y_j)_{j\in J};Z\big) \arrow[r, "F"] & \mathsf{D}\big((FX_i)_{i\in I},(FY_j)_{j\in J};FZ\big)
\end{tikzcd} \tag{1.3.13}$$

commutes, for each $I, J \in \operatorname{Ob}\mathcal{O}$ and objects $X_i, Y_j, Z \in \operatorname{Ob}\mathsf{C}$, $i \in I$, $j \in J$.

Proof. Pushing an arbitrary morphism $g : (Y_j)_{j\in J} \to \underline{\mathsf{C}}((X_i)_{i\in I};Z)$ along the top-right path produces the composite

$$\big[(FX_i)_{i\in I},(FY_j)_{j\in J} \xrightarrow{(1)_I,Fg} (FX_i)_{i\in I}, F\underline{\mathsf{C}}((X_i)_{i\in I};Z) \xrightarrow{(1)_I,\underline{F}_{(X_i);Z}} \\ (FX_i)_{i\in I}, \mathsf{D}((FX_i)_{i\in I};FZ) \xrightarrow{\mathrm{ev}^{\mathsf{D}}_{(FX_i);FZ}} FZ\big].$$

The last two arrows compose to $F\,\mathrm{ev}^{\mathsf{C}}_{(X_i);Z}$ by the definition of $\underline{F}_{(X_i);Z}$. Since F preserves composition, the above composite equals

$$F\big(((1)_I,g)\cdot \mathrm{ev}^{\mathsf{C}}_{(X_i);Z}\big) = F\big(\varphi_{(Y_j)_{j\in J};(X_i)_{i\in I};Z}(g)\big),$$

hence the assertion. □

1.3.29. Corollary. *For $J = \varnothing$ we get the following relation between F and $\underline{F}$:*

$$\begin{tikzcd}
\mathsf{C}\big(;\underline{\mathsf{C}}((X_i)_{i\in I};Z)\big) \arrow[r, "F"] \arrow[d, "\varphi_{;(X_i)_{i\in I};Z}"'] & \mathsf{D}\big(;F\underline{\mathsf{C}}((X_i)_{i\in I};Z)\big) \arrow[r, "\mathsf{D}(;\underline{F}_{(X_i);Z})"] & \mathsf{D}\big(;\underline{\mathsf{D}}((FX_i)_{i\in I};FZ)\big) \arrow[d, "\varphi_{;(FX_i)_{i\in I};FZ}"] \\
\mathsf{C}\big((X_i)_{i\in I};Z\big) \arrow[rr, "F"] & & \mathsf{D}\big((FX_i)_{i\in I};FZ\big)
\end{tikzcd}$$

Let $F : \mathsf{C} \to \mathsf{D}$ be a symmetric multifunctor. According to Section 1.3.9, it gives rise to a **Cat**-functor F_* from the 2-category of symmetric C-multicategories to the 2-category of symmetric D-multicategories. The **Cat**-functor F_* takes the symmetric C-multicategory $\underline{\mathsf{C}}$ to the symmetric D-multicategory $F_*\underline{\mathsf{C}}$ whose objects are those of C, and whose objects of morphisms are $(F_*\underline{\mathsf{C}})((X_i)_{i\in I};Y) = F\underline{\mathsf{C}}((X_i)_{i\in I};Y)$, for each $I \in \operatorname{Ob}\mathcal{O}$ and $X_i, Y \in \operatorname{Ob}\mathsf{C}$, $i \in I$. Composition in $F_*\underline{\mathsf{C}}$ is given by

$$F\mu_\phi^{\underline{\mathsf{C}}} : (F\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J}, F\underline{\mathsf{C}}((Y_j)_{j\in J};Z) \to F\underline{\mathsf{C}}((X_i)_{i\in I};Z).$$

The identity of an object X is $1_X^{F_*\underline{\mathsf{C}}} = F1_X^{\underline{\mathsf{C}}} : () \to F\underline{\mathsf{C}}(X;X)$.

1.3.30. Proposition. *There is a* D*-multifunctor* $\underline{F} : F_*\underline{\mathsf{C}} \to \underline{\mathsf{D}}$ *such that* $\operatorname{Ob}\underline{F} = \operatorname{Ob}F$, *and* $\underline{F}_{(X_i)_{i\in I};Y} : F\underline{\mathsf{C}}((X_i)_{i\in I};Y) \to \underline{\mathsf{D}}((FX_i)_{i\in I};FY)$ *is the closing transformation, for each* $I \in \operatorname{Ob}\mathcal{O}$ *and* $X_i, Y \in \operatorname{Ob}\mathsf{C}$, $i \in I$.

Proof. It follows from Corollary 1.3.29 that $\underline{F}$ preserves identities. Indeed, take $I = \mathbf{1}$, $X_1 = Z = X \in \operatorname{Ob}\mathsf{C}$. Starting with the element $1_X^{\underline{\mathsf{C}}}$ of the source and tracing it along the sides of the pentagon, we obtain that the composite

$$() \xrightarrow{F1_X^{\underline{\mathsf{C}}}} F\underline{\mathsf{C}}(X;X) \xrightarrow{\underline{F}_{X;X}} \mathsf{D}(FX;FX)$$

is mapped by $\varphi_{;FX;FX}$ to $F(\varphi_{;X;X}(1_X^{\underline{\mathsf{C}}})) = F1_X^{\mathsf{C}} = 1_{FX}^{\mathsf{D}} = \varphi_{;FX;FX}(1_{FX}^{\underline{\mathsf{D}}})$. Since $\varphi_{;X;X}$ is bijective, it follows that

$$\bigl[() \xrightarrow{F1_X^{\underline{\mathsf{C}}}} F\underline{\mathsf{C}}(X;X) \xrightarrow{\underline{F}_{X;X}} \mathsf{D}(FX;FX)\bigr] = 1_{FX}^{\underline{\mathsf{D}}},$$

so that $\underline{F}$ preserves identities.

To show $\underline{F}$ preserves composition, we must show that, for each map $\phi : I \to J$ in $\operatorname{Mor}\mathcal{S}$, the diagram

$$\begin{tikzcd}
(F\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J}, F\underline{\mathsf{C}}((Y_j)_{j\in J};Z) \arrow[r, "F\mu_\phi^{\underline{\mathsf{C}}}"] \arrow[d, "(\underline{F}_{(X_i);Y_j})_{j\in J}, \underline{F}_{(Y_j);Z}"'] & F\underline{\mathsf{C}}((X_i)_{i\in I};Z) \arrow[d, "\underline{F}_{(X_i);Z}"] \\
(\underline{\mathsf{D}}((FX_i)_{i\in\phi^{-1}j};FY_j))_{j\in J}, \underline{\mathsf{D}}((FY_j)_{j\in J};FZ) \arrow[r, "\mu_\phi^{\underline{\mathsf{D}}}"] & \underline{\mathsf{D}}((FX_i)_{i\in I};FZ)
\end{tikzcd} \tag{1.3.14}$$

commutes. This follows from Diagram 1.9(a). The lower diamond is the definition of the composition morphism $\mu_\phi^{\underline{\mathsf{D}}}$. The exterior commutes by the definition of $\mu_\phi^{\underline{\mathsf{C}}}$ and since F preserves composition. The left upper diamond and both triangles commute by the definition of closing transformation. □

1.3.31. Lemma. *Let* $F : \mathsf{C} \to \mathsf{D}$ *be a symmetric multifunctor. Let* $f : (Y_j)_{j\in J} \to Z$ *be a morphism in* C, $X_i \in \operatorname{Ob}\mathsf{C}$, $i \in I$ *a family of objects, and* $\phi : I \to J$ *a map in* $\operatorname{Mor}\mathcal{S}$. *Then the diagram*

$$\begin{tikzcd}
(F\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J} \arrow[r, "F\underline{\mathsf{C}}(\phi;f)"] \arrow[d, "(\underline{F}_{(X_i);Y_j})_{j\in J}"'] & F\underline{\mathsf{C}}((X_i)_{i\in I};Z) \arrow[d, "\underline{F}_{(X_i);Z}"] \\
(\underline{\mathsf{D}}((FX_i)_{i\in\phi^{-1}j};FY_j))_{j\in J} \arrow[r, "\underline{\mathsf{D}}(\phi;Ff)"] & \underline{\mathsf{D}}((FX_i)_{i\in I};FZ)
\end{tikzcd} \tag{1.3.15}$$

commutes.

$$\begin{tikzcd}
& \begin{array}{l} (FX_i)_{i\in I}, \\ \big(F\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J}, \\ F\underline{\mathsf{C}}((Y_j)_{j\in J};Z) \end{array} \arrow[dl, "(F\,\mathrm{ev}^{\mathsf{C}}_{(X_i);Y_j})_{j\in J},1"'] \arrow[dr, "F\mu^{\mathsf{C}}_{\phi}"] \arrow[dd, "(1)_I,(\underline{F}_{(X_i);Y_j})_{j\in J},\underline{F}_{(Y_j);Z}"] & \\
\begin{array}{l} (FY_j)_{j\in J}, \\ F\underline{\mathsf{C}}((Y_j)_{j\in J};Z) \end{array} \arrow[dd, "(1)_J,\underline{F}_{(Y_j);Z}"] \arrow[dddr, bend right=60, "F\,\mathrm{ev}^{\mathsf{C}}_{(Y_j);Z}"'] & & \begin{array}{l} (FX_i)_{i\in I}, \\ F\underline{\mathsf{C}}((X_i)_{i\in I};Z) \end{array} \arrow[dd, "(1)_I,\underline{F}_{(X_i);Z}"'] \arrow[dddl, bend left=60, "F\,\mathrm{ev}^{\mathsf{C}}_{(X_i);Z}"] \\
& \begin{array}{l} (FX_i)_{i\in I}, \\ \big(\underline{\mathsf{D}}((FX_i)_{i\in\phi^{-1}j};FY_j)\big)_{j\in J}, \\ \underline{\mathsf{D}}((FY_j)_{j\in J};FZ) \end{array} \arrow[dl, "(\mathrm{ev}^{\mathsf{D}}_{(FX_i);FY_j})_{j\in J},1"] \arrow[dr, "(1)_I,\mu^{\mathsf{D}}_{\phi}"'] & \\
\begin{array}{l} (FY_j)_{j\in J}, \\ \underline{\mathsf{D}}((FY_j)_{j\in J};FZ) \end{array} \arrow[dr, "\mathrm{ev}^{\mathsf{D}}_{(FY_j);FZ}"] & & \begin{array}{l} (FX_i)_{i\in I}, \\ \underline{\mathsf{D}}((FX_i)_{i\in I};FZ) \end{array} \arrow[dl, "\mathrm{ev}^{\mathsf{D}}_{(FX_i);FZ}"'] \\
& FZ &
\end{tikzcd}$$

(a)

$$\begin{tikzcd}
& \begin{array}{l} (FW_k)_{k\in K}, \\ F\underline{\mathsf{C}}((X_i)_{i\in I};Z) \end{array} \arrow[dl, "(Ff_i)_{i\in I},1"'] \arrow[dr, "(1)_K,F\underline{\mathsf{C}}((f_i)_{i\in I};1)"] \arrow[dd, "(1)_K,\underline{F}_{(X_i);Z}"] & \\
\begin{array}{l} (FX_i)_{i\in I}, \\ F\underline{\mathsf{C}}((X_i)_{i\in I};Z) \end{array} \arrow[dd, "(1)_I,\underline{F}_{(X_i);Z}"] \arrow[dddr, bend right=60, "F\,\mathrm{ev}^{\mathsf{C}}_{(X_i);Z}"'] & & \begin{array}{l} (FW_k)_{k\in K}, \\ F\underline{\mathsf{C}}((W_k)_{k\in K};Z) \end{array} \arrow[dd, "(1)_K,\underline{F}_{(W_k);Z}"'] \arrow[dddl, bend left=60, "F\,\mathrm{ev}^{\mathsf{C}}_{(W_k);Z}"] \\
& \begin{array}{l} (FW_k)_{k\in K}, \\ \underline{\mathsf{D}}((FX_i)_{i\in I};FZ) \end{array} \arrow[dl, "(Ff_i)_{i\in I},1"'] \arrow[dr, "(1)_K,\underline{\mathsf{D}}((Ff_i)_{i\in I};1)"] & \\
\begin{array}{l} (FX_i)_{i\in I}, \\ \underline{\mathsf{D}}((FX_i)_{i\in I};FZ) \end{array} \arrow[dr, "\mathrm{ev}^{\mathsf{D}}_{(FX_i);FZ}"] & & \begin{array}{l} (FW_k)_{k\in K}, \\ \underline{\mathsf{D}}((FW_k)_{k\in K};FZ) \end{array} \arrow[dl, "\mathrm{ev}^{\mathsf{D}}_{(FW_k);FZ}"'] \\
& FZ &
\end{tikzcd}$$

(b)

DIAGRAM 1.9.

Proof. The claim follows from the diagram

$$\begin{tikzcd}
\begin{array}{l}(FX_i)_{i\in I},\\ \big(F\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J}\end{array} \arrow[r, "(1)_I{,}F\underline{\mathsf{C}}(\phi;f)"] \arrow[d, "(1)_I{,}(\underline{F}_{(X_i);Y_j})_{j\in J}"] \arrow[dd, bend right=60, "(F\,\mathrm{ev}^{\mathsf{C}}_{(X_i);Y_j})_{j\in J}"'] & \begin{array}{l}(FX_i)_{i\in I},\\ F\underline{\mathsf{C}}((X_i)_{i\in I};Z)\end{array} \arrow[d, "(1)_I{,}\underline{F}_{(X_i);Z}"'] \arrow[dd, bend left=60, "F\,\mathrm{ev}^{\mathsf{C}}_{(X_i);Z}"] \\
\begin{array}{l}(FX_i)_{i\in I},\\ \big(\underline{\mathsf{D}}((FX_i)_{i\in\phi^{-1}j};FY_j)\big)_{j\in J}\end{array} \arrow[r, "(1)_I{,}\underline{\mathsf{D}}(\phi;Ff)"] \arrow[d, "(\mathrm{ev}^{\mathsf{D}}_{(X_i);Y_j})_{j\in J}"] & \begin{array}{l}(FX_i)_{i\in I},\\ \underline{\mathsf{D}}((FX_i)_{i\in I};FZ)\end{array} \arrow[d, "\mathrm{ev}^{\mathsf{D}}_{(X_i);Z}"'] \\
(FY_j)_{j\in J} \arrow[r, "Ff"] & FZ
\end{tikzcd}$$

The lower square is the definition of the morphism $\underline{\mathsf{D}}(\phi;Ff)$. The exterior commutes by the definition of $\underline{\mathsf{C}}(\phi;f)$ and since F preserves composition. The triangles are commutative by the definition of closing transformation. ☐

1.3.32. Lemma. *Let* $\psi : K \to I$ *be a map in* $\operatorname{Mor}\mathcal{S}$. *Let* $f_i : (W_k)_{k\in\psi^{-1}i} \to X_i$, $i \in I$, *be morphisms in* C, *and let* $F : \mathsf{C} \to \mathsf{D}$ *be a symmetric multifunctor. The following diagram commutes:*

$$\begin{tikzcd}
F\underline{\mathsf{C}}((X_i)_{i\in I};Z) \arrow[r, "F\underline{\mathsf{C}}((f_i)_{i\in I};1)"] \arrow[d, "\underline{F}_{(X_i);Z}"'] & F\underline{\mathsf{C}}((W_k)_{k\in K};Z) \arrow[d, "\underline{F}_{(W_k);Z}"] \\
\underline{\mathsf{D}}((FX_i)_{i\in I};FZ) \arrow[r, "\underline{\mathsf{D}}((Ff_i)_{i\in I};1)"] & \underline{\mathsf{D}}((FW_k)_{k\in K};FZ)
\end{tikzcd} \tag{1.3.16}$$

Proof. The claim follows from Diagram 1.9(b). The lower diamond is the definition of the morphism $\underline{\mathsf{D}}((Ff_i)_{i\in I};1)$. The exterior is commutative by the definition of $\underline{\mathsf{C}}((f_i)_{i\in I};1)$ and since F preserves composition. The triangles commute by the definition of closing transformation. ☐

1.3.33. Lemma. *Given a multinatural transformation* $\nu : F \to G : \mathsf{C} \to \mathsf{D}$, *the diagram*

$$\begin{tikzcd}
F\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[rr, "\underline{F}_{(X_i);Y}"] \arrow[d, "\nu_{\underline{\mathsf{C}}((X_i)_{i\in I};Y)}"'] & & \underline{\mathsf{D}}((FX_i)_{i\in I};FY) \arrow[d, "\underline{\mathsf{D}}(\triangleright;\nu_Y)"] \\
G\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[r, "\underline{G}_{(X_i);Y}"] & \underline{\mathsf{D}}((GX_i)_{\in I};GY) \arrow[r, "\underline{\mathsf{D}}((\nu_{X_i})_{i\in I};1)"] & \underline{\mathsf{D}}((FX_i)_{i\in I};GY)
\end{tikzcd} \tag{1.3.17}$$

is commutative.

Proof. The claim follows from Diagram 1.10. Its exterior expresses the multinaturality of ν. The square is the definition of $\underline{\mathsf{D}}((\nu_{X_i})_{i\in I};1)$. The right diamond is the definition of $\underline{\mathsf{D}}(\triangleright;\nu_Y)$. The triangles commute by the definition of closing transformation. ☐

1.3.34. Lemma. *Let* C, D, E *be closed symmetric multicategories. Let* $\mathsf{C} \xrightarrow{F} \mathsf{D} \xrightarrow{G} \mathsf{E}$ *be symmetric multifunctors. Then*

$$\underline{G\circ F} = \big[GF\underline{\mathsf{C}}((X_i)_{i\in I};Y) \xrightarrow{G\underline{F}_{(X_i);Y}} G\underline{\mathsf{D}}((FX_i)_{i\in I};FY) \xrightarrow{\underline{G}_{(FX_i);FY}} \underline{\mathsf{E}}((GFX_i)_{i\in I};GFY)\big].$$

$$\begin{tikzcd}[column sep=small]
 & & \begin{array}{c}(FX_i)_{i\in I}\\ F\underline{\mathsf{C}}((X_i)_{i\in I};Y)\end{array} \arrow[dll, "{(1)_I,\nu_{\underline{\mathsf{C}}((X_i)_{i\in I};Y)}}"'] \arrow[dr, "{(1)_I,\underline{F}_{(X_i);Y}}"] \arrow[dddr, bend left, "F\,\mathrm{ev}^{\mathsf{C}}_{(X_i);Y}"] & \\
\begin{array}{c}(FX_i)_{i\in I}\\ G\underline{\mathsf{C}}((X_i)_{i\in I};Y)\end{array} \arrow[dd, "{(\nu_{X_i})_{i\in I},1}"'] \arrow[dr, "{(1)_I,\underline{G}_{(X_i);Y}}"] & & & \begin{array}{c}(FX_i)_{i\in I}\\ \underline{\mathsf{D}}((FX_i)_{i\in I};FY)\end{array} \arrow[dl, "{(1)_I,\underline{\mathsf{D}}(\triangleright;\nu_Y)}"'] \arrow[dd, "\mathrm{ev}^{\mathsf{D}}_{(FX_i);FY}"'] \\
 & \begin{array}{c}(FX_i)_{i\in I},\\ \underline{\mathsf{D}}((GX_i)_{i\in I};GY)\end{array} \arrow[r, "{(1)_I,\underline{\mathsf{D}}((\nu_{X_i})_{i\in I};1)}"] \arrow[dd, "{(\nu_{X_i})_{i\in I},1}"] & \begin{array}{c}(FX_i)_{i\in I},\\ \underline{\mathsf{D}}((FX_i)_{i\in I};GY)\end{array} \arrow[dd, "\mathrm{ev}^{\mathsf{D}}_{(FX_i);GY}"'] & \\
\begin{array}{c}(GX_i)_{i\in I},\\ G\underline{\mathsf{C}}((X_i)_{i\in I};Y)\end{array} \arrow[dr, "{(1)_I,\underline{G}_{(X_i);Y}}"] \arrow[drr, bend right, "G\,\mathrm{ev}^{\mathsf{C}}_{(X_i);Y}"'] & & & FY \arrow[dl, "\nu_Y"] \\
 & \begin{array}{c}(GX_i)_{i\in I},\\ \underline{\mathsf{D}}((GX_i)_{i\in I};GY)\end{array} \arrow[r, "\mathrm{ev}^{\mathsf{D}}_{(GX_i);GY}"] & GY &
\end{tikzcd}$$

Diagram 1.10.

Proof. This follows from the commutative diagram

$$\begin{tikzcd}
(GFX_i)_{i\in I}, G\underline{\mathsf{D}}((FX_i)_{i\in I};FY) \arrow[r, "{(1)_I,\underline{G}_{(FX_i);FY}}"] \arrow[dr, "G\,\mathrm{ev}^{\mathsf{D}}_{(FX_i);FY}"] & (GFX_i)_{i\in I}, \underline{\mathsf{E}}((GFX_i)_{i\in I};GFY) \arrow[d, "\mathrm{ev}^{\mathsf{E}}_{(GFX_i);GFY}"] \\
(GFX_i)_{i\in I}, GF\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[u, "{(1)_I,G\underline{F}_{(X_i);Y}}"] \arrow[r, "GF\,\mathrm{ev}^{\mathsf{C}}_{(X_i);Y}"] & GFY
\end{tikzcd}$$

The upper triangle is the definition of $\underline{G}_{(FX_i);FY}$, the lower triangle commutes by the definition of $\underline{F}_{(X_i);Y}$ and since G preserves composition. □

1.3.35. Example. Let $\mathcal{C} = (\mathcal{C}, \otimes^I, \lambda^f)$ be a symmetric Monoidal category. Then for each $J \in \operatorname{Ob} \mathcal{O}$ the category $\mathcal{C}^J$ has a natural symmetric Monoidal structure $(\mathcal{C}^J, \otimes^I_{\mathcal{C}^J}, \lambda^f_{\mathcal{C}^J})$. Here

$$\otimes^I_{\mathcal{C}^J} = \big[(\mathcal{C}^J)^I \xrightarrow{\sim} (\mathcal{C}^I)^J \xrightarrow{(\otimes^I)^J} \mathcal{C}^J\big],$$

and for a map $f : I \to K$

$$\begin{tikzcd}
 & \prod_{k\in K}(\mathcal{C}^J)^{f^{-1}k} \arrow[rr, "\sim"] & & \prod_{k\in K}(\mathcal{C}^{f^{-1}k})^J \arrow[rr, "\prod_{k\in K}(\otimes^{f^{-1}k})^J"] \arrow[d, "\wr"'] & & \prod_{k\in K}\mathcal{C}^J \arrow[r, "\sim"] & (\mathcal{C}^J)^K \arrow[d, "\wr"] \\
\lambda^f_{\mathcal{C}^J} = & & = & \left(\prod_{k\in K}\mathcal{C}^{f^{-1}k}\right)^J \arrow[rrr, "(\prod_{k\in K}\otimes^{f^{-1}k})^J"] & & & (\mathcal{C}^K)^J \arrow[d, "(\otimes^K)^J"] \\
 & (\mathcal{C}^J)^I \arrow[uu, "\wr"] \arrow[rr, "\sim"] & & (\mathcal{C}^I)^J \arrow[u, "\wr"] \arrow[urrr, "(\otimes^f)^J"] \arrow[rrr, "(\otimes^I)^J"'] & & & \mathcal{C}^J
\arrow[from=2-5, to=3-5, phantom, "="]
\arrow[from=3-6, to=2-7, Rightarrow, "(\lambda^f)^J"']
\end{tikzcd}$$

Equation (1.1.1) holds true due to the coherence principle of Lemma 1.1.7 and Remark 1.1.8.

The functor $\otimes^J : \mathcal{C}^J \to \mathcal{C}$ equipped with natural isomorphism

$$\begin{tikzcd}
 & \mathcal{C}^{I\times J} \arrow[dd, "\wr"'] \arrow[ddrr, "\otimes^{I\times J}"] & (\mathcal{C}^J)^I \arrow[l, "\sim"'] \arrow[r, "(\otimes^J)^I"] & \mathcal{C}^I \arrow[dd, "\otimes^I"] \\
\sigma_{(12)} = & & & \\
 & (\mathcal{C}^I)^J \arrow[r, "(\otimes^I)^J"] & \mathcal{C}^J \arrow[r, "\otimes^J"] & \mathcal{C}
\arrow[from=1-4, to=2-3, Rightarrow, "(\lambda^{\mathrm{pr}_1:I\times J\to I})^{-1}"']
\arrow[from=2-3, to=3-2, Rightarrow, "\lambda^{\mathrm{pr}_2:I\times J\to J}"']
\end{tikzcd}$$

is a symmetric Monoidal functor. Explicitly, $\sigma_{(12)}$ is given by (1.1.4). Equation (1.1.2) holds due to the mentioned coherence principle.

Assume that $\mathcal{C}$ is closed, then $\mathcal{C}^J$ is closed as well. Thus the symmetric Monoidal functor $(\otimes^J, \sigma_{(12)}) : (\mathcal{C}^J, \otimes^I_{\mathcal{C}^J}, \lambda^f_{\mathcal{C}^J}) \to (\mathcal{C}, \otimes^I, \lambda^f)$ determines the closing transformation

$$\underline{\otimes}^J : \otimes^{j\in J}\underline{\mathcal{C}}(X_j, Y_j) \to \underline{\mathcal{C}}(\otimes^{j\in J}X_j, \otimes^{j\in J}Y_j),$$

which is the only solution of the equation

$$\begin{tikzcd}
(\otimes^{j\in J}X_j) \otimes (\otimes^{j\in J}\underline{\mathcal{C}}(X_j, Y_j)) \arrow[r, "1\otimes\underline{\otimes}^J"] \arrow[d, "\sigma_{(12)}"'] & (\otimes^{j\in J}X_j) \otimes \underline{\mathcal{C}}(\otimes^{j\in J}X_j, \otimes^{j\in J}Y_j) \arrow[d, "\mathrm{ev}^{\mathcal{C}}"] \\
\otimes^{j\in J}(X_j \otimes \underline{\mathcal{C}}(X_j, Y_j)) \arrow[r, "\otimes^J \mathrm{ev}^{\mathcal{C}}"] & \otimes^{j\in J}Y_j
\end{tikzcd} \tag{1.3.18}$$

These transformations turn $\underline{\mathcal{C}}$ into a symmetric Monoidal $\mathcal{C}$-category. For each $f : I \to J$, the isomorphism $\lambda^f_{\underline{\mathcal{C}}} : \mathbb{1}_{\mathcal{C}} \to \underline{\mathcal{C}}(\otimes^{i\in I}X_i, \otimes^{j\in J} \otimes^{i\in f^{-1}j} X_i)$ is the morphism $\dot{\lambda}^f$. Equation (1.1.5) for $\lambda^f_{\underline{\mathcal{C}}}$ follows from similar equation (1.1.1) for λ^f.

CHAPTER 2

Serre functors for enriched categories

The notion of Serre functor was introduced by Bondal and Kapranov and motivated by the Serre-Grothendieck duality for coherent sheaves on a smooth projective variety [**4**]. It proved useful in many other contexts, see e.g. Bondal and Orlov [**6**], Kapranov [**24**], Reiten and van den Bergh [**47**], Mazorchuk and Stroppel [**45**]. Serre functors play an important role in the theory of cluster categories; in fact, the notion of Serre functor enters the very definition of cluster category, see e.g. Caldero and Keller [**9**].

We extend the definition of Serre functor to enriched categories. The motivation here is as follows. We will see in Section 3.4 that unital A_∞-categories form a $\Bbbk$-$\mathbf{Cat}$-multicategory A_∞^u, and that there is symmetric $\Bbbk$-$\mathbf{Cat}$-multifunctor k from A_∞^u to the $\Bbbk$-$\mathbf{Cat}$-multicategory of categories enriched in $\mathcal{K}$, the homotopy category of complexes of $\Bbbk$-modules. On the one hand, k can be thought as a kind of forgetful multifunctor since it neglects higher homotopies. On the other hand, it forgets not too much: we will see, for example, that it reflects isomorphisms of A_∞-functors and A_∞-equivalences. It is therefore natural to expect that a Serre A_∞-functor in a unital A_∞-category $\mathcal{A}$, whatever it is, should induce a Serre $\mathcal{K}$-functor in the $\mathcal{K}$-category $\mathsf{k}\mathcal{A}$. This motivates us to introduce Serre $\mathcal{K}$-functors. In fact, we define Serre $\mathcal{V}$-functors for arbitrary $\mathcal{V}$-categories, where $\mathcal{V}$ is a closed symmetric Monoidal category. Taking $\mathcal{V} = \Bbbk$-$\mathbf{Mod}$, the category of $\Bbbk$-modules, we recover the definition of ordinary Serre functor. There is also another reason for such generality. Namely, we would like to find conditions that assure the existence of a Serre $\mathcal{K}$-functor in a $\mathcal{K}$-category $\mathcal{C}$ provided that an ordinary Serre functor in the 0^{th} cohomology $H^0(\mathcal{C})$ exists. It is hard to do directly, that is why we introduce an intermediate graded category $H^\bullet\mathcal{C}$, the full cohomology. We prove that if a graded category $\mathcal{C}$ is closed under shifts and its 0^{th} part $\mathcal{C}^0$ admits an ordinary Serre functor, then $\mathcal{C}$ admits a graded Serre functor, provided that the ground ring $\Bbbk$ is a field. Furthermore, we prove that if $\Bbbk$ is a field, then a $\mathcal{K}$-category $\mathcal{C}$ admits a Serre $\mathcal{K}$-functor if and only if $H^\bullet(\mathcal{C})$ admits a graded Serre functor.

The contents of the chapter can be found in [**41**]. The idea to introduce Serre functors for enriched categories as a bridge between Serre A_∞-functors and ordinary Serre functors is due to the author. Section 2.2.10 was suggested by Volodymyr Lyubashenko. It was also his idea that a graded category may possess a graded Serre functor as soon as it is closed under shifts and its 0^{th} component admits an ordinary Serre functor. The remaining results have been proven by the author.

2.1. Preliminaries on enriched categories

Let $\mathcal{V} = (\mathcal{V}, \otimes^I, \lambda^f)$ be a closed symmetric Monoidal category. Its unit object is denoted by $\mathbb{1}$. The main examples we are interested in are $\mathcal{V} = \mathcal{K}$, the homotopy category of complexes of $\Bbbk$-modules, and $\mathcal{V} = \mathbf{gr}$, the category of graded $\Bbbk$-modules. To simplify the notation, we often ignore isomorphisms λ^f for order-preserving maps f and work with the category $\mathcal{V}$ as if it were strict. This is justified by the coherence theorem, see e.g.

[**37**, Theorem 1.2.7]. Let $c = \lambda^{\mathsf{X}} : X \otimes Y \to Y \otimes X$ denote the symmetry of $\mathcal{V}$, where $\mathsf{X} = (12) : \mathbf{2} \to \mathbf{2}$. For a permutation $\pi \in \mathfrak{S}_n$, denote by

$$\pi_c : X_1 \otimes \cdots \otimes X_n \to X_{\pi^{-1}(1)} \otimes \cdots \otimes X_{\pi^{-1}(n)}$$

the action of π in tensor products via the symmetry c. In particular, the symmetry c coincides with the morphism $(12)_c : X \otimes Y \to Y \otimes X$. Sometimes, when the permutation of factors reads clearly, we write simply perm for the corresponding canonical isomorphism. We stick with the notation of Section 1.3.10. Evaluation and coevaluation morphisms are denoted by $\mathrm{ev}^{\mathcal{V}}$ and $\mathrm{coev}^{\mathcal{V}}$ respectively. As usual, $\underline{\mathcal{V}}$ denotes the $\mathcal{V}$-category arising from the closed structure of $\mathcal{V}$. The category of unital (resp. non-unital) $\mathcal{V}$-categories is denoted $\mathcal{V}$-$\mathbf{Cat}$ (resp. $\mathcal{V}$-$\mathbf{Cat}^{nu}$). We refer the reader to [**29**, Chapter 1] for the basic theory of enriched categories.

2.1.1. Opposite $\mathcal{V}$-categories. Let $\mathcal{A}$ be a $\mathcal{V}$-category, not necessarily unital. Its opposite $\mathcal{A}^{\mathrm{op}}$ is defined in the standard way. Namely, $\mathrm{Ob}\,\mathcal{A}^{\mathrm{op}} = \mathrm{Ob}\,\mathcal{A}$, and for each pair of objects $X, Y \in \mathrm{Ob}\,\mathcal{A}$, $\mathcal{A}^{\mathrm{op}}(X,Y) = \mathcal{A}(Y,X)$. Composition in $\mathcal{A}^{\mathrm{op}}$ is given by

$$\mu_{\mathcal{A}^{\mathrm{op}}} = \big[\mathcal{A}^{\mathrm{op}}(X,Y) \otimes \mathcal{A}^{\mathrm{op}}(Y,Z) = \mathcal{A}(Y,X) \otimes \mathcal{A}(Z,Y) \xrightarrow{c} \mathcal{A}(Z,Y) \otimes \mathcal{A}(Y,X) \xrightarrow{\mu_{\mathcal{A}}} \mathcal{A}(Z,X) = \mathcal{A}^{\mathrm{op}}(X,Z)\big].$$

More generally, for each $n \geqslant 1$, the iterated n-ary composition in $\mathcal{A}^{\mathrm{op}}$ is

$$\mu^{\mathbf{n}}_{\mathcal{A}^{\mathrm{op}}} = \big[\otimes^{i\in\mathbf{n}} \mathcal{A}^{\mathrm{op}}(X_{i-1}, X_i) = \otimes^{i\in\mathbf{n}} \mathcal{A}(X_i, X_{i-1}) \xrightarrow{\omega^0_c} \otimes^{i\in\mathbf{n}} \mathcal{A}(X_{n-i+1}, X_{n-i}) \xrightarrow{\mu^{\mathbf{n}}_{\mathcal{A}}} \mathcal{A}(X_n, X_0) = \mathcal{A}^{\mathrm{op}}(X_0, X_n)\big], \quad (2.1.1)$$

where the permutation $\omega^0 = \left(\begin{smallmatrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \end{smallmatrix}\right)$ is the longest element of $\mathfrak{S}_n$. Note that if $\mathcal{A}$ is unital, then so is $\mathcal{A}^{\mathrm{op}}$, with the same identity morphisms.

Let $f : \mathcal{A} \to \mathcal{B}$ be a $\mathcal{V}$-functor, not necessarily unital. It gives rise to a $\mathcal{V}$-functor $f^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ with $\mathrm{Ob}\, f^{\mathrm{op}} = \mathrm{Ob}\, f : \mathrm{Ob}\,\mathcal{A} \to \mathrm{Ob}\,\mathcal{B}$, and

$$f^{\mathrm{op}}_{X,Y} = f_{Y,X} : \mathcal{A}^{\mathrm{op}}(X,Y) = \mathcal{A}(Y,X) \to \mathcal{B}(Yf, Xf) = \mathcal{B}^{\mathrm{op}}(Xf, Yf), \quad X, Y \in \mathrm{Ob}\,\mathcal{A}.$$

Note that f^{op} is a unital $\mathcal{V}$-functor if so is f. Clearly, the correspondences $\mathcal{A} \mapsto \mathcal{A}^{\mathrm{op}}$, $f \mapsto f^{\mathrm{op}}$ define a functor $-^{\mathrm{op}} : \mathcal{V}\text{-}\mathbf{Cat}^{nu} \to \mathcal{V}\text{-}\mathbf{Cat}^{nu}$ which restricts to a functor $-^{\mathrm{op}} : \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Cat}$. The functor $-^{\mathrm{op}}$ is symmetric Monoidal. More precisely, for arbitrary $\mathcal{V}$-categories $\mathcal{A}_i$, $i \in \mathbf{n}$, the equation $\boxtimes^{i\in\mathbf{n}} \mathcal{A}_i^{\mathrm{op}} = (\boxtimes^{i\in\mathbf{n}} \mathcal{A}_i)^{\mathrm{op}}$ holds. Indeed, the sets of objects and objects of morphisms of both $\mathcal{V}$-categories coincide, and so do identity morphisms if the $\mathcal{V}$-categories $\mathcal{A}_i$, $i \in \mathbf{n}$, are unital. Composition in $\boxtimes^{i\in\mathbf{n}} \mathcal{A}_i^{\mathrm{op}}$ is given by

$$\mu_{\boxtimes^{i\in\mathbf{n}} \mathcal{A}_i^{\mathrm{op}}} = \big[\big(\otimes^{i\in\mathbf{n}} \mathcal{A}_i^{\mathrm{op}}(X_i, Y_i)\big) \otimes \big(\otimes^{i\in\mathbf{n}} \mathcal{A}_i^{\mathrm{op}}(Y_i, Z_i)\big) \xrightarrow{\sigma_{(12)}} \otimes^{i\in\mathbf{n}} (\mathcal{A}_i(Y_i, X_i) \otimes \mathcal{A}_i(Z_i, Y_i)) \xrightarrow{\otimes^{i\in\mathbf{n}} c} \otimes^{i\in\mathbf{n}} (\mathcal{A}_i(Z_i, Y_i) \otimes \mathcal{A}_i(Y_i, X_i)) \xrightarrow{\otimes^{i\in\mathbf{n}} \mu_{\mathcal{A}_i}} \otimes^{i\in\mathbf{n}} \mathcal{A}_i(Z_i, X_i) = \otimes^{i\in\mathbf{n}} \mathcal{A}_i^{\mathrm{op}}(X_i, Z_i)\big].$$

Composition in $(\boxtimes^{i\in\mathbf{n}} \mathcal{A}_i)^{\mathrm{op}}$ is given by

$$\mu_{(\boxtimes^{i\in\mathbf{n}} \mathcal{A}_i)^{\mathrm{op}}} = \big[(\boxtimes^{i\in\mathbf{n}} \mathcal{A}_i)^{\mathrm{op}}((X_i)_{i\in\mathbf{n}}, (Y_i)_{i\in\mathbf{n}}) \otimes (\boxtimes^{i\in\mathbf{n}} \mathcal{A}_i)^{\mathrm{op}}((Y_i)_{i\in\mathbf{n}}, (Z_i)_{i\in\mathbf{n}}) = \big(\otimes^{i\in\mathbf{n}} \mathcal{A}_i(Y_i, X_i)\big) \otimes \big(\otimes^{i\in\mathbf{n}} \mathcal{A}_i(Z_i, Y_i)\big) \xrightarrow{c} \big(\otimes^{i\in\mathbf{n}} \mathcal{A}_i(Z_i, Y_i)\big) \otimes \big(\otimes^{i\in\mathbf{n}} \mathcal{A}_i(Y_i, X_i)\big) \xrightarrow{\sigma_{(12)}} \otimes^{i\in\mathbf{n}} (\mathcal{A}_i(Z_i, Y_i) \otimes \mathcal{A}_i(Y_i, X_i)) \xrightarrow{\otimes^{i\in\mathbf{n}} \mu_{\mathcal{A}_i}} \otimes^{i\in\mathbf{n}} \mathcal{A}_i(Z_i, X_i) = \otimes^{i\in\mathbf{n}} \mathcal{A}_i^{\mathrm{op}}(X_i, Z_i)\big].$$

The equation $\mu_{\boxtimes^{i\in\mathbf{n}}\mathcal{A}_i^{\mathrm{op}}} = \mu_{(\boxtimes^{i\in\mathbf{n}}\mathcal{A}_i)^{\mathrm{op}}}$ follows from the following equation in $\mathcal{V}$:

$$
\begin{aligned}
&\big[\big(\otimes^{i\in\mathbf{n}}\mathcal{A}_i(Y_i,X_i)\big)\otimes\big(\otimes^{i\in\mathbf{n}}\mathcal{A}_i(Z_i,Y_i)\big)\xrightarrow{\sigma_{(12)}} \\
&\qquad\qquad \otimes^{i\in\mathbf{n}}\big(\mathcal{A}_i(Y_i,X_i)\otimes\mathcal{A}_i(Z_i,Y_i)\big)\xrightarrow{\otimes^{i\in\mathbf{n}}c}\otimes^{i\in\mathbf{n}}\big(\mathcal{A}_i(Z_i,Y_i)\otimes\mathcal{A}_i(Y_i,X_i)\big)\big] \\
&= \big[\big(\otimes^{i\in\mathbf{n}}\mathcal{A}_i(Y_i,X_i)\big)\otimes\big(\otimes^{i\in\mathbf{n}}\mathcal{A}_i(Z_i,Y_i)\big)\xrightarrow{c}\big(\otimes^{i\in\mathbf{n}}\mathcal{A}_i(Z_i,Y_i)\big)\otimes\big(\otimes^{i\in\mathbf{n}}\mathcal{A}_i(Y_i,X_i)\big) \\
&\qquad\qquad\qquad\qquad \xrightarrow{\sigma_{(12)}}\otimes^{i\in\mathbf{n}}\big(\mathcal{A}_i(Z_i,Y_i)\otimes\mathcal{A}_i(Y_i,X_i)\big)\big],
\end{aligned}
$$

which is a consequence of coherence principle of the Lemma 1.1.7 and Remark 1.1.8. Therefore, $-^{\mathrm{op}}$ induces a symmetric multifunctor $-^{\mathrm{op}} : \widehat{\mathcal{V}\text{-}\mathbf{Cat}}^{nu} \to \widehat{\mathcal{V}\text{-}\mathbf{Cat}}^{nu}$ which restricts to a symmetric multifunctor $-^{\mathrm{op}} : \widehat{\mathcal{V}\text{-}\mathbf{Cat}} \to \widehat{\mathcal{V}\text{-}\mathbf{Cat}}$.

2.1.2. Hom-**functor.** A $\mathcal{V}$-category $\mathcal{A}$ gives rise to a $\mathcal{V}$-functor $\mathrm{Hom}_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}}\boxtimes\mathcal{A}\to\underline{\mathcal{V}}$ which maps a pair of objects $(X,Y)\in\mathrm{Ob}\,\mathcal{A}\times\mathrm{Ob}\,\mathcal{A}$ to $\mathcal{A}(X,Y)\in\mathrm{Ob}\,\underline{\mathcal{V}}$, and whose action on morphisms is given by

$$
\begin{aligned}
&\mathrm{Hom}_{\mathcal{A}} = \big[\mathcal{A}^{\mathrm{op}}(X,Y)\otimes\mathcal{A}(U,V) = \mathcal{A}(Y,X)\otimes\mathcal{A}(U,V)\xrightarrow{\mathrm{coev}^{\mathcal{V}}} \\
&\underline{\mathcal{V}}(\mathcal{A}(X,U),\mathcal{A}(X,U)\otimes\mathcal{A}(Y,X)\otimes\mathcal{A}(U,V))\xrightarrow{\underline{\mathcal{V}}(1,(c\otimes1)\mu^{\mathbf{3}}_{\mathcal{A}})}\underline{\mathcal{V}}(\mathcal{A}(X,U),\mathcal{A}(Y,V))\big].
\end{aligned}
\tag{2.1.2}
$$

Equivalently, $\mathrm{Hom}_{\mathcal{A}}$ is found by closedness of $\mathcal{V}$ from the diagram

$$
\begin{tikzcd}
\mathcal{A}(X,U)\otimes\mathcal{A}(Y,X)\otimes\mathcal{A}(U,V) \arrow[r, "1\otimes\mathrm{Hom}_{\mathcal{A}}"] \arrow[d, "c\otimes1"'] & \mathcal{A}(X,U)\otimes\underline{\mathcal{V}}(\mathcal{A}(X,U),\mathcal{A}(Y,V)) \arrow[d, "\mathrm{ev}^{\mathcal{V}}"] \\
\mathcal{A}(Y,X)\otimes\mathcal{A}(X,U)\otimes\mathcal{A}(U,V) \arrow[r, "\mu^{\mathbf{3}}_{\mathcal{A}}"] & \mathcal{A}(Y,V)
\end{tikzcd}
\tag{2.1.3}
$$

2.1.3. Lemma. *Let $\mathcal{A}$ be a $\mathcal{V}$-category. Then*

$$
\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}} = \big[\mathcal{A}\boxtimes\mathcal{A}^{\mathrm{op}}\xrightarrow{c}\mathcal{A}^{\mathrm{op}}\boxtimes\mathcal{A}\xrightarrow{\mathrm{Hom}_{\mathcal{A}}}\underline{\mathcal{V}}\big].
$$

Proof. Using (2.1.1), we obtain:

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}} &= \big[\mathcal{A}(X,Y)\otimes\mathcal{A}^{\mathrm{op}}(U,V) = \mathcal{A}^{\mathrm{op}}(Y,X)\otimes\mathcal{A}^{\mathrm{op}}(U,V) \\
&\qquad \xrightarrow{\mathrm{coev}^{\mathcal{V}}}\underline{\mathcal{V}}(\mathcal{A}^{\mathrm{op}}(X,U),\mathcal{A}^{\mathrm{op}}(X,U)\otimes\mathcal{A}^{\mathrm{op}}(Y,X)\otimes\mathcal{A}^{\mathrm{op}}(U,V)) \\
&\qquad\qquad \xrightarrow{\underline{\mathcal{V}}(1,(c\otimes1)\mu^{\mathbf{3}}_{\mathcal{A}^{\mathrm{op}}})}\underline{\mathcal{V}}(\mathcal{A}^{\mathrm{op}}(X,U),\mathcal{A}^{\mathrm{op}}(Y,V))\big] \\
&= \big[\mathcal{A}(X,Y)\otimes\mathcal{A}(V,U)\xrightarrow{\mathrm{coev}^{\mathcal{V}}}\underline{\mathcal{V}}(\mathcal{A}(U,X),\mathcal{A}(U,X)\otimes\mathcal{A}(X,Y)\otimes\mathcal{A}(V,U)) \\
&\qquad\qquad \xrightarrow{\underline{\mathcal{V}}(1,(c\otimes1)\omega^0_c\mu^{\mathbf{3}}_{\mathcal{A}})}\underline{\mathcal{V}}(\mathcal{A}(U,X),\mathcal{A}(V,Y))\big],
\end{aligned}
$$

where $\omega^0 = (13) \in \mathfrak{S}_3$. Clearly, $(c\otimes 1)\omega^0_c = (1\otimes c)(c\otimes 1)$, therefore

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}} &= \big[\mathcal{A}(X,Y)\otimes\mathcal{A}(V,U) \xrightarrow{\mathrm{coev}^{\mathcal{V}}} \underline{\mathcal{V}}(\mathcal{A}(U,X),\mathcal{A}(U,X)\otimes\mathcal{A}(X,Y)\otimes\mathcal{A}(V,U)) \\
&\qquad\qquad\qquad \xrightarrow{\underline{\mathcal{V}}(1,(1\otimes c)(c\otimes 1)\mu^{\mathbf{3}}_{\mathcal{A}})} \underline{\mathcal{V}}(\mathcal{A}(U,X),\mathcal{A}(V,Y))\big] \\
&= \big[\mathcal{A}(X,Y)\otimes\mathcal{A}(V,U) \xrightarrow{c} \mathcal{A}(V,U)\otimes\mathcal{A}(X,Y) \\
&\quad \xrightarrow{\mathrm{coev}^{\mathcal{V}}} \underline{\mathcal{V}}(\mathcal{A}(U,X),\mathcal{A}(U,X)\otimes\mathcal{A}(V,U)\otimes\mathcal{A}(X,Y)) \\
&\qquad\qquad\qquad \xrightarrow{\underline{\mathcal{V}}(1,(c\otimes 1)\mu^{\mathbf{3}}_{\mathcal{A}})} \underline{\mathcal{V}}(\mathcal{A}(U,X),\mathcal{A}(V,Y))\big] \\
&= \big[\mathcal{A}(X,Y)\otimes\mathcal{A}(V,U) \xrightarrow{c} \mathcal{A}(V,U)\otimes\mathcal{A}(X,Y) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}} \underline{\mathcal{V}}(\mathcal{A}(U,X),\mathcal{A}(V,Y))\big].
\end{aligned}$$

The lemma is proven. □

An object X of $\mathcal{A}$ defines a $\mathcal{V}$-functor $X : \mathbb{1} \to \mathcal{A}$, $* \mapsto X$, $\mathbb{1}(*,*) = \mathbb{1} \xrightarrow{1^{\mathcal{A}}_X} \mathcal{A}(X,X)$, whose source $\mathbb{1}$ is a $\mathcal{V}$-category with one object $*$. This $\mathcal{V}$-category is a unit of tensor multiplication $\boxtimes$. The $\mathcal{V}$-functors $\mathcal{A}(-,Y) = \mathrm{Hom}_{\mathcal{A}}(-,Y) : \mathcal{A}^{\mathrm{op}} \to \underline{\mathcal{V}}$ and $\mathcal{A}(X,-) = \mathrm{Hom}_{\mathcal{A}}(X,-) : \mathcal{A} \to \underline{\mathcal{V}}$ are defined as follows:

$$\begin{aligned}
\mathcal{A}(-,Y) &= \big[\mathcal{A}^{\mathrm{op}} \xrightarrow{\sim} \mathcal{A}^{\mathrm{op}}\boxtimes\mathbb{1} \xrightarrow{1\boxtimes Y} \mathcal{A}^{\mathrm{op}}\boxtimes\mathcal{A} \xrightarrow{\mathrm{Hom}_{\mathcal{A}}} \underline{\mathcal{V}}\big], \\
\mathcal{A}(X,-) &= \big[\mathcal{A} \xrightarrow{\sim} \mathbb{1}\boxtimes\mathcal{A} \xrightarrow{X\boxtimes 1} \mathcal{A}^{\mathrm{op}}\boxtimes\mathcal{A} \xrightarrow{\mathrm{Hom}_{\mathcal{A}}} \underline{\mathcal{V}}\big].
\end{aligned}$$

Thus, the $\mathcal{V}$-functor $\mathcal{A}(-,Y)$ maps an object X to $\mathcal{A}(X,Y)$, and its action on morphisms is given by

$$\begin{aligned}
\mathcal{A}(-,Y) = \big[\mathcal{A}^{\mathrm{op}}(W,X) = \mathcal{A}(X,W) &\xrightarrow{\mathrm{coev}^{\mathcal{V}}} \underline{\mathcal{V}}(\mathcal{A}(W,Y),\mathcal{A}(W,Y)\otimes\mathcal{A}(X,W)) \\
&\qquad \xrightarrow{\underline{\mathcal{V}}(1,c\mu_{\mathcal{A}})} \underline{\mathcal{V}}(\mathcal{A}(W,Y),\mathcal{A}(X,Y))\big]. \qquad (2.1.4)
\end{aligned}$$

Similarly, the $\mathcal{V}$-functor $\mathcal{A}(X,-)$ maps an object Y to $\mathcal{A}(X,Y)$, and its action on morphisms is given by

$$\begin{aligned}
\mathcal{A}(X,-) = \big[\mathcal{A}(Y,Z) &\xrightarrow{\mathrm{coev}^{\mathcal{V}}} \underline{\mathcal{V}}(\mathcal{A}(X,Y),\mathcal{A}(X,Y)\otimes\mathcal{A}(Y,Z)) \\
&\qquad \xrightarrow{\underline{\mathcal{V}}(1,\mu_{\mathcal{A}})} \underline{\mathcal{V}}(\mathcal{A}(X,Y),\mathcal{A}(X,Z))\big]. \qquad (2.1.5)
\end{aligned}$$

2.1.4. Duality functor. The unit object $\mathbb{1}$ of $\mathcal{V}$ defines the duality $\mathcal{V}$-functor

$$\underline{\mathcal{V}}(-,\mathbb{1}) = \mathrm{Hom}_{\underline{\mathcal{V}}}(-,\mathbb{1}) : \underline{\mathcal{V}}^{\mathrm{op}} \to \underline{\mathcal{V}}.$$

The functor $\underline{\mathcal{V}}(-,\mathbb{1})$ maps an object M to its dual $\underline{\mathcal{V}}(M,\mathbb{1})$, and its action on morphisms is given by

$$\begin{aligned}
\underline{\mathcal{V}}(-,\mathbb{1}) = \big[\underline{\mathcal{V}}^{\mathrm{op}}(M,N) = \underline{\mathcal{V}}(N,M) &\xrightarrow{\mathrm{coev}^{\mathcal{V}}} \underline{\mathcal{V}}(\underline{\mathcal{V}}(M,\mathbb{1}),\underline{\mathcal{V}}(M,\mathbb{1})\otimes\underline{\mathcal{V}}(N,M)) \xrightarrow{\underline{\mathcal{V}}(1,c)} \\
\underline{\mathcal{V}}(\underline{\mathcal{V}}(M,\mathbb{1}),\underline{\mathcal{V}}(N,M)\otimes\underline{\mathcal{V}}(M,\mathbb{1})) &\xrightarrow{\underline{\mathcal{V}}(1,\mu_{\underline{\mathcal{V}}})} \underline{\mathcal{V}}(\underline{\mathcal{V}}(M,\mathbb{1}),\underline{\mathcal{V}}(N,\mathbb{1}))\big]. \qquad (2.1.6)
\end{aligned}$$

For each object M there is a natural morphism $e : M \to \underline{\mathcal{V}}(\underline{\mathcal{V}}(M,\mathbb{1}),\mathbb{1})$ which is a unique solution of the following equation in $\mathcal{V}$:

$$\begin{tikzcd}
\underline{\mathcal{V}}(M,\mathbb{1})\otimes M \arrow[r, "c"] \arrow[d, "1\otimes e"'] & M\otimes\underline{\mathcal{V}}(M,\mathbb{1}) \arrow[d, "\mathrm{ev}^{\mathcal{V}}"] \\
\underline{\mathcal{V}}(M,\mathbb{1})\otimes\underline{\mathcal{V}}(\underline{\mathcal{V}}(M,\mathbb{1}),\mathbb{1}) \arrow[r, "\mathrm{ev}^{\mathcal{V}}"] & \mathbb{1}
\end{tikzcd}$$

Explicitly,

$$e = \big[M \xrightarrow{\mathrm{coev}^{\mathcal{V}}} \underline{\mathcal{V}}(\underline{\mathcal{V}}(M,\mathbb{1}),\underline{\mathcal{V}}(M,\mathbb{1})\otimes M) \xrightarrow{\underline{\mathcal{V}}(1,c)} \underline{\mathcal{V}}(\underline{\mathcal{V}}(M,\mathbb{1}),M\otimes\underline{\mathcal{V}}(M,\mathbb{1})) \xrightarrow{\underline{\mathcal{V}}(1,\mathrm{ev}^{\mathcal{V}})} \underline{\mathcal{V}}(\underline{\mathcal{V}}(M,\mathbb{1}),\mathbb{1})\big].$$

An object M is *reflexive* if e is an isomorphism in $\mathcal{V}$.

2.1.5. Representability. Let us state for the record the following proposition.

2.1.6. Proposition (Weak Yoneda Lemma). *Let $F : \mathcal{A} \to \underline{\mathcal{V}}$ be a $\mathcal{V}$-functor, X an object of $\mathcal{A}$. There is a bijection between elements of $F(X)$,* i.e., *morphisms $t : \mathbb{1} \to F(X)$, and $\mathcal{V}$-natural transformations $\mathcal{A}(X,-) \to F : \mathcal{A} \to \underline{\mathcal{V}}$ defined as follows: with an element $t : \mathbb{1} \to F(X)$ a $\mathcal{V}$-natural transformation is associated whose components are given by*

$$\mathcal{A}(X,Z) \cong \mathbb{1}\otimes\mathcal{A}(X,Z) \xrightarrow{t\otimes F_{X,Z}} F(X)\otimes\underline{\mathcal{V}}(F(X),F(Z)) \xrightarrow{\mathrm{ev}^{\mathcal{V}}} F(Z), \quad Z \in \mathrm{Ob}\,\mathcal{A}.$$

In particular, F is representable if and only if there is an object $X \in \mathrm{Ob}\,\mathcal{A}$ and an element $t : \mathbb{1} \to F(X)$ such that for each object $Z \in \mathrm{Ob}\,\mathcal{A}$ the above composite is invertible.

Proof. Standard, see [**29**, Section 1.9]. □

2.2. Basic properties of Serre functors

Let $\mathcal{C}$ be a $\mathcal{V}$-category, $S : \mathcal{C} \to \mathcal{C}$ a $\mathcal{V}$-functor. Consider a $\mathcal{V}$-natural transformation ψ as in the diagram below:

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}}\boxtimes\mathcal{C} \arrow[r, "1\boxtimes S"] \arrow[d, "\mathrm{Hom}^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}}"'] & \mathcal{C}^{\mathrm{op}}\boxtimes\mathcal{C} \arrow[d, "\mathrm{Hom}_{\mathcal{C}}"] \arrow[dl, Rightarrow, "\psi"'] \\
\underline{\mathcal{V}}^{\mathrm{op}} \arrow[r, "\underline{\mathcal{V}}(-{,}\mathbb{1})"] & \underline{\mathcal{V}}
\end{tikzcd} \tag{2.2.1}$$

The $\mathcal{V}$-natural transformation ψ is a collection of morphisms of $\mathcal{V}$

$$\psi_{X,Y} : \mathbb{1} \to \underline{\mathcal{V}}(\mathcal{C}(X,YS),\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1})), \quad X,Y \in \mathrm{Ob}\,\mathcal{C}.$$

Equivalently, ψ is given by a collection of morphisms $\psi_{X,Y} : \mathcal{C}(X,YS) \to \underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1})$ of $\mathcal{V}$, for $X,Y \in \mathrm{Ob}\,\mathcal{C}$. The $\mathcal{V}$-naturality of ψ may be verified variable-by-variable.

2.2.1. Definition. Let $\mathcal{C}$ be a $\mathcal{V}$-category. A $\mathcal{V}$-functor $S : \mathcal{C} \to \mathcal{C}$ is called a *right Serre $\mathcal{V}$-functor* if there exists a natural isomorphism ψ as in (2.2.1). If moreover S is a self–equivalence, it is called a *Serre $\mathcal{V}$-functor*.

Taking $\mathcal{V} = \Bbbk$-**Mod**, the category of $\Bbbk$-modules, we recover the definition of ordinary Serre functor. The terminology agrees with the conventions of Mazorchuk and Stroppel [**45**] and up to taking dual spaces with the terminology of Reiten and van den Bergh [**47**].

2.2.2. Lemma. *Let $S : \mathcal{C} \to \mathcal{C}$ be a $\mathcal{V}$-functor. Fix an object Y of $\mathcal{C}$. A collection of morphisms $(\psi_{X,Y} : \mathcal{C}(X,YS) \to \underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}))_{X\in\operatorname{Ob}\mathcal{C}}$ of $\mathcal{V}$ is $\mathcal{V}$-natural in X if and only if*

$$\psi_{X,Y} = \bigl[\mathcal{C}(X,YS) \xrightarrow{\mathrm{coev}^{\mathcal{V}}} \underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathcal{C}(Y,X)\otimes\mathcal{C}(X,YS)) \xrightarrow{\underline{\mathcal{V}}(1,\mu_{\mathcal{C}})} \underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathcal{C}(Y,YS)) \xrightarrow{\underline{\mathcal{V}}(1,\tau_Y)} \underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1})\bigr], \qquad (2.2.2)$$

where

$$\tau_Y = \bigl[\mathcal{C}(Y,YS) \xrightarrow{1^{\mathcal{C}}_Y\otimes 1} \mathcal{C}(Y,Y)\otimes\mathcal{C}(Y,YS) \xrightarrow{1\otimes\psi_{Y,Y}} \mathcal{C}(Y,Y)\otimes\underline{\mathcal{V}}(\mathcal{C}(Y,Y),\mathbb{1}) \xrightarrow{\mathrm{ev}^{\mathcal{V}}} \mathbb{1}\bigr]. \qquad (2.2.3)$$

The morphisms τ_Y are called *trace functionals* in the sequel.

Proof. The collection $(\psi_{X,Y})_{X\in\operatorname{Ob}\mathcal{C}}$ is a $\mathcal{V}$-natural transformation

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[rr, "\mathcal{C}(-,YS)"] \arrow[dr, "\mathcal{C}(Y,-)^{\mathrm{op}}"'] & {} \arrow[d, Rightarrow, "\psi_{-,Y}"] & \underline{\mathcal{V}} \\
& \underline{\mathcal{V}}^{\mathrm{op}} \arrow[ur, "\underline{\mathcal{V}}(-,\mathbb{1})"'] &
\end{tikzcd}$$

if the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C}(Z,X) \arrow[r, "\mathcal{C}(-,YS)"] \arrow[dd, "\mathcal{C}(Y,-)"'] & \underline{\mathcal{V}}(\mathcal{C}(X,YS),\mathcal{C}(Z,YS)) \arrow[d, "\underline{\mathcal{V}}(1,\psi_{Z,Y})"] \\
& \underline{\mathcal{V}}(\mathcal{C}(X,YS),\underline{\mathcal{V}}(\mathcal{C}(Y,Z),\mathbb{1})) \\
\underline{\mathcal{V}}(\mathcal{C}(Y,Z),\mathcal{C}(Y,X)) \arrow[r, "\underline{\mathcal{V}}(-,\mathbb{1})"] & \underline{\mathcal{V}}(\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}),\underline{\mathcal{V}}(\mathcal{C}(Y,Z),\mathbb{1})) \arrow[u, "\underline{\mathcal{V}}(\psi_{X,Y},1)"']
\end{tikzcd}$$

By the closedness of $\mathcal{V}$, this is equivalent to the commutativity of the exterior of the following diagram:

$$\begin{tikzcd}
\mathcal{C}(X,YS)\otimes\mathcal{C}(Z,X) \arrow[rr, "1\otimes\mathcal{C}(-,YS)"] \arrow[dr, "c"] \arrow[dd, "\psi_{X,Y}\otimes 1"'] & & \mathcal{C}(X,YS)\otimes\otimes\underline{\mathcal{V}}(\mathcal{C}(X,YS),\mathcal{C}(Z,YS)) \arrow[dd, "\mathrm{ev}^{\mathcal{V}}"] \\
& \mathcal{C}(Z,X)\otimes\mathcal{C}(X,YS) \arrow[dr, "\mu_{\mathcal{C}}"] \arrow[dd, "1\otimes\psi_{X,Y}"] & \\
\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1})\otimes\mathcal{C}(Z,X) \arrow[dr, "c"] \arrow[dd, "1\otimes\mathcal{C}(Y,-)"'] & & \mathcal{C}(Z,YS) \arrow[dddd, "\psi_{Z,Y}"] \\
& \mathcal{C}(Z,X)\otimes\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}) \arrow[dd, "\mathcal{C}(Y,-)\otimes 1"] & \\
\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1})\otimes\underline{\mathcal{V}}(\mathcal{C}(Y,Z),\mathcal{C}(Y,X)) \arrow[dr, "c"] \arrow[dd, "1\otimes\underline{\mathcal{V}}(-,\mathbb{1})"'] & & \\
& \underline{\mathcal{V}}(\mathcal{C}(Y,Z),\mathcal{C}(Y,X))\otimes\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}) \arrow[dr, "\mu_{\underline{\mathcal{V}}}"] & \\
\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1})\otimes\underline{\mathcal{V}}(\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}),\underline{\mathcal{V}}(\mathcal{C}(Y,Z),\mathbb{1})) \arrow[rr, "\mathrm{ev}^{\mathcal{V}}"] & & \underline{\mathcal{V}}(\mathcal{C}(Y,Z),\mathbb{1})
\end{tikzcd}$$

The right upper quadrilateral and the left lower quadrilateral commute by the definition of $\mathcal{C}(-,YS)$ and $\underline{\mathcal{V}}(-,\mathbb{1})$ respectively, see (2.1.4) and its particular case (2.1.6). Since c

is an isomorphism, the commutativity of the exterior is equivalent to the commutativity of the pentagon. Again, by closedness, this is equivalent to the commutativity of the exterior of the following diagram:

$$\begin{tikzcd}
\mathcal{C}(Y,Z)\otimes\mathcal{C}(Z,X)\otimes\mathcal{C}(X,YS) \arrow[r, "1\otimes\mu_{\mathcal{C}}"] \arrow[d, "1\otimes1\otimes\psi_{X,Y}"'] & \mathcal{C}(Y,Z)\otimes\mathcal{C}(Z,YS) \arrow[d, "1\otimes\psi_{Z,Y}"] \\
\mathcal{C}(Y,Z)\otimes\mathcal{C}(Z,X)\otimes\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}) \arrow[dd, "1\otimes\mathcal{C}(Y,-)\otimes1"'] \arrow[ddr, "\mu_{\mathcal{C}}\otimes1"] & \mathcal{C}(Y,Z)\otimes\underline{\mathcal{V}}(\mathcal{C}(Y,Z),\mathbb{1}) \arrow[d, "\mathrm{ev}^{\mathcal{V}}"] \\
 & \mathbb{1} \\
\mathcal{C}(Y,Z)\otimes\underline{\mathcal{V}}(\mathcal{C}(Y,Z),\mathcal{C}(Y,X))\otimes\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}) \arrow[r, "\mathrm{ev}^{\mathcal{V}}\otimes1"] & \mathcal{C}(Y,X)\otimes\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}) \arrow[u, "\mathrm{ev}^{\mathcal{V}}"']
\end{tikzcd}$$

The triangle commutes by definition of $\mathcal{C}(Y,-)$, see (2.1.5). It follows that the $\mathcal{V}$-naturality of $\psi_{-,Y}$ is equivalent to the commutativity of the hexagon:

$$\begin{tikzcd}
\mathcal{C}(Y,Z)\otimes\mathcal{C}(Z,X)\otimes\mathcal{C}(X,YS) \arrow[r, "1\otimes\mu_{\mathcal{C}}\psi_{Z,Y}"] \arrow[d, "\mu_{\mathcal{C}}\otimes\psi_{X,Y}"'] & \mathcal{C}(Y,Z)\otimes\underline{\mathcal{V}}(\mathcal{C}(Y,Z),\mathbb{1}) \arrow[d, "\mathrm{ev}^{\mathcal{V}}"] \\
\mathcal{C}(Y,X)\otimes\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}) \arrow[r, "\mathrm{ev}^{\mathcal{V}}"] & \mathbb{1}
\end{tikzcd} \tag{2.2.4}$$

Assume that $\psi_{-,Y}$ is $\mathcal{V}$-natural, so the above diagram commutes, and consider a particular case, $Z = Y$. Composing both paths of the diagram with the morphism $1^{\mathcal{C}}_Y\otimes1\otimes1 : \mathcal{C}(Y,X)\otimes\mathcal{C}(X,YS)\to\mathcal{C}(Y,Y)\otimes\mathcal{C}(Y,X)\otimes\mathcal{C}(X,YS)$, we obtain:

$$\begin{tikzcd}
\mathcal{C}(Y,X)\otimes\mathcal{C}(X,YS) \arrow[r, "\mu_{\mathcal{C}}"] \arrow[d, "1\otimes\psi_{X,Y}"'] & \mathcal{C}(Y,YS) \arrow[d, "\tau_Y"] \\
\mathcal{C}(Y,X)\otimes\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}) \arrow[r, "\mathrm{ev}^{\mathcal{V}}"] & \mathbb{1}
\end{tikzcd} \tag{2.2.5}$$

where τ_Y is given by expression (2.2.3). By closedness, the above equation admits a unique solution $\psi_{X,Y}$, namely (2.2.2).

Assume now that $\psi_{X,Y}$ is given by (2.2.2). Then (2.2.5) holds true. Plugging it into (2.2.4), whose commutativity is to be proven, we obtain the equation

$$\begin{tikzcd}
\mathcal{C}(Y,Z)\otimes\mathcal{C}(Z,X)\otimes\mathcal{C}(X,YS) \arrow[r, "1\otimes\mu_{\mathcal{C}}"] \arrow[d, "\mu_{\mathcal{C}}\otimes1"'] & \mathcal{C}(Y,Z)\otimes\mathcal{C}(Z,YS) \arrow[r, "\mu_{\mathcal{C}}"] & \mathcal{C}(Y,YS) \arrow[d, "\tau_Y"] \\
\mathcal{C}(Y,X)\otimes\mathcal{C}(X,YS) \arrow[r, "\mu_{\mathcal{C}}"] & \mathcal{C}(Y,YS) \arrow[r, "\tau_Y"] & \mathbb{1}
\end{tikzcd}$$

which holds true by the associativity of composition. □

2.2.3. Lemma. *Let $S : \mathcal{C}\to\mathcal{C}$ be a $\mathcal{V}$-functor. Fix an object X of $\mathcal{C}$. A collection of morphisms $(\psi_{X,Y} : \mathcal{C}(X,YS)\to\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}))_{Y\in\mathrm{Ob}\,\mathcal{C}}$ of $\mathcal{V}$ is $\mathcal{V}$-natural in Y if and only if* *for each* $Y \in \operatorname{Ob} \mathcal{C}$

$$\psi_{X,Y} = \big[\mathcal{C}(X, YS) \xrightarrow{\mathrm{coev}^{\mathcal{V}}} \underline{\mathcal{V}}(\mathcal{C}(Y, X), \mathcal{C}(Y, X) \otimes \mathcal{C}(X, YS)) \xrightarrow{\underline{\mathcal{V}}(1,S\otimes 1)} \\ \underline{\mathcal{V}}(\mathcal{C}(Y, X), \mathcal{C}(YS, XS) \otimes \mathcal{C}(X, YS)) \xrightarrow{\underline{\mathcal{V}}(1,c\mu_{\mathcal{C}})} \underline{\mathcal{V}}(\mathcal{C}(Y, X), \mathcal{C}(X, XS)) \\ \xrightarrow{\underline{\mathcal{V}}(1,\tau_X)} \underline{\mathcal{V}}(\mathcal{C}(Y, X), \mathbb{1})\big], \quad (2.2.6)$$

where τ_X *is given by* (2.2.3).

Proof. The $\mathcal{V}$-naturality of $\psi_{X,-}$ presented by the square

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "S"] \arrow[d, "\mathcal{C}(-,X)^{\mathrm{op}}"'] & \mathcal{C} \arrow[d, "\mathcal{C}(X,-)"] \arrow[dl, Rightarrow, "\psi_{X,-}"'] \\
\underline{\mathcal{V}}^{\mathrm{op}} \arrow[r, "\underline{\mathcal{V}}(-,\mathbb{1})"] & \underline{\mathcal{V}}
\end{tikzcd}$$

is expressed by the commutativity in $\mathcal{V}$ of the following diagram:

$$\begin{tikzcd}
\mathcal{C}(Y, Z) \arrow[r, "S"] \arrow[d, "\mathcal{C}(-,X)"'] & \mathcal{C}(YS, ZS) \arrow[d, "\mathcal{C}(X,-)"] \\
\underline{\mathcal{V}}(\mathcal{C}(Z, X), \mathcal{C}(Y, X)) \arrow[d, "\underline{\mathcal{V}}(-,\mathbb{1})"'] & \underline{\mathcal{V}}(\mathcal{C}(X, YS), \mathcal{C}(X, ZS)) \arrow[d, "\underline{\mathcal{V}}(1,\psi_{X,Z})"] \\
\underline{\mathcal{V}}(\underline{\mathcal{V}}(\mathcal{C}(Y, X), \mathbb{1}), \underline{\mathcal{V}}(\mathcal{C}(Z, X), \mathbb{1})) \arrow[r, "\underline{\mathcal{V}}(\psi_{X,Y},1)"] & \underline{\mathcal{V}}(\mathcal{C}(X, YS), \underline{\mathcal{V}}(\mathcal{C}(Z, X), \mathbb{1}))
\end{tikzcd}$$

By closedness, the latter is equivalent to the commutativity of the exterior of Diagram 2.1. Since c is an isomorphism, it follows that the polygon marked by $\boxed{*}$ must be commutative. By closedness, this is equivalent to the commutativity of the exterior of the following diagram:

$$\begin{tikzcd}
\mathcal{C}(Z, X) \otimes \mathcal{C}(X, YS) \otimes \mathcal{C}(Y, Z) \arrow[r, "1\otimes 1\otimes S"] \arrow[d, "1\otimes c"'] & \mathcal{C}(Z, X) \otimes \mathcal{C}(X, YS) \otimes \mathcal{C}(YS, ZS) \arrow[d, "1\otimes\mu_{\mathcal{C}}"] \\
\mathcal{C}(Z, X) \otimes \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, YS) \arrow[d, "1\otimes 1\otimes\psi_{X,Y}"'] & \mathcal{C}(Z, X) \otimes \mathcal{C}(X, ZS) \arrow[d, "1\otimes\psi_{X,Z}"] \\
\mathcal{C}(Z, X) \otimes \mathcal{C}(Y, Z) \otimes \underline{\mathcal{V}}(\mathcal{C}(Y, X), \mathbb{1}) \arrow[dd, "1\otimes\mathcal{C}(-,X)\otimes 1"'] \arrow[ddr, "c\mu_{\mathcal{C}}\otimes 1"] & \mathcal{C}(Z, X) \otimes \underline{\mathcal{V}}(\mathcal{C}(Z, X), \mathbb{1}) \arrow[d, "\mathrm{ev}^{\mathcal{V}}"] \\
 & \mathbb{1} \\
\mathcal{C}(Z, X) \otimes \underline{\mathcal{V}}(\mathcal{C}(Z, X), \mathcal{C}(Y, X)) \otimes \underline{\mathcal{V}}(\mathcal{C}(Y, X), \mathbb{1}) \arrow[r, "\mathrm{ev}^{\mathcal{V}}\otimes 1"] & \mathcal{C}(Y, X) \otimes \underline{\mathcal{V}}(\mathcal{C}(Y, X), \mathbb{1}) \arrow[u, "\mathrm{ev}^{\mathcal{V}}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}(X,YS) \otimes \mathcal{C}(Y,Z) \arrow[r, "1\otimes S"] \arrow[dd, "\psi_{X,Y}\otimes 1"'] \arrow[dr, "c"] & \mathcal{C}(X,YS) \otimes \mathcal{C}(YS,ZS) \arrow[r, "1\otimes\mathcal{C}(X,-)"] \arrow[dr, "\mu_{\mathcal{C}}"] & \mathcal{C}(X,YS) \otimes \underline{\mathcal{V}}(\mathcal{C}(X,YS),\mathcal{C}(X,ZS)) \arrow[d, "\mathrm{ev}^{\mathcal{V}}"] \\
 & \mathcal{C}(Y,Z) \otimes \mathcal{C}(X,YS) \arrow[dd, "1\otimes\psi_{X,Y}"] & \mathcal{C}(X,ZS) \arrow[ddddd, "\psi_{X,Z}"] \\
\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}) \otimes \mathcal{C}(Y,Z) \arrow[dd, "1\otimes\mathcal{C}(-,X)"'] \arrow[dr, "c"] & & \\
 & \mathcal{C}(Y,Z) \otimes \underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}) \qquad \boxed{*} \arrow[dd, "\mathcal{C}(-,X)\otimes 1"] & \\
\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}) \otimes \underline{\mathcal{V}}(\mathcal{C}(Z,X),\mathcal{C}(Y,X)) \arrow[dd, "1\otimes\underline{\mathcal{V}}(-,\mathbb{1})"'] \arrow[dr, "c"] & & \\
 & \underline{\mathcal{V}}(\mathcal{C}(Z,X),\mathcal{C}(Y,X)) \otimes \underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}) \arrow[dr, "\mu_{\underline{\mathcal{V}}}"] & \\
\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}) \otimes \underline{\mathcal{V}}(\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}),\underline{\mathcal{V}}(\mathcal{C}(Z,X),\mathbb{1})) \arrow[rr, "\mathrm{ev}^{\mathcal{V}}"] & & \underline{\mathcal{V}}(\mathcal{C}(Z,X),\mathbb{1})
\end{tikzcd}$$

DIAGRAM 2.1.

The triangle commutes by (2.1.4). Therefore, the remaining polygon is commutative as well:

$$\begin{tikzcd}
\mathcal{C}(Z,X)\otimes\mathcal{C}(X,YS)\otimes\mathcal{C}(Y,Z) \arrow[r, "1\otimes1\otimes S"] \arrow[d, "(123)_c"'] & \mathcal{C}(Z,X)\otimes\mathcal{C}(X,YS)\otimes\mathcal{C}(YS,ZS) \arrow[d, "1\otimes\mu_{\mathcal{C}}"] \\
\mathcal{C}(Y,Z)\otimes\mathcal{C}(Z,X)\otimes\mathcal{C}(X,YS) \arrow[d, "\mu_{\mathcal{C}}\otimes1"'] & \mathcal{C}(Z,X)\otimes\mathcal{C}(X,ZS) \arrow[d, "1\otimes\psi_{X,Z}"] \\
\mathcal{C}(Y,X)\otimes\mathcal{C}(X,YS) \arrow[d, "1\otimes\psi_{X,Y}"'] & \mathcal{C}(Z,X)\otimes\underline{\mathcal{V}}(\mathcal{C}(Z,X),\mathbb{1}) \arrow[d, "\mathrm{ev}^{\mathcal{V}}"] \\
\mathcal{C}(Y,X)\otimes\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}) \arrow[r, "\mathrm{ev}^{\mathcal{V}}"] & \mathbb{1}
\end{tikzcd} \tag{2.2.7}$$

Suppose that the collection of morphisms $(\psi_{X,Y}:\mathcal{C}(X,YS)\to\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}))_{Y\in\operatorname{Ob}\mathcal{C}}$ is natural in Y. Consider diagram (2.2.7) with $Z=X$. Composing both paths with the morphism $1^{\mathcal{C}}_X\otimes1\otimes1:\mathcal{C}(X,YS)\otimes\mathcal{C}(Y,X)\to\mathcal{C}(X,X)\otimes\mathcal{C}(X,YS)\otimes\mathcal{C}(Y,X)$ gives an equation:

$$\begin{tikzcd}
\mathcal{C}(Y,X)\otimes\mathcal{C}(X,YS) \arrow[r, "S\otimes1"] \arrow[d, "1\otimes\psi_{X,Y}"'] & \mathcal{C}(YS,XS)\otimes\mathcal{C}(X,YS) \arrow[r, "c\mu_{\mathcal{C}}"] & \mathcal{C}(X,XS) \arrow[d, "\tau_X"] \\
\mathcal{C}(Y,X)\otimes\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}) \arrow[rr, "\mathrm{ev}^{\mathcal{V}}"] & & \mathbb{1}
\end{tikzcd} \tag{2.2.8}$$

The only solution to the above equation is given by (2.2.6).

Conversely, suppose equation (2.2.8) holds. It suffices to prove that diagram (2.2.7) is commutative. Plugging in the expressions for $(1\otimes\psi_{X,Y})\,\mathrm{ev}^{\mathcal{V}}$ and $(1\otimes\psi_{X,Z})\,\mathrm{ev}^{\mathcal{V}}$ into (2.2.7), we obtain (cancelling a common permutation of the factors of the source object):

$$\begin{tikzcd}
\mathcal{C}(X,YS)\otimes\mathcal{C}(Y,Z)\otimes\mathcal{C}(Z,X) \arrow[r, "1\otimes S\otimes S"] \arrow[d, "1\otimes\mu_{\mathcal{C}}"'] & \mathcal{C}(X,YS)\otimes\mathcal{C}(YS,ZS)\otimes\mathcal{C}(ZS,XS) \arrow[d, "\mu_{\mathcal{C}}\otimes1"] \\
\mathcal{C}(X,YS)\otimes\mathcal{C}(Y,X) \arrow[d, "1\otimes S"'] & \mathcal{C}(X,ZS)\otimes\mathcal{C}(ZS,XS) \arrow[d, "\mu_{\mathcal{C}}"] \\
\mathcal{C}(X,YS)\otimes\mathcal{C}(YS,XS) \arrow[d, "\mu_{\mathcal{C}}"'] & \mathcal{C}(X,XS) \arrow[d, "\tau_X"] \\
\mathcal{C}(X,XS) \arrow[r, "\tau_X"] & \mathbb{1}
\end{tikzcd}$$

The commutativity of the diagram follows from the associativity of $\mu_{\mathcal{C}}$ and the fact that S is a $\mathcal{V}$-functor. The lemma is proven. □

2.2.4. Proposition. *Assume that $S : \mathcal{C} \to \mathcal{C}$ is a $\mathcal{V}$-functor, and ψ is a $\mathcal{V}$-natural transformation as in* (2.2.1)*. Then the following diagram commutes (in $\mathcal{V}$):*

$$\begin{tikzcd}
\mathcal{C}(Y,X) \arrow[r, "e"] \arrow[d, "S"'] & \underline{\mathcal{V}}(\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}),\mathbb{1}) \arrow[d, "\underline{\mathcal{V}}(\psi_{X,Y},\mathbb{1})"] \\
\mathcal{C}(YS,XS) \arrow[r, "\psi_{YS,X}"] & \underline{\mathcal{V}}(\mathcal{C}(X,YS),\mathbb{1})
\end{tikzcd}$$

In particular, if for each pair of objects $X, Y \in \operatorname{Ob}\mathcal{C}$ the object $\mathcal{C}(Y,X)$ is reflexive, and ψ is an isomorphism, then S is fully faithful.

Proof. By closedness, it suffices to prove the commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{C}(X,YS)\otimes\mathcal{C}(Y,X) \arrow[r, "1\otimes e"] \arrow[d, "1\otimes S"'] & \mathcal{C}(X,YS)\otimes\underline{\mathcal{V}}(\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}),\mathbb{1}) \arrow[d, "1\otimes\underline{\mathcal{V}}(\psi_{X,Y},\mathbb{1})"] \\
\mathcal{C}(X,YS)\otimes\mathcal{C}(YS,XS) \arrow[d, "1\otimes\psi_{YS,X}"'] & \mathcal{C}(X,YS)\otimes\underline{\mathcal{V}}(\mathcal{C}(X,YS),\mathbb{1}) \arrow[d, "\mathrm{ev}^{\mathcal{V}}"] \\
\mathcal{C}(X,YS)\otimes\underline{\mathcal{V}}(\mathcal{C}(X,YS),\mathbb{1}) \arrow[r, "\mathrm{ev}^{\mathcal{V}}"] & \mathbb{1}
\end{tikzcd}$$

Using (2.2.5) and the definition of e, the above diagram can be transformed as follows:

$$\begin{tikzcd}
\mathcal{C}(X,YS)\otimes\mathcal{C}(Y,X) \arrow[r, "c"] \arrow[d, "1\otimes S"'] & \mathcal{C}(Y,X)\otimes\mathcal{C}(X,YS) \arrow[d, "1\otimes\psi_{X,Y}"] \\
\mathcal{C}(X,YS)\otimes\mathcal{C}(YS,XS) \arrow[d, "\mu_{\mathcal{C}}"'] & \mathcal{C}(Y,X)\otimes\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}) \arrow[d, "\mathrm{ev}^{\mathcal{V}}"] \\
\mathcal{C}(X,XS) \arrow[r, "\tau_X"] & \mathbb{1}
\end{tikzcd}$$

It is commutative by (2.2.8). □

Proposition 2.2.4 implies that a right Serre functor is fully faithful if and only if $\mathcal{C}$ is Hom-*reflexive*, i.e., if $\mathcal{C}(X,Y)$ is a reflexive object of $\mathcal{V}$, for each pair of objects $X, Y \in \operatorname{Ob}\mathcal{C}$. If this is the case, a right Serre functor will be a Serre functor if and only if it is essentially surjective on objects. The most natural reason for Hom-reflexivity is, of course, $\Bbbk$ being a field. When $\Bbbk$ is a field, an object C of **gr** is reflexive if and only if all spaces C^n are finite-dimensional. The ring $\Bbbk$ being a field, the homology functor $H^\bullet : \mathcal{K} \to$ **gr** is an equivalence (see e.g. [**17**, Chapter III, § 2, Proposition 4]). Hence, an object C of $\mathcal{K}$ is reflexive iff all homology spaces H^nC are finite-dimensional. A projective module of finite rank over an arbitrary commutative ring $\Bbbk$ is reflexive as an object of a rigid monoidal category [**12**, Example 1.23]. Thus, an object C of **gr** whose components C^n are projective $\Bbbk$-modules of finite rank is reflexive.

2.2.5. Proposition. *Suppose $\mathcal{C}$ is a $\mathcal{V}$-category. There exists a right Serre $\mathcal{V}$-functor $S : \mathcal{C} \to \mathcal{C}$ if and only if for each object $Y \in \operatorname{Ob}\mathcal{C}$ the $\mathcal{V}$-functor*

$$\operatorname{Hom}_{\mathcal{C}}(Y,-)^{\mathrm{op}} \cdot \underline{\mathcal{V}}(-,\mathbb{1}) = \underline{\mathcal{V}}(\mathcal{C}(Y,-)^{\mathrm{op}},\mathbb{1}) : \mathcal{C}^{\mathrm{op}} \to \underline{\mathcal{V}}$$

is representable.

Proof. Standard, see [**29**, Section 1.10]. □

2.2.6. Commutation with equivalences. Let $\mathcal{C}$ and $\mathcal{C}'$ be $\mathcal{V}$-categories with right Serre functors $S : \mathcal{C} \to \mathcal{C}$ and $S' : \mathcal{C}' \to \mathcal{C}'$ respectively. Let ψ and ψ' be isomorphisms as in (2.2.1). For objects $Y \in \operatorname{Ob} \mathcal{C}$, $Z \in \operatorname{Ob} \mathcal{C}'$, define τ_Y, τ'_Z by (2.2.3). Let $T : \mathcal{C} \to \mathcal{C}'$ be a $\mathcal{V}$-functor, and suppose that T is fully faithful. Then there is a $\mathcal{V}$-natural transformation $\varkappa : ST \to TS'$ such that, for each object $Y \in \operatorname{Ob} \mathcal{C}$, the following equation holds:

$$\big[\mathcal{C}(Y,YS) \xrightarrow{T} \mathcal{C}'(YT,YST) \xrightarrow{\mathcal{C}'(YT,\varkappa)} \mathcal{C}'(YT,YTS') \xrightarrow{\tau'_{YT}} \mathbb{1}\big] = \tau_Y. \tag{2.2.9}$$

Indeed, the left hand side of equation (2.2.9) equals

$$\big[\mathcal{C}(Y,YS) \xrightarrow{T} \mathcal{C}'(YT,YST) \xrightarrow{1\otimes\varkappa_Y} \mathcal{C}'(YT,YST) \otimes \mathcal{C}'(YST,YTS') \\ \xrightarrow{\mu_{\mathcal{C}'}} \mathcal{C}'(YT,YTS') \xrightarrow{\tau'_{YT}} \mathbb{1}\big].$$

Using relation (2.2.5) between τ'_{YT} and $\psi'_{YST,YT}$, we get:

$$\big[\mathcal{C}(Y,YS) \xrightarrow{T} \mathcal{C}'(YT,YST) \xrightarrow{1\otimes\varkappa_Y} \mathcal{C}'(YT,YST) \otimes \mathcal{C}'(YST,YTS') \\ \xrightarrow{1\otimes\psi'_{YST,YT}} \mathcal{C}'(YT,YST) \otimes \underline{\mathcal{V}}(\mathcal{C}'(YT,YST),\mathbb{1}) \xrightarrow{\mathrm{ev}^{\mathcal{V}}} \mathbb{1}\big].$$

Therefore, equation (2.2.9) is equivalent to the following equation:

$$\big[\mathcal{C}(Y,YS) \xrightarrow{1\otimes\varkappa_Y} \mathcal{C}(Y,YS) \otimes \mathcal{C}'(YST,YTS') \xrightarrow{1\otimes\psi'_{YST,YT}} \\ \mathcal{C}(Y,YS) \otimes \underline{\mathcal{V}}(\mathcal{C}'(YT,YST),\mathbb{1}) \xrightarrow{1\otimes\underline{\mathcal{V}}(T,\mathbb{1})} \mathcal{C}(Y,YS) \otimes \underline{\mathcal{V}}(\mathcal{C}(Y,YS),\mathbb{1}) \xrightarrow{\mathrm{ev}^{\mathcal{V}}} \mathbb{1}\big] = \tau_Y.$$

It implies that the composite

$$\mathbb{1} \xrightarrow{\varkappa_Y} \mathcal{C}'(YST,YTS') \xrightarrow{\psi'_{YST,YT}} \underline{\mathcal{V}}(\mathcal{C}'(YT,YST),\mathbb{1}) \xrightarrow{\underline{\mathcal{V}}(T,\mathbb{1})} \underline{\mathcal{V}}(\mathcal{C}(Y,YS),\mathbb{1})$$

is equal to $\dot{\tau}_Y : \mathbb{1} \to \underline{\mathcal{V}}(\mathcal{C}(Y,YS),\mathbb{1})$, the morphism that corresponds to τ_Y by the closedness of the category $\mathcal{V}$. Since the morphisms $\psi'_{YST,YT}$ and $\underline{\mathcal{V}}(T,\mathbb{1})$ are invertible, the morphism $\varkappa_Y : \mathbb{1} \to \mathcal{C}'(YST,YTS')$ is uniquely determined.

2.2.7. Lemma. *The $\mathcal{V}$-natural transformation $\varkappa$ satisfies the following equation:*

$$\psi_{X,Y} = \big[\mathcal{C}(X,YS) \xrightarrow{T} \mathcal{C}'(XT,YST) \xrightarrow{\mathcal{C}'(XT,\varkappa)} \mathcal{C}(XT,YTS') \\ \xrightarrow{\psi'_{XT,YT}} \underline{\mathcal{V}}(\mathcal{C}'(YT,XT),\mathbb{1}) \xrightarrow{\underline{\mathcal{V}}(T,\mathbb{1})} \underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1})\big],$$

for each pair of objects $X, Y \in \operatorname{Ob} \mathcal{C}$.

Proof. The exterior of the following diagram commutes:

$$\begin{tikzcd}
& \mathcal{C}(Y,YS) \arrow[rr, "\tau_Y"] \arrow[d, "T"] & & \mathbb{1} \\
\mathcal{C}(Y,X)\otimes\mathcal{C}(X,YS) \arrow[ur, "\mu_{\mathcal{C}}"] \arrow[dr, "T\otimes T"'] & \mathcal{C}'(YT,YST) \arrow[rr, "\mathcal{C}'(YT,\varkappa)"] & & \mathcal{C}'(YT,YTS') \arrow[u, "\tau'_{YT}"'] \\
& \mathcal{C}'(YT,XT)\otimes\mathcal{C}'(XT,YST) \arrow[u, "\mu_{\mathcal{C}'}"'] \arrow[rr, "1\otimes\mathcal{C}'(XT,\varkappa)"] & & \mathcal{C}'(YT,XT)\otimes\mathcal{C}'(XT,YTS') \arrow[u, "\mu_{\mathcal{C}'}"']
\end{tikzcd}$$

The right upper square commutes by the definition of $\varkappa$, the commutativity of the lower square is a consequence of the associativity of $\mu_{\mathcal{C}'}$. The left quadrilateral is commutative since T is a $\mathcal{V}$-functor. Transforming both paths with the help of equation (2.2.5) yields the following equation:

$$\begin{aligned}
&\bigl[\mathcal{C}(Y,X)\otimes\mathcal{C}(X,YS) \xrightarrow{1\otimes\psi_{X,Y}} \mathcal{C}(Y,X)\otimes\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}) \xrightarrow{\mathrm{ev}^{\mathcal{V}}} \mathbb{1}\bigr] \\
&= \bigl[\mathcal{C}(Y,X)\otimes\mathcal{C}(X,YS) \xrightarrow{T\otimes T} \mathcal{C}'(YT,XT)\otimes\mathcal{C}'(XT,YST) \xrightarrow{1\otimes\mathcal{C}'(XT,\varkappa)} \\
&\qquad \mathcal{C}'(YT,XT)\otimes\mathcal{C}'(XT,YTS') \xrightarrow{1\otimes\psi'_{XT,YT}} \mathcal{C}'(YT,XT)\otimes\underline{\mathcal{V}}(\mathcal{C}'(YT,XT),\mathbb{1}) \xrightarrow{\mathrm{ev}^{\mathcal{V}}} \mathbb{1}\bigr] \\
&= \bigl[\mathcal{C}(Y,X)\otimes\mathcal{C}(X,YS) \xrightarrow{1\otimes T} \mathcal{C}(Y,X)\otimes\mathcal{C}'(XT,YST) \xrightarrow{1\otimes\mathcal{C}'(XT,\varkappa)} \\
&\qquad\qquad \mathcal{C}(Y,X)\otimes\mathcal{C}'(XT,YTS') \xrightarrow{1\otimes\psi'_{XT,YT}} \mathcal{C}(Y,X)\otimes\underline{\mathcal{V}}(\mathcal{C}'(YT,XT),\mathbb{1}) \\
&\qquad\qquad\qquad \xrightarrow{1\otimes\underline{\mathcal{V}}(T,\mathbb{1})} \mathcal{C}(Y,X)\otimes\underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1}) \xrightarrow{\mathrm{ev}^{\mathcal{V}}} \mathbb{1}\bigr].
\end{aligned}$$

The required equation follows by the closedness of $\mathcal{V}$. ☐

2.2.8. Corollary. *If T is an equivalence, then the $\mathcal{V}$-natural transformation $\varkappa : ST \to TS'$ is an isomorphism.*

Proof. Lemma 2.2.7 implies that $\mathcal{C}'(XT,\varkappa) : \mathcal{C}'(XT,YST) \to \mathcal{C}'(XT,XTS')$ is an isomorphism, for each $X \in \operatorname{Ob}\mathcal{C}$. Since T is essentially surjective, it follows that the morphism $\mathcal{C}'(Z,\varkappa) : \mathcal{C}'(Z,YST) \to \mathcal{C}'(Z,YTS')$ is invertible, for each $Z \in \operatorname{Ob}\mathcal{C}'$, thus $\varkappa$ is an isomorphism. ☐

2.2.9. Corollary. *A right Serre $\mathcal{V}$-functor is unique up to isomorphism.*

Proof. Suppose $S, S' : \mathcal{C} \to \mathcal{C}$ are right Serre functors. Applying Corollary 2.2.8 to the functor $T = \mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ yields a natural isomorphism $\varkappa : S \to S'$. ☐

2.2.10. Trace functionals determine the Serre functor. Combining for a $\mathcal{V}$-natural transformation ψ diagrams (2.2.5) and (2.2.8) we get the equation

$$\begin{tikzcd}
\mathcal{C}(Y,X)\otimes\mathcal{C}(X,YS) \arrow[r, "\mu_{\mathcal{C}}"] \arrow[d, "S\otimes 1"'] & \mathcal{C}(Y,YS) \arrow[r, "\tau_Y"] \arrow[d, phantom, "="] & \mathbb{1} \\
\mathcal{C}(YS,XS)\otimes\mathcal{C}(X,YS) \arrow[r, "c"] & \mathcal{C}(X,YS)\otimes\mathcal{C}(YS,XS) \arrow[r, "\mu_{\mathcal{C}}"] & \mathcal{C}(X,XS) \arrow[u, "\tau_X"']
\end{tikzcd} \qquad (2.2.10)$$

The above diagram can be written as the equation

$$\begin{tikzcd}
\mathcal{C}(X,YS)\otimes\mathcal{C}(Y,X) \arrow[r, "1\otimes S"] \arrow[d, "c"'] \arrow[rd, phantom, "="] & \mathcal{C}(X,YS)\otimes\mathcal{C}(YS,XS) \arrow[d, "\phi_{YS,X}"] \\
\mathcal{C}(Y,X)\otimes\mathcal{C}(X,YS) \arrow[r, "\phi_{X,Y}"] & \mathbb{1}
\end{tikzcd} \tag{2.2.11}$$

where

$$\phi_{X,Y} = \bigl[\mathcal{C}(Y,X)\otimes\mathcal{C}(X,YS) \xrightarrow{\mu_{\mathcal{C}}} \mathcal{C}(Y,YS) \xrightarrow{\tau_Y} \mathbb{1}\bigr]. \tag{2.2.12}$$

When S is a fully faithful right Serre functor, pairing (2.2.12) is perfect. Namely, the induced by it morphism $\psi_{X,Y} : \mathcal{C}(X,YS) \to \underline{\mathcal{V}}(\mathcal{C}(Y,X),\mathbb{1})$ is invertible, and the morphism $\psi' : \mathcal{C}(Y,X) \to \underline{\mathcal{V}}(\mathcal{C}(X,YS),\mathbb{1})$, induced by the pairing

$$c\cdot\phi_{X,Y} = \bigl[\mathcal{C}(X,YS)\otimes\mathcal{C}(Y,X) \xrightarrow{c} \mathcal{C}(Y,X)\otimes\mathcal{C}(X,YS) \xrightarrow{\phi_{X,Y}} \mathbb{1}\bigr],$$

is invertible. In fact, diagram (2.2.11) implies that

$$\psi' = \bigl[\mathcal{C}(Y,X) \xrightarrow{S} \mathcal{C}(YS,XS) \xrightarrow{\psi_{YS,X}} \underline{\mathcal{V}}(\mathcal{C}(X,YS),\mathbb{1})\bigr].$$

Diagram (2.2.11) allows to restore the morphisms $S : \mathcal{C}(Y,X) \to \mathcal{C}(YS,XS)$ unambiguously from $\operatorname{Ob} S$ and trace functionals τ, due to $\psi_{YS,X}$ being isomorphisms.

2.2.11. Proposition. *A map* $\operatorname{Ob} S$ *and trace functionals* τ_X, $X \in \operatorname{Ob}\mathcal{C}$, *such that the induced* $\psi_{X,Y}$ *from* (2.2.2) *are invertible, define a unique right Serre* $\mathcal{V}$*-functor* $(S,\psi_{X,Y})$.

Proof. Let us show that the obtained morphisms $S : \mathcal{C}(Y,X) \to \mathcal{C}(YS,XS)$ preserve composition in $\mathcal{C}$. In fact, due to the associativity of composition we have

$$\begin{aligned}
&\bigl[\mathcal{C}(X,ZS)\otimes\mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,X) \xrightarrow{1\otimes S\otimes S} \mathcal{C}(X,ZS)\otimes\mathcal{C}(ZS,YS)\otimes\mathcal{C}(YS,XS) \\
&\qquad\qquad \xrightarrow{1\otimes\mu_{\mathcal{C}}} \mathcal{C}(X,ZS)\otimes\mathcal{C}(ZS,XS) \xrightarrow{\phi_{ZS,X}} \mathbb{1}\bigr] \\
&= \bigl[\mathcal{C}(X,ZS)\otimes\mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,X) \xrightarrow{1\otimes S\otimes S} \mathcal{C}(X,ZS)\otimes\mathcal{C}(ZS,YS)\otimes\mathcal{C}(YS,XS) \\
&\qquad\qquad \xrightarrow{\mu_{\mathcal{C}}\otimes 1} \mathcal{C}(X,YS)\otimes\mathcal{C}(YS,XS) \xrightarrow{\mu_{\mathcal{C}}} \mathcal{C}(X,XS) \xrightarrow{\tau_X} \mathbb{1}\bigr] \\
&= \bigl[\mathcal{C}(X,ZS)\otimes\mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,X) \xrightarrow{(1\otimes S\otimes 1)(\mu_{\mathcal{C}}\otimes 1)} \mathcal{C}(X,YS)\otimes\mathcal{C}(Y,X) \\
&\qquad\qquad \xrightarrow{1\otimes S} \mathcal{C}(X,YS)\otimes\mathcal{C}(YS,XS) \xrightarrow{\mu_{\mathcal{C}}} \mathcal{C}(X,XS) \xrightarrow{\tau_X} \mathbb{1}\bigr] \\
&= \bigl[\mathcal{C}(X,ZS)\otimes\mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,X) \xrightarrow{(1\otimes S\otimes 1)(\mu_{\mathcal{C}}\otimes 1)} \mathcal{C}(X,YS)\otimes\mathcal{C}(Y,X) \\
&\qquad\qquad \xrightarrow{c} \mathcal{C}(Y,X)\otimes\mathcal{C}(X,YS) \xrightarrow{\mu_{\mathcal{C}}} \mathcal{C}(Y,YS) \xrightarrow{\tau_Y} \mathbb{1}\bigr] \\
&= \bigl[\mathcal{C}(X,ZS)\otimes\mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,X) \xrightarrow{(123)_c(1\otimes 1\otimes S)} \mathcal{C}(Y,X)\otimes\mathcal{C}(X,ZS)\otimes\mathcal{C}(ZS,YS) \\
&\qquad\qquad \xrightarrow{1\otimes\mu_{\mathcal{C}}} \mathcal{C}(Y,X)\otimes\mathcal{C}(X,YS) \xrightarrow{\mu_{\mathcal{C}}} \mathcal{C}(Y,YS) \xrightarrow{\tau_Y} \mathbb{1}\bigr] \\
&= \bigl[\mathcal{C}(X,ZS)\otimes\mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,X) \xrightarrow{(123)_c(1\otimes 1\otimes S)} \mathcal{C}(Y,X)\otimes\mathcal{C}(X,ZS)\otimes\mathcal{C}(ZS,YS) \\
&\qquad\qquad \xrightarrow{\mu_{\mathcal{C}}\otimes 1} \mathcal{C}(Y,ZS)\otimes\mathcal{C}(ZS,YS) \xrightarrow{\mu_{\mathcal{C}}} \mathcal{C}(Y,YS) \xrightarrow{\tau_Y} \mathbb{1}\bigr] \\
&= \bigl[\mathcal{C}(X,ZS)\otimes\mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,X) \xrightarrow{(123)_c(\mu_{\mathcal{C}}\otimes 1)} \mathcal{C}(Y,ZS)\otimes\mathcal{C}(Z,Y) \\
&\qquad\qquad \xrightarrow{1\otimes S} \mathcal{C}(Y,ZS)\otimes\mathcal{C}(ZS,YS) \xrightarrow{\mu_{\mathcal{C}}} \mathcal{C}(Y,YS) \xrightarrow{\tau_Y} \mathbb{1}\bigr]
\end{aligned}$$

$$\begin{aligned}
&= \big[\mathcal{C}(X,ZS)\otimes\mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,X) \xrightarrow{(123)_c(\mu_{\mathcal{C}}\otimes 1)} \mathcal{C}(Y,ZS)\otimes\mathcal{C}(Z,Y) \\
&\qquad\qquad \xrightarrow{c} \mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,ZS) \xrightarrow{\mu_{\mathcal{C}}} \mathcal{C}(Z,ZS) \xrightarrow{\tau_Z} \mathbb{1}\big] \\
&= \big[\mathcal{C}(X,ZS)\otimes\mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,X) \xrightarrow{(321)_c} \mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,X)\otimes\mathcal{C}(X,ZS) \\
&\qquad\qquad \xrightarrow{1\otimes\mu_{\mathcal{C}}} \mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,ZS) \xrightarrow{\mu_{\mathcal{C}}} \mathcal{C}(Z,ZS) \xrightarrow{\tau_Z} \mathbb{1}\big].
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\big[\mathcal{C}(X,ZS)\otimes\mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,X) \xrightarrow{1\otimes\mu_{\mathcal{C}}} \mathcal{C}(X,ZS)\otimes\mathcal{C}(Z,X) \\
&\qquad\qquad \xrightarrow{1\otimes S} \mathcal{C}(X,ZS)\otimes\mathcal{C}(ZS,XS) \xrightarrow{\phi_{ZS,X}} \mathbb{1}\big] \\
&= \big[\mathcal{C}(X,ZS)\otimes\mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,X) \xrightarrow{1\otimes\mu_{\mathcal{C}}} \mathcal{C}(X,ZS)\otimes\mathcal{C}(Z,X) \\
&\qquad\qquad \xrightarrow{c} \mathcal{C}(Z,X)\otimes\mathcal{C}(X,ZS) \xrightarrow{\phi_{X,Z}} \mathbb{1}\big] \\
&= \big[\mathcal{C}(X,ZS)\otimes\mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,X) \xrightarrow{(321)_c} \mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,X)\otimes\mathcal{C}(X,ZS) \\
&\qquad\qquad \xrightarrow{\mu_{\mathcal{C}}\otimes 1} \mathcal{C}(Z,X)\otimes\mathcal{C}(X,ZS) \xrightarrow{\mu_{\mathcal{C}}} \mathcal{C}(Z,ZS) \xrightarrow{\tau_Z} \mathbb{1}\big] \\
&= \big[\mathcal{C}(X,ZS)\otimes\mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,X) \xrightarrow{(321)_c} \mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,X)\otimes\mathcal{C}(X,ZS) \\
&\qquad\qquad \xrightarrow{1\otimes\mu_{\mathcal{C}}} \mathcal{C}(Z,Y)\otimes\mathcal{C}(Y,ZS) \xrightarrow{\mu_{\mathcal{C}}} \mathcal{C}(Z,ZS) \xrightarrow{\tau_Z} \mathbb{1}\big].
\end{aligned}$$

The last lines of both expressions coincide, hence $(S\otimes S)\mu_{\mathcal{C}} = \mu_{\mathcal{C}}S$.

Let us prove that the morphisms $S : \mathcal{C}(X,X) \to \mathcal{C}(XS,XS)$ of $\mathcal{V}$ preserve identities. Indeed, the exterior of the following diagram commutes:

$$\begin{tikzcd}
{} & \mathcal{C}(X,XS) \arrow[r, "\tau_X"] & \mathbb{1} \\
\mathcal{C}(X,XS) \arrow[ur, equal] \arrow[r, "\lambda^{\cdot\mathsf{I}}(1_X\otimes 1)"] \arrow[d, "\lambda^{\mathsf{I}\cdot}"', "\wr"] & \mathcal{C}(X,X)\otimes\mathcal{C}(X,XS) \arrow[u, "\mu_{\mathcal{C}}"'] \arrow[ul, phantom, "="] & {} \\
\mathcal{C}(X,XS)\otimes\mathbb{1} \arrow[r, "1\otimes 1_X"] \arrow[dr, "1\otimes?", "1\otimes 1_{XS}"'] \arrow[ur, phantom, "="] & \mathcal{C}(X,XS)\otimes\mathcal{C}(X,X) \arrow[u, "c"'] \arrow[d, "1\otimes S"'] & {} \\
{} & \mathcal{C}(X,XS)\otimes\mathcal{C}(XS,XS) \arrow[r, "\mu_{\mathcal{C}}"] \arrow[uur, phantom, "="] & \mathcal{C}(X,XS) \arrow[uuu, "\tau_X"']
\end{tikzcd}$$

Therefore, both paths from $\mathcal{C}(X,XS)$ to $\mathbb{1}$, going through the isomorphism $\lambda^{\mathsf{I}\cdot}$, sides of triangle marked '1⊗?', $\mu_{\mathcal{C}}$ and τ_X, compose to the same morphism τ_X. The invertibility of $\psi_{X,X}$ implies that the origin '?' of the mentioned triangle commutes, that is,

$$1_{XS} = \big[\mathbb{1} \xrightarrow{1_X} \mathcal{C}(X,X) \xrightarrow{S} \mathcal{C}(XS,XS)\big].$$

Summing up, the constructed $S : \mathcal{C} \to \mathcal{C}$ is a $\mathcal{V}$-functor. Applying Lemma 2.2.2 we deduce that $\psi_{-,Y}$ is $\mathcal{V}$-natural in the first argument for all objects Y of $\mathcal{C}$. Recall that $\psi_{X,Y}$ is a unique morphism which makes diagram (2.2.5) commutative. Due to equation (2.2.10), $\psi_{X,Y}$ makes commutative also diagram (2.2.8). This means that $\psi_{X,Y}$ can be presented also in the form (2.2.6). Applying Lemma 2.2.3 we deduce that $\psi_{X,-}$ is

$\mathcal{V}$-natural in the second argument for all objects X of $\mathcal{C}$. Being $\mathcal{V}$-natural in each variable ψ is $\mathcal{V}$-natural as a whole [**29**, Section 1.4]. $\square$

2.3. Serre functors and base change

Let $(B,\beta^I):(\mathcal{V},\otimes^I_{\mathcal{V}},\lambda^f_{\mathcal{V}})\to(\mathcal{W},\otimes^I_{\mathcal{W}},\lambda^f_{\mathcal{W}})$ be a lax symmetric Monoidal functor between closed symmetric Monoidal categories. Denote by $\widehat{B}:\widehat{\mathcal{V}}\to\widehat{\mathcal{W}}$ the corresponding multifunctor. According to Section 1.1.14, (B,β^I) gives rise to a lax symmetric Monoidal **Cat**-functor $(B_*,\beta^I_*):\mathcal{V}\text{-}\mathbf{Cat}\to\mathcal{W}\text{-}\mathbf{Cat}$. Since the multicategories $\widehat{\mathcal{V}}$ and $\widehat{\mathcal{W}}$ are closed, the multifunctor $\widehat{B}$ determines the closing transformation $\underline{\widehat{B}}$. In particular , we have a $\mathcal{W}$-functor $B_*\underline{\mathcal{V}}\to\underline{\mathcal{W}}$, $X\mapsto BX$, which is denoted by $\underline{\widehat{B}}$ by abuse of notation, see also Proposition 1.3.30. Its action on morphisms is found from the following equation in $\mathcal{W}$:

$$\bigl[BX\otimes B(\underline{\mathcal{V}}(X,Y))\xrightarrow{1\otimes\underline{\widehat{B}}}BX\otimes\underline{\mathcal{W}}(BX,BY)\xrightarrow{\mathrm{ev}^{\mathcal{W}}}BY\bigr]=\widehat{B}(\mathrm{ev}^{\mathcal{V}}). \tag{2.3.1}$$

Let $\widehat{B_*}:\widehat{\mathcal{V}\text{-}\mathbf{Cat}}\to\widehat{\mathcal{W}\text{-}\mathbf{Cat}}$ denote the symmetric **Cat**-multifunctor that corresponds to the lax symmetric Monoidal **Cat**-functor (B_*,β^I_*). Clearly, $\widehat{B_*}$ commutes with taking opposite.

We are going to investigate the following question: given a $\mathcal{V}$-category $\mathcal{C}$ together with a Serre $\mathcal{V}$-functor $S:\mathcal{C}\to\mathcal{C}$, is the $\mathcal{W}$-functor $B_*S:B_*\mathcal{C}\to B_*\mathcal{C}$ a Serre $\mathcal{W}$-functor in the $\mathcal{W}$-category $B_*\mathcal{C}$ obtained by the base change? In general, the answer is negative, however it is affirmative in the cases of interest.

In the sequel, the tensor product in the categories $\mathcal{V}$ and $\mathcal{W}$ is denoted by $\otimes$, the unit objects in both categories are denoted by $\mathbb{1}$.

Let $\mathcal{A}$ be a $\mathcal{V}$-category. We claim that the $\mathcal{W}$-functor

$$\widehat{B_*}\operatorname{Hom}_{\mathcal{A}}\cdot\underline{\widehat{B}}=\bigl[(B_*\mathcal{A})^{\mathrm{op}}\boxtimes B_*\mathcal{A}=B_*(\mathcal{A}^{\mathrm{op}})\boxtimes B_*\mathcal{A}\xrightarrow{\widehat{B_*}\operatorname{Hom}_{\mathcal{A}}}B_*\underline{\mathcal{V}}\xrightarrow{\underline{\widehat{B}}}\underline{\mathcal{W}}\bigr]$$

coincides with $\operatorname{Hom}_{B_*\mathcal{A}}$. Indeed, both $\mathcal{W}$-functors map a pair $(X,Y)\in\operatorname{Ob}\mathcal{A}\times\operatorname{Ob}\mathcal{A}$ to the object $B(\mathcal{A}(X,Y))=(B_*\mathcal{A})(X,Y)$ of $\mathcal{W}$. Applying $\widehat{B}$ to equation (2.1.3) yields a commutative diagram

$$\begin{tikzcd}
B\mathcal{A}(X,U)\otimes B\mathcal{A}(Y,X)\otimes B\mathcal{A}(U,V) \arrow[r, "1\otimes\widehat{B}\operatorname{Hom}_{\mathcal{A}}"] \arrow[d, "c\otimes1"'] & B\mathcal{A}(X,U)\otimes B\underline{\mathcal{V}}(\mathcal{A}(X,U),\mathcal{A}(Y,V)) \arrow[d, "\widehat{B}(\mathrm{ev}^{\mathcal{V}})"] \\
B\mathcal{A}(Y,X)\otimes B\mathcal{A}(X,U)\otimes B\mathcal{A}(U,V) \arrow[r, "\substack{\widehat{B}(\mu^{\mathbf{3}}_{\mathcal{A}})\\ \|}", "\mu^{\mathbf{3}}_{B_*\mathcal{A}}"'] & B\mathcal{A}(Y,V)
\end{tikzcd}$$

Expanding $\widehat{B}(\mathrm{ev}^{\mathcal{V}})$ according to (2.3.1) we transform the above diagram as follows:

$$\begin{tikzcd}
B\mathcal{A}(X,U)\otimes B\mathcal{A}(Y,X)\otimes B\mathcal{A}(U,V) \arrow[r, "1\otimes\widehat{B}\operatorname{Hom}_{\mathcal{A}}\cdot\underline{\widehat{B}}"] \arrow[d, "c\otimes1"'] & B\mathcal{A}(X,U)\otimes\underline{\mathcal{W}}(B\mathcal{A}(X,U),B\mathcal{A}(Y,V)) \arrow[d, "\mathrm{ev}^{\mathcal{W}}"] \\
B\mathcal{A}(Y,X)\otimes B\mathcal{A}(X,U)\otimes B\mathcal{A}(U,V) \arrow[r, "\mu^{\mathbf{3}}_{B_*\mathcal{A}}"] & B\mathcal{A}(Y,V)
\end{tikzcd}$$

It follows that the $\mathcal{W}$-functors $\widehat{B_*}\operatorname{Hom}_{\mathcal{A}}\cdot\underline{\widehat{B}}$ and $\operatorname{Hom}_{B_*\mathcal{A}}$ are solutions to the same equation, therefore they must coincide by the closedness of $\mathcal{W}$.

There is a $\mathcal{W}$-natural transformation ζ' as in the diagram below:

$$\begin{tikzcd}
B_*\underline{\mathcal{V}}^{\mathrm{op}} \arrow[r, "B_*\underline{\mathcal{V}}(-{,}\mathbb{1})"] \arrow[d, "(\underline{\widehat{B}})^{\mathrm{op}}"'] & B_*\underline{\mathcal{V}} \arrow[d, "\underline{\widehat{B}}"] \arrow[dl, Rightarrow, "\zeta'"'] \\
\underline{\mathcal{W}}^{\mathrm{op}} \arrow[r, "\underline{\mathcal{W}}(-{,}B\mathbb{1})"] & \underline{\mathcal{W}}
\end{tikzcd}$$

For each object X, the morphism $\zeta'_X : B(\underline{\mathcal{V}}(X, \mathbb{1})) \to \underline{\mathcal{W}}(BX, B\mathbb{1})$ in $\mathcal{W}$ comes from the map $\widehat{B}(\mathrm{ev}^{\mathcal{V}}) : BX \otimes B\underline{\mathcal{V}}(X, \mathbb{1}) \to B\mathbb{1}$ by the closedness of $\mathcal{W}$. In other words, $\zeta'_X = \underline{\widehat{B}}_{X,\mathbb{1}}$. The $\mathcal{W}$-naturality of ζ' is expressed by the following equation in $\mathcal{W}$:

$$\begin{tikzcd}
B\underline{\mathcal{V}}(Y, X) \arrow[r, "B\underline{\mathcal{V}}(-{,}\mathbb{1})"] \arrow[d, "\underline{\widehat{B}}"'] & B\underline{\mathcal{V}}(\underline{\mathcal{V}}(X, \mathbb{1}), \underline{\mathcal{V}}(Y, \mathbb{1})) \arrow[d, "\underline{\widehat{B}}"] \\
\underline{\mathcal{W}}(BY, BX) \arrow[d, "\underline{\mathcal{W}}(-{,}B\mathbb{1})"'] & \underline{\mathcal{W}}(B\underline{\mathcal{V}}(X, \mathbb{1}), B\underline{\mathcal{V}}(Y, \mathbb{1})) \arrow[d, "\underline{\mathcal{W}}(1{,}\zeta'_Y)"] \\
\underline{\mathcal{W}}(\underline{\mathcal{W}}(BX, B\mathbb{1}), \underline{\mathcal{W}}(BY, B\mathbb{1})) \arrow[r, "\underline{\mathcal{W}}(\zeta'_X{,}1)"] & \underline{\mathcal{W}}(B\underline{\mathcal{V}}(X, \mathbb{1}), \underline{\mathcal{W}}(BY, B\mathbb{1}))
\end{tikzcd}$$

By the closedness of $\mathcal{W}$, it is equivalent to the following equation:

$$\begin{tikzcd}
B\underline{\mathcal{V}}(X, \mathbb{1}) \otimes B\underline{\mathcal{V}}(Y, X) \arrow[r, "1\otimes B\underline{\mathcal{V}}(-{,}\mathbb{1})"] \arrow[d, "\zeta'_X\otimes\underline{\widehat{B}}"'] & B\underline{\mathcal{V}}(X, \mathbb{1}) \otimes B\underline{\mathcal{V}}(\underline{\mathcal{V}}(X, \mathbb{1}), \underline{\mathcal{V}}(Y, \mathbb{1})) \arrow[d, "\widehat{B}(\mathrm{ev}^{\mathcal{V}})"] \\
\underline{\mathcal{W}}(BX, B\mathbb{1}) \otimes \underline{\mathcal{W}}(BY, BX) \arrow[d, "1\otimes\underline{\mathcal{W}}(-{,}B\mathbb{1})"'] & B\underline{\mathcal{V}}(Y, \mathbb{1}) \arrow[d, "\zeta'_Y"] \\
\underline{\mathcal{W}}(BX, B\mathbb{1}) \otimes \underline{\mathcal{W}}(\underline{\mathcal{W}}(BX, B\mathbb{1}), \underline{\mathcal{W}}(BY, B\mathbb{1})) \arrow[r, "\mathrm{ev}^{\mathcal{W}}"] & \underline{\mathcal{W}}(BY, B\mathbb{1})
\end{tikzcd}$$

By (2.1.4), the above equation reduces to the equation

$$\begin{tikzcd}
B\underline{\mathcal{V}}(Y, X) \otimes B\underline{\mathcal{V}}(X, \mathbb{1}) \arrow[r, "\widehat{B}(\mu_{\underline{\mathcal{V}}})", "\parallel" description, "\mu_{B_*\underline{\mathcal{V}}}"'] \arrow[d, "\underline{\widehat{B}}\otimes\zeta'_X" ', "=" description, "\underline{\widehat{B}}_{Y,X}\otimes\underline{\widehat{B}}_{X,\mathbb{1}}"] & B\underline{\mathcal{V}}(Y, \mathbb{1}) \arrow[d, "\underline{\widehat{B}}_{Y,\mathbb{1}}"', "=" description, "\zeta'_Y"] \\
\underline{\mathcal{W}}(BY, BX) \otimes \underline{\mathcal{W}}(BX, B\mathbb{1}) \arrow[r, "\mu_{\underline{\mathcal{W}}}"] & \underline{\mathcal{W}}(BY, B\mathbb{1})
\end{tikzcd}$$

which expresses the fact that $\underline{\widehat{B}} : B_*\underline{\mathcal{V}} \to \underline{\mathcal{W}}$ is a $\mathcal{W}$-functor.

Suppose that $\beta^{\varnothing} : \mathbb{1} \to B\mathbb{1}$ is an isomorphism. Then there is a $\mathcal{W}$-natural isomorphism

$$\underline{\mathcal{W}}(1, (\beta^{\varnothing})^{-1}) : \underline{\mathcal{W}}(-, B\mathbb{1}) \to \underline{\mathcal{W}}(-, \mathbb{1}) : \underline{\mathcal{W}}^{\mathrm{op}} \to \underline{\mathcal{W}}.$$

Pasting it with ζ' gives a $\mathcal{W}$-natural transformation ζ as in the diagram below:

$$\begin{tikzcd}
B_*\underline{\mathcal{V}}^{\mathrm{op}} \arrow[r, "B_*\underline{\mathcal{V}}(-,\mathbb{1})"] \arrow[d, "(\widehat{\underline{B}})^{\mathrm{op}}"'] & B_*\underline{\mathcal{V}} \arrow[d, "\widehat{\underline{B}}"] \arrow[dl, Rightarrow, "\zeta"'] \\
\underline{\mathcal{W}}^{\mathrm{op}} \arrow[r, "\underline{\mathcal{W}}(-,\mathbb{1})"] & \underline{\mathcal{W}}
\end{tikzcd} \tag{2.3.2}$$

2.3.1. Proposition. *Suppose ζ is an isomorphism. Let $\mathcal{C}$ be a $\mathcal{V}$-category, and suppose $S : \mathcal{C} \to \mathcal{C}$ is a right Serre $\mathcal{V}$-functor. Then $B_*S : B_*\mathcal{C} \to B_*\mathcal{C}$ is a right Serre $\mathcal{W}$-functor.*

Proof. Let ψ be a $\mathcal{V}$-natural isomorphism as in (2.2.1). Applying the **Cat**-multifunctor $\widehat{B_*}$ and patching the result with diagram (2.3.2) yields the following diagram:

$$\begin{tikzcd}
B_*(\mathcal{C})^{\mathrm{op}} \boxtimes B_*(\mathcal{C}) \arrow[r, "1\boxtimes B_*(S)"] \arrow[d, "\widehat{B_*}(\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}})^{\mathrm{op}}"'] & B_*(\mathcal{C})^{\mathrm{op}} \boxtimes B_*(\mathcal{C}) \arrow[d, "\widehat{B_*}\,\mathrm{Hom}_{\mathcal{C}}"] \arrow[dl, Rightarrow, "\widehat{B_*}(\psi)"'] \\
B_*\underline{\mathcal{V}}^{\mathrm{op}} \arrow[r, "B_*\underline{\mathcal{V}}(-,\mathbb{1})"] \arrow[d, "(\widehat{\underline{B}})^{\mathrm{op}}"'] & B_*\underline{\mathcal{V}} \arrow[d, "\widehat{\underline{B}}"] \arrow[dl, Rightarrow, "\zeta"'] \\
\underline{\mathcal{W}}^{\mathrm{op}} \arrow[r, "\underline{\mathcal{W}}(-,\mathbb{1})"] & \underline{\mathcal{W}}
\end{tikzcd} \tag{2.3.3}$$

Since $\widehat{B_*}\,\mathrm{Hom}_{\mathcal{C}} \cdot \widehat{\underline{B}} = \mathrm{Hom}_{B_*\mathcal{C}}$ and $\widehat{B_*}\,\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}} \cdot \widehat{\underline{B}} = \mathrm{Hom}_{B_*\mathcal{C}^{\mathrm{op}}}$, we obtain a $\mathcal{W}$-natural transformation

$$\begin{tikzcd}
B_*\mathcal{C}^{\mathrm{op}} \boxtimes B_*\mathcal{C} \arrow[r, "1\boxtimes B_*S"] \arrow[d, "\mathrm{Hom}^{\mathrm{op}}_{B_*\mathcal{C}^{\mathrm{op}}}"'] & B_*\mathcal{C}^{\mathrm{op}} \boxtimes B_*\mathcal{C} \arrow[d, "\mathrm{Hom}_{B_*\mathcal{C}}"] \arrow[dl, Rightarrow] \\
\underline{\mathcal{W}}^{\mathrm{op}} \arrow[r, "\underline{\mathcal{W}}(-,\mathbb{1})"] & \underline{\mathcal{W}}
\end{tikzcd}$$

It is invertible since so are ψ and ζ. It follows that a right Serre $\mathcal{V}$-functor $S : \mathcal{C} \to \mathcal{C}$ induces a right Serre $\mathcal{W}$-functor $B_*S : B_*\mathcal{C} \to B_*\mathcal{C}$. □

We are going to apply the above results according to the scheme

$$\mathcal{K} \xrightarrow{(H^\bullet,\kappa^I)} \mathbf{gr} \xrightarrow{(N,\nu^I)} \Bbbk\text{-}\mathbf{Mod},$$

where the first functor is given by the total cohomology, and the second functor singles out the 0th component. We assume that the reader is familiar with the fact that the involved symmetric Monoidal categories are closed; details can be found in Section 3.1.

2.3.2. From $\mathcal{K}$-categories to gr-categories. Consider the lax symmetric Monoidal base change functor $(H^\bullet,\kappa^I) : \mathcal{K} \to \mathbf{gr}$, $X \mapsto H^\bullet X = (H^n X)_{n\in\mathbb{Z}}$, where for each $I \in \mathrm{Ob}\,\mathcal{O}$ the morphism $\kappa^I : \otimes^{i\in I} H^\bullet X_i \to H^\bullet \otimes^{i\in I} X_i$ is the Künneth map. There is a **gr**-functor $\widehat{\underline{H^\bullet}} : H^\bullet_*\underline{\mathcal{K}} \to \underline{\mathbf{gr}}$, $X \mapsto H^\bullet X$, that acts on morphisms via the map

$$\mathcal{K}(X[-n],Y) = H^n\underline{\mathcal{K}}(X,Y) \to \underline{\mathbf{gr}}(H^\bullet X, H^\bullet Y)^n = \prod_{d\in\mathbb{Z}} \underline{\Bbbk\text{-}\mathbf{Mod}}(H^{d-n}X, H^dY)$$

which sends the homotopy class of a chain map $f : X[-n] \to Y$ to $(H^d(f))_{d\in\mathbb{Z}}$. Here $\underline{\Bbbk\text{-}\mathbf{Mod}}(M,N)$ denotes the internal Hom-object in the category $\Bbbk$-**Mod**. As a set, it coincides with $\Bbbk$-**Mod**(M, N). Note that $H^\bullet$ preserves the unit object, therefore there is a **gr**-natural transformation

$$\begin{tikzcd}
H^\bullet_* \underline{\mathcal{K}}^{\mathrm{op}} \arrow[r, "H^\bullet_* \underline{\mathcal{K}}(-{,}\Bbbk)"] \arrow[d, "(\underline{\widehat{H^\bullet}})^{\mathrm{op}}"'] & H^\bullet_* \underline{\mathcal{K}} \arrow[d, "\underline{\widehat{H^\bullet}}"] \arrow[dl, Rightarrow, "\zeta"'] \\
\underline{\mathbf{gr}}^{\mathrm{op}} \arrow[r, "\underline{\mathbf{gr}}(-{,}\Bbbk)"] & \underline{\mathbf{gr}}
\end{tikzcd}$$

Explicitly, the map $\zeta_X = \underline{\widehat{H^\bullet}}_{X,\Bbbk} : H^\bullet(\underline{\mathcal{K}}(X, \Bbbk)) \to \underline{\mathbf{gr}}(H^\bullet X, H^\bullet \Bbbk) = \underline{\mathbf{gr}}(H^\bullet X, \Bbbk)$ is given by its components

$$\mathcal{K}(X[-n], \Bbbk) = H^n \underline{\mathcal{K}}(X, \Bbbk) \to \underline{\mathbf{gr}}(H^\bullet X, \Bbbk)^n = \underline{\Bbbk\text{-}\mathbf{Mod}}(H^{-n} X, \Bbbk), \quad f \mapsto H^0(f).$$

In general, ζ is not invertible. However, if $\Bbbk$ is a field, ζ is an isomorphism. In fact, in this case $H^\bullet : \mathcal{K} \to \mathbf{gr}$ is an equivalence. A quasi-inverse is given by the functor $F : \mathbf{gr} \to \mathcal{K}$ which equips a graded $\Bbbk$-module with the trivial differential.

2.3.3. Proposition. *Suppose $\Bbbk$ is a field. Let $S : \mathcal{C} \to \mathcal{C}$ be a (right) Serre $\mathcal{K}$-functor. Then $H^\bullet_*(S) : H^\bullet_*(\mathcal{C}) \to H^\bullet_*(\mathcal{C})$ is a (right) Serre* **gr***-functor. Moreover, $H^\bullet_*$ reflects (right) Serre functors: if $H^\bullet_*(\mathcal{C})$ admits a (right) Serre* **gr***-functor, then $\mathcal{C}$ admits a (right) Serre $\mathcal{K}$-functor.*

Proof. The first assertion follows from Proposition 2.3.1. For the proof of the second, note that the symmetric Monoidal functor $F : \mathbf{gr} \to \mathcal{K}$ induces a symmetric Monoidal **Cat**-functor $F_* : \widehat{\mathbf{gr}\text{-}\mathbf{Cat}} \to \widehat{\mathcal{K}\text{-}\mathbf{Cat}}$. The corresponding $\mathcal{K}$-functor $\underline{\widehat{F}} : F_* \underline{\mathbf{gr}} \to \underline{\mathcal{K}}$ acts as the identity on morphisms (the complex $\underline{\mathcal{K}}(FX, FY)$ carries the trivial differential and coincides with $\underline{\mathbf{gr}}(X, Y)$ as a graded $\Bbbk$-module). Furthermore, F preserves the unit object, therefore Proposition 2.3.1 applies. It follows that if $\bar{S} : H^\bullet_*(\mathcal{C}) \to H^\bullet_*(\mathcal{C})$ is a right Serre **gr**-functor, then $F_*(\bar{S}) : F_* H^\bullet_*(\mathcal{C}) \to F_* H^\bullet_*(\mathcal{C})$ is a right Serre $\mathcal{K}$-functor. Since the $\mathcal{K}$-category $F_* H^\bullet_*(\mathcal{C})$ is isomorphic to $\mathcal{C}$, the right Serre $\mathcal{K}$-functor $F_*(\bar{S})$ translates to a right Serre $\mathcal{K}$-functor on $\mathcal{C}$. ☐

2.3.4. From gr-categories to $\Bbbk$-categories. Consider a lax symmetric Monoidal base change functor $(N, \nu^I) : \mathbf{gr} \to \Bbbk\text{-}\mathbf{Mod}$, $X = (X^n)_{n \in \mathbb{Z}} \mapsto X^0$, where for each $I \in \operatorname{Ob} \mathcal{O}$ the map

$$\nu^I : \otimes^{i \in I} N X_i = \otimes^{i \in I} X_i^0 \to N \otimes^{i \in I} X_i = \bigoplus_{\sum_{i \in I} n_i = 0} X_i^{n_i}$$

is the natural embedding. The $\Bbbk$-**Mod**-functor $\underline{\widehat{N}} : N_* \underline{\mathbf{gr}} \to \underline{\Bbbk\text{-}\mathbf{Mod}}$, $X \mapsto NX = X^0$, acts on morphisms via the projection

$$N \underline{\mathbf{gr}}(X, Y) = \underline{\mathbf{gr}}(X, Y)^0 = \prod_{d \in \mathbb{Z}} \underline{\Bbbk\text{-}\mathbf{Mod}}(X^d, Y^d)$$
$$\to \underline{\Bbbk\text{-}\mathbf{Mod}}(X^0, Y^0) = \underline{\Bbbk\text{-}\mathbf{Mod}}(NX, NY).$$

The functor N preserves the unit object, therefore there exists a $\Bbbk$-**Mod**-natural transformation

$$\begin{tikzcd} N_*\underline{\mathbf{gr}}^{\mathrm{op}} \arrow[r, "N_*\underline{\mathbf{gr}}(-{,}\Bbbk)"] \arrow[d, "(\widehat{\underline{N}})^{\mathrm{op}}"'] & N_*\underline{\mathbf{gr}} \arrow[d, "\widehat{\underline{N}}"] \arrow[dl, Rightarrow, "\zeta"'] \\ \underline{\Bbbk\text{-}\mathbf{Mod}}^{\mathrm{op}} \arrow[r, "\underline{\Bbbk\text{-}\mathbf{Mod}}(-{,}\Bbbk)"] & \underline{\Bbbk\text{-}\mathbf{Mod}} \end{tikzcd}$$

Explicitly, the map $\zeta_X = \widehat{\underline{N}}_{X,\Bbbk}$ is the identity map

$$N\underline{\mathbf{gr}}(X,\Bbbk) = \underline{\mathbf{gr}}(X,\Bbbk)^0 \to \underline{\Bbbk\text{-}\mathbf{Mod}}(X^0,\Bbbk) = \underline{\Bbbk\text{-}\mathbf{Mod}}(NX,\Bbbk).$$

2.3.5. Corollary (to Proposition 2.3.1)**.** *Suppose $S : \mathcal{C} \to \mathcal{C}$ is a right Serre* **gr***-functor. Then $N_*(S) : N_*(\mathcal{C}) \to N_*(\mathcal{C})$ is a right Serre $\Bbbk$-***Mod***-functor (i.e., an ordinary Serre functor).*

If $N_*(\mathcal{C})$ possess a right Serre $\Bbbk$-functor, it does not imply, in general, that $\mathcal{C}$ has a right Serre **gr**-functor. However, this is the case if $\mathcal{C}$ is closed under shifts, as explained in the next section.

2.3.6. Categories closed under shifts. We are going to define $\mathcal{K}$-categories and **gr**-categories closed under shifts. Since the construction is the same in both cases, we treat them simultaneously.

Let $\mathcal{C}$ be a $\mathcal{V}$-category, where $\mathcal{V}$ is either $\mathcal{K}$ or **gr**. Define a $\mathcal{V}$-category $\mathcal{C}^{[]}$ as follows. The set of objects of $\mathcal{C}^{[]}$ is $\operatorname{Ob}\mathcal{C} \times \mathbb{Z}$. Thus, an object of $\mathcal{C}^{[]}$ is a pair (X,m), where X is an object of $\mathcal{C}$ and m is an integer. It is thought as a formal translation of X by m. For objects (X,m), (Y,n), the object of morphisms $\mathcal{C}^{[]}((X,m),(Y,n))$ is $\mathcal{C}(X,Y)[n-m]$. In the case $\mathcal{V} = \mathcal{K}$, the graded $\Bbbk$-module $\mathcal{C}^{[]}((X,m),(Y,n))$ is equipped with the differential

$$d_{\mathcal{C}^{[]}} : \mathcal{C}^{[]}((X,m),(Y,n)) \to \mathcal{C}^{[]}((X,m),(Y,n))$$

given by $fd_{\mathcal{C}^{[]}} = (-)^{n-m} fd_{\mathcal{C}}$, for each $f \in \mathcal{C}^{[]}((X,m),(Y,n))^k = \mathcal{C}(X,Y)^{n-m+k}$, $k \in \mathbb{Z}$, where $d_{\mathcal{C}}$ is the differential in $\mathcal{C}(X,Y)$. For each triple (X,m),(Y,n),(Z,p) of objects of $\mathcal{C}$, the composition

$$\mu_{\mathcal{C}^{[]}} : \mathcal{C}^{[]}((X,m),(Y,n))^k \otimes \mathcal{C}^{[]}((Y,n),(Z,p))^l \to \mathcal{C}^{[]}((X,m),(Z,p))^{k+l}$$

is given by

$$(-)^{(m-n)(p-m+l)} \mu_{\mathcal{C}} : \mathcal{C}(X,Y)^{m-n+k} \otimes \mathcal{C}(Y,Z)^{p-m+l} \to \mathcal{C}(X,Z)^{p-m+k+l},$$

for $k,l \in \mathbb{Z}$. The identity of an object (X,n) is simply

$$1_X^{\mathcal{C}^{[]}} = 1_X^{\mathcal{C}} : \Bbbk \to \mathcal{C}^{[]}((X,n),(X,n)) = \mathcal{C}(X,X).$$

We leave it to the reader to check that $\mathcal{C}^{[]}$ is a $\mathcal{V}$-category. The $\mathcal{V}$-category $\mathcal{C}$ embeds fully faithfully into $\mathcal{C}^{[]}$ via $u_{[]} : \mathcal{C} \to \mathcal{C}^{[]}$, $X \mapsto (X,0)$.

2.3.7. Definition. We say that a $\mathcal{V}$-category $\mathcal{C}$ is *closed under shifts* if every object (X,n) of $\mathcal{C}^{[]}$ is isomorphic in $\mathcal{C}^{[]}$ to some object $(Y,0)$, $Y \in \operatorname{Ob}\mathcal{C}$. We write $Y = X[n]$.

One finds immediately that a $\mathcal{V}$-category $\mathcal{C}$ is closed under shifts if and only if the $\mathcal{V}$-functor $u_{[]} : \mathcal{C} \to \mathcal{C}^{[]}$ is an equivalence.

The lax symmetric Monoidal base change functor $(H^\bullet, \kappa^I) : \mathcal{K} \to \mathbf{gr}$ gives a lax symmetric Monoidal **Cat**-functor $(H^\bullet_*, \kappa^I_*) : \mathcal{K}\text{-}\mathbf{Cat} \to \mathbf{gr}\text{-}\mathbf{Cat}$. It is easy to see that $(H^\bullet_*\mathcal{C})^{[]} \cong H^\bullet_*(\mathcal{C}^{[]})$, and that the embedding $u_{[]} : H^\bullet_*\mathcal{C} \to (H^\bullet_*\mathcal{C})^{[]}$ identifies with the

gr-functor $H^\bullet_*(u_{[]}) : H^\bullet_*\mathcal{C} \to H^\bullet_*(\mathcal{C}^{[]})$. Therefore, if $\mathcal{C}$ is a $\mathcal{K}$-category closed under shifts, then $H^\bullet_*\mathcal{C}$ is a **gr**-category closed under shifts.

For a **gr**-category $\mathcal{C}$, the components of the graded $\Bbbk$-module $\mathcal{C}(X,Y)$ are denoted by $\mathcal{C}(X,Y)^n = \mathcal{C}^n(X,Y)$, $X,Y \in \operatorname{Ob}\mathcal{C}$, $n \in \mathbb{Z}$. The $\Bbbk$-linear category $N_*(\mathcal{C})$ is denoted by $\mathcal{C}^0$.

2.3.8. Proposition. *Let $\mathcal{C}$ be a* **gr***-category closed under shifts. Suppose $S^0 : \mathcal{C}^0 \to \mathcal{C}^0$ is an ordinary right Serre functor. Then there exists a right Serre* **gr***-functor $S : \mathcal{C} \to \mathcal{C}$ such that $N_*(S) = S^0$.*

Proof. Let $\psi^0 = (\psi^0_{X,Y} : \mathcal{C}^0(X,YS^0) \to \underline{\Bbbk\text{-}\mathbf{Mod}}(\mathcal{C}^0(Y,X),\Bbbk))_{X,Y\in\operatorname{Ob}\mathcal{C}}$ be a natural isomorphism. Let $\phi^0_{X,Y} : \mathcal{C}^0(Y,X)\otimes\mathcal{C}^0(X,YS) \to \Bbbk$, $X,Y \in \operatorname{Ob}\mathcal{C}$, denote the corresponding pairings from (2.2.12). Define trace functionals $\tau^0_X : \mathcal{C}^0(X,XS) \to \Bbbk$, $X \in \operatorname{Ob}\mathcal{C}$, by formula (2.2.3). We are going to apply Proposition 2.2.11. For this we need to specify a map $\operatorname{Ob}S : \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{C}$ and trace functionals $\tau_X : \mathcal{C}(X,XS) \to \Bbbk$, $X \in \operatorname{Ob}\mathcal{C}$. Set $\operatorname{Ob}S = \operatorname{Ob}S^0$. Let the 0th component of τ_X be equal to the map τ^0_X, the other components necessarily vanish since $\Bbbk$ is concentrated in degree 0. Let us prove that the pairings $\phi_{X,Y}$ given by (2.2.12) are perfect. For $n \in \mathbb{Z}$, the restriction of $\phi_{X,Y}$ to the summand $\mathcal{C}^n(Y,X)\otimes\mathcal{C}^{-n}(X,YS)$ is given by

$$\phi_{X,Y} = \bigl[\mathcal{C}^n(Y,X)\otimes\mathcal{C}^{-n}(X,YS) \xrightarrow{\mu_{\mathcal{C}}} \mathcal{C}^0(Y,YS) \xrightarrow{\tau^0_Y} \Bbbk\bigr].$$

It can be written as follows:

$$\begin{aligned}\phi_{X,Y} = \bigl[\mathcal{C}^n(Y,X)\otimes\mathcal{C}^{-n}(X,YS) &= \mathcal{C}^{[]}((Y,0),(X,n))^0\otimes\mathcal{C}^{[]}((X,n),(YS,0))^0\\ &\xrightarrow{(-)^n\mu_{\mathcal{C}^{[]}}} \mathcal{C}^{[]}((Y,0),(YS,0))^0 = \mathcal{C}^0(Y,YS) \xrightarrow{\tau^0_Y} \Bbbk\bigr].\end{aligned}$$

Since $\mathcal{C}$ is closed under shifts, there exist an object $X[n] \in \operatorname{Ob}\mathcal{C}$ and an isomorphism $\alpha : (X,n) \to (X[n],0)$ in $\mathcal{C}^{[]}$. Using the associativity of $\mu_{\mathcal{C}^{[]}}$, we obtain:

$$\begin{aligned}\phi_{X,Y} &= \bigl[\mathcal{C}^n(Y,X)\otimes\mathcal{C}^{-n}(X,YS) = \mathcal{C}^{[]}((Y,0),(X,n))^0\otimes\mathcal{C}^{[]}((X,n),(YS,0))^0\\ &\qquad\xrightarrow{\mathcal{C}^{[]}(1,\alpha)^0\otimes\mathcal{C}^{[]}(\alpha^{-1},1)^0} \mathcal{C}^{[]}((Y,0),(X[n],0))^0\otimes\mathcal{C}^{[]}((X[n],0),(YS,0))^0\\ &\qquad\xrightarrow{(-)^n\mu_{\mathcal{C}^{[]}}} \mathcal{C}^{[]}((Y,0),(YS,0))^0 = \mathcal{C}^0(Y,YS) \xrightarrow{\tau^0_Y} \Bbbk\bigr]\\ &= \bigl[\mathcal{C}^n(Y,X)\otimes\mathcal{C}^{-n}(X,YS) = \mathcal{C}^{[]}((Y,0),(X,n))^0\otimes\mathcal{C}^{[]}((X,n),(YS,0))^0\\ &\qquad\xrightarrow{\mathcal{C}^{[]}(1,\alpha)^0\otimes\mathcal{C}^{[]}(\alpha^{-1},1)^0} \mathcal{C}^{[]}((Y,0),(X[n],0))^0\otimes\mathcal{C}^{[]}((X[n],0),(YS,0))^0\\ &\qquad= \mathcal{C}^0(Y,X[n])\otimes\mathcal{C}^0(X[n],YS) \xrightarrow{(-)^n\mu_{\mathcal{C}^0}} \mathcal{C}^0(Y,YS) \xrightarrow{\tau^0_Y} \Bbbk\bigr]\\ &= \bigl[\mathcal{C}^n(Y,X)\otimes\mathcal{C}^{-n}(X,YS) = \mathcal{C}^{[]}((Y,0),(X,n))^0\otimes\mathcal{C}^{[]}((X,n),(YS,0))^0\\ &\qquad\xrightarrow{\mathcal{C}^{[]}(1,\alpha)^0\otimes\mathcal{C}^{[]}(\alpha^{-1},1)^0} \mathcal{C}^{[]}((Y,0),(X[n],0))^0\otimes\mathcal{C}^{[]}((X[n],0),(YS,0))^0\\ &\qquad= \mathcal{C}^0(Y,X[n])\otimes\mathcal{C}^0(X[n],YS) \xrightarrow{(-)^n\phi^0_{X[n],Y}} \Bbbk\bigr].\end{aligned}$$

Since $\phi^0_{X[n],Y}$ is a perfect pairing and the maps $\mathcal{C}^{[]}(1,\alpha)^0$ and $\mathcal{C}^{[]}(\alpha^{-1},1)^0$ are invertible, the pairing $\phi_{X,Y}$ is perfect as well. Indeed, it is easy to see that the corresponding maps

$\psi^{-n}_{X,Y}$ and $\psi^{0}_{X[n],Y}$ are related as follows:

$$\psi^{-n}_{X,Y} = \bigl[\mathcal{C}^{-n}(X,YS) \xrightarrow{\mathcal{C}^{[\,]}(\alpha^{-1},1)^0} \mathcal{C}^0(X[n],YS) \xrightarrow{(-)^n\psi^0_{X[n],Y}} \\ \underline{\Bbbk\text{-}\mathbf{Mod}}(\mathcal{C}^0(Y,X[n]),\Bbbk) \xrightarrow{\underline{\Bbbk\text{-}\mathbf{Mod}}(\mathcal{C}^{[\,]}(1,\alpha))^0,1)} \underline{\Bbbk\text{-}\mathbf{Mod}}(\mathcal{C}^n(Y,X),\Bbbk)\bigr].$$

Proposition 2.2.11 implies that there is a right Serre **gr**-functor $S : \mathcal{C} \to \mathcal{C}$. Its components are determined unambiguously by equation (2.2.10). Applying the multifunctor $\widehat{N}$ to it we find that the functor $N_*(S) : \mathcal{C}^0 \to \mathcal{C}^0$ satisfies the same equation the functor $S^0 : \mathcal{C}^0 \to \mathcal{C}^0$ does. By the uniqueness of the solution, $N_*(S) = S^0$. $\square$

CHAPTER 3

A_∞-categories

The notion of A_∞-algebra goes back to Stasheff [**52**]. He introduced it as a linearization of the notion of A_∞-space, a topological space equipped with a product operation which is associative up to homotopy, and the homotopy which makes the product associative can be chosen so that it satisfies a collection of higher coherence conditions. For a long time, A_∞-algebras have been a subject of algebraic topology. Their applications have been studied, e.g., by Smirnov [**49**] and Kadeishvili [**22, 23**]. Over the past decade, A_∞-algebras have experienced a recommencement of interest due to applications in non-commutative geometry, see e.g. Getzler and Jones [**18**], Hamilton and Lazarev [**20**], Kontsevich and Soibelman [**31**]; in algebraic geometry, see e.g. Penkava and Schwarz [**46**]; in representation theory, see e.g. Keller [**27**], Palmieri et al. [**36**], etc.

A_∞-categories are to A_∞-algebras as linear categories to algebras. On the other hand, A_∞-categories generalize differential graded categories, or **dg**-categories, categories enriched in the category **dg** of complexes of modules. In contrast to **dg**-categories, composition in A_∞-categories is associative only up to homotopy that satisfies certain equation up to another homotopy, and so on. The notion of A_∞-category appeared in the work of Fukaya on Floer homology [**15**]. Its relation to mirror symmetry became apparent after Kontsevich's prominent talk at ICM '94 [**30**]. The basic notions of A_∞-category and A_∞-functor have been studied by Fukaya [**16**], Keller [**26**], Lefèvre-Hasegawa [**34**], Soibelman [**51**]. Lyubashenko initiated a 2-category approach to A_∞-categories [**38**]. He introduced the notions of (weakly) unital A_∞-category and A_∞-functor, and proved that these together with certain equivalence classes of natural A_∞-transformations form a 2-category. In particular, the notions of isomorphism of A_∞-functors, of A_∞-equivalence, etc. make sense. According to the recent paper of Kontsevich and Soibelman [**31**], A_∞-categories may be regarded as models for non-commutative varieties. Non-commutative geometry of A_∞-categories brings new insights to the theory and makes many results more transparent, including A_∞-structure on A_∞-functors and the theory of Hochschild complexes, both chain and cochain.

Here we present a different approach to A_∞-categories that is being developed in the book in progress by Bespalov, Lyubashenko, and the author [**3**]. It is based on the observation that (unital) A_∞-categories constitute a closed symmetric multicategory. Since A_∞-categories are defined as differential graded coalgebras of special form, and coalgebras form a symmetric Monoidal category, it is not surprising that A_∞-categories form a symmetric multicategory. Its closedness can be derived easily from existing results about A_∞-categories of A_∞-functors. However, the mere fact of closedness does not help too much if we do not have an explicit description of internal Hom-objects and evaluations. Such a description is obtained in [**3**]. It became possible after the elaboration of an appropriate language, namely that of multicomonads and Kleisli multicategories. These exciting topics lie, however, apart from the subject of the dissertation, and for the reason of size, it seems impossible to discuss them here. Therefore, we have chosen the following compromise: we give a short summary of results from [**3**] that are used in the sequel;

however further results concerning unital A_∞-categories, which are more relevant for the dissertation, are treated at full length. In particular, we give a detailed proof of closedness of the multicategory of unital A_∞-categories.

The chapter is organized as follows. In Section 3.1, we explain our conventions concerning graded modules and complexes, introduce some notation, and establish some basic identities in the closed symmetric Monoidal category of complexes. In Section 3.2.1, we recall the notions of graded span and graded quiver. In Section 3.2.2, we briefly review the definitions of coalgebra, morphism of coalgebras, and coderivation, and work out the main example of interest, the tensor coalgebra of a quiver. We also spend some time studying morphisms and coderivations from a tensor product of tensor coalgebras to another tensor coalgebra, since it is important for understanding A_∞-functors and A_∞-transformations. The symmetric multicategory A_∞ of A_∞-categories is introduced in Section 3.2.16.

Section 3.3 starts with a quick but indirect proof of closedness of the multicategory A_∞. In the rest of the section we discuss the closed structure of A_∞ following [**3**]. In particular, we give explicit formulas for internal Hom-objects and evaluations, and describe the corresponding symmetric A_∞-multicategory $\underline{\mathsf{A}_\infty}$.

Unital A_∞-categories are introduced in Section 3.4. We extend the definition of unital A_∞-functor from [**38**] to A_∞-functors of several arguments. We prove that unital A_∞-categories and unital A_∞-functors form a closed symmetric multicategory $\mathsf{A}^{\mathrm{u}}_\infty$. We extend the functor $\mathsf{k} : \mathsf{A}^{\mathrm{u}}_\infty \to \mathcal{K}$-**Cat** from [**38**] to a symmetric multifunctor. We demonstrate that the multicategory $\mathsf{A}^{\mathrm{u}}_\infty$ may be viewed as a multicategory enriched in the category $\Bbbk$-**Cat** of $\Bbbk$-linear categories, and extend the multifunctor k to a $\Bbbk$-**Cat**-multifunctor. Finally, we relate the closed multicategory approach to unital A_∞-categories with the 2-category approach undertaken in [**38, 39, 40**].

Section 3.5 introduces the operation of dualization for A_∞-categories. To each (unital) A_∞-category $\mathcal{A}$, there is an opposite (unital) A_∞-category $\mathcal{A}^{\mathrm{op}}$. We extend the correspondence $\mathcal{A} \mapsto \mathcal{A}^{\mathrm{op}}$ to a multifunctor $-^{\mathrm{op}} : \mathsf{A}_\infty \to \mathsf{A}_\infty$ and compute its closing transformation.

Section 3.1 does not contain new results. Section 3.2 collects standard material about graded quivers, coalgebras, A_∞-categories, and A_∞-functors. To a large extend, it can be found in [**28**] or [**34**]. The only new points here are the definition of A_∞-functor of several arguments and symmetric multicategory of A_∞-categories. As was mentioned above, Section 3.3 is a short account of the theory of A_∞-categories developed in [**3**]. The results of Section 3.4 can also be found in [*loc. cit.*]. However, these results are crucial for the dissertation, so we decided to include complete proofs. The definition of the multifunctor k (Propositions 3.4.2 and 3.4.3) is due to Volodymyr Lyubashenko. The criterion of unitality of Proposition 3.4.8 and the closedness of the symmetric multicategory of unital categories (Proposition 3.4.12) have been proven by joint efforts of the author and Prof. Lyubashenko. It was the author's observation that the multifunctor k is symmetric. It led to the conclusion that the multicategory of unital A_∞-categories can be regarded as a multicategory enriched in $\Bbbk$-**Cat**, and to an extension of k to a symmetric $\Bbbk$-**Cat**-multifunctor, which is crucial for our treatment of Serre A_∞-functors. Finally, Section 3.5 is entirely an author's contribution.

3.1. Graded modules and complexes

3.1.1. Graded modules. Let us explain our conventions concerning graded modules extending Examples 1.2.9 and 1.2.23. A *$\mathbb{Z}$-graded $\Bbbk$-module* is a sequence of $\Bbbk$-modules $X = (X^n)_{n\in\mathbb{Z}}$. Since no other gradings occur in this dissertation, in the sequel 'graded' will always mean '$\mathbb{Z}$-graded'. A *morphism of graded $\Bbbk$-modules* $f : X \to Y$ of degree d is a sequence of $\Bbbk$-linear maps $f^n : X^n \to Y^{n+d}$, $n \in \mathbb{Z}$. A morphism of degree d is also called a $\Bbbk$-linear map of degree d. Let **gr** denote the category whose objects are graded modules and whose morphisms are $\Bbbk$-linear maps of degree 0. Quite often, a graded $\Bbbk$-module is defined as a $\Bbbk$-module X with a decomposition into a direct sum $X = \bigoplus_{n\in\mathbb{Z}} X^n$. We find this definition somewhat misleading, since, for example, the product of graded $\Bbbk$-modules X_i, $i \in I$, in the category **gr** is given by

$$\bigl(\textstyle\prod_{i\in I} X_i\bigr)^n = \prod_{i\in I} X_i^n, \quad n \in \mathbb{Z}.$$

It differs from the product of $\Bbbk$-modules $\bigoplus_{n\in\mathbb{Z}} X_i^n$, $i \in I$, in the category $\Bbbk$-**Mod** of $\Bbbk$-modules. We need the former definition of the product.

Abusing notation, we write $x \in X$ meaning that $x \in X^n$ for some n. An element $x \in X^n$ is assigned the degree $\deg x = n$.

The category **gr** has the structure of a symmetric Monoidal category described in Example 1.2.23. Moreover, it is closed. For graded $\Bbbk$-modules X and Y, the internal Hom-object $\underline{\mathbf{gr}}(X,Y)$ is a graded $\Bbbk$-module whose d^{th} component $\underline{\mathbf{gr}}(X,Y)^d$ consists of $\Bbbk$-linear maps of degree d. Thus,

$$\underline{\mathbf{gr}}(X,Y)^d = \prod_{n\in\mathbb{Z}} \underline{\Bbbk\text{-}\mathbf{Mod}}(X^n, Y^{n+d}).$$

Here $\underline{\Bbbk\text{-}\mathbf{Mod}}(M,N)$ is the internal Hom-object in the category of $\Bbbk$-modules. As a set, it coincides with $\Bbbk$-$\mathbf{Mod}(M,N)$, the set of $\Bbbk$-linear maps from M to N. The evaluation map $\mathrm{ev}^{\mathbf{gr}} : X \otimes \underline{\mathbf{gr}}(X,Y) \to Y$ is given by

$$\bigl[X^n \otimes \underline{\mathbf{gr}}(X,Y)^d \xrightarrow{1\otimes\mathrm{pr}} X^n \otimes \underline{\Bbbk\text{-}\mathbf{Mod}}(X^n, Y^{n+d}) \xrightarrow{\mathrm{ev}^{\Bbbk\text{-}\mathbf{Mod}}} Y^{n+d}\bigr].$$

Less formally, $\mathrm{ev}^{\mathbf{gr}}$ assigns to an element $x \in X$ of degree n and a $\Bbbk$-linear map $f : X \to Y$ of degree d the element $xf \in Y$ of degree $n + d$.

The closed symmetric Monoidal category **gr** gives rise to a symmetric Monoidal **gr**-category $\underline{\mathbf{gr}}$ in the standard way described Example 1.3.35. In particular, we have a $\Bbbk$-linear map

$$\underline{\otimes} : \underline{\mathbf{gr}}(X,Y) \otimes \underline{\mathbf{gr}}(U,V) \to \underline{\mathbf{gr}}(X \otimes U, Y \otimes V),$$

for arbitrary graded $\Bbbk$-modules X, Y, U, V. It is found from the following equation in **gr**:

$$\begin{tikzcd}
(X \otimes U) \otimes (\underline{\mathbf{gr}}(X,Y) \otimes \mathbf{gr}(U,V)) \arrow[r, "1\otimes\underline{\otimes}"] \arrow[d, "\sigma_{(12)}"'] & (X \otimes Y) \otimes \underline{\mathbf{gr}}(X \otimes U, Y \otimes V) \arrow[d, "\mathrm{ev}^{\mathbf{gr}}"] \\
(X \otimes \underline{\mathbf{gr}}(X,Y)) \otimes (U \otimes \underline{\mathbf{gr}}(U,V)) \arrow[r, "\mathrm{ev}^{\mathbf{gr}} \otimes \mathrm{ev}^{\mathbf{gr}}"] & U \otimes V
\end{tikzcd}$$

It follows that, for $\Bbbk$-linear maps $f : X \to Y$ and $g : U \to V$ of certain degrees, the tensor product $f \underline{\otimes} g$ is given by

$$(x \otimes y)(f \underline{\otimes} g) = (-1)^{\deg f \cdot \deg y} xf \otimes yg,$$

for each $x \in X$ and $y \in Y$. Abusing notation, we denote the tensor product $f \underline{\otimes} g$ simply by $f \otimes g$. This is justified by the fact that if f, g are of degree 0, $f \underline{\otimes} g$ agrees with the tensor product $f \otimes g$ in **gr**. In the sequel, the notation $(-1)^{\deg f \cdot \deg y}$ is abbreviated to $(-)^{fy}$. Similarly, $(-)^x$ means $(-1)^{\deg x}$, $(-)^{x+y}$ means $(-1)^{\deg x + \deg y}$, etc.

3.1.2. Complexes. Let $\mathsf{C}_\Bbbk = \mathbf{dg}$ denote the category of complexes of $\Bbbk$-modules. An object of $\mathsf{C}_\Bbbk$ is a graded $\Bbbk$-module X equipped with a $\Bbbk$-linear map of degree 1, a differential, $d : X \to X$ such that $d^2 = 0$. A morphism $f : (X, d) \to (Y, d)$ in $\mathsf{C}_\Bbbk$ is a $\Bbbk$-linear map $f : X \to Y$ of degree 0 that preserves the differential. A morphism in $\mathsf{C}_\Bbbk$ is also called a chain map. The category $\mathsf{C}_\Bbbk$ is symmetric Monoidal. Its structure is described in Example 1.2.24. Using our conventions about graded $\Bbbk$-modules, the differential in the tensor product $\otimes^{i\in I} X_i$ of complexes (X_i, d), $i \in I$, can be written as $d = \sum_{j\in I} \otimes^I [(1)_{i<j}, d, (1)_{i>j}] : \otimes^{i\in I} X_i \to \otimes^{i\in I} X_i$. Moreover, the category $\mathsf{C}_\Bbbk$ is closed. For each pair of complexes X and Y, the complex $\underline{\mathsf{C}}_\Bbbk(X, Y)$ is the graded $\Bbbk$-module $\underline{\mathbf{gr}}(X, Y)$ equipped with the differential d given by

$$(f^n)_{n\in\mathbb{Z}} \mapsto (f^n d^{n+k} - (-)^k d^n f^{n+1})_{n\in\mathbb{Z}},$$

for each $(f^n)_{n\in\mathbb{Z}} \in \prod_{n\in\mathbb{Z}} \underline{\Bbbk\textbf{-Mod}}(X^n, Y^{n+k})$. The evaluation $\operatorname{ev}^{\mathsf{C}_\Bbbk} : X \otimes \underline{\mathsf{C}}_\Bbbk(X, Y) \to Y$ coincides with $\operatorname{ev}^{\mathbf{gr}}$, which is a chain map. The closed Monoidal category $\mathsf{C}_\Bbbk$ gives rise to a $\mathsf{C}_\Bbbk$-category $\underline{\mathsf{C}}_\Bbbk$ is the standard way. Categories enriched in the category $\mathsf{C}_\Bbbk = \mathbf{dg}$ are also called differential graded categories or simply $\mathbf{dg}$-categories. Thus, $\underline{\mathsf{C}}_\Bbbk$ is a differential graded category. Its objects are complexes of $\Bbbk$-modules. For a pair of complexes X and Y, we have the complex $\underline{\mathsf{C}}_\Bbbk(X, Y)$ described above. The composition

$$\mu_{\underline{\mathsf{C}}_\Bbbk} : \underline{\mathsf{C}}_\Bbbk(X, Y) \otimes \underline{\mathsf{C}}_\Bbbk(Y, Z) \to \underline{\mathsf{C}}_\Bbbk(X, Z)$$

in $\underline{\mathsf{C}}_\Bbbk$ coincides with the composition in $\underline{\mathbf{gr}}$. It is induced by composition of $\Bbbk$-linear maps of certain degree. For each complex X, we have the identity morphism $1_X : \Bbbk \to \underline{\mathsf{C}}_\Bbbk(X, X)$, $1 \mapsto (1_{X^n})_{n\in\mathbb{Z}}$. It is convenient to denote the differential in $\underline{\mathsf{C}}_\Bbbk$ by $m_1 = m_1^{\underline{\mathsf{C}}_\Bbbk}$ and composition by $m_2 = m_2^{\underline{\mathsf{C}}_\Bbbk}$. The reason for this will become clear soon.

Since internal Hom-objects in the categories $\mathbf{gr}$ and $\underline{\mathsf{C}}_\Bbbk$ coincide as graded $\Bbbk$-modules, for each pair of complexes X and Y, we often confuse the notation and write $\underline{\mathsf{C}}_\Bbbk(X, Y)$ even when $\underline{\mathbf{gr}}(X, Y)$ seems more appropriate. This is reflected by the following bit of notation.

Given a complex Z and an element $a \in \underline{\mathsf{C}}_\Bbbk(X, Y)$, we assign to it elements

$$1 \otimes a \in \underline{\mathsf{C}}_\Bbbk(Z \otimes X, Z \otimes Y), \qquad (z \otimes x)(1 \otimes a) = z \otimes xa,$$
$$a \otimes 1 \in \underline{\mathsf{C}}_\Bbbk(X \otimes Z, Y \otimes Z), \qquad (x \otimes z)(a \otimes 1) = (-)^{za} xa \otimes z.$$

Clearly, $(1\otimes a)c = c(a\otimes 1) \in \underline{\mathsf{C}}_\Bbbk(Z\otimes X, Y\otimes Z)$ and $(a\otimes 1)c = c(1\otimes a) \in \underline{\mathsf{C}}_\Bbbk(X\otimes Z, Z\otimes Y)$. If $g \in \underline{\mathsf{C}}_\Bbbk(Z, W)$, then we have $(1 \otimes a)(g \otimes 1) = (-)^{ag}(g \otimes 1)(1 \otimes a) \in \underline{\mathsf{C}}_\Bbbk(Z \otimes X, W \otimes Y)$.

A complex Z gives rise to a chain map

$$\underline{\mathsf{C}}_\Bbbk(Z, -) : \underline{\mathsf{C}}_\Bbbk(X, Y) \to \underline{\mathsf{C}}_\Bbbk(\underline{\mathsf{C}}_\Bbbk(Z, X), \underline{\mathsf{C}}_\Bbbk(Z, Y))$$

found from the following equation in $\mathsf{C}_\Bbbk$:

$$\bigl[\underline{\mathsf{C}}_\Bbbk(Z, X)\otimes\underline{\mathsf{C}}_\Bbbk(X, Y) \xrightarrow{1\otimes\underline{\mathsf{C}}_\Bbbk(Z,-)} \underline{\mathsf{C}}_\Bbbk(Z, X)\otimes\underline{\mathsf{C}}_\Bbbk(\underline{\mathsf{C}}_\Bbbk(Z, X), \underline{\mathsf{C}}_\Bbbk(Z, Y)) \xrightarrow{\operatorname{ev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(Z, Y)\bigr] = m_2^{\underline{\mathsf{C}}_\Bbbk},$$

cf. Example 1.3.18. It maps an element $a \in \underline{\mathsf{C}}_\Bbbk(X, Y)$ to the element $\underline{\mathsf{C}}_\Bbbk(Z, a) = \underline{\mathsf{C}}_\Bbbk(1, a)$ of $\underline{\mathsf{C}}_\Bbbk(\underline{\mathsf{C}}_\Bbbk(Z, X), \underline{\mathsf{C}}_\Bbbk(Z, Y))$. Despite that the map a is not a chain map we write this element as $a : X \to Y$, and we write $\underline{\mathsf{C}}_\Bbbk(1, a)$ as

$$\underline{\mathsf{C}}_\Bbbk(1, a) : \underline{\mathsf{C}}_\Bbbk(Z, X) \to \underline{\mathsf{C}}_\Bbbk(Z, Y), \qquad (f^i)_{i\in\mathbb{Z}} \mapsto (f^i a^{i+\deg f})_{i\in\mathbb{Z}}.$$

It is not a chain map, only a $\Bbbk$-linear map of degree $\deg a$. Similarly, a complex X gives rise to a chain map

$$\underline{\mathsf{C}}_\Bbbk(-, X) : \underline{\mathsf{C}}_\Bbbk(W, Z) \to \underline{\mathsf{C}}_\Bbbk(\underline{\mathsf{C}}_\Bbbk(Z, X), \underline{\mathsf{C}}_\Bbbk(W, X))$$

found from the following equation in $\underline{\mathsf{C}}_\Bbbk$:

$$\begin{tikzcd}
\underline{\mathsf{C}}_\Bbbk(Z,X)\otimes\underline{\mathsf{C}}_\Bbbk(W,Z) \arrow[r,"{1,\underline{\mathsf{C}}_\Bbbk(-,X)}"] \arrow[d,"c"'] & \underline{\mathsf{C}}_\Bbbk(Z,X)\otimes\underline{\mathsf{C}}_\Bbbk(\underline{\mathsf{C}}_\Bbbk(Z,X),\underline{\mathsf{C}}_\Bbbk(W,X)) \arrow[d,"\mathrm{ev}^{\mathsf{C}_\Bbbk}"] \\
\underline{\mathsf{C}}_\Bbbk(W,Z)\otimes\underline{\mathsf{C}}_\Bbbk(Z,X) \arrow[r,"m_2^{\mathsf{C}_\Bbbk}"] & \underline{\mathsf{C}}_\Bbbk(W,X)
\end{tikzcd}$$

cf. Example 1.3.19. It takes an element $g\in\underline{\mathsf{C}}_\Bbbk(W,Z)$ to the element $\underline{\mathsf{C}}_\Bbbk(g,X)=\underline{\mathsf{C}}_\Bbbk(g,1)$ of $\underline{\mathsf{C}}_\Bbbk(\underline{\mathsf{C}}_\Bbbk(Z,X),\underline{\mathsf{C}}_\Bbbk(W,X))$. Although the map $\underline{\mathsf{C}}_\Bbbk(g,1)$ is not a chain map, only a $\Bbbk$-linear map of degree $\deg g$, we write it as

$$\underline{\mathsf{C}}_\Bbbk(g,1):\underline{\mathsf{C}}_\Bbbk(Z,X)\to\underline{\mathsf{C}}_\Bbbk(W,X),\qquad (f^i)_{i\in\mathbb{Z}}\mapsto\big((-)^{\deg f\cdot\deg g}g^if^{i+\deg g}\big)_{i\in\mathbb{Z}}.$$

For each pair of elements $a\in\underline{\mathsf{C}}_\Bbbk(X,Y)$, $g\in\underline{\mathsf{C}}_\Bbbk(W,Z)$ we have

$$\begin{aligned}\big[\underline{\mathsf{C}}_\Bbbk(Z,X)\xrightarrow{\underline{\mathsf{C}}_\Bbbk(Z,a)}\underline{\mathsf{C}}_\Bbbk(Z,Y)\xrightarrow{\underline{\mathsf{C}}_\Bbbk(g,Y)}\underline{\mathsf{C}}_\Bbbk(W,Y)\big]&\\ =(-)^{ag}\big[\underline{\mathsf{C}}_\Bbbk(Z,X)\xrightarrow{\underline{\mathsf{C}}_\Bbbk(g,X)}\underline{\mathsf{C}}_\Bbbk(W,X)&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(W,a)}\underline{\mathsf{C}}_\Bbbk(W,Y)\big].\end{aligned}$$

This equation follows from one of the standard identities in closed symmetric monoidal categories [**14**], and can be verified directly. We also have $\underline{\mathsf{C}}_\Bbbk(1,a)\underline{\mathsf{C}}_\Bbbk(1,h)=\underline{\mathsf{C}}_\Bbbk(1,ah)$ and $\underline{\mathsf{C}}_\Bbbk(g,1)\underline{\mathsf{C}}_\Bbbk(e,1)=(-)^{ge}\underline{\mathsf{C}}_\Bbbk(eg,1)$, whenever these maps are defined.

It is easy to see that the differential $m_1^{\underline{\mathsf{C}}_\Bbbk}:\underline{\mathsf{C}}_\Bbbk(X,Y)\to\underline{\mathsf{C}}_\Bbbk(X,Y)$ coincides with $\underline{\mathsf{C}}_\Bbbk(1,d_Y)-\underline{\mathsf{C}}_\Bbbk(d_X,1)$, where $d_X:X\to X$ and $d_Y:Y\to Y$ are the differentials in complexes X and Y respectively.

Let $f:A\otimes X\to B$, $g:B\otimes Y\to C$ be $\Bbbk$-linear maps of arbitrary degrees. Then the following holds:

$$\begin{aligned}\big[X\otimes Y\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}_{A,X}\otimes\mathrm{coev}^{\mathsf{C}_\Bbbk}_{B,Y}}\underline{\mathsf{C}}_\Bbbk(A,A\otimes X)\otimes\underline{\mathsf{C}}_\Bbbk(B,B\otimes Y)&\\ \xrightarrow{\underline{\mathsf{C}}_\Bbbk(A,f)\otimes\underline{\mathsf{C}}_\Bbbk(B,g)}\underline{\mathsf{C}}_\Bbbk(A,B)\otimes\underline{\mathsf{C}}_\Bbbk(B,C)&\xrightarrow{m_2^{\mathsf{C}_\Bbbk}}\underline{\mathsf{C}}_\Bbbk(A,C)\big]\\ =\big[X\otimes Y\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}_{A,X\otimes Y}}\underline{\mathsf{C}}_\Bbbk(A,A\otimes X\otimes Y)\xrightarrow{\underline{\mathsf{C}}_\Bbbk(A,f\otimes1)}\underline{\mathsf{C}}_\Bbbk(A,B\otimes Y)&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(A,g)}\underline{\mathsf{C}}_\Bbbk(A,C)\big].\end{aligned}\tag{3.1.1}$$

Indeed, $(\mathrm{coev}_{A,X}\otimes\mathrm{coev}_{B,Y})(\underline{\mathsf{C}}_\Bbbk(A,f)\otimes\underline{\mathsf{C}}_\Bbbk(B,g))=(\mathrm{coev}_{A,X}\underline{\mathsf{C}}_\Bbbk(A,f)\otimes\mathrm{coev}_{B,Y}\underline{\mathsf{C}}_\Bbbk(B,g))$, for coev has degree 0. Denote $\bar f=\mathrm{coev}_{A,X}\underline{\mathsf{C}}_\Bbbk(A,f)$, $\bar g=\mathrm{coev}_{B,Y}\underline{\mathsf{C}}_\Bbbk(B,g)$. The morphisms $\bar f$ and $\bar g$ correspond to f and g by adjunction (1.3.5). Further, the morphism $m_2^{\underline{\mathsf{C}}_\Bbbk}$ comes by adjunction from the following map:

$$A\otimes\underline{\mathsf{C}}_\Bbbk(A,B)\otimes\underline{\mathsf{C}}_\Bbbk(B,C)\xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}_{A,B}\otimes1}B\otimes\underline{\mathsf{C}}_\Bbbk(B,C)\xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}_{B,C}}C.$$

Thus we have a commutative diagram

$$\begin{tikzcd}
A \otimes X \otimes Y \arrow[r, "1\otimes\bar{f}\otimes 1"] & A \otimes \underline{\mathsf{C}}_\Bbbk(A,B) \otimes Y \arrow[r, "\operatorname{ev}^{\mathsf{C}_\Bbbk}_{A,B}\otimes 1"] \arrow[d, "1\otimes 1\otimes\bar{g}"'] & B \otimes Y \arrow[d, "1\otimes\bar{g}"] \\
& A \otimes \underline{\mathsf{C}}_\Bbbk(A,B) \otimes \underline{\mathsf{C}}_\Bbbk(B,C) \arrow[r, "\operatorname{ev}^{\mathsf{C}_\Bbbk}_{A,B}\otimes 1"] \arrow[d, "1\otimes m_2^{\mathsf{C}_\Bbbk}"'] & B \otimes \underline{\mathsf{C}}_\Bbbk(B,C) \arrow[d, "\operatorname{ev}^{\mathsf{C}_\Bbbk}_{B,C}"] \\
& A \otimes \underline{\mathsf{C}}_\Bbbk(A,C) \arrow[r, "\operatorname{ev}^{\mathsf{C}_\Bbbk}_{A,C}"] & C
\end{tikzcd}$$

The top-right composite coincides with $(f\otimes 1)g$ by adjunction (1.3.5), thus so does the top-left-bottom composite. On the other hand, $(1\otimes\bar{f}\otimes 1)(1\otimes 1\otimes\bar{g})(1\otimes m_2^{\mathsf{C}_\Bbbk})\operatorname{ev}^{\mathsf{C}_\Bbbk}_{A,C}$ is the image of $(\bar{f}\otimes\bar{g})m_2$ under adjunction (1.3.5), so that $(\bar{f}\otimes\bar{g})m_2^{\mathsf{C}_\Bbbk} = \operatorname{coev}_{A,X\otimes Y}\underline{\mathsf{C}}_\Bbbk(A,(f\otimes 1)g)$, and we are done.

One verifies similarly the following assertion: for each homogeneous $a \in \underline{\mathsf{C}}_\Bbbk(X,A)$, the equation

$$\begin{aligned}
&\big[\underline{\mathsf{C}}_\Bbbk(A,B)\otimes\underline{\mathsf{C}}_\Bbbk(B,C) \xrightarrow{m_2^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(A,C) \xrightarrow{\underline{\mathsf{C}}_\Bbbk(a,C)} \underline{\mathsf{C}}_\Bbbk(X,C)\big] \\
&\quad = \big[\underline{\mathsf{C}}_\Bbbk(A,B)\otimes\underline{\mathsf{C}}_\Bbbk(B,C) \xrightarrow{\underline{\mathsf{C}}_\Bbbk(a,B)\otimes 1} \underline{\mathsf{C}}_\Bbbk(X,B)\otimes\underline{\mathsf{C}}_\Bbbk(B,C) \xrightarrow{m_2^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(X,C)\big]. \qquad (3.1.2)
\end{aligned}$$

holds true.

Let $\mathcal{K}$ denote the homotopy category of complexes of $\Bbbk$-modules. Tensor product of complexes in $\mathsf{C}_\Bbbk$ induces a tensor product in $\mathcal{K}$, and $\mathcal{K}$ becomes a symmetric Monoidal category. Moreover, it is closed. For each pair of complexes X and Y, the internal Hom-object $\underline{\mathcal{K}}(X,Y)$ is the same as the internal Hom-complex $\underline{\mathsf{C}}_\Bbbk(X,Y)$ in the closed symmetric Monoidal category $\mathsf{C}_\Bbbk$. The evaluation morphism $\operatorname{ev}^{\mathcal{K}} : X\otimes\underline{\mathcal{K}}(X,Y)\to Y$ and the coevaluation morphism $\operatorname{coev}^{\mathcal{K}} : Y\to\underline{\mathcal{K}}(X,X\otimes Y)$ in $\mathcal{K}$ are the homotopy classes of the evaluation morphism $\operatorname{ev}^{\mathsf{C}_\Bbbk} : X\otimes\underline{\mathsf{C}}_\Bbbk(X,Y)\to Y$ and the coevaluation morphism $\operatorname{coev}^{\mathsf{C}_\Bbbk} : Y\to\underline{\mathsf{C}}_\Bbbk(X,X\otimes Y)$ in $\mathsf{C}_\Bbbk$ respectively.

3.2. A_∞-categories and A_∞-functors

The notion of graded quiver underlies the definition of A_∞-category in the same way as the notion of quiver (or graph) underlies the definition of category. Roughly speaking, a graded quiver is a graded module distributed over a set of points. We begin by giving precise definitions and discussing Monoidal structures on quivers. In Chapter 4, where we treat A_∞-bimodules, we will also need a generalization of graded quivers, called graded spans. We discuss both concepts simultaneously.

3.2.1. Spans and quivers. A *graded span* $\mathcal{A}$ consists of a set of source objects $\operatorname{Ob}_s\mathcal{A}$, a set of target objects $\operatorname{Ob}_t\mathcal{A}$, and for each $X\in\operatorname{Ob}_s\mathcal{A}$, $Y\in\operatorname{Ob}_t\mathcal{A}$, a graded $\Bbbk$-module of morphisms $\mathcal{A}(X,Y)$. A morphisms of graded spans $f:\mathcal{A}\to\mathcal{B}$ of degree n consists of functions

$$\operatorname{Ob}_s f : \operatorname{Ob}_s\mathcal{A}\to\operatorname{Ob}_s\mathcal{B}, \quad \operatorname{Ob}_t f : \operatorname{Ob}_t\mathcal{A}\to\operatorname{Ob}_t\mathcal{B},$$

and for each $X\in\operatorname{Ob}_s\mathcal{A}$, $Y\in\operatorname{Ob}_t\mathcal{B}$, a $\Bbbk$-linear map

$$f = f_{X,Y} : \mathcal{A}(X,Y)\to\mathcal{B}(X.\operatorname{Ob}_s f, Y.\operatorname{Ob}_t f)$$

of degree n. With obvious composition and identities, graded spans constitute a category. It admits the structure of a symmetric Monoidal category. The tensor product of graded spans $\mathcal{Q}_i$, $i \in I$, is given by

$$\operatorname{Ob}_s \boxtimes^{i\in I}\mathcal{Q}_i = \prod_{i\in I}\operatorname{Ob}_s \mathcal{Q}_i, \quad \operatorname{Ob}_t \boxtimes^{i\in I}\mathcal{Q}_i = \prod_{i\in I}\operatorname{Ob}_t \mathcal{Q}_i,$$

and $(\boxtimes^{i\in I}\mathcal{Q}_i)((X_i)_{i\in I},(Y_i)_{i\in I}) = \otimes^{i\in I}\mathcal{Q}_i(X_i,Y_i)$, $X_i \in \operatorname{Ob}_s \mathcal{Q}_i$, $Y_i \in \operatorname{Ob}_t \mathcal{Q}_i$ $i \in I$. For each map $f : I \to J$ in $\operatorname{Mor}\mathcal{S}$, the isomorphism $\lambda^f : \boxtimes^{i\in I}\mathcal{Q}_i \xrightarrow{\sim} \boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}\mathcal{Q}_i$ consists of the bijections

$$\lambda^f_{\mathbf{Set}} : \prod_{i\in I}\operatorname{Ob}_s \mathcal{Q}_i \xrightarrow{\sim} \prod_{j\in J}\prod_{i\in f^{-1}j}\operatorname{Ob}_s \mathcal{Q}_i, \quad \lambda^f_{\mathbf{Set}} : \prod_{i\in I}\operatorname{Ob}_t \mathcal{Q}_i \xrightarrow{\sim} \prod_{j\in J}\prod_{i\in f^{-1}j}\operatorname{Ob}_t \mathcal{Q}_i$$

on objects and of isomorphisms

$$\lambda^f_{\mathbf{gr}} : \otimes^{i\in I}\mathcal{Q}_i(X_i,Y_i) \xrightarrow{\sim} \otimes^{j\in J}\otimes^{i\in f^{-1}j}\mathcal{Q}_i(X_i,Y_i)$$

on morphisms. The unit object is the graded span $\mathbb{1}$ with $\operatorname{Ob}_s \mathbb{1} = \operatorname{Ob}_t \mathbb{1} = \{*\}$ and $\mathbb{1}(*,*) = \mathbb{k}$.

If $f, g : \mathcal{A} \to \mathcal{B}$ are morphisms of graded spans of the same degree and such that $\operatorname{Ob}_s f = \operatorname{Ob}_s g$, $\operatorname{Ob}_t f = \operatorname{Ob}_t g$, then the sum $f + g : \mathcal{A} \to \mathcal{B}$ is a well-defined morphism of graded spans.

If $\mathcal{P}$ and $\mathcal{Q}$ are graded spans such that $\operatorname{Ob}_t \mathcal{P} = \operatorname{Ob}_s \mathcal{Q}$, then there exists a tensor product $\mathcal{P}\otimes\mathcal{Q}$, a graded span given by $\operatorname{Ob}_s \mathcal{P}\otimes\mathcal{Q} = \operatorname{Ob}_s \mathcal{P}$, $\operatorname{Ob}_t \mathcal{P}\otimes\mathcal{Q} = \operatorname{Ob}_t \mathcal{Q}$, and

$$(\mathcal{P}\otimes\mathcal{Q})(X,Z) = \bigoplus_{Y\in\operatorname{Ob}_t \mathcal{P}=\operatorname{Ob}_s \mathcal{Q}} \mathcal{P}(X,Y)\otimes\mathcal{Q}(Y,Z),$$

for each $X \in \operatorname{Ob}_s \mathcal{P}$, $Z \in \operatorname{Ob}_t \mathcal{Q}$. Similarly, if $f : \mathcal{M} \to \mathcal{P}$ and $g : \mathcal{N} \to \mathcal{Q}$ are morphisms of graded spans such that

$$\operatorname{Ob}_t f = \operatorname{Ob}_s g : \operatorname{Ob}_t \mathcal{M} = \operatorname{Ob}_s \mathcal{N} \to \operatorname{Ob}_t \mathcal{P} = \operatorname{Ob}_s \mathcal{Q},$$

then there exists a tensor product $f\otimes g : \mathcal{M}\otimes\mathcal{N} \to \mathcal{P}\otimes\mathcal{Q}$ given by the functions

$$\operatorname{Ob}_s f\otimes g = \operatorname{Ob}_s f : \operatorname{Ob}_s \mathcal{M} \to \operatorname{Ob}_s \mathcal{P}, \quad \operatorname{Ob}_t f\otimes g = \operatorname{Ob}_t g : \operatorname{Ob}_t \mathcal{N} \to \operatorname{Ob}_t \mathcal{Q},$$

and by $\mathbb{k}$-linear maps

$$\begin{tikzcd}
\bigoplus_{Y\in\operatorname{Ob}_t \mathcal{M}=\operatorname{Ob}_s \mathcal{N}}\mathcal{M}(X,Y)\otimes\mathcal{N}(Y,Z) \arrow[d, "\sum f_{X,Y}\otimes g_{Y,Z}"] \\
\bigoplus_{Y\in\operatorname{Ob}_t \mathcal{M}=\operatorname{Ob}_s \mathcal{N}}\mathcal{P}(X.\operatorname{Ob}_s f, Y.\operatorname{Ob}_t f)\otimes\mathcal{Q}(Y.\operatorname{Ob}_s g, Z.\operatorname{Ob}_t g) \arrow[d, hook] \\
\bigoplus_{U\in\operatorname{Ob}_t \mathcal{P}=\operatorname{Ob}_s \mathcal{Q}}\mathcal{P}(X.\operatorname{Ob}_s f, U)\otimes\mathcal{Q}(U, Z.\operatorname{Ob}_t g),
\end{tikzcd}$$

for each $X \in \operatorname{Ob}_s \mathcal{M}$, $Y \in \operatorname{Ob}_t \mathcal{N}$. Extension to an arbitrary positive number of factors is straightforward.

A *graded quiver* $\mathcal{A}$ is a graded span such that $\operatorname{Ob}_s \mathcal{A} = \operatorname{Ob}_t \mathcal{A} = \operatorname{Ob}\mathcal{A}$. For graded quivers $\mathcal{A}$, $\mathcal{B}$, a morphism of graded quivers $f : \mathcal{A} \to \mathcal{B}$ is a morphism of graded spans of degree 0 such that $\operatorname{Ob}_s f = \operatorname{Ob}_t f = \operatorname{Ob} f$. We write Xf instead of $X.\operatorname{Ob} f$. Let $\mathscr{Q}$ denote the category of graded quivers. It inherits the structure of a symmetric Monoidal category $(\mathscr{Q}, \boxtimes^I, \lambda^f)$.

An arbitrary set S gives rise to a quiver $\mathbb{k}S$ given by $\operatorname{Ob}\mathbb{k}S = S$, $\mathbb{k}S(X,X) = \mathbb{k}$ and $\mathbb{k}S(X,Y) = 0$ if $X \neq Y$, $X, Y \in S$. A map $f : S \to R$ of sets gives rise to a morphism of quivers $\Bbbk f : \Bbbk S \to \Bbbk R$ given by $\operatorname{Ob} \Bbbk f = f$, $f_{X,X} = \mathrm{id}_\Bbbk : \Bbbk S(X,X) \to \Bbbk R(Xf, Xf)$ and $f_{X,Y} = 0 : \Bbbk S(X,Y) = 0 \to \Bbbk R(Xf, Yf)$ if $X \neq Y$, $X, Y \in S$. Given a quiver $\mathcal{A}$, we abbreviate $\Bbbk \operatorname{Ob} \mathcal{A}$ to $\Bbbk \mathcal{A}$. For a morphism of quivers $f : \mathcal{A} \to \mathcal{B}$, the morphism $\Bbbk \operatorname{Ob} f$ is abbreviated to $\Bbbk f$.

For a set S, denote by $\mathscr{Q}/S$ the subcategory of $\mathscr{Q}$ whose objects are graded quivers $\mathcal{A}$ such that $\operatorname{Ob} \mathcal{A} = S$, and whose morphisms are morphisms of graded quivers $f : \mathcal{A} \to \mathcal{B}$ such that $\operatorname{Ob} f = \mathrm{id}_S$. The abelian $\Bbbk$-linear category $\mathscr{Q}/S$ is Monoidal. The tensor product of quivers $\mathcal{A}_i$, $i \in \mathbf{n}$, is given by

$$(\otimes^{i \in \mathbf{n}} \mathcal{A}_i)(X, Z) = \bigoplus_{Y_1, \dots, Y_{n-1} \in S}^{X = Y_0, Y_n = Z} \otimes^{i \in \mathbf{n}} \mathcal{A}_i(Y_{i-1}, Y_i), \quad X, Z \in S.$$

Isomorphisms λ^f extend linearly those of $\mathbf{gr}$. The unit object is the graded quiver $\Bbbk S$. If S is a 1-element set, the category $\mathscr{Q}/S$ is naturally equivalent (as a Monoidal category) to the category of graded $\Bbbk$-modules.

In particular, for each graded quiver $\mathcal{A}$ and $n \geqslant 0$, there is an n-fold tensor product $T^n \mathcal{A} = \otimes^{\mathbf{n}} \mathcal{A}$ in $\mathscr{Q}/ \operatorname{Ob} \mathcal{A}$. For example, $T^0 \mathcal{A} = \Bbbk \mathcal{A}$, $T^1 \mathcal{A} = \mathcal{A}$, $T^2 \mathcal{A} = \mathcal{A} \otimes \mathcal{A}$ etc. A morphism of quivers $f : \mathcal{A} \to \mathcal{B}$ gives rise to a morphism

$$T^n f = \otimes^{\mathbf{n}} f : T^n \mathcal{A} = \otimes^{\mathbf{n}} \mathcal{A} \to T^n \mathcal{B} = \otimes^{\mathbf{n}} \mathcal{B},$$

thus we obtain a functor $T^n : \mathscr{Q} \to \mathscr{Q}$. For a graded quiver $\mathcal{A}$ and a sequence of objects $(X_0, \dots, X_n)$ of $\mathcal{A}$ we use the notation

$$\begin{aligned} \bar{T}^n \mathcal{A}(X_0, \dots, X_n) &= \mathcal{A}(X_0, X_1) \otimes \dots \otimes \mathcal{A}(X_{n-1}, X_n), \qquad (3.2.1) \\ T^n \mathcal{A}(X_0, X_n) &= \bigoplus_{X_1, \dots, X_{n-1} \in \operatorname{Ob} \mathcal{C}} \bar{T}^n \mathcal{A}(X_0, \dots, X_n). \end{aligned}$$

When the list of arguments is obvious we abbreviate the notation $\bar{T}^n \mathcal{A}(X_0, \dots, X_n)$ to $\bar{T}^n \mathcal{A}(X_0, X_n)$.

Suppose that $\mathcal{A}_i^j$ are graded quivers, $i \in I$, $j \in \mathbf{m}$, and that $\operatorname{Ob} \mathcal{A}_i^j = S_i$ does not depend on j. Define $S = \prod_{i \in I} S_i$. Denote by $\otimes_{S_i}$ the tensor product in $\mathscr{Q}/S_i$. There is an isomorphism of graded quivers

$$\overline{\varkappa} : \otimes_S^{j \in \mathbf{m}} \boxtimes^{i \in I} \mathcal{A}_i^j \xrightarrow{\sim} \boxtimes^{i \in I} \otimes_{S_i}^{j \in \mathbf{m}} \mathcal{A}_i^j, \qquad (3.2.2)$$

identity on objects, which is a direct sum of permutation isomorphisms

$$\sigma_{(12)} : \otimes^{j \in \mathbf{m}} \otimes^{i \in I} \mathcal{A}_i^j(X_i^{j-1}, X_i^j) \xrightarrow{\sim} \otimes^{i \in I} \otimes^{j \in \mathbf{m}} \mathcal{A}_i^j(X_i^{j-1}, X_i^j),$$

see (1.1.4), where $X_i^j \in S_i$, $0 \leqslant j \leqslant m$. In the particular case $\mathcal{A}_i^j = \mathcal{A}_i$ we get isomorphisms

$$\overline{\varkappa} : T^m \boxtimes^{i \in I} \mathcal{A}_i \xrightarrow{\sim} \boxtimes^{i \in I} T^m \mathcal{A}_i.$$

3.2.2. Coalgebras. A_∞-categories are defined as augmented differential graded coalgebras of special form. We adopt the following definition.

3.2.3. Definition. A *graded coalgebra* is a graded quiver $\mathcal{C}$ equipped with morphisms $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and $\varepsilon : \mathcal{C} \to \Bbbk \mathcal{C}$ in $\mathscr{Q}/ \operatorname{Ob} \mathcal{C}$ such that the triple $(\mathcal{C}, \Delta, \varepsilon)$ is a coassociative counital coalgebra in the Monoidal category $\mathscr{Q}/ \operatorname{Ob} \mathcal{C}$. Thus, the morphism $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$, the *comultiplication*, makes the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathcal{C} \otimes \mathcal{C} \arrow[d, "1\otimes\Delta"] \\
\mathcal{C} \otimes \mathcal{C} \arrow[r, "\Delta\otimes 1"] & \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}
\end{tikzcd}$$

commute, and the morphism $\varepsilon : \mathcal{C} \to \Bbbk\mathcal{C}$, the *counit*, satisfies the equations

$$\bigl[\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{1\otimes\varepsilon} \mathcal{C}\bigr] = \mathrm{id}_{\mathcal{C}}, \quad \bigl[\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{\varepsilon\otimes 1} \mathcal{C}\bigr] = \mathrm{id}_{\mathcal{C}}.$$

We are committing abuse of notation by silently identifying $\mathcal{C} \otimes (\mathcal{C} \otimes \mathcal{C})$ and $(\mathcal{C} \otimes \mathcal{C}) \otimes \mathcal{C}$ with $\mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}$. The detailed form of the coassociativity axiom is

$$\bigl[\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{1\otimes\Delta} \mathcal{C} \otimes (\mathcal{C} \otimes \mathcal{C}) \xrightarrow[\sim]{(\lambda^{\mathsf{IV}})^{-1}} \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}\bigr]$$
$$= \bigl[\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{\Delta\otimes 1} (\mathcal{C} \otimes \mathcal{C}) \otimes \mathcal{C} \xrightarrow[\sim]{(\lambda^{\mathsf{VI}})^{-1}} \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}\bigr].$$

Similarly, $\mathcal{C} \otimes \Bbbk\mathcal{C}$ and $\Bbbk\mathcal{C} \otimes \mathcal{C}$ are identified with $\mathcal{C}$. More precisely, the counit axiom reads as follows:

$$\bigl[\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{1\otimes\varepsilon} \mathcal{C} \otimes \Bbbk\mathcal{C} \xrightarrow[\sim]{(\lambda^{\bullet\mathsf{I}})^{-1}} \mathcal{C}\bigr] = \mathrm{id}_{\mathcal{C}},$$
$$\bigl[\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{\varepsilon\otimes 1} \Bbbk\mathcal{C} \otimes \mathcal{C} \xrightarrow[\sim]{(\lambda^{\mathsf{I}\bullet})^{-1}} \mathcal{C}\bigr] = \mathrm{id}_{\mathcal{C}}.$$

A morphism of graded coalgebras $f : (\mathcal{C}, \Delta, \varepsilon) \to (\mathcal{D}, \Delta, \varepsilon)$ is a morphism of graded quivers $f : \mathcal{C} \to \mathcal{D}$ that preserves the comultiplication and the counit, meaning that

$$\bigl[\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{\Delta} \mathcal{D} \otimes \mathcal{D}\bigr] = \bigl[\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{f\otimes f} \mathcal{D} \otimes \mathcal{D}\bigr], \quad \bigl[\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{\varepsilon} \Bbbk\mathcal{D}\bigr] = \bigl[\mathcal{C} \xrightarrow{\varepsilon} \Bbbk\mathcal{C} \xrightarrow{\Bbbk f} \Bbbk\mathcal{D}\bigr].$$

3.2.4. Example. For a set S, the graded quiver $\Bbbk S$ is turned into a graded coalgebra by defining the comultiplication to be the isomorphism $\lambda^{\varnothing\to\mathbf{2}} : \Bbbk S \xrightarrow{\sim} \Bbbk S \otimes \Bbbk S$ in $\mathscr{Q}/S$ and the counit to be the identity $\Bbbk S \to \Bbbk S$. For a map $f : S \to R$, the induced morphisms of quivers $\Bbbk f : \Bbbk S \to \Bbbk R$ is a morphism of graded coalgebras.

3.2.5. Definition. Suppose $(\mathcal{C}, \Delta, \varepsilon)$ is a graded coalgebra. An *augmentation* is a morphism of graded coalgebras $\eta : \Bbbk\mathcal{C} \to \mathcal{C}$. Thus, the diagram

$$\begin{tikzcd}
\Bbbk\mathcal{C} \arrow[r, "\eta"] \arrow[d, "\lambda^{\varnothing\to\mathbf{2}}"', "\wr"] & \mathcal{C} \arrow[d, "\Delta"] \\
\Bbbk\mathcal{C} \otimes \Bbbk\mathcal{C} \arrow[r, "\eta\otimes\eta"] & \mathcal{C} \otimes \mathcal{C}
\end{tikzcd}$$

commutes, and the composite $\Bbbk\mathcal{C} \xrightarrow{\eta} \mathcal{C} \xrightarrow{\varepsilon} \Bbbk\mathcal{C}$ is the identity of the graded quiver $\Bbbk\mathcal{C}$. An *augmented graded coalgebra* is a graded coalgebra equipped with an augmentation. A morphism of augmented graded coalgebras is a morphism of graded coalgebras that preserves the augmentation. Let $\mathrm{ac}\mathscr{Q}$ denote the category of augmented graded coalgebras.

3.2.6. Example. For a set S, the graded coalgebra $\Bbbk S$ becomes an augmented graded coalgebra when equipped with an augmentation given by the identity $\Bbbk S \to \Bbbk S$. Clearly, for a map $f : S \to R$, the induced morphism of graded coalgebras $\Bbbk f : \Bbbk S \to \Bbbk R$ preserves the augmentation.

3.2.7. Example. Let $\mathcal{A}$ be a graded quiver. The tensor quiver $T\mathcal{A} = \bigoplus_{n=0}^{\infty} T^n\mathcal{A}$ equipped with a cut comultiplication $\Delta_0 : T\mathcal{A} \to T\mathcal{A} \otimes T\mathcal{A}$, the counit $\varepsilon = \mathrm{pr}_0 : T\mathcal{A} \to T^0\mathcal{A} = \Bbbk\mathcal{A}$, and the augmentation $\eta = \mathrm{in}_0 : \Bbbk\mathcal{A} = T^0\mathcal{A} \to T\mathcal{A}$ is an augmented graded coalgebra, the *tensor coalgebra* of the graded quiver $\mathcal{A}$. The comultiplication Δ_0 is given by

$$\Delta_0|_{T^n\mathcal{A}} : T^n\mathcal{A} \to \bigoplus_{p+q=n} T^p\mathcal{A} \otimes T^q\mathcal{A},$$

$$f_1 \otimes \dots \otimes f_n \mapsto \sum_{i=0}^{n} f_1 \otimes \dots \otimes f_i \bigotimes f_{i+1} \otimes \dots \otimes f_n.$$

More precisely, the graded quiver $T\mathcal{A} \otimes T\mathcal{A}$ decomposes into a direct sum

$$T\mathcal{A} \otimes T\mathcal{A} = \bigoplus_{p,q \geqslant 0} T^p\mathcal{A} \otimes T^q\mathcal{A},$$

and for each $n, p, q \geqslant 0$, the matrix coefficient

$$(\Delta_0)_{n;p,q} \overset{\text{def}}{=} \bigl[T^n\mathcal{A} \xrightarrow{\mathrm{in}_n} T\mathcal{A} \xrightarrow{\Delta} T\mathcal{A} \otimes T\mathcal{A} \xrightarrow{\mathrm{pr}_p \otimes \mathrm{pr}_q} T^p\mathcal{A} \otimes T^q\mathcal{A}\bigr]$$

vanishes if $n \neq p+q$; otherwise, $(\Delta_0)_{n;p,q}$ is the isomorphism

$$\lambda^g : T^n\mathcal{A} = \otimes^{\mathbf{n}}\mathcal{A} \xrightarrow{\sim} \otimes^{i\in\mathbf{2}} \otimes^{g^{-1}i} \mathcal{A} = (\otimes^{g^{-1}1}\mathcal{A}) \otimes (\otimes^{g^{-1}2}\mathcal{A}) = T^p\mathcal{A} \otimes T^q\mathcal{A},$$

where $g : \mathbf{n} \to \mathbf{2}$ corresponds to the partition $n = p+q$, $p = |g^{-1}1|$, $q = |g^{-1}2|$.

Let $(\mathcal{C}, \Delta, \varepsilon, \eta)$ be an augmented graded coalgebra. Denote by $\Delta^{(n)} : \mathcal{C} \to \otimes^{\mathbf{n}}\mathcal{C}$ the comultiplication iterated $n-1$ times, so that

$$\Delta^{(0)} = \varepsilon, \quad \Delta^{(1)} = \mathrm{id}_{\mathcal{C}}, \quad \Delta^{(2)} = \Delta, \quad \Delta^{(3)} = \Delta(\Delta \otimes 1) = \Delta(1 \otimes \Delta),$$

and so on.

3.2.8. Example. The iterations $\Delta_0^{(m)} : T\mathcal{A} \to \otimes^{\mathbf{m}}T\mathcal{A}$ of the cut comultiplication Δ_0 are described as follows. The m-fold tensor product $\otimes^{\mathbf{m}}T\mathcal{A}$ decomposes into a direct sum

$$\otimes^{\mathbf{m}}T\mathcal{A} = \bigoplus_{(n_i)_{i\in\mathbf{m}} \in \mathbb{N}^m} \otimes^{i\in\mathbf{m}}T^{n_i}\mathcal{A}_i.$$

For each $n, n_1, \dots, n_m \geqslant 0$, the matrix coefficient

$$(\Delta_0^{(m)})_{n;n_1,\dots,n_m} \overset{\text{def}}{=} \bigl[T^n\mathcal{A} \xrightarrow{\mathrm{in}_n} T\mathcal{A} \xrightarrow{\Delta_0^{(m)}} \otimes^{i\in\mathbf{m}}T\mathcal{A} \xrightarrow{\otimes^{i\in\mathbf{m}} \mathrm{pr}_{n_i}} \otimes^{i\in\mathbf{m}}T^{n_i}\mathcal{A}\bigr]$$

vanishes if $n \neq n_1 + \dots + n_m$; otherwise, $(\Delta_0^{(m)})_{n;n_1,\dots,n_m}$ is the isomorphism

$$\lambda^g : T^n\mathcal{A} = \otimes^{\mathbf{n}}\mathcal{A} \xrightarrow{\sim} \otimes^{i\in\mathbf{m}} \otimes^{g^{-1}i} \mathcal{A} = \otimes^{i\in\mathbf{m}}T^{n_i}\mathcal{A},$$

where $g : \mathbf{n} \to \mathbf{m}$ corresponds to the partition $n = n_1 + \dots + n_m$, $n_i = |g^{-1}i|$, $i \in \mathbf{m}$.

3.2.9. Proposition. *The category* ac$\mathcal{Q}$ *is symmetric Monoidal. The tensor product of graded coalgebras* $(\mathcal{C}_i, \Delta_i, \varepsilon_i, \eta_i)$, $i \in I$, *is the graded quiver* $\boxtimes^{i\in I}\mathcal{C}_i$ *equipped with the comultiplication, the counit, and the augmentation given by*

$$\Delta = \bigl[\boxtimes^{i\in I}\mathcal{C}_i \xrightarrow{\boxtimes^{i\in I}\Delta_i} \boxtimes^{i\in I}(\mathcal{C}_i \otimes \mathcal{C}_i) \xrightarrow{\varkappa^{-1}} (\boxtimes^{i\in I}\mathcal{C}_i) \otimes (\boxtimes^{i\in I}\mathcal{C}_i)\bigr], \tag{3.2.3}$$

$$\varepsilon = \bigl[\boxtimes^{i\in I}\mathcal{C}_i \xrightarrow{\boxtimes^{i\in I}\varepsilon_i} \boxtimes^{i\in I}\Bbbk\mathcal{C}_i \xrightarrow{\varkappa^{-1}} \Bbbk \boxtimes^{i\in I} \mathcal{C}_i\bigr], \tag{3.2.4}$$

$$\eta = \bigl[\Bbbk \boxtimes^{i\in I} \mathcal{C}_i \xrightarrow{\varkappa} \boxtimes^{i\in I}\Bbbk\mathcal{C}_i \xrightarrow{\boxtimes^{i\in I}\eta_i} \boxtimes^{i\in I}\mathcal{C}_i\bigr]. \tag{3.2.5}$$

For each map $f : I \to J$ *in* Mor $\mathcal{S}$, *the isomorphism* $\lambda^f : \boxtimes^{i\in I}\mathcal{C}_i \xrightarrow{\sim} \boxtimes^{j\in J} \boxtimes^{i\in f^{-1}j} \mathcal{C}_i$ *is that of* $\mathcal{Q}$.

Proof. The proof is by a straightforward computation. □

3.2.10. Example. Let $(\mathcal{C}_i, \Delta_i, \varepsilon_i, \eta_i)$, $i \in I$, be graded coalgebras. The iterated comultiplication $\Delta^{(n)}$ of the tensor product $\boxtimes^{i\in I}\mathcal{C}_i$ is given by

$$\Delta^{(n)} = \bigl[\boxtimes^{i\in I}\mathcal{C}_i \xrightarrow{\boxtimes^{i\in I}\Delta_i^{(n)}} \boxtimes^{i\in I} \otimes^{\mathbf{n}} \mathcal{C}_i \xrightarrow{\overline{\varkappa}^{-1}} \otimes^{\mathbf{n}} \boxtimes^{i\in I} \mathcal{C}_i\bigr].$$

Take $\mathcal{C}_i = T\mathcal{A}_i$, $i \in I$, with the structure of an augmented graded coalgebra described in Example 3.2.7. Then using Example 3.2.8, we can write explicit formulas for matrix coefficients of $\Delta^{(n)}$. Namely, the graded quiver $\otimes^{\mathbf{n}} \boxtimes^{i\in I} T\mathcal{A}_i$ decomposes into a direct sum

$$\otimes^{\mathbf{n}} \boxtimes^{i\in I} T\mathcal{A}_i = \bigoplus_{(m_{i,p})_{(i,p)\in I\times\mathbf{n}}\in\mathbb{N}^{I\times\mathbf{n}}} \otimes^{p\in\mathbf{n}} \boxtimes^{i\in I} T^{m_{i,p}}\mathcal{A}_i.$$

For $m_i, m_{i,p} \geqslant 0$, $i \in I$, $p \in \mathbf{n}$, define $(\Delta^{(n)})_{(m_i);(m_{i,p})}$ by the composite

$$\bigl[\boxtimes^{i\in I}T^{m_i}\mathcal{A}_i \xrightarrow{\boxtimes^{i\in I}\operatorname{in}_{m_i}} \boxtimes^{i\in I}T\mathcal{A}_i \xrightarrow{\Delta^{(n)}} \otimes^{p\in\mathbf{n}} \boxtimes^{i\in I} T\mathcal{A}_i \xrightarrow{\otimes^{p\in\mathbf{n}}\boxtimes^{i\in I}\operatorname{pr}_{m_{i,p}}} \otimes^{p\in\mathbf{n}} \boxtimes^{i\in I} T^{m_{i,p}}\mathcal{A}_i\bigr].$$

Then $(\Delta^{(n)})_{(m_i);(m_{i,p})}$ vanishes if $m_i \neq m_{i,1} + \dots + m_{i,n}$ for some $i \in I$; otherwise, it is equal to the isomorphism

$$\bigl[\boxtimes^{i\in I}T^{m_i}\mathcal{A}_i \underset{\sim}{\xrightarrow{\boxtimes^{i\in I}\lambda^{g_i}}} \boxtimes^{i\in I} \otimes^{p\in\mathbf{n}} \otimes^{g_i^{-1}p}\mathcal{A}_i = \boxtimes^{i\in I} \otimes^{p\in\mathbf{n}} T^{m_{i,p}}\mathcal{A}_i \underset{\sim}{\xrightarrow{\sigma_{(12)}}} \otimes^{p\in\mathbf{n}} \boxtimes^{i\in I} T^{m_{i,p}}\mathcal{A}_i\bigr],$$

where $g_i : \mathbf{m}_i \to \mathbf{n}$ corresponds to the partition $m_i = m_{i,1} + \dots + m_{i,n}$, $m_{i,p} = |g_i^{-1}p|$, $p \in \mathbf{n}$, for each $i \in I$.

3.2.11. Remark. Note that, for each graded quiver $\mathcal{A}$, the graded quiver $T\mathcal{A}$ admits also the structure of an algebra in the Monoidal category $\mathscr{Q}/\operatorname{Ob}\mathcal{A}$. The multiplication $\mu : T\mathcal{A} \otimes T\mathcal{A} \to T\mathcal{A}$ removes brackets in tensor products of the form

$$(x_1 \otimes \dots \otimes x_p) \bigotimes (y_1 \otimes \dots \otimes y_q).$$

More precisely, for each $p, q, n \geqslant 0$, the matrix coefficient

$$\mu_{p,q;n} \overset{\text{def}}{=} \bigl[T^p\mathcal{A} \otimes T^q\mathcal{A} \xrightarrow{\operatorname{in}_p \otimes \operatorname{in}_q} T\mathcal{A} \otimes T\mathcal{A} \xrightarrow{\mu} T\mathcal{A} \xrightarrow{\operatorname{pr}_n} T^n\mathcal{A}\bigr]$$

vanishes if $p + q \neq n$; otherwise, $\mu_{p,q;n}$ is equal to the isomorphism

$$(\lambda^g)^{-1} : T^p\mathcal{A} \otimes T^q\mathcal{A} = (\otimes^{g^{-1}1}\mathcal{A}) \otimes (\otimes^{g^{-1}2}\mathcal{A}) = \otimes^{i\in\mathbf{2}} \otimes^{g^{-1}i} \mathcal{A} \xrightarrow{\sim} \otimes^{\mathbf{n}}\mathcal{A} = T^n\mathcal{A},$$

where $g : \mathbf{n} \to \mathbf{2}$ encodes the partition $p + q = n$, $p = |g^{-1}1|$, $q = |g^{-1}2|$. The unit is the embedding $\eta = \operatorname{in}_0 : \Bbbk\mathcal{A} = T^0\mathcal{A} \hookrightarrow T\mathcal{A}$. The iterated multiplication $\mu^{(n)} : \otimes^{\mathbf{n}}T\mathcal{A} \to T\mathcal{A}$ removes brackets in tensor products of the form

$$(a_1^1 \otimes \dots \otimes a_{k_1}^1) \bigotimes (a_1^2 \otimes \dots \otimes a_{k_2}^2) \bigotimes \dots \bigotimes (a_1^n \otimes \dots \otimes a_{k_n}^n).$$

More precisely, for $k_1, \dots, k_n, k \geqslant 0$, the matrix coefficient

$$\mu^{(n)}_{k_1,\dots,k_n;k} \overset{\text{def}}{=} \bigl[\otimes^{i\in\mathbf{n}}T^{k_i}\mathcal{A} \xrightarrow{\otimes^{i\in\mathbf{n}}\operatorname{in}_{k_i}} \otimes^{i\in\mathbf{n}}T\mathcal{A} \xrightarrow{\mu^{(n)}} T\mathcal{A} \xrightarrow{\operatorname{pr}_k} T^k\mathcal{A}\bigr]$$

of $\mu^{(n)}$ vanishes if $k_1 + \dots + k_n \neq k$ and is equal to the isomorphism

$$(\lambda^g)^{-1} : \otimes^{i\in\mathbf{n}}T^{k_i}\mathcal{A} = \otimes^{i\in\mathbf{n}} \otimes^{g^{-1}i} \mathcal{A} \xrightarrow{\sim} \otimes^{\mathbf{k}}\mathcal{A} = T^k\mathcal{A},$$

otherwise, where $g : \mathbf{k} \to \mathbf{n}$ corresponds to the partition $k_1 + \dots + k_n = k$, $k_i = |g^{-1}i|$, $i \in \mathbf{n}$.

Let $r : \boxtimes^{i\in I} T\mathcal{A}_i \to T\mathcal{B}$ be a morphism of graded spans of a certain degree. Denote by

$$r_{(m_i)_{i\in I};k} \overset{\text{def}}{=} \bigl[\boxtimes^{i\in I} T^{m_i}\mathcal{A}_i \xrightarrow{\boxtimes^{i\in I}\operatorname{in}_{m_i}} \boxtimes^{i\in I} T\mathcal{A}_i \xrightarrow{r} T\mathcal{B} \xrightarrow{\operatorname{pr}_k} T^k\mathcal{B}\bigr]$$

matrix coefficients of r, for $m_i, k \geqslant 0$, $i \in I$. Matrix coefficients $r_{(m_i)_{i\in I};1}$ are called *components* of r and abbreviated to $r_{(m_i)_{i\in I}}$. The morphism of graded spans

$$\check{r} \overset{\text{def}}{=} r \cdot \operatorname{pr}_1 : \boxtimes^{i\in I} T\mathcal{A}_i \to \mathcal{B}$$

collects components of r, meaning that $r_{(m_i)_{i\in I}} = \check{r}|_{\boxtimes^{i\in I}T^{m_i}\mathcal{A}_i} = \boxtimes^{i\in I}\operatorname{in}_{m_i} \cdot \check{r}$.

The following statement was obtained by Keller [**28**, Lemma 5.2] in a more general form for morphisms out of so called *cocomplete* graded coalgebra to the tensor coalgebra of a quiver. Here we provide a direct proof sufficient for our purposes.

3.2.12. Proposition. *Let $\mathcal{A}_i, \mathcal{B}$, $i \in I$, be graded quivers. Then the map*

$$\operatorname{ac}\mathscr{Q}(\boxtimes^{i\in I} T\mathcal{A}_i, T\mathcal{B}) \to \mathscr{Q}(\boxtimes^{i\in I} T\mathcal{A}_i, \mathcal{B}), \quad f \mapsto \check{f},$$

establishes a bijection between the set of morphisms of augmented graded coalgebras $f : \boxtimes^{i\in I} T\mathcal{A}_i \to T\mathcal{B}$ and the set of morphisms of graded quivers $g : \boxtimes^{i\in I} T\mathcal{A}_i \to \mathcal{B}$ such that $g|_{\boxtimes^{i\in I}T^0\mathcal{A}_i} = 0$.

Proof. Note the following property of the cut comultiplication:

$$\bigl[T\mathcal{B} \xrightarrow{\Delta_0^{(n)}} \otimes^{\mathbf{n}} T\mathcal{B} \xrightarrow{\otimes^{\mathbf{n}}\operatorname{pr}_1} \otimes^{\mathbf{n}}\mathcal{B} = T^n\mathcal{B}\bigr] = \operatorname{pr}_n, \quad n \geqslant 0.$$

Indeed, by Example 3.2.8, the restriction of the left hand side to the summand $T^m\mathcal{B}$ of the source is the matrix coefficient $\Delta^{(n)}_{m;1,\dots,1} : T^m\mathcal{B} \to T^n\mathcal{B}$. It vanishes if $m \neq n$ and equals $\lambda^g = \mathrm{id} : T^n\mathcal{B} \to T^n\mathcal{B}$ if $m = n$; here $g = \mathrm{id} : \mathbf{n} \to \mathbf{n}$ corresponds to the partition $n = 1 + \dots + 1$ (n summands).

Let Δ denote the comultiplication in $\boxtimes^{i\in I} T\mathcal{A}_i$. Suppose $f : \boxtimes^{i\in I} T\mathcal{A}_i \to T\mathcal{B}$ is a morphism of augmented graded coalgebras. Since f preserves the augmentation

$$\eta = \bigl[T^0 \boxtimes^{i\in I} \mathcal{A}_i \xrightarrow{\varkappa} \boxtimes^{i\in I} T^0\mathcal{A}_i \xrightarrow{\boxtimes^{i\in I}\operatorname{in}_0} \boxtimes^{i\in I} T\mathcal{A}_i\bigr],$$

it follows that $\check{f}|_{\boxtimes^{i\in I}T^0\mathcal{A}_i} = f_{(0,\dots,0)} = 0$. The morphism f preserves the counit and the comultiplication, i.e., $f \cdot \varepsilon = \varepsilon \cdot \Bbbk f$ and $f \cdot \Delta_0 = \Delta \cdot (f \otimes f)$. The former equation can be written as $f \cdot \Delta_0^{(0)} = \Delta^{(0)} \cdot \otimes^{\mathbf{0}} f$. By induction, $f \cdot \Delta_0^{(n)} = \Delta^{(n)} \cdot \otimes^{\mathbf{n}} f$, $n \geqslant 0$. It follows that

$$f \cdot \operatorname{pr}_n = f \cdot \Delta_0^{(n)} \cdot \otimes^{\mathbf{n}} \operatorname{pr}_1 = \Delta^{(n)} \cdot \otimes^{\mathbf{n}} f \cdot \otimes^{\mathbf{n}} \operatorname{pr}_1 = \Delta^{(n)} \cdot \otimes^{\mathbf{n}} \check{f}.$$

Therefore, the morphism f can be recovered from $\check{f}$ as

$$f = f \cdot \sum_{n\geqslant 0} \operatorname{pr}_n \cdot \operatorname{in}_n = \sum_{n\geqslant 0} f \cdot \operatorname{pr}_n \cdot \operatorname{in}_n = \sum_{n\geqslant 0} \Delta^{(n)} \cdot \otimes^{\mathbf{n}} \check{f} \cdot \operatorname{in}_n, \tag{3.2.6}$$

so that the map $f \mapsto \check{f}$ is injective. Conversely, suppose $g : \boxtimes^{i\in I} T\mathcal{A}_i \to \mathcal{B}$ is a morphism of graded quivers such that $g|_{\boxtimes^{i\in I}T^0\mathcal{A}_i} = 0$. Denote by

$$g_{(m_i)_{i\in I}} = g|_{\boxtimes^{i\in I}T^{m_i}\mathcal{A}_i} = \bigl[\boxtimes^{i\in I} T^{m_i}\mathcal{A}_i \xrightarrow{\boxtimes^{i\in I}\operatorname{in}_{m_i}} \boxtimes^{i\in I} T\mathcal{A}_i \xrightarrow{g} \mathcal{B}\bigr]$$

components of g. Define a morphism f by

$$f = \sum_{n\geqslant 0} \bigl[\boxtimes^{i\in I} T\mathcal{A}_i \xrightarrow{\Delta^{(n)}} \otimes^{\mathbf{n}} \boxtimes^{i\in I} T\mathcal{A}_i \xrightarrow{\otimes^{\mathbf{n}} g} \otimes^{\mathbf{n}}\mathcal{B} = T^n\mathcal{B} \xrightarrow{\operatorname{in}_n} T\mathcal{B}\bigr].$$

It is well-defined. Indeed, it suffices to make sure that for each $m_i \geqslant 0$, $i \in I$, the restriction of the right hand side to the summand $\boxtimes^{i\in I} T^{m_i}\mathcal{A}_i$ of the source is well-defined.

Using formulas for matrix coefficients of $\Delta^{(n)}$ obtained in Example 3.2.10, the restriction can be written as follows:

$$\sum_{n\geqslant 0} \sum_{(m_{i,p})_{(i,p)\in I\times\mathbf{n}}\in\mathbb{N}^{I\times\mathbf{n}}}^{m_i=m_{i,1}+\dots+m_{i,n},\, i\in I} \bigl[\boxtimes^{i\in I} T^{m_i}\mathcal{A}_i \xrightarrow[\sim]{\boxtimes^{i\in I}\lambda^{g_i}} \boxtimes^{i\in I} \otimes^{p\in\mathbf{n}} T^{m_{i,p}}\mathcal{A}_i \xrightarrow[\sim]{\overline{\varkappa}} \otimes^{p\in\mathbf{n}} \boxtimes^{i\in I} T^{m_{i,p}}\mathcal{A}_i \xrightarrow{\otimes^{p\in\mathbf{n}} g_{(m_{i,p})_{i\in I}}} \otimes^{p\in\mathbf{n}}\mathcal{B} = T^n\mathcal{B} \xrightarrow{\mathrm{in}_n} T\mathcal{B}\bigr],$$

where $g_i : \mathbf{m}_i \to \mathbf{n}$ corresponds to a partition $m_i = m_{i,1} + \dots + m_{i,p}$, $m_{i,p} = |g_i^{-1}p|$, $p \in \mathbf{n}$, for each $i \in I$. It is easy to see that if $n > \sum_{i\in I} m_i$, then for arbitrary partitions $m_i = m_{i,1} + \dots + m_{i,n}$, $i \in I$, there exists $p \in \mathbf{n}$ such that $m_{i,p} = 0$, for each $i \in I$. The corresponding term of the sum vanishes since $g_{(m_{i,p})_{i\in I}} = 0$. Therefore the sum is finite.

The morphism f clearly preserves the counit and the augmentation. The comultiplication is also preserved. Indeed, to check the equation $f \cdot \Delta_0 = \Delta \cdot (f \otimes f)$, it suffices to show that

$$f \cdot \Delta_0 \cdot (\mathrm{pr}_m \otimes \mathrm{pr}_n) = \Delta \cdot (f \otimes f) \cdot (\mathrm{pr}_m \otimes \mathrm{pr}_n) : \boxtimes^{i\in I} T\mathcal{A}_i \to T^m\mathcal{B} \otimes T^n\mathcal{B},$$

for each $m, n \geqslant 0$. Denote $\mathcal{C} = \boxtimes^{i\in I} T\mathcal{A}_i$. Expanding out the left hand side, we obtain

$$\bigl[\mathcal{C} \xrightarrow{\Delta^{(m+n)}} \otimes^{\mathbf{m}+\mathbf{n}}\mathcal{C} \xrightarrow{\otimes^{\mathbf{m}+\mathbf{n}} g} \otimes^{\mathbf{m}+\mathbf{n}}\mathcal{B} \xrightarrow{\lambda^g} (\otimes^{\mathbf{m}}\mathcal{B}) \otimes (\otimes^{\mathbf{n}}\mathcal{B})\bigr],$$

where $g : \mathbf{m} + \mathbf{n} \to \mathbf{2}$ correspond to the partition of $m + n$, $m = |g^{-1}1|$, $n = |g^{-1}2|$, while the right hand side becomes

$$\bigl[\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{\Delta^{(m)}\otimes\Delta^{(n)}} (\otimes^{\mathbf{m}}\mathcal{C}) \otimes (\otimes^{\mathbf{n}}\mathcal{C}) \xrightarrow{(\otimes^{\mathbf{m}}g)\otimes(\otimes^{\mathbf{n}}g)} (\otimes^{\mathbf{m}}\mathcal{B}) \otimes (\otimes^{\mathbf{n}}\mathcal{B})\bigr].$$

The obtained expressions coincide by the naturality of λ^g and due to the equation

$$\bigl[\mathcal{C} \xrightarrow{\Delta} \mathcal{C}\otimes\mathcal{C} \xrightarrow{\Delta^{(m)}\otimes\Delta^{(n)}} (\otimes^{\mathbf{n}}\mathcal{C})\otimes(\otimes^{\mathbf{m}}\mathcal{C})\bigr] = \bigl[\mathcal{C} \xrightarrow{\Delta^{(m+n)}} \otimes^{\mathbf{m}+\mathbf{n}}\mathcal{C} \xrightarrow{\lambda^g} (\otimes^{\mathbf{n}}\mathcal{C})\otimes(\otimes^{\mathbf{m}}\mathcal{C})\bigr], \quad (3.2.7)$$

which holds true by the coassociativity of Δ.

Clearly, $\check{f} = g$, so that the map $f \mapsto \check{f}$ is surjective, hence the assertion. □

It follows from the proof of the preceding proposition that a morphism of augmented graded coalgebras $f : \boxtimes^{i\in I} T\mathcal{A}_i \to T\mathcal{B}$ is unambiguously determined by its components $f_{(m_i)_{i\in I}} : \boxtimes^{i\in I} T^{m_i}\mathcal{A}_i \to \mathcal{B}$. For each $m_i, n \geqslant 0$, $i \in I$, the matrix coefficient $f_{(m_i)_{i\in I};n}$ is given by

$$f_{(m_i)_{i\in I};n} = \sum_{(m_{i,p})_{(i,p)\in I\times\mathbf{n}}\in\mathbb{N}^{I\times\mathbf{n}}}^{m_i=m_{i,1}+\dots+m_{i,n},\, i\in I} \bigl[\boxtimes^{i\in I} T^{m_i}\mathcal{A}_i \xrightarrow[\sim]{\boxtimes^{i\in I}\lambda^{g_i}} \boxtimes^{i\in I} \otimes^{p\in\mathbf{n}} T^{m_{i,p}}\mathcal{A}_i \xrightarrow[\sim]{\overline{\varkappa}} \otimes^{p\in\mathbf{n}} \boxtimes^{i\in I} T^{m_{i,p}}\mathcal{A}_i \xrightarrow{\otimes^{p\in\mathbf{n}} f_{(m_{i,p})_{i\in I}}} \otimes^{p\in\mathbf{n}}\mathcal{B} = T^n\mathcal{B}\bigr], \quad (3.2.8)$$

where $g_i : \mathbf{m}_i \to \mathbf{n}$ corresponds to a partition $m_i = m_{i,1} + \dots + m_{i,n}$, $m_{i,p} = |g_i^{-1}p|$, $p \in \mathbf{n}$, $i \in I$. In particular, if I is a 1-element set, a morphism of augmented graded coalgebras $f : T\mathcal{A} \to T\mathcal{B}$ is recovered from its components via the formula

$$f_{m;n} = \sum_{m_1+\dots+m_n=m} \bigl[T^m\mathcal{A} \xrightarrow[\sim]{\lambda^g} \otimes^{p\in\mathbf{n}} T^{m_p}\mathcal{A} \xrightarrow{\otimes^{p\in\mathbf{n}} f_{m_p}} \otimes^{p\in\mathbf{n}}\mathcal{B} = T^n\mathcal{B}\bigr],$$

for each $m, n \geqslant 0$; here $g : \mathbf{m} \to \mathbf{n}$ encodes a partition $m = m_1 + \dots + m_n$, $m_p = |g^{-1}p|$, $p \in \mathbf{n}$. We often omit isomorphisms λ^g and write the matrix coefficient $f_{m;n}$ in a more simple form:

$$f_{m;n} = \sum_{m_1+\dots+m_n=m} f_{m_1} \otimes \dots \otimes f_{m_n} : T^m\mathcal{A} \to T^n\mathcal{B}. \tag{3.2.9}$$

In particular, $f_{m;n}$ vanishes if $m < n$. The coefficient $f_{0;0}$ equals $T^0 f : T^0\mathcal{A} \to T^0\mathcal{B}$.

3.2.13. Definition. For morphisms of graded coalgebras $f, g : \mathcal{C} \to \mathcal{D}$, an (f, g)-*coderivation* of degree n is a morphism of graded spans $r : \mathcal{C} \to \mathcal{D}$ of degree n such that $\operatorname{Ob}_s r = \operatorname{Ob} f$, $\operatorname{Ob}_t r = \operatorname{Ob} g$, and $r\Delta = \Delta(f \otimes r + r \otimes g)$.

3.2.14. Remark. Suppose $r : \mathcal{C} \to \mathcal{D}$ is an (f, g)-coderivation. Then $r\varepsilon = 0$. Indeed, by the counit axiom $\varepsilon = \Delta(\varepsilon \otimes \varepsilon)$. Since f and g preserve the counit, it follows that

$$r\varepsilon = r\Delta(\varepsilon \otimes \varepsilon) = \Delta(f\varepsilon \otimes r\varepsilon + r\varepsilon \otimes g\varepsilon) = \Delta \cdot (\varepsilon \otimes 1) \cdot r\varepsilon + \Delta \cdot (1 \otimes \varepsilon) \cdot r\varepsilon = r\varepsilon + r\varepsilon,$$

hence the assertion.

3.2.15. Proposition. *Suppose $f, g : \boxtimes^{i\in I} T\mathcal{A}_i \to T\mathcal{B}$ are morphisms of augmented graded coalgebras. Then there exists a bijection between (f, g)-coderivations $r : \boxtimes^{i\in I} T\mathcal{A}_i \to T\mathcal{B}$ of degree d and morphisms of graded spans $u : \boxtimes^{i\in I} T\mathcal{A}_i \to \mathcal{B}$ of degree d such that $\operatorname{Ob}_s u = \operatorname{Ob} f$, $\operatorname{Ob}_t u = \operatorname{Ob} g$. The bijection is given by assigning to an (f, g)-coderivation r the morphism of graded spans $\check{r}$.*

Proof. The proof is quite similar to the proof of Proposition 3.2.12. We provide it for the sake of completeness.

Suppose $r : \boxtimes^{i\in I} T\mathcal{A}_i \to T\mathcal{B}$ is an (f, g)-coderivation. Then $r \cdot \Delta_0 = \Delta \cdot (f \otimes r + r \otimes g)$, and by induction

$$r \cdot \Delta_0^{(n)} = \Delta^{(n)} \cdot \sum_{j=1}^{n} \otimes^{p\in\mathbf{n}}((f)_{p<j}, r, (g)_{p>j}), \quad n \geqslant 1. \tag{3.2.10}$$

It follows that

$$\begin{aligned} r \cdot \mathrm{pr}_n = r \cdot \Delta_0^{(n)} \cdot \otimes^{\mathbf{n}} \mathrm{pr}_1 &= \Delta^{(n)} \cdot \sum_{j=1}^{n} \otimes^{p\in\mathbf{n}}((f \cdot \mathrm{pr}_1)_{p<j}, r \cdot \mathrm{pr}_1, (g \cdot \mathrm{pr}_1)_{p>j}) \\ &= \Delta^{(n)} \cdot \sum_{j=1}^{n} \otimes^{p\in\mathbf{n}}((\check{f})_{p<j}, \check{r}, (\check{g})_{p>j}), \end{aligned}$$

for each $n \geqslant 1$. By Remark 3.2.14, $r \cdot \mathrm{pr}_0 = 0$. Therefore r can be recovered from $\check{r}$ as

$$r = r \cdot \sum_{n\geqslant 0} \mathrm{pr}_n \cdot \mathrm{in}_n = \sum_{n\geqslant 1} r \cdot \mathrm{pr}_n \cdot \mathrm{in}_n = \sum_{n\geqslant 1}\sum_{j=1}^{n} \Delta^{(n)} \cdot \otimes^{p\in\mathbf{n}}((\check{f})_{p<j}, \check{r}, (\check{g})_{p>j}) \cdot \mathrm{in}_n,$$

so that the map $r \mapsto \check{r}$ is injective. Conversely, suppose $u : \boxtimes^{i\in I} T\mathcal{A}_i \to \mathcal{B}$ is a morphism of graded spans such that $\operatorname{Ob}_s u = \operatorname{Ob} f$, $\operatorname{Ob}_t u = \operatorname{Ob} g$. Denote by

$$u_{(m_i)_{i\in I}} = u|_{\boxtimes^{i\in I} T^{m_i}\mathcal{A}_i} = \bigl[\boxtimes^{i\in I} T^{m_i}\mathcal{A}_i \xrightarrow{\boxtimes^{i\in I}\mathrm{in}_{m_i}} \boxtimes^{i\in I} T\mathcal{A}_i \xrightarrow{u} \mathcal{B}\bigr]$$

components of u. Define a morphism r by

$$r = \sum_{n\geqslant 1}\sum_{j=1}^{n} \bigl[\boxtimes^{i\in I} T\mathcal{A}_i \xrightarrow{\Delta^{(n)}} \otimes^{p\in\mathbf{n}} \boxtimes^{i\in I} T\mathcal{A}_i \xrightarrow{\otimes^{p\in\mathbf{n}}((\check{f})_{p<j}, u, (\check{g})_{p>j})} \otimes^{p\in\mathbf{n}}\mathcal{B} = T^n\mathcal{B} \xrightarrow{\mathrm{in}_n} T\mathcal{B}\bigr].$$

To see that it is well-defined, it suffices to check that, for each $m_i \geqslant 0$, $i \in I$, the restriction of the right hand side to the summand $\boxtimes^{i\in I} T^{m_i}\mathcal{A}_i$ of the source is given by a finite sum. Indeed, the restriction can be written as follows:

$$\sum_{n\geqslant 1}\sum_{j=1}^{n}\sum_{(m_{i,p})_{(i,p)\in I\times\mathbf{n}}\in\mathbb{N}^{I\times\mathbf{n}}}^{m_i=m_{i,1}+\dots+m_{i,n},\, i\in I} \big[\boxtimes^{i\in I}T^{m_i}\mathcal{A}_i \xrightarrow[\sim]{\boxtimes^{i\in I}\lambda^{g_i}} \boxtimes^{i\in I}\otimes^{p\in\mathbf{n}} T^{m_{i,p}}\mathcal{A}_i \\ \xrightarrow[\sim]{\sigma_{(12)}} \otimes^{p\in\mathbf{n}}\boxtimes^{i\in I} T^{m_{i,p}}\mathcal{A}_i \\ \xrightarrow{\otimes^{p\in\mathbf{n}}((f_{(m_{i,p})_{i\in I}})_{p<j},u_{(m_{i,j})_{i\in I}},(g_{(m_{i,p})_{i\in I}})_{p>j})} \otimes^{p\in\mathbf{n}}\mathcal{B} = T^n\mathcal{B} \xrightarrow{\mathrm{in}_n} T\mathcal{B}\big],$$

where $g_i : \mathbf{m}_i \to \mathbf{n}$ corresponds to a partition $m_i = m_{i,1} + \dots + m_{i,p}$, $m_{i,p} = |g_i^{-1}p|$, $p \in \mathbf{n}$, for each $i \in I$. It is easy to see that if $n > \sum_{i\in I} m_i + 1$, then for arbitrary partitions $m_i = m_{i,1} + \dots + m_{i,n}$, $i \in I$, there exist $k, \ell \in \mathbf{n}$, $k < \ell$, such that $m_{i,k} = m_{i,\ell} = 0$, for each $i \in I$. The corresponding term of the sum vanishes since either $k < j$, and then $f_{(m_{i,k})_{i\in I}} = 0$, or $\ell > k \geqslant j$, and then $g_{(m_{i,\ell})_{i\in I}} = 0$. Therefore the sum is finite.

To check that r is an (f, g)-coderivation, it suffices to show

$$r \cdot \Delta_0 \cdot (\mathrm{pr}_m \otimes \mathrm{pr}_n) = \Delta \cdot (f \otimes r + r \otimes g) \cdot (\mathrm{pr}_m \otimes \mathrm{pr}_n) : \boxtimes^{i\in I} T\mathcal{A}_i \to T^m\mathcal{B} \otimes T^n\mathcal{B},$$

for each $m, n \geqslant 0$. Denote $\mathcal{C} = \boxtimes^{i\in I} T\mathcal{A}_i$. The left hand side equals

$$\sum_{j=1}^{m+n}\big[\mathcal{C} \xrightarrow{\Delta^{(m+n)}} \otimes^{\mathbf{m}+\mathbf{n}}\mathcal{C} \xrightarrow{\otimes^{p\in\mathbf{m}+\mathbf{n}}((\check{f})_{p<j},u,(\check{g})_{p>j})} \otimes^{\mathbf{m}+\mathbf{n}}\mathcal{B} \xrightarrow{\lambda^g} (\otimes^{\mathbf{m}}\mathcal{B}) \otimes (\otimes^{\mathbf{n}}\mathcal{B})\big],$$

where $g : \mathbf{m} + \mathbf{n} \to \mathbf{2}$ encodes the partition of $m + n$, $m = |g^{-1}1|$, $n = |g^{-1}2|$. Expanding out the right hand side using equation (3.2.6), we obtain

$$\big[\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \otimes \mathcal{C} \xrightarrow{\Delta^{(m)}\otimes\Delta^{(n)}} (\otimes^{\mathbf{m}}\mathcal{C}) \otimes (\otimes^{\mathbf{n}}\mathcal{C}) \xrightarrow{S} (\otimes^{\mathbf{m}}\mathcal{B}) \otimes (\otimes^{\mathbf{n}}\mathcal{B})\big],$$

where

$$S = \sum_{j=1}^{n}(\otimes^{\mathbf{m}}\check{f}) \otimes \big(\otimes^{p\in\mathbf{n}}\big((\check{f})_{p<j}, u, (\check{g})_{p>j}\big)\big) + \sum_{k=1}^{m}\big(\otimes^{q\in\mathbf{m}}\big((\check{f})_{q<k}, u, (\check{g})_{q>k}\big)\big) \otimes (\otimes^{\mathbf{n}}\check{g}).$$

The obtained expressions coincide by the naturality of λ^g and by equation (3.2.7).

Clearly, $\check{r} = u$, so that the map $r \mapsto \check{r}$ is surjective, hence the assertion. ☐

Proposition 3.2.12 implies that an arbitrary (f, g)-coderivation $r : \boxtimes^{i\in I} T\mathcal{A}_i \to T\mathcal{B}$ is unambiguously determined by its components $r_{(m_i)_{i\in I}} : \boxtimes^{i\in I} T^{m_i}\mathcal{A}_i \to \mathcal{B}$. For each $m_i, n \geqslant 0$, $i \in I$, the matrix coefficient $r_{(m_i)_{i\in I};n}$ is given by

$$\sum_{j=1}^{n}\sum_{(m_{i,p})_{(i,p)\in I\times\mathbf{n}}\in\mathbb{N}^{I\times\mathbf{n}}}^{m_i=m_{i,1}+\dots+m_{i,n},\, i\in I} \big[\boxtimes^{i\in I}T^{m_i}\mathcal{A}_i \xrightarrow[\sim]{\boxtimes^{i\in I}\lambda^{g_i}} \boxtimes^{i\in I}\otimes^{p\in\mathbf{n}} T^{m_{i,p}}\mathcal{A}_i \\ \xrightarrow[\sim]{\overline{\varkappa}} \otimes^{p\in\mathbf{n}}\boxtimes^{i\in I} T^{m_{i,p}}\mathcal{A}_i \\ \xrightarrow{\otimes^{p\in\mathbf{n}}((f_{(m_{i,p})_{i\in I}})_{p<j},r_{(m_{i,j})_{i\in I}},(g_{(m_{i,p})_{i\in I}})_{p>j})} \otimes^{p\in\mathbf{n}}\mathcal{B} = T^n\mathcal{B}\big], \quad (3.2.11)$$

where $g_i : \mathbf{m}_i \to \mathbf{n}$ corresponds to a partition $m_i = m_{i,1} + \dots + m_{i,p}$, $m_{i,p} = |g_i^{-1}p|$, $p \in \mathbf{n}$, for each $i \in I$. In particular, if I is a 1-element set and $f, g : T\mathcal{A} \to T\mathcal{B}$ are augmented coalgebra morphisms, then an (f, g)-coderivation $r : T\mathcal{A} \to T\mathcal{B}$ is recovered from its components via the formula

$$r_{m;n} = \sum_{j=1}^{n} \sum_{m_1+\dots+m_n=m} \bigl[T^m\mathcal{A} \xrightarrow[\sim]{\lambda^g} \otimes^{p\in\mathbf{n}} T^{m_p}\mathcal{A} \xrightarrow{\otimes^{p\in\mathbf{n}}((f_{m_p})_{p<j}, r_{m_j}, (g_{m_p})_{p>j})} \otimes^{p\in\mathbf{n}}\mathcal{B} = T^n\mathcal{B}\bigr],$$

where $g : \mathbf{m} \to \mathbf{n}$ encodes a partition $m = m_1 + \dots + m_n$, $m_p = |g^{-1}p|$, $p \in \mathbf{n}$. We often omit isomorphisms λ^g and write the matrix coefficient $r_{m;n}$ in the following form:

$$r_{m;n} = \sum_{i_1+\dots+i_p+j+k_1+\dots+k_q=m}^{p+1+q=n} f_{i_1} \otimes \dots \otimes f_{i_p} \otimes r_j \otimes g_{k_1} \otimes \dots \otimes g_{k_q} : T^m\mathcal{A} \to T^n\mathcal{B}. \tag{3.2.12}$$

In particular, $r_{m;n}$ vanish unless $1 \leqslant n \leqslant m + 1$.

The notions of graded span, graded quiver, graded coalgebra etc. admit obvious differential graded analogs. For a graded span $\mathcal{A}$, a *differential* is a morphism of graded spans $d : \mathcal{A} \to \mathcal{A}$ of degree 1 such that $\operatorname{Ob}_s d = \mathrm{id}_{\operatorname{Ob}_s \mathcal{A}}$, $\operatorname{Ob}_t d = \mathrm{id}_{\operatorname{Ob}_t \mathcal{A}}$, and $d^2 = 0$. A *differential graded span* is a graded span $\mathcal{A}$ equipped with a differential. Equivalently, a differential graded span $\mathcal{A}$ consists of a set of source objects $\operatorname{Ob}_s \mathcal{A}$, a set of target objects $\operatorname{Ob}_t \mathcal{A}$, and for each pair $X \in \operatorname{Ob}_s \mathcal{A}$, $Y \in \operatorname{Ob}_t \mathcal{A}$, a differential graded $\Bbbk$-module (i.e., a complex of $\Bbbk$-modules) $(\mathcal{A}(X, Y), d)$. A morphism of differential graded spans $f : (\mathcal{A}, d) \to (\mathcal{B}, d)$ is a morphism of graded spans $f : \mathcal{A} \to \mathcal{B}$ of degree 0 that preserves the differential. A *differential graded quiver* is a graded quiver equipped with a differential. A morphism of differential graded quivers is a morphism of graded quivers that preserves the differential. Let ${}^d\mathscr{Q}$ denote the category of differential graded quivers. It is symmetric Monoidal. The tensor product of differential graded quivers $(\mathcal{Q}_i, d^{\mathcal{Q}_i})$, $i \in I$, is the graded quiver $\boxtimes^{i\in I}\mathcal{Q}_i$ equipped with the differential

$$d = \sum_{k\in I} \boxtimes^{i\in I}((\mathrm{id}_{\mathcal{Q}_i})_{i<k}, d^{\mathcal{Q}_k}, (\mathrm{id}_{\mathcal{Q}_i})_{i>k}). \tag{3.2.13}$$

Isomorphisms λ^f of ${}^d\mathscr{Q}$ are those of $\mathscr{Q}$.

For a set S, denote by ${}^d\mathscr{Q}/S$ the subcategory of ${}^d\mathscr{Q}$ whose objects are differential graded quivers $\mathcal{A}$ such that $\operatorname{Ob}\mathcal{A} = S$ and whose morphisms are morphisms of differential graded quivers $f : \mathcal{A} \to \mathcal{B}$ such that $\operatorname{Ob} f = \mathrm{id}_S$. The category ${}^d\mathscr{Q}$ is Monoidal. The tensor product of differential graded quivers $(\mathcal{A}_i, d^{\mathcal{A}_i})$, $i \in I$, is the graded quivers $\otimes^{i\in I}\mathcal{A}_i$ equipped with the differential

$$d = \sum_{k\in I} \otimes^{i\in I}((\mathrm{id}_{\mathcal{A}_i})_{i<k}, d^{\mathcal{A}_k}, (\mathrm{id}_{\mathcal{A}_i})_{i>k}).$$

Isomorphisms λ^f of ${}^d\mathscr{Q}/S$ are those of $\mathscr{Q}/S$. The unit object is the graded quiver $\Bbbk S$ equipped with the trivial differential.

A *differential graded coalgebra* is a differential graded quiver $(\mathcal{C}, d)$ equipped with morphisms $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and $\varepsilon : \mathcal{C} \to \Bbbk\mathcal{C}$ in ${}^d\mathscr{Q}/\operatorname{Ob}\mathcal{C}$ such that the triple $(\mathcal{C}, \Delta, \varepsilon)$ is a coassociative counital coalgebra in the Monoidal category ${}^d\mathscr{Q}/\operatorname{Ob}\mathcal{C}$. Equivalently, a differential graded coalgebra is a graded coalgebra $(\mathcal{C}, \Delta, \varepsilon)$ equipped with a differential $d : \mathcal{C} \to \mathcal{C}$ such that Δ and ε are morphisms of differential graded quivers. The fact that Δ preserves the differential is expressed by the equation $d \cdot \Delta = \Delta \cdot (\mathrm{id}_{\mathcal{C}} \otimes d + d \otimes \mathrm{id}_{\mathcal{C}})$, which means that $d : \mathcal{C} \to \mathcal{C}$ is an $(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}})$-coderivation of degree 1. Remark 3.2.14 implies that $d \cdot \varepsilon = 0$. In other words, ε preserves the differential as soon as Δ does. A morphism of differential graded coalgebras is a morphism of graded coalgebras that preserves the differential. An *augmented differential graded coalgebra* is a differential graded coalgebra equipped with an augmentation which is a morphism of differential graded quivers. Equivalently, an augmented differential graded coalgebra is an augmented graded coalgebra $(\mathcal{C}, \Delta, \varepsilon, \eta)$ equipped with an $(\mathrm{id}_\mathcal{C}, \mathrm{id}_\mathcal{C})$-coderivation $d : \mathcal{C} \to \mathcal{C}$ of degree 1 such that $d^2 = 0$ and $\eta{\cdot}d = 0$. A morphism of differential graded augmented coalgebras is a morphism of augmented graded coalgebras that preserves the differential. Let $\mathrm{ac}^d\mathscr{Q}$ denote the category of augmented differential graded coalgebras. It is symmetric Monoidal. The tensor product of augmented differential graded coalgebras $(\mathcal{A}_i, d^{\mathcal{A}_i}, \Delta_i, \varepsilon_i, \eta_i)$, $i \in I$, is the graded quiver $\boxtimes^{i\in I}\mathcal{A}_i$ equipped with differential (3.2.13), comultiplication (3.2.3), counit (3.2.4), and augmentation (3.2.5). Isomorphisms λ^f of $\mathrm{ac}^d\mathscr{Q}$ are those of $\mathrm{ac}\mathscr{Q}$.

3.2.16. Multicategory of A_∞-categories. For a graded quiver $\mathcal{A}$, denote by $s\mathcal{A}$ its *suspension.* Thus, $\mathrm{Ob}\, s\mathcal{A} = \mathrm{Ob}\,\mathcal{A}$, and $(s\mathcal{A}(X,Y)) = s(\mathcal{A}(X,Y)) = \mathcal{A}(X,Y)[1]$, for $X, Y \in \mathrm{Ob}\,\mathcal{A}$. The 'identity' morphism $\mathcal{A} \to s\mathcal{A}$ of degree -1 is denoted by s.

A_∞-categories and A_∞-functors form a full submulticategory of the multicategory $\widehat{\mathrm{ac}^d\mathscr{Q}}$ of augmented differential graded coalgebras.

3.2.17. Definition. An *A_∞-category* consist of a graded quiver $\mathcal{A}$ and a differential $b : Ts\mathcal{A} \to Ts\mathcal{A}$ that turns $(Ts\mathcal{A}, \Delta_0, \mathrm{pr}_0, \mathrm{in}_0)$ into an augmented differential graded coalgebra. Thus, $b : Ts\mathcal{A} \to Ts\mathcal{A}$ is an $(\mathrm{id}_{Ts\mathcal{A}}, \mathrm{id}_{Ts\mathcal{A}})$-coderivation of degree 1 such that $b^2 = 0$ and $b|_{T^0 s\mathcal{A}} = \mathrm{in}_0 \cdot b = 0$.

Since $b : Ts\mathcal{A} \to Ts\mathcal{A}$ is a coderivation and a differential, it may be called a *codifferential.* By Proposition 3.2.15, it is unambiguously determined by its components

$$b_n = \bigl[T^n s\mathcal{A} \xrightarrow{\mathrm{in}_n} Ts\mathcal{A} \xrightarrow{b} Ts\mathcal{A} \xrightarrow{\mathrm{pr}_1} s\mathcal{A}\bigr], \quad n \geqslant 0.$$

The condition $b|_{T^0 s\mathcal{A}} = 0$ implies that b_0 vanishes. It is easy to see that b^2 is an $(\mathrm{id}_{Ts\mathcal{A}}, \mathrm{id}_{Ts\mathcal{A}})$-coderivation of degree 2. By Proposition 3.2.12, the equation $b^2 = 0$ is equivalent to

$$b^2 \cdot \mathrm{pr}_1 = b \cdot \check{b} = 0 : Ts\mathcal{A} \to s\mathcal{A}.$$

Note that the identity morphism $\mathrm{id}_{Ts\mathcal{A}} : Ts\mathcal{A} \to Ts\mathcal{A}$ has a unique non-trivial component, namely $(\mathrm{id}_{Ts\mathcal{A}})_1 = \mathrm{id}_{s\mathcal{A}} : s\mathcal{A} \to s\mathcal{A}$. Using formula (3.2.12), we can write the condition $b \cdot \check{b} = 0$ as follows:

$$\sum_{p+k+q=m} (1^{\otimes p} \otimes b_k \otimes 1^{\otimes q}) b_{p+1+q} = 0 : T^m s\mathcal{A} \to s\mathcal{A}, \quad m \geqslant 1. \tag{3.2.14}$$

Thus, an A_∞-category consists of a graded quiver $\mathcal{A}$, and for each $n \geqslant 1$ and a sequence of objects $X_0, X_1, \dots, X_n$ of $\mathcal{A}$, a $\Bbbk$-linear map

$$b_n : s\mathcal{A}(X_0, X_1) \otimes \dots \otimes s\mathcal{A}(X_{n-1}, X_n) \to s\mathcal{A}(X_0, X_n)$$

of degree 1, subject to identities (3.2.14). Using another, more traditional, form of components of b:

$$m_n = \bigl(\mathcal{A}^{\otimes n} \xrightarrow{s^{\otimes n}} (s\mathcal{A})^{\otimes n} \xrightarrow{b_n} s\mathcal{A} \xrightarrow{s^{-1}} \mathcal{A}\bigr)$$

we rewrite (3.2.14) as follows:

$$\sum_{p+k+q=m} (-)^{q+pk} (1^{\otimes p} \otimes m_k \otimes 1^{\otimes q}) m_{p+1+q} = 0 : T^m \mathcal{A} \to \mathcal{A}. \tag{3.2.15}$$

3.2.18. Example. A differential graded category is an example of A_∞-category for which $m_n = 0$ for $n \geqslant 3$. The components m_1 and m_2 are the differential and composition respectively.

3.2.19. Definition. For A_∞-categories $\mathcal{A}_i, \mathcal{B}$, $i \in I$, an *A_∞-functor* $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ is a morphism of augmented differential graded coalgebras $f : \boxtimes^{i\in I} Ts\mathcal{A}_i \to Ts\mathcal{B}$. Equivalently, f is a morphism of augmented graded coalgebras and preserves the differential, meaning that

$$\sum_{j\in I} \boxtimes^{i\in I}((\mathrm{id}_{Ts\mathcal{A}_i})_{i<j}, b, (\mathrm{id}_{Ts\mathcal{A}_i})_{i>j}) \cdot f = f \cdot b. \tag{3.2.16}$$

By Proposition 3.2.12, an A_∞-functor f is unambiguously determined by its components $f_{(m_i)_{i\in I}} : \boxtimes^{i\in I} T^{m_i} s\mathcal{A}_i \to s\mathcal{B}$. It is easy to see that both sides of equation (3.2.16) are (f, f)-coderivations, therefore, by Proposition 3.2.15, equation (3.2.16) is equivalent to

$$\sum_{j\in I} \boxtimes^{i\in I}((\mathrm{id}_{Ts\mathcal{A}_i})_{i<j}, b, (\mathrm{id}_{Ts\mathcal{A}_i})_{i>j}) \cdot f \cdot \mathrm{pr}_1 = f \cdot b \cdot \mathrm{pr}_1 : \boxtimes^{i\in I} Ts\mathcal{A}_i \to s\mathcal{B},$$

or equivalently

$$\sum_{j\in I} \boxtimes^{i\in I}((\mathrm{id}_{Ts\mathcal{A}_i})_{i<j}, b, (\mathrm{id}_{Ts\mathcal{A}_i})_{i>j}) \cdot \check{f} = f \cdot \check{b} : \boxtimes^{i\in I} Ts\mathcal{A}_i \to s\mathcal{B}. \tag{3.2.17}$$

Using equations (3.2.8) and (3.2.11), the above condition can be written as follows:

$$\begin{aligned}
&\sum \Big[\boxtimes^{i\in I} T^{m_i} s\mathcal{A}_i \xrightarrow[\sim]{\boxtimes^{i\in I}[(1)_{i<j},\lambda^{\phi_j},(1)_{i>j}]} \\
&\qquad \boxtimes^{i\in I} \big[(T^{m_i} s\mathcal{A}_i)_{i<j}, \otimes^{\mathbf{p}}[(s\mathcal{A}_j)_{q\in\mathbf{r}}, T^k s\mathcal{A}_j, (s\mathcal{A}_j)_{q\in\mathbf{t}}], (T^{m_i} s\mathcal{A}_i)_{i>j}\big] \\
&\qquad \xrightarrow{\boxtimes^{i\in I}[(1)_{i<j},\otimes^{\mathbf{p}}[(1)_{q\in\mathbf{r}},b_k,(1)_{q\in\mathbf{t}}],(1)_{i>j}]} \boxtimes^{i\in I} \big[(T^{m_i} s\mathcal{A}_i)_{i<j}, T^p s\mathcal{A}_j, (T^{m_i} s\mathcal{A}_i)_{i>j}\big] \\
&\qquad\qquad \xrightarrow{f_{(m_i)_{i<j},p,(m_i)_{i>j}}} s\mathcal{B}\Big] \\
&= \sum \Big[\boxtimes^{i\in I} T^{m_i} s\mathcal{A}_i \xrightarrow{\boxtimes^{i\in I}\lambda^{g_i}} \boxtimes^{i\in I} \otimes^{l\in\mathbf{n}} T^{m_{i,l}} s\mathcal{A}_i \\
&\qquad \xrightarrow[\sim]{\overline{\varkappa}} \otimes^{l\in\mathbf{n}} \boxtimes^{i\in I} T^{m_{i,l}} s\mathcal{A}_i \xrightarrow{\otimes^{l\in\mathbf{n}} f_{(m_{i,l})_{i\in I}}} \otimes^{l\in\mathbf{n}} s\mathcal{B} \xrightarrow{b_n} s\mathcal{B}\Big],
\end{aligned}$$

for each $m_i \geqslant 0$, $i \in I$. The summation in the right hand side extends over $j \in I$ and over partitions

$$m_j = \underbrace{1 + \dots + 1}_{r \text{ summands}} + k + \underbrace{1 + \dots + 1}_{t \text{ summands}},$$

encoded by an order-preserving map $\phi_j : \mathbf{m}_j \to \mathbf{p}$, $p = r + 1 + t$, $|\phi_j^{-1} q| = 1$, $q \neq r + 1$, $|\phi_j^{-1}(r + 1)| = k$. The summation in the right hand side extends over $n \geqslant 1$ and over partitions

$$m_i = m_{i,1} + \dots + m_{i,n}, \quad i \in I,$$

encoded by order-preserving maps $g_i : \mathbf{m}_i \to \mathbf{n}$, $|g_i^{-1} l| = m_{i,l}$, $l \in \mathbf{n}$, for each $i \in I$. In particular, if I is a 1-element set, omitting isomorphisms λ, the above equation can be written as

$$\sum_{r+k+t=m} (1^{\otimes r} \otimes b_k \otimes 1^{\otimes t}) f_{r+1+t} = \sum_{m_1+\dots+m_n=m} (f_{m_1} \otimes \dots \otimes f_{m_n}) b_n : T^m s\mathcal{A} \to s\mathcal{B},$$

for each $m \geqslant 1$.

3.2.20. Definition. An A_∞-functor $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ is called *strict* if $f_n = 0$ for each $n \in \mathbb{N}^I$ unless $|n| = 1$.

3.2.21. Example. Suppose $\mathcal{C}$ and $\mathcal{D}$ are **dg**-categories. Then a **dg**-functor $F : \mathcal{C} \to \mathcal{D}$ gives rise to a strict A_∞-functor $f : \mathcal{C} \to \mathcal{D}$ such that $\operatorname{Ob} f = \operatorname{Ob} F$ and

$$f_1 = s^{-1}Fs : s\mathcal{C}(X,Y) \to s\mathcal{D}(XF,YF),$$

for each pair $X, Y \in \operatorname{Ob}\mathcal{C}$.

3.2.22. Example (Shifts as **dg**-functors)**.** Let $f : X \to Y$ be a $\Bbbk$-linear map of certain degree $d = \deg f$. Define

$$f^{[n]} = (-)^{fn}s^{-n}fs^n = (-)^{dn}\bigl(X[n]^i = X^{i+n} \xrightarrow{f^{i+n}} Y^{i+n+d} = Y[n]^{i+d}\bigr),$$

which is an element of $\underline{\mathsf{C}}_\Bbbk(X[n],Y[n])$ of the same degree $\deg f$. Define the shift differential graded functor $[n] : \underline{\mathsf{C}}_\Bbbk \to \underline{\mathsf{C}}_\Bbbk$ as follows. It takes a complex (X,d) to the complex $(X[n], d^{[n]})$, $d^{[n]} = (-)^n s^{-n} d s^n$. On morphisms it acts via

$$\underline{\mathsf{C}}_\Bbbk(s^{-n},1)\cdot\underline{\mathsf{C}}_\Bbbk(1,s^n) : \underline{\mathsf{C}}_\Bbbk(X,Y) \to \underline{\mathsf{C}}_\Bbbk(X[n],Y[n]), \quad f \mapsto f^{[n]}.$$

Clearly, $[n]\cdot[m] = [n+m]$.

3.2.23. Definition. The symmetric multicategory A_∞ is the full submulticategory of $\widehat{\mathrm{ac}^d\mathcal{Q}}$ whose objects are A_∞-categories.

Thus, for each map $\phi : I \to J$ in $\operatorname{Mor}\mathcal{S}$ and A_∞-functors $f_j : (\mathcal{A}_i)_{i\in\phi^{-1}j} \to \mathcal{B}_j$, $j \in J$, $g : (\mathcal{B}_j)_{j\in J} \to \mathcal{C}$, the composite $(f_j)_{j\in J} \cdot_\phi g$ equals

$$\bigl[\boxtimes^{i\in I} Ts\mathcal{A}_i \xrightarrow{\lambda^\phi} \boxtimes^{j\in J}\boxtimes^{i\in\phi^{-1}j} Ts\mathcal{A}_i \xrightarrow{\boxtimes^{j\in J}f_j} \boxtimes^{j\in J} Ts\mathcal{B}_j \xrightarrow{g} Ts\mathcal{C}\bigr]. \tag{3.2.18}$$

For an A_∞-category $\mathcal{A}$, the identity A_∞-functor $\mathrm{id}_\mathcal{A} : \mathcal{A} \to \mathcal{A}$ is represented by the identity morphism of augmented differential graded coalgebras $\mathrm{id}_{Ts\mathcal{A}} : Ts\mathcal{A} \to Ts\mathcal{A}$. Its only non-trivial component is $(\mathrm{id}_\mathcal{A})_1 = \mathrm{id}_{s\mathcal{A}} : s\mathcal{A} \to s\mathcal{A}$.

3.2.24. Remark. Note that there is a bijection between A_∞-functors $() \to \mathcal{B}$ and objects of $\mathcal{B}$. Indeed, an A_∞-functor $f : () \to \mathcal{B}$ is a morphism of augmented graded coalgebras $f : \mathbb{1} \to Ts\mathcal{B}$ that preserves the differential. It amounts to a mapping $* \mapsto U \in \operatorname{Ob}\mathcal{B}$ on objects and to the identity $\mathbb{1}(*,*) = \Bbbk \to \Bbbk = T^0 s\mathcal{B}(U,U)$ on morphisms. Commutation with the differential holds true automatically.

Let $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ be an A_∞-functor, and let $J \subset I$ be a subset. Choose a family of objects $(X_i)_{i\in I\smallsetminus J} \in \prod_{i\in I\smallsetminus J}\operatorname{Ob}\mathcal{A}_i$. Viewing them as A_∞-functors $X_i : () \to \mathcal{A}_i$, define an A_∞-functor $f|_J^{(X_i)_{i\in I\smallsetminus J}} : (\mathcal{A}_j)_{j\in J} \to \mathcal{B}$, the restriction of f to arguments in J, as the composite

$$\bigl((\mathrm{id}_{\mathcal{A}_j})_{j\in J}, (X_i)_{i\in I\smallsetminus J}\bigr)\cdot_{J\hookrightarrow I} f : (\mathcal{A}_j)_{j\in J} \to \mathcal{B}$$

in A_∞. The A_∞-functor $f|_J^{(X_i)_{i\in I\smallsetminus J}}$ takes an object $(X_j)_{j\in J}$ to $((X_i)_{i\in I})f \in \operatorname{Ob}\mathcal{B}$. For each $k = (k_j)_{j\in J} \in \mathbb{N}^J$, the component $\bigl(f|_J^{(X_i)_{i\in I\smallsetminus J}}\bigr)_k$ of $f|_J^{(X_i)_{i\in I\smallsetminus J}}$ is given by the composite

$$\bigl[\otimes^{j\in J} T^{k_j} s\mathcal{A}_j(X_j,Y_j) \xrightarrow{\lambda^{J\hookrightarrow I}} \otimes^{i\in I} T^{\overline{k}_i} s\mathcal{A}_i(X_i,Y_i) \xrightarrow{f_{\overline{k}}} s\mathcal{B}\bigl(((X_i)_{i\in I})f, ((Y_i)_{i\in I})f\bigr)\bigr], \tag{3.2.19}$$

where $\overline{k} = (\overline{k}_i)_{i\in I} \in \mathbb{N}^I$, $\overline{k}_i = k_i$ if $i \in J$, $\overline{k}_i = 0$ and $Y_i = X_i$ if $i \in I \smallsetminus J$.

In particular, for $J = \{j\} \hookrightarrow I = \mathbf{n}$ we have an A_∞-functor $f|_j^{(X_i)_{i\neq j}} : \mathcal{A}_j \to \mathcal{B}$, the restriction of f to the j^{th} argument. It takes an object $X_j \in \operatorname{Ob}\mathcal{A}_j$ to the object

$((X_i)_{i\in\mathbf{n}})f \in \operatorname{Ob}\mathcal{B}$. The k^{th} component $\bigl(f|_j^{(X_i)_{i\neq j}}\bigr)_k$ is given by the composite

$$\bigl[T^k s\mathcal{A}_j(X_j,Y_j) \cong \otimes^{i\in\mathbf{n}}\bigl[(T^0 s\mathcal{A}_i(X_i,X_i))_{i<j}, T^k s\mathcal{A}_j(X_j,Y_j), (T^0 s\mathcal{A}_i(X_i,X_i))_{i>j}\bigr] \\ \xrightarrow{f_{ke_j}} s\mathcal{B}\bigl((X_1,\dots,X_n)f, (X_1,\dots,X_{j-1},Y_j,X_{j+1},\dots,X_n)f\bigr)\bigr],$$

where $ke_j = (0,\dots,0,k,0,\dots,0) \in \mathbb{N}^n$ has k at the j^{th} place. The first component $\bigl(f|_j^{(X_i)_{i\neq j}}\bigr)_1 = f_{e_j}$ commutes with b_1, and there are chain maps

$$sf_{e_j}s^{-1} : \mathcal{A}_j(X_j,Y_j) \to \mathcal{B}\bigl(((Y_i)_{i<j},(X_i)_{i\geqslant j})f, ((Y_i)_{i\leqslant j},(X_i)_{i>j})f\bigr), \tag{3.2.20}$$

for each $X_j, Y_j \in \operatorname{Ob}\mathcal{A}_j$, $j \in J$.

3.3. A_∞-categories of A_∞-functors

We are going to show that the symmetric multicategory A_∞ is closed. It is well-known that, for each pair of A_∞-categories $\mathcal{A}$, $\mathcal{B}$, there exists an A_∞-category of A_∞-functors $\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})$. These A_∞-categories have been studied by many authors, e.g. Fukaya [**16**], Kontsevich and Soibelman [**32**], Lefèvre-Hasegawa [**34**], Lyubashenko [**38**]. We begin by recalling the construction of $\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})$ following [**38**], where it was denoted by $A_\infty(\mathcal{A},\mathcal{B})$. An object of $\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})$ is an A_∞-functor $\mathcal{A}\to\mathcal{B}$. For each pair of A_∞-functors $f,g:\mathcal{A}\to\mathcal{B}$, the k^{th} component of the graded $\Bbbk$-module of morphisms $\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})(f,g)$ consists of (f,g)-coderivations $r : Ts\mathcal{A}\to Ts\mathcal{B}$ of degree $k-1$. By Proposition 3.2.15, an arbitrary (f,g)-coderivation $r : Ts\mathcal{A}\to Ts\mathcal{B}$ is uniquely determined by its components $r_n : T^n s\mathcal{A}\to s\mathcal{B}$, $n\geqslant 0$. Therefore, the graded $\Bbbk$-module $s\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})(f,g)$ is identified with the product

$$\prod_{X,Y\in\operatorname{Ob}\mathcal{A}} \underline{\mathbf{gr}}(Ts\mathcal{A}(X,Y), s\mathcal{B}(Xf,Yg)) \cong \prod_{n\geqslant 0}\prod_{X,Y\in\operatorname{Ob}\mathcal{A}} \underline{\mathbf{gr}}(T^n s\mathcal{A}(X,Y), s\mathcal{B}(Xf,Yg)).$$

The differential $B_1 : s\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})(f,g) \to s\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})(f,g)$ maps an (f,g)-coderivation $r : Ts\mathcal{A}\to Ts\mathcal{B}$ of degree $\deg r$ to an (f,g)-coderivation

$$rB_1 = [r,b] = rb - (-)^r br : Ts\mathcal{A}\to Ts\mathcal{B}$$

of degree $\deg r + 1$. The higher components

$$B_n : s\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})(f^0,f^1)\otimes\cdots\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})(f^{n-1},f^n) \to s\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})(f^0,f^n), \quad n\geqslant 2,$$

are defined as follows. For (f^{i-1},f^i)-coderivations $r^i : Ts\mathcal{A}\to Ts\mathcal{B}$, $i\in\mathbf{n}$, the (f^0,f^n)-coderivation $r = (r^1\otimes\cdots\otimes r^n)B_n$ is given by its components

$$r_k = \sum (f^0_{i^0_1}\otimes\cdots\otimes f^0_{i^0_{m_0}}\otimes r^1_{j_1}\otimes f^1_{i^1_1}\otimes\cdots\otimes f^1_{i^1_{m_1}}\otimes\cdots\otimes r^n_{j_n}\otimes f^n_{i^n_1}\otimes\cdots\otimes f^n_{i^n_{m_n}})b_{m_0+\dots+m_n+n},$$

where the summation is taken over all partitions

$$i^0_1+\dots+i^0_{m_0}+j_1+i^1_1+\dots+i^1_{m_1}+j_n+i^n_1+\dots+i^n_{m_n} = k.$$

There is an A_∞-functor $\operatorname{ev}^{\mathsf{A}_\infty} = \operatorname{ev}^{\mathsf{A}_\infty}_{\mathcal{A},\mathcal{B}} : Ts\mathcal{A}\boxtimes Ts\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}) \to Ts\mathcal{B}$, denoted in [**38**] by α. It maps a pair (X,f) consisting of an object $X\in\operatorname{Ob}\mathcal{A}$ and an A_∞-functor $f:\mathcal{A}\to\mathcal{B}$ to the object Xf of $\mathcal{B}$. The only non-trivial components of $\operatorname{ev}^{\mathsf{A}_\infty}$ are

$$\operatorname{ev}^{\mathsf{A}_\infty}_{n;0} = \bigl[T^n s\mathcal{A}(X,Y)\otimes T^0 s\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})(f,f) \xrightarrow{\sim} T^n s\mathcal{A}(X,Y) \xrightarrow{f_n} s\mathcal{B}(Xf,Yf)\bigr]$$

for $n \geqslant 1$, and

$$\operatorname{ev}^{\mathsf{A}_\infty}_{n;1} = \bigl[T^n s\mathcal{A}(X,Y) \otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})(f,g) \xrightarrow{1\otimes\operatorname{pr}^n_{X,Y}} \\ T^n s\mathcal{A}(X,Y) \otimes \underline{\mathbf{gr}}(T^n s\mathcal{A}(X,Y), s\mathcal{B}(Xf,Yg)) \xrightarrow{\operatorname{ev}^{\mathbf{gr}}} s\mathcal{B}(Xf,Yg)\bigr],$$

for $n \geqslant 0$. Proposition 5.5 of [**38**] asserts that, for each A_∞-functor $f : \mathcal{A}, \mathcal{C}_1, \dots, \mathcal{C}_q \to \mathcal{B}$, there exists a unique A_∞-functor $g : \mathcal{C}_1, \dots, \mathcal{C}_q \to \underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})$ such that

$$f = \bigl[\mathcal{A}, \mathcal{C}_1, \dots, \mathcal{C}_q \xrightarrow{1,g} \mathcal{A}, \underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}) \xrightarrow{\operatorname{ev}^{\mathsf{A}_\infty}_{\mathcal{A};\mathcal{B}}} \mathcal{B}\bigr].$$

Therefore, the function

$$\varphi^{\mathsf{A}_\infty}_{\mathcal{C}_1,\dots,\mathcal{C}_q;\mathcal{A};\mathcal{B}} : \mathsf{A}_\infty(\mathcal{C}_1, \dots, \mathcal{C}_q, \underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})) \to \mathsf{A}_\infty(\mathcal{A}, \mathcal{C}_1, \dots, \mathcal{C}_q;\mathcal{B})$$

is a bijection, so that the assumptions of Proposition 1.3.13 are satisfied. Hence, A_∞ is a closed symmetric multicategory.

The presented argument is short, and it convinces that the assertion is true. However, internal Hom-objects and evaluations constructed as in the proof of Proposition 1.3.13 are rather inconvenient for practical computations. As we have mentioned at the beginning of the chapter, a conceptual approach to the closedness of the symmetric multicategory A_∞ is developed in [**3**]. It leads to a different (but, of course, isomorphic) choice of the structure of a closed multicategory for A_∞. We are going to use it in the sequel, so it is briefly discussed below. The reader is referred to [**3**] for details.

For A_∞-categories $\mathcal{A}_i, \mathcal{B}$, $i \in I$, there exists an A_∞-category $\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ whose objects are A_∞-functors $(\mathcal{A}_i)_{i\in I} \to \mathcal{B}$, i.e., morphisms of augmented graded coalgebras $\boxtimes^{i\in I} Ts\mathcal{A}_i \to Ts\mathcal{B}$ that preserve the differential. For each pair of A_∞-functors $f, g : \boxtimes^{i\in I} Ts\mathcal{A}_i \to Ts\mathcal{B}$, the k^{th} component of the graded $\Bbbk$-module $\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g)$ consists of (f,g)-coderivations $\boxtimes^{i\in I} Ts\mathcal{A}_i \to Ts\mathcal{B}$ of degree $k-1$. By Proposition 3.2.15, the graded $\Bbbk$-module $s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ is identified with the product

$$\prod^{i\in I}_{X_i,Y_i\in\operatorname{Ob}\mathcal{A}_i} \underline{\mathbf{gr}}(\otimes^{i\in I} Ts\mathcal{A}_i(X_i,Y_i), s\mathcal{B}((X_i)_{i\in I}f, (Y_i)_{i\in I}g)) \\ \cong \prod_{(n_i)_{i\in I}\in\mathbb{N}^I} \prod^{i\in I}_{X_i,Y_i\in\operatorname{Ob}\mathcal{A}_i} \underline{\mathbf{gr}}(\otimes^{i\in I} T^{n_i} s\mathcal{A}_i(X_i,Y_i), s\mathcal{B}((X_i)_{i\in I}f, (Y_i)_{i\in I}g)).$$

For (f^{p-1}, f^p)-coderivations $r^p : \boxtimes^{i\in I} Ts\mathcal{A}_i \to Ts\mathcal{B}$, $p \in \mathbf{n}$, define a morphism of graded spans $(r^1 \otimes \dots \otimes r^n)\theta : \boxtimes^{i\in I} Ts\mathcal{A}_i \to Ts\mathcal{B}$ by its matrix coefficients

$$\sum \bigl[\boxtimes^{i\in I} T^{k_i} s\mathcal{A}_i \xrightarrow{\boxtimes^{i\in I}\lambda^{g_i:\mathbf{k}_i\to N}} \boxtimes^{i\in I} \otimes^N (T^{l_i^0} s\mathcal{A}_i, T^{j_i^1} s\mathcal{A}_i, T^{l_i^1} s\mathcal{A}_i, \dots, T^{l_i^n} s\mathcal{A}_i) \\ \xrightarrow{\overline{\varkappa}^{-1}} \otimes^N (\boxtimes^{i\in I} T^{l_i^0} s\mathcal{A}_i, \boxtimes^{i\in I} T^{j_i^1} s\mathcal{A}_i, \boxtimes^{i\in I} T^{l_i^1} s\mathcal{A}_i, \dots, \boxtimes^{i\in I} T^{l_i^n} s\mathcal{A}_i) \\ \xrightarrow{\otimes^N[f^0_{(l_i^0);m_0}, r^1_{(j_i^1)}, f^1_{(l_i^1);m_1}, \dots, f^n_{(l_i^n);m_n}]} T^m s\mathcal{B}\bigr], \tag{3.3.1}$$

where $N = \{1, 2, \dots, 2n+1\}$, the summation is taken over all partitions

$$l_i^0 + j_i^1 + l_i^1 + \dots + l_i^n = k_i, \quad i \in I, \quad m_0 + \dots + m_n + n = m,$$

the mapping $g_i : \mathbf{k}_i \to N$ encodes the partition $l_i^0 + j_i^1 + l_i^1 + \dots + l_i^n = k_i$, $i \in I$, and $f^p_{(l_i^p);m_p} : \boxtimes^{i\in I} T^{l_i^p} s\mathcal{A}_i \to T^{m_p} s\mathcal{B}$ are matrix coefficients of f^p, $0 \leqslant p \leqslant n$. More concisely, the morphism $(r^1\otimes\cdots\otimes r^n)\theta$ is given by the composite

$$(r^1\otimes\cdots\otimes r^n)\theta=\bigl[\boxtimes^{i\in I}Ts\mathcal{A}_i\xrightarrow{\Delta^{(2n+1)}}\otimes^N\boxtimes^{i\in I}Ts\mathcal{A}_i\xrightarrow{\otimes^N(f^0,\check{r}^1,f^1,\dots,f^n)}\\ \otimes^N(Ts\mathcal{B},s\mathcal{B},Ts\mathcal{B},\dots,Ts\mathcal{B})\xrightarrow{\mu^{(2n+1)}}Ts\mathcal{B}\bigr],\quad(3.3.2)$$

where μ is the multiplication in the tensor quiver $Ts\mathcal{B}$, see Remark 3.2.11. In particular, if $n=1$, then $(r^1)\theta=r^1$. If $n=0$, then $()\theta=f^0$. The differential

$$B_1:s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g)\to s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g)$$

maps an (f,g)-coderivation $r:\boxtimes^{i\in I}Ts\mathcal{A}_i\to Ts\mathcal{B}$ of degree $\deg r$ to an (f,g)-coderivation

$$rB_1=[r,b]=rb-(-)^r\sum_{k\in I}\boxtimes^{i\in I}((1)_{i<k},b,(1)_{i>k})r:\boxtimes^{i\in I}Ts\mathcal{A}_i\to Ts\mathcal{B}\quad(3.3.3)$$

of degree $\deg r+1$. The higher components

$$B_n:s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f^0,f^1)\otimes\cdots\otimes s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f^{n-1},f^n)\to s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f^0,f^n),$$

for $n\geqslant 2$, are defined as follows. For (f^{p-1},f^p)-coderivations $r^p:\boxtimes^{i\in I}Ts\mathcal{A}_i\to Ts\mathcal{B}$, $p\in\mathbf{n}$, the (f^0,f^n)-coderivation $r=(r^1\otimes\cdots\otimes r^n)B_n$ is given by

$$\check{r}=\bigl[\boxtimes^{i\in I}Ts\mathcal{A}_i\xrightarrow{(r^1\otimes\cdots\otimes r^n)\theta}Ts\mathcal{B}\xrightarrow{\check{b}}s\mathcal{B}\bigr].\quad(3.3.4)$$

In other words, for each $(k_i)_{i\in I}\in\mathbb{N}^I$,

$$[(r^1\otimes\cdots\otimes r^n)B_n]_{(k_i)_{i\in I}}=\sum_{m=1}^{\infty}[(r^1\otimes\cdots\otimes r^n)\theta]_{(k_i)_{i\in I};m}b_m.\quad(3.3.5)$$

For A_∞-functors $f,g:(\mathcal{A}_i)_{i\in I}\to\mathcal{B}$, an (f,g)-coderivation $r:\boxtimes^{i\in I}Ts\mathcal{A}_i\to Ts\mathcal{B}$ is also called an *A_∞-transformation* and denoted $r:f\to g:(\mathcal{A}_i)_{i\in I}\to\mathcal{B}$. An A_∞-transformation $r:f\to g:(\mathcal{A}_i)_{i\in I}\to\mathcal{B}$ is called *natural* if $\deg r=-1$ and $rB_1=0$. Two A_∞-transformations $r,t:f\to g:(\mathcal{A}_i)_{i\in I}\to\mathcal{B}$ of the same degree d are called *equivalent* and denoted $r\equiv t$ if they differ by a boundary, i.e., if there exists an A_∞-transformation $v:f\to g:(\mathcal{A}_i)_{i\in I}\to\mathcal{B}$ of degree $d-1$ such that $r-t=vB_1$. Later we will give an interpretation of these notions in terms of base change.

3.3.1. Remark. Suppose $\mathcal{B}$ is a **dg**-category. Then $B_n=0$ for $n\geqslant 3$. Indeed, it follows from formula (3.3.1) that the matrix coefficient $[(r^1\otimes\cdots\otimes r^n)\theta]_{(k_i)_{i\in I};m}$ vanishes if $m<n$. Formula (3.3.5) implies that $(r^1\otimes\cdots\otimes r^n)B_n$ vanishes if $n\geqslant 3$.

The evaluation A_∞-functor $\mathrm{ev}^{\mathsf{A}_\infty}=\mathrm{ev}^{\mathsf{A}_\infty}_{(\mathcal{A}_i)_{i\in I};\mathcal{B}}:(\mathcal{A}_i)_{i\in I},\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\to\mathcal{B}$ maps a tuple $((X_i)_{i\in I},f)$ consisting of objects $X_i\in\operatorname{Ob}\mathcal{A}_i$, $i\in I$, and an A_∞-functor $f:(\mathcal{A}_i)_{i\in I}\to\mathcal{B}$ to the object $(X_i)_{i\in I}f$ of $\mathcal{B}$. The only non-vanishing components of $\mathrm{ev}^{\mathsf{A}_\infty}$ are

$$\mathrm{ev}^{\mathsf{A}_\infty}_{(n_i)_{i\in I},0}=\bigl[\otimes^{i\in I}T^{n_i}s\mathcal{A}(X_i,Y_i)\otimes T^0s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\xrightarrow{\sim}\\ \otimes^{i\in I}T^{n_i}s\mathcal{A}(X_i,Y_i)\xrightarrow{f_{(n_i)_{i\in I}}}s\mathcal{B}((X_i)_{i\in I}f,(Y_i)_{i\in I}f)\bigr],\quad(3.3.6)$$

for $(n_i)_{i\in I} \in \mathbb{N}^I$, $(n_i)_{i\in I} \neq 0$, and

$$\begin{aligned}\mathrm{ev}^{\mathsf{A}_\infty}_{(n_i)_{i\in I},1} = \bigl[(\otimes^{i\in I}T^{n_i}s\mathcal{A}(X_i,Y_i)) \otimes s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g) &\xrightarrow{1\otimes \mathrm{pr}^{(n_i)_{i\in I}}_{(X_i)_{i\in I},(Y_i)_{i\in I}}} \\ (\otimes^{i\in I}T^{n_i}s\mathcal{A}(X_i,Y_i)) \otimes \underline{\mathbf{gr}}(\otimes^{i\in I}T^{n_i}s\mathcal{A}_i(X_i,Y_i), s\mathcal{B}((X_i)_{i\in I}f,(Y_i)_{i\in I}g)) \\ \xrightarrow{\mathrm{ev}^{\mathbf{gr}}} s\mathcal{B}((X_i)_{i\in I}f,(Y_i)_{i\in I}g)\bigr], \quad (3.3.7)\end{aligned}$$

for $(n_i)_{i\in I} \in \mathbb{N}^I$. If I is a 1-element set, we recover the definitions from the beginning of the section.

Note that $\underline{\mathsf{A}_\infty}(;\mathcal{B}) \cong \mathcal{B}$, as it should be in a closed multicategory. Indeed, the isomorphism $\underline{\mathsf{A}_\infty}(;\mathcal{B}) \xrightarrow{\sim} \mathcal{B}$ extends the bijection of Remark 3.2.24. It is a strict A_∞-functor with the first component given by the isomorphisms

$$s\underline{\mathsf{A}_\infty}(;\mathcal{B})(f,f) = \underline{\mathbf{gr}}(\mathbb{1}(*,*), s\mathcal{B}(U,U)) = \underline{\mathbf{gr}}(\Bbbk, s\mathcal{B}(U,U)) \xrightarrow{\sim} s\mathcal{B}(U,U),$$

where $U = (*)f \in \operatorname{Ob}\mathcal{B}$. In the sequel, we tacitly identify $\underline{\mathsf{A}_\infty}(;\mathcal{B})$ with $\mathcal{B}$.

It is easy to see that the bijection

$$\varphi^{\mathsf{A}_\infty}_{;(\mathcal{A}_i)_{i\in I};\mathcal{B}} : \mathsf{A}_\infty(;\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})) \xrightarrow{\sim} \mathsf{A}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})$$

identifies with the bijection

$$\mathsf{A}_\infty(;\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})) \xrightarrow{\sim} \operatorname{Ob}\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B}) = \mathsf{A}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})$$

of Remark 3.2.24.

3.3.2. Inversion formulas. We are going to compute explicitly the inverse of the function

$$\varphi^{\mathsf{A}_\infty} : \mathsf{A}_\infty((\mathcal{B}_j)_{j\in J};\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})) \to \mathsf{A}_\infty((\mathcal{A}_i)_{i\in I},(\mathcal{B}_j)_{j\in J};\mathcal{C}), \quad f \mapsto ((\mathrm{id}_{\mathcal{A}_i})_{i\in I},f)\,\mathrm{ev}^{\mathsf{A}_\infty}.$$

Suppose $g : (\mathcal{A}_i)_{i\in I},(\mathcal{B}_j)_{j\in J} \to \mathcal{C}$ is an A_∞-functor. There exists a unique A_∞-functor $f = (\varphi^{\mathsf{A}_\infty})^{-1}(g) : (\mathcal{B}_j)_{j\in J} \to \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})$ such that

$$g = \bigl[(\mathcal{A}_i)_{i\in I},(\mathcal{B}_j)_{j\in J} \xrightarrow{(\mathrm{id}_{\mathcal{A}_i})_{i\in I},f} (\mathcal{A}_i)_{i\in I},\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C}) \xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty}} \mathcal{C}\bigr]. \qquad (3.3.8)$$

As a morphism of augmented graded coalgebras, f can be restored from equation (3.3.8) unambiguously. Indeed, it follows immediately that, for each $U_j \in \operatorname{Ob}\mathcal{B}_j$, $j \in J$, the image $(U_j)_{j\in J}f$ is an A_∞-functor $(U_j)_{j\in J}f : (\mathcal{A}_i)_{i\in I} \to \mathcal{C}$ that takes an object $(X_i)_{i\in I}$ to $(X_i)_{i\in I}(U_j)_{j\in J}f = ((X_i)_{i\in I},(U_j)_{j\in J})g$. To compute components of the A_∞-functor $(U_j)_{j\in J}f$, pick $m = (m_i)_{i\in I} \in \mathbb{N}^I$, $m \neq 0$, objects $X_i, Y_i \in \operatorname{Ob}\mathcal{A}_i$, $i \in I$, and consider the restriction of equation (3.3.8) to the appropriate direct summand of the source:

$$\begin{aligned}g_{m,0} = \bigl[\bigl(\otimes^{i\in I}T^{m_i}s\mathcal{A}_i(X_i,Y_i)\bigr) \otimes \bigl(\otimes^{j\in J}T^0 s\mathcal{B}_j(U_j,U_j)\bigr) \\ \cong \otimes^{i\in I}T^{m_i}s\mathcal{A}_i(X_i,Y_i) \xrightarrow{1\otimes((U_j)_{j\in J}f)_m} s\mathcal{C}((X_i)_{i\in I}(U_j)_{j\in J}f,(Y_i)_{i\in I}(U_j)_{j\in J}f)\bigr].\end{aligned}$$

Comparing the result with formula (3.2.19), we conclude that $(U_j)_{j\in J}f = g|_I^{(U_j)_{j\in J}}$. To find components of f, take $m = (m_i)_{i\in I} \in \mathbb{N}^I$, $n = (n_j)_{j\in J} \in \mathbb{N}^J$, objects $X_i, Y_i \in \operatorname{Ob}\mathcal{A}_i$, $i \in I$, $U_j, V_j \in \operatorname{Ob}\mathcal{B}_j$, $j \in J$, and consider the restriction of (3.3.8) to the direct summand

$\bigl(\otimes^{i\in I}T^{m_i}s\mathcal{A}_i(X_i,Y_i)\bigr)\otimes\bigl(\otimes^{j\in J}T^{n_j}s\mathcal{B}_j(U_j,V_j)\bigr)$ of the source:

$$\begin{aligned}g_{m,n}=\Bigl[\bigl(\otimes^{i\in I}T^{m_i}s\mathcal{A}_i(X_i,Y_i)\bigr)&\otimes\bigl(\otimes^{j\in J}T^{n_j}s\mathcal{B}_j(U_j,V_j)\bigr)\\ &\xrightarrow{1\otimes f_n}\bigl(\otimes^{i\in I}T^{m_i}s\mathcal{A}_i(X_i,Y_i)\bigr)\otimes\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})((U_j)_{j\in J}f,(V_j)_{j\in J}f)\\ &\xrightarrow{1\otimes\mathrm{pr}}\bigl(\otimes^{i\in I}T^{m_i}s\mathcal{A}_i(X_i,Y_i)\bigr)\otimes\underline{\mathbf{gr}}(\otimes^{i\in I}T^{m_i}s\mathcal{A}_i(X_i,Y_i),\\ &\qquad\qquad s\mathcal{C}((X_i)_{i\in I}(U_j)_{j\in J}f,(Y_i)_{i\in I}(V_j)_{j\in J}f))\\ &\xrightarrow{\mathrm{ev}^{\mathbf{gr}}}s\mathcal{C}(((X_i)_{i\in I},(U_j)_{j\in J})g,((Y_i)_{i\in I},(V_j)_{j\in J})g)\Bigr].\end{aligned}$$

Therefore, somewhat informal, f_n maps an element $\otimes^{j\in J}\otimes^{l_j\in\mathbf{n}_j}q_j^{l_j}$ of $\otimes^{j\in J}T^{n_j}s\mathcal{B}_j(U_j,V_j)$ to an $(g|_I^{(U_j)_{j\in J}},g|_I^{(V_j)_{j\in J}})$-coderivation $r:\boxtimes^{i\in I}Ts\mathcal{A}_i\to Ts\mathcal{C}$, whose m^{th} component r_m takes an element $\otimes^{i\in I}\otimes^{k_i\in\mathbf{m}_i}p_i^{k_i}$ of $\otimes^{i\in I}T^{m_i}s\mathcal{A}_i(X_i,Y_i)$ to

$$\bigl((\otimes^{i\in I}\otimes^{k_i\in\mathbf{m}_i}p_i^{k_i})\otimes(\otimes^{j\in J}\otimes^{l_j\in\mathbf{n}_j}q_j^{l_j})\bigr)g_{m,n}\in s\mathcal{C}(((X_i)_{i\in I},(U_j)_{j\in J})g,((Y_i)_{i\in I},(V_j)_{j\in J})g).$$

See [**3**] for the proof that f is an A_∞-functor.

According to Proposition 1.3.16, there exists an isomorphism of A_∞-categories

$$\underline{\varphi}^{\mathsf{A}_\infty}:\underline{\mathsf{A}_\infty}((\mathcal{B}_j)_{j\in J};\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C}))\to\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I},(\mathcal{B}_j)_{j\in J};\mathcal{C}).\tag{3.3.9}$$

By Corollary 1.3.17, $\operatorname{Ob}\underline{\varphi}^{\mathsf{A}_\infty}=\varphi^{\mathsf{A}_\infty}$. Indeed, in the commutative diagram

$$\begin{tikzcd}\mathsf{A}_\infty(;\underline{\mathsf{A}_\infty}((\mathcal{B}_j)_{j\in J};\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C}))) \arrow[r, "\mathsf{A}_\infty(;\underline{\varphi}^{\mathsf{A}_\infty})"] \arrow[d, "\varphi^{\mathsf{A}_\infty}"'] & \mathsf{A}_\infty(;\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I},(\mathcal{B}_j)_{j\in J};\mathcal{C})) \arrow[d, "\varphi^{\mathsf{A}_\infty}"] \\ \mathsf{A}_\infty((\mathcal{B}_j)_{j\in J};\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})) \arrow[r, "\varphi^{\mathsf{A}_\infty}"] & \mathsf{A}_\infty((\mathcal{A}_i)_{i\in I},(\mathcal{B}_j)_{j\in J};\mathcal{C})\end{tikzcd}$$

the bottom arrow identifies with the mapping $\operatorname{Ob}\underline{\varphi}^{\mathsf{A}_\infty}$, by Remark 3.2.24. The isomorphisms $\underline{\varphi}^{\mathsf{A}_\infty}$ allows to compare the internal Hom-objects $\underline{\mathsf{A}_\infty}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{B})$ with those defined inductively according to the recipe of Proposition 1.3.13. Namely, we have a chain of isomorphisms

$$\begin{aligned}\underline{\mathsf{A}_\infty}(\mathcal{A}_n;\underline{\mathsf{A}_\infty}(\mathcal{A}_{n-1};\dots\underline{\mathsf{A}_\infty}(\mathcal{A}_1;\mathcal{B})\dots))&\xrightarrow[\sim]{\underline{\varphi}^{\mathsf{A}_\infty}}\underline{\mathsf{A}_\infty}(\mathcal{A}_{n-1},\mathcal{A}_n;\underline{\mathsf{A}_\infty}(\mathcal{A}_{n-2};\dots\underline{\mathsf{A}_\infty}(\mathcal{A}_1;\mathcal{B})\dots))\\ &\dots\\ &\xrightarrow[\sim]{\underline{\varphi}^{\mathsf{A}_\infty}}\underline{\mathsf{A}_\infty}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{B}).\end{aligned}$$

We will not need explicit formulas for $\underline{\varphi}^{\mathsf{A}_\infty}$, though it is not difficult to show that it is a strict A_∞-functor.

3.3.3. Composition. According to the general recipe of Proposition 1.3.14, for each map $\phi:I\to J$ in $\operatorname{Mor}\mathcal{S}$ and A_∞-categories $\mathcal{A}_i$, $\mathcal{B}_j$, $\mathcal{C}$, $i\in I$, $j\in J$, there is a unique A_∞-functor

$$M=\mu_\phi^{\underline{\mathsf{A}_\infty}}:\bigl(\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in\phi^{-1}j};\mathcal{B}_j)\bigr)_{j\in J},\underline{\mathsf{A}_\infty}((\mathcal{B}_j)_{j\in J};\mathcal{C})\to\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})$$

that makes the diagram

$$\begin{tikzcd}
(\mathcal{A}_i)_{i\in I}, \bigl(\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in\phi^{-1}j};\mathcal{B}_j)\bigr)_{j\in J}, \underline{\mathsf{A}_\infty}((\mathcal{B}_j)_{j\in J};\mathcal{C}) \arrow[r, "(\mathrm{id}_{\mathcal{A}_i})_{i\in I}{,}M"] \arrow[d, "(\mathrm{ev}^{\mathsf{A}_\infty}_{(\mathcal{A}_i)_{i\in\phi^{-1}j};\mathcal{B}_j})_{j\in J}{,}\mathrm{id}_{\underline{\mathsf{A}_\infty}((\mathcal{B}_j)_{j\in J};\mathcal{C})}"'] & (\mathcal{A}_i)_{i\in I}, \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C}) \arrow[d, "\mathrm{ev}^{\mathsf{A}_\infty}_{(\mathcal{A}_i)_{i\in I};\mathcal{C}}"] \\
(\mathcal{B}_j)_{j\in J}, \underline{\mathsf{A}_\infty}((\mathcal{B}_j)_{j\in J};\mathcal{C}) \arrow[r, "\mathrm{ev}^{\mathsf{A}_\infty}_{(\mathcal{B}_j)_{j\in J};\mathcal{C}}"] & \mathcal{C}
\end{tikzcd}$$

commute. The mapping $\operatorname{Ob} M$ is composition of A_∞-functors. It is not difficult to deduce from the diagram that components

$$M_{(m_i)_{i\in I},n} : \boxtimes^{i\in I} T^{m_i} s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in\phi^{-1}j};\mathcal{B}_j) \boxtimes T^n s\underline{\mathsf{A}_\infty}((\mathcal{B}_j)_{j\in J};\mathcal{C}) \to s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})$$

vanish unless $n = 0$ or $n = 1$. For each $(m_i)_{i\in I} \in \mathbb{N}^I$, the component

$$\begin{aligned} M_{(m_i)_{i\in I},0} : \otimes^{j\in J} T^{m_i} s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in\phi^{-1}j};\mathcal{B}_j)(f_j,g_j) \otimes T^0 s\underline{\mathsf{A}_\infty}((\mathcal{B}_j)_{j\in J};\mathcal{C})(h,h) \\ \to s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})((f_j)_{j\in J}\cdot_\phi h,(g_j)_{j\in J}\cdot_\phi h) \end{aligned}$$

maps an element $\otimes^{j\in J}(r^{j1}\otimes\dots\otimes r^{jm_j})$ of the source to an $((f_j)_{j\in J}\cdot_\phi h,(g_j)_{j\in J}\cdot_\phi h)$-coderivation p given by

$$\check{p} = \bigl[\boxtimes^{i\in I} Ts\mathcal{A}_i \xrightarrow[\sim]{\lambda^\phi} \boxtimes^{j\in J}\boxtimes^{i\in\phi^{-1}j} Ts\mathcal{A}_i \xrightarrow{\boxtimes^{j\in J}(r^{j1}\otimes\dots\otimes r^{jm_j})\theta} \boxtimes^{j\in J} Ts\mathcal{B}_j \xrightarrow{\check{h}} Ts\mathcal{C}\bigr].$$

Similarly, the component

$$\begin{aligned} M_{(m_i)_{i\in I},1} : \otimes^{j\in J} T^{m_i} s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in\phi^{-1}j};\mathcal{B}_j)(f_j,g_j) \otimes s\underline{\mathsf{A}_\infty}((\mathcal{B}_j)_{j\in J};\mathcal{C})(h,k) \\ \to s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})((f_j)_{j\in J}\cdot_\phi h,(g_j)_{j\in J}\cdot_\phi k) \end{aligned}$$

maps an element $\otimes^{j\in J}(r^{j1}\otimes\dots\otimes r^{jm_j})\otimes t$ to an $((f_j)_{j\in J}\cdot_\phi h,(g_j)_{j\in J}\cdot_\phi k)$-coderivation q given by

$$\check{q} = \bigl[\boxtimes^{i\in I} Ts\mathcal{A}_i \xrightarrow[\sim]{\lambda^\phi} \boxtimes^{j\in J}\boxtimes^{i\in\phi^{-1}j} Ts\mathcal{A}_i \xrightarrow{\boxtimes^{j\in J}(r^{j1}\otimes\dots\otimes r^{jm_j})\theta} \boxtimes^{j\in J} Ts\mathcal{B}_j \xrightarrow{\check{t}} Ts\mathcal{C}\bigr].$$

3.3.4. Example. Suppose that $f^1 : \mathcal{A}_1^1,\dots,\mathcal{A}_1^{m_1} \to \mathcal{B}_1$, …, $f^n : \mathcal{A}_n^1,\dots,\mathcal{A}_n^{m_n} \to \mathcal{B}_n$ are A_∞-functors. Let $r : g \to h : \mathcal{B}_1,\dots,\mathcal{B}_n \to \mathcal{C}$ be a coderivation. Then

$$(f^1\boxtimes\dots\boxtimes f^n\boxtimes r)M_{0\dots01} = (f^1\boxtimes\dots\boxtimes f^n)\cdot r$$

as $\bigl((f^1\boxtimes\dots\boxtimes f^n)\cdot g,(f^1\boxtimes\dots\boxtimes f^n)\cdot h\bigr)$-coderivations. Indeed, it is easy to see that the right hand side is an $\bigl((f^1\boxtimes\dots\boxtimes f^n)\cdot g,(f^1\boxtimes\dots\boxtimes f^n)\cdot h\bigr)$-coderivation. Moreover, by definition

$$[(f^1\boxtimes\dots\boxtimes f^n\boxtimes r)M_{0\dots01}]^\vee = (f^1\boxtimes\dots\boxtimes f^n)\cdot\check{r} = [(f^1\boxtimes\dots\boxtimes f^n)\cdot r]^\vee.$$

The claimed equality follows from Proposition 3.2.15.

3.3.5. Example. Suppose that $f^1 : \mathcal{A}_1^1,\dots,\mathcal{A}_1^{m_1} \to \mathcal{B}_1$, …, $f^n : \mathcal{A}_n^1,\dots,\mathcal{A}_n^{m_n} \to \mathcal{B}_n$, $g : \mathcal{B}_1,\dots,\mathcal{B}_n \to \mathcal{C}$ and $h^i : \mathcal{A}_i^1,\dots,\mathcal{A}_i^{m_i} \to \mathcal{B}_i$ for some i, $1 \leqslant i \leqslant n$, are A_∞-functors. Let $p : f^i \to h^i : \mathcal{A}_i^1,\dots,\mathcal{A}_i^{m_i} \to \mathcal{B}_i$ be a coderivation. Then similarly to the previous example

$$(f^1\boxtimes\dots\boxtimes f^{i-1}\boxtimes p\boxtimes f^{i+1}\boxtimes\dots\boxtimes f^n\boxtimes g)M_{e_i0} = (f^1\boxtimes\dots\boxtimes f^{i-1}\boxtimes p\boxtimes f^{i+1}\boxtimes\dots\boxtimes f^n)\cdot g$$

as $\bigl((f^1\boxtimes\dots\boxtimes f^i\boxtimes\dots\boxtimes f^n)\cdot g,(f^1\boxtimes\dots\boxtimes h^i\boxtimes\dots\boxtimes f^n)\cdot g\bigr)$-coderivations. Here $e_i = (0,\dots,0,1,0,\dots,0) \in \mathbb{N}^n$ (1 at i^{th} spot).

3.4. Unital A_∞-categories

Denote by $\mathcal{K}$ the homotopy category of complexes of $\Bbbk$-modules. We will consider non-unital categories and functors, enriched in $\mathcal{K}$. They form a category $\mathcal{K}$-$\mathbf{Cat}^{nu}$. Unital $\mathcal{K}$-categories and $\mathcal{K}$-functors form a smaller category $\mathcal{K}$-$\mathbf{Cat}$. Abusing notation, denote the underlying category of the multicategory A_∞ by the same symbol. There is a functor $\mathsf{k} : \mathsf{A}_\infty \to \mathcal{K}\text{-}\mathbf{Cat}^{nu}$, constructed in [**38**, Proposition 8.6]. It assigns to an A_∞-category $\mathcal{C}$ the $\mathcal{K}$-category $\mathsf{k}\mathcal{C}$ with the same set of objects $\operatorname{Ob}\mathsf{k}\mathcal{C} = \operatorname{Ob}\mathcal{C}$, the same graded $\Bbbk$-module of morphisms $\mathsf{k}\mathcal{C}(X,Y) = \mathcal{C}(X,Y)$, equipped with the differential $m_1 = sb_1s^{-1}$. Composition $\mu_{\mathsf{k}\mathcal{C}}$ in $\mathsf{k}\mathcal{C}$ is given by (the homotopy equivalence class of) $m_2 = (s\otimes s)b_2s^{-1} : \mathcal{C}(X,Y)\otimes\mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$. Thus the functor k can be thought as a kind of forgetful functor that truncates the structure of an A_∞-category at the level 2, forgetting higher homotopies.

3.4.1. Example. It is easy to see that $\mu_{\underline{\mathcal{K}}} = m_2^{\underline{\mathsf{C}}_\Bbbk}$ and $1_X^{\underline{\mathcal{K}}} = 1_X^{\underline{\mathsf{C}}_\Bbbk}$, therefore $\underline{\mathcal{K}} = \mathsf{k}\underline{\mathsf{C}}_\Bbbk$.

We are going to extend the functor k to a sort of multifunctor $\mathsf{k} : \mathsf{A}_\infty \to \widehat{\mathcal{K}\text{-}\mathbf{Cat}}^{nu}$. The mapping $\operatorname{Ob}\mathsf{k}$ which assigns the $\mathcal{K}$-category $\mathsf{k}\mathcal{C}$ to an A_∞-category $\mathcal{C}$ is described above.

Let $f : (\mathcal{A}_j)_{j\in J} \to \mathcal{B}$ be an A_∞-functor. Define a (non-unital) $\mathcal{K}$-functor $\mathsf{k}f : \boxtimes^{j\in J}\mathsf{k}\mathcal{A}_j \to \mathsf{k}\mathcal{B}$ on objects by $\operatorname{Ob}\mathsf{k}f = \operatorname{Ob}f : \prod_{j\in J}\operatorname{Ob}\mathcal{A}_j \to \operatorname{Ob}\mathcal{B}$. On morphisms we set

$$\begin{aligned}\mathsf{k}f = \bigl[\otimes^{j\in J}\mathsf{k}\mathcal{A}_j(X_j,Y_j) &\xrightarrow{\otimes^{j\in J}sf_{e_j}s^{-1}} \otimes^{j\in J}\mathsf{k}\mathcal{B}\bigl(((Y_i)_{i<j},(X_i)_{i\geqslant j})f,((Y_i)_{i\leqslant j},(X_i)_{i>j})f\bigr) \\ &\xrightarrow{\mu^J_{\mathsf{k}\mathcal{B}}} \mathsf{k}\mathcal{B}\bigl((X_i)_{i\in J}f,(Y_i)_{i\in J}f\bigr)\bigr], \qquad (3.4.1)\end{aligned}$$

where the chain maps $sf_{e_j}s^{-1}$ are given by (3.2.20), and $\mu^J_{\mathsf{k}\mathcal{B}}$ is the composition of $|J|$ composable arrows in $\mathsf{k}\mathcal{B}$.

3.4.2. Proposition. *$\mathsf{k}f$ is a (non-unital) $\mathcal{K}$-functor.*

Proof. Let X_i, Y_i, Z_i be objects of $\mathcal{A}_i$, $i\in I$. We must prove the following equation in $\mathcal{K}$:

$$\begin{aligned}&\bigl[\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i)\otimes\otimes^{i\in I}\mathcal{A}_i(Y_i,Z_i) \\ &\quad\xrightarrow{\otimes^{i\in I}sf_{e_i}s^{-1}\otimes\otimes^{i\in I}sf_{e_i}s^{-1}} \otimes^{i\in I}\mathcal{B}\bigl(((Y_j)_{j<i},(X_j)_{j\geqslant i})f,((Y_j)_{j\leqslant i},(X_j)_{j>i})f\bigr) \\ &\qquad\qquad\otimes\otimes^{i\in I}\mathcal{B}\bigl(((Z_j)_{j<i},(Y_j)_{j\geqslant i})f,((Z_j)_{j\leqslant i},(Y_j)_{j>i})f\bigr) \\ &\quad\xrightarrow{\mu^I_{\mathsf{k}\mathcal{B}}\otimes\mu^I_{\mathsf{k}\mathcal{B}}} \mathcal{B}((X_i)_{i\in I}f,(Y_i)_{i\in I}f)\otimes\mathcal{B}((Y_i)_{i\in I}f,(Z_i)_{i\in I}f) \xrightarrow{\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{B}}} \mathcal{B}((X_i)_{i\in I}f,(Z_i)_{i\in I}f)\bigr] \\ &= \bigl[\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i)\otimes\otimes^{i\in I}\mathcal{A}_i(Y_i,Z_i) \\ &\quad\xrightarrow{\sigma_{(12)}} \otimes^{i\in I}(\mathcal{A}_i(X_i,Y_i)\otimes\mathcal{A}_i(Y_i,Z_i)) \xrightarrow{\otimes^{i\in I}\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{A}_i}} \otimes^{i\in I}\mathcal{A}_i(X_i,Z_i) \\ &\quad\xrightarrow{\otimes^{i\in I}sf_{e_i}s^{-1}} \otimes^{i\in I}\mathcal{B}\bigl(((Z_j)_{j<i},(X_j)_{j\geqslant i})f,((Z_j)_{j\leqslant i},(X_j)_{j>i})f\bigr) \\ &\qquad\qquad\xrightarrow{\mu^I_{\mathsf{k}\mathcal{B}}} \mathcal{B}((X_i)_{i\in I}f,(Z_i)_{i\in I}f)\bigr], \qquad (3.4.2)\end{aligned}$$

In particular case $I = \mathbf{1}$ it takes the form

$$\bigl[\mathcal{A}(X,Y)\otimes\mathcal{A}(Y,Z)\xrightarrow{sf_1s^{-1}\otimes sf_1s^{-1}}\mathcal{B}(Xf,Yf)\otimes\mathcal{B}(Yf,Zf)\xrightarrow{\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{B}}}\mathcal{B}(Xf,Zf)\bigr]$$
$$=\bigl[\mathcal{A}(X,Y)\otimes\mathcal{A}(Y,Z)\xrightarrow{\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{A}}}\mathcal{A}(X,Z)\xrightarrow{sf_1s^{-1}}\mathcal{B}(Xf,Zf)\bigr]. \quad (3.4.3)$$

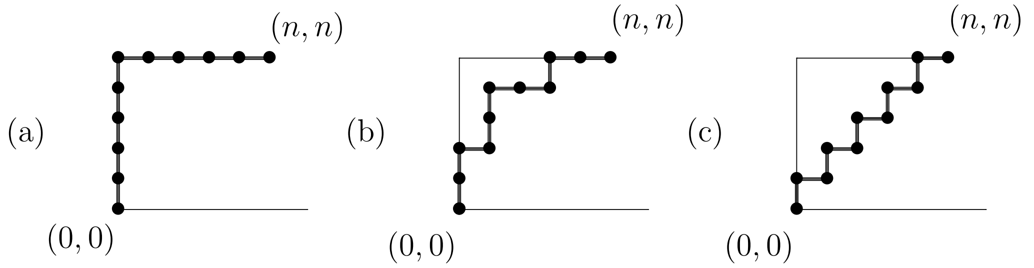
This equation in $\mathcal{K}$ is proven in [**38**, Proposition 8.6]. In fact, the difference of the right and the left hand sides is the boundary of $(s\otimes s)f_2s^{-1}$.

Since $f_{e_i} : \mathcal{A}_i \to \mathcal{B}$, $i \in I$, are A_∞-functors, they satisfy equation (3.4.3). Therefore, (3.4.2) is equivalent to the following equation in $\mathcal{K}$:

$$\bigl[\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i)\otimes\otimes^{i\in I}\mathcal{A}_i(Y_i,Z_i)$$
$$\xrightarrow{\otimes^{i\in I}sf_{e_i}s^{-1}\otimes\otimes^{i\in I}sf_{e_i}s^{-1}}\otimes^{i\in I}\mathcal{B}\bigl(((Y_j)_{j<i},(X_j)_{j\geqslant i})f,((Y_j)_{j\leqslant i},(X_j)_{j>i})f\bigr)$$
$$\otimes\otimes^{i\in I}\mathcal{B}\bigl(((Z_j)_{j<i},(Y_j)_{j\geqslant i})f,((Z_j)_{j\leqslant i},(Y_j)_{j>i})f\bigr)$$
$$\xrightarrow{\mu^I_{\mathsf{k}\mathcal{B}}\otimes\mu^I_{\mathsf{k}\mathcal{B}}}\mathcal{B}((X_i)_{i\in I}f,(Y_i)_{i\in I}f)\otimes\mathcal{B}((Y_i)_{i\in I}f,(Z_i)_{i\in I}f)\xrightarrow{\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{B}}}\mathcal{B}((X_i)_{i\in I}f,(Z_i)_{i\in I}f)\bigr]$$
$$=\bigl[\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i)\otimes\otimes^{i\in I}\mathcal{A}_i(Y_i,Z_i)$$
$$\xrightarrow{\otimes^{i\in I}sf_{e_i}s^{-1}\otimes\otimes^{i\in I}sf_{e_i}s^{-1}}\otimes^{i\in I}\mathcal{B}(((Z_j)_{j<i},(X_j)_{j\geqslant i})f,((Z_j)_{j<i},Y_i,(X_j)_{j>i})f)$$
$$\otimes\otimes^{i\in I}\mathcal{B}(((Z_j)_{j<i},Y_i,(X_j)_{j>i})f,((Z_j)_{j\leqslant i},(X_j)_{j>i})f)$$
$$\xrightarrow{\sigma_{(12)}}\otimes^{i\in I}\bigl(\mathcal{B}(((Z_j)_{j<i},(X_j)_{j\geqslant i})f,((Z_j)_{j<i},Y_i,(X_j)_{j>i})f)$$
$$\otimes\mathcal{B}(((Z_j)_{j<i},Y_i,(X_j)_{j>i})f,((Z_j)_{j\leqslant i},(X_j)_{j>i})f)\bigr)$$
$$\xrightarrow{\otimes^{i\in I}\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{B}}}\otimes^{i\in I}\mathcal{B}(((Z_j)_{j<i},(X_j)_{j\geqslant i})f,((Z_j)_{j\geqslant i},(X_j)_{j>i})f)$$
$$\xrightarrow{\mu^I_{\mathsf{k}\mathcal{B}}}\mathcal{B}((X_i)_{i\in I}f,(Z_i)_{i\in I}f)\bigr], \quad (3.4.4)$$

which we are going to prove. We may assume that $I = \mathbf{n}$. Two parts of equation (3.4.4) are particular cases of the following construction.

Consider *staircases* defined as connected subsets S of the plane which are unions of $2n$ segments of the form $[(k-1,i-1),(k-1,i)]$ or $[(k-1,i),(k,i)]$ for integers $1 \leqslant k \leqslant i \leqslant n$. We assume also that $(0,0) \in S$ and $(n,n) \in S$, see examples with $n = 5$ below.



(3.4.5)

With a staircase S two non-decreasing functions $l, k : \mathbf{n} \to \mathbf{n}$ are associated. Namely, $l(p) = l_S(p)$ is the smallest l such that $(p,l) \in S$, and $k(i) = k_S(i)$ is the smallest k such that $(k-1,i) \in S$. Notice that $l_S(p) \geqslant p$ and $k_S(i) \leqslant i$. Moreover, any non-decreasing function $k : \mathbf{n} \to \mathbf{n}$ (resp. $l : \mathbf{n} \to \mathbf{n}$) such that $k(i) \leqslant i$ (resp. $l(p) \geqslant p$) determines a unique staircase S such that $k_S = k$ (resp. $l_S = l$).

Let $(W_{p,l})_{0\leqslant p\leqslant l\leqslant n}$ be objects of $\mathcal{B}$. The staircase S gives rise to a map

$$\begin{aligned}\otimes^{i\in\mathbf{n}}\,\mathcal{B}(W_{k(i)-1,i-1},W_{k(i)-1,i})\otimes\otimes^{p\in\mathbf{n}}\mathcal{B}(W_{p-1,l(p)},W_{p,l(p)})&\\ \xrightarrow{\mathrm{sh}_S}\otimes^{\mathbf{2n}}\mathcal{B}(W_{\bullet\bullet},W_{\bullet\bullet})\xrightarrow{\mu^{\mathbf{2n}}_{\Bbbk\mathcal{B}}}\mathcal{B}(W_{0,0},W_{n,n}),&\quad(3.4.6)\end{aligned}$$

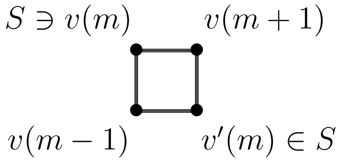
where the signed (n,n)-shuffle sh_S is associated with the staircase S. Namely, if the m-th segment of S, $1\leqslant m\leqslant 2n$ (starting from the segment $[(0,0),(0,1)]$) is vertical (resp. horizontal), then the m-th factor of $\otimes^{\mathbf{2n}}\mathcal{B}(W_{\bullet\bullet},W_{\bullet\bullet})$ comes from the first (resp. the last) n factors of the source. Thus the intermediate tensor product has the form $\otimes^{m\in\mathbf{2n}}\mathcal{B}(W_{v(m-1)},W_{v(m)})$, where $(v(m))_{m=0}^{2n}\subset\mathbb{Z}^2$ is the list of all adjacent integer vertices belonging to S, $v(0)=(0,0)$, $v(2n)=(n,n)$. In particular, the composition $\mu^{\mathbf{2n}}_{\Bbbk\mathcal{B}}$ makes sense.

Let us take $W_{p,l}=((Z_j)_{j\leqslant p},(Y_j)_{p<j\leqslant l},(X_j)_{j>l})f$, $0\leqslant p\leqslant l\leqslant n$. Given a staircase S, we extend mapping (3.4.6) to the following:

$$\begin{aligned}\varkappa(f,S)=\bigl[\otimes^{i\in\mathbf{n}}\mathcal{A}(X_i,Y_i)\otimes\otimes^{p\in\mathbf{n}}\mathcal{A}_p(Y_p,Z_p)&\xrightarrow{\otimes^{i\in\mathbf{n}}sf_{e_i}s^{-1}\otimes\otimes^{p\in\mathbf{n}}sf_{e_p}s^{-1}}\\ \otimes^{i\in\mathbf{n}}\,\mathcal{B}\bigl(((Z_j)_{j<k(i)},(Y_j)^{j<i}_{j\geqslant k(i)},(X_j)_{j\geqslant i})f,((Z_j)_{j<k(i)},(Y_j)^{j\leqslant i}_{j\geqslant k(i)},(X_i)_{j>i})f\bigr)&\\ \otimes\otimes^{p\in\mathbf{n}}\mathcal{B}\bigl(((Z_j)_{j<p},(Y_j)^{j\leqslant l(p)}_{j\geqslant p},(X_j)_{j>l(p)})f,((Z_j)_{j\leqslant p},(Y_j)^{j\leqslant l(p)}_{j>p},(X_j)_{j>l(p)})f\bigr)&\\ =\otimes^{i\in\mathbf{n}}\mathcal{B}(W_{k(i)-1,i-1},W_{k(i)-1,i})\otimes\otimes^{p\in\mathbf{n}}\mathcal{B}(W_{p-1,l(p)},W_{p,l(p)})&\\ \xrightarrow{\mathrm{sh}_S}\otimes^{\mathbf{2n}}\mathcal{B}(W_{v(m-1)},W_{v(m)})\xrightarrow{\mu^{\mathbf{2n}}_{\Bbbk\mathcal{B}}}\mathcal{B}(W_{0,0},W_{n,n})=\mathcal{B}((X_j)_{j\in\mathbf{n}}f,(Z_j)_{j\in\mathbf{n}}f)\bigr].&\end{aligned}$$

The left hand side of (3.4.4) equals $\varkappa(f,S_a)$, where S_a from (3.4.5)(a) gives $\mathrm{sh}_{S_a}=\mathrm{id}$, $k(i)=1$, $l(p)=n$. The right hand side of (3.4.4) equals $\varkappa(f,S_c)$, where S_c from (3.4.5)(c) gives $\mathrm{sh}_{S_c}=\varkappa$, $k(i)=i$, $l(p)=p$.

We claim that the composition $\varkappa(f,S)$ does not depend on the staircase S. Indeed, consider two staircases S, S' which coincide everywhere except in m^{th} and $(m+1)^{\text{st}}$ segments, $0<m<2m-1$, as drawn:



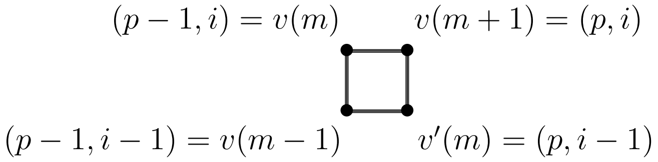
Then the corresponding shuffles are related by the equation $\mathrm{sh}_{S'}=\mathrm{sh}_S\cdot(m,m+1)_c$. Let $i\in\mathbf{n}$ (resp. $p\in\mathbf{n}$) index the factors which come to m^{th} (resp. $(m+1)^{\text{st}}$) place after application of sh_S. Expressions

$$\mathcal{B}(W_{k(i)-1,i-1},W_{k(i)-1,i})\otimes\mathcal{B}(W_{p-1,l(p)},W_{p,l(p)})$$

and

$$\mathcal{B}(W_{v(m-1)},W_{v(m)})\otimes\mathcal{B}(W_{v(m)},W_{v(m+1)})$$

are identical. This implies $p=k(i)$ and $i=l(p)$ and gives coordinates of the four points:



In particular, $p \leqslant i-1$. The expression for $\varkappa(f,S')$ differs from that for $\varkappa(f,S)$ by an extra factor

$$(12)_c : \mathcal{B}(W_{v'(m)}, W_{v(m+1)}) \otimes \mathcal{B}(W_{v(m-1)}, W_{v'(m)}) \\ \to \mathcal{B}(W_{v(m-1)}, W_{v'(m)}) \otimes \mathcal{B}(W_{v'(m)}, W_{v(m+1)}).$$

Thus, the equation $\varkappa(f,S) = \varkappa(f,S')$ follows from the equation in $\mathcal{K}$

$$\big[\mathcal{A}_i(X_i,Y_i) \otimes \mathcal{A}_p(Y_p,Z_p) \xrightarrow{sf_{e_i}s^{-1} \otimes sf_{e_p}s^{-1}} \\ \mathcal{B}(W_{p-1,i-1}, W_{p-1,i}) \otimes \mathcal{B}(W_{p-1,i}, W_{p,i}) \xrightarrow{\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{B}}} \mathcal{B}(W_{p-1,i-1}, W_{p,i})\big] \\ = \big[\mathcal{A}_i(X_i,Y_i) \otimes \mathcal{A}_p(Y_p,Z_p) \xrightarrow{sf_{e_i}s^{-1} \otimes sf_{e_p}s^{-1}} \mathcal{B}(W_{p,i-1}, W_{p,i}) \otimes \mathcal{B}(W_{p-1,i-1}, W_{p,i-1}) \\ \xrightarrow{(12)_c} \mathcal{B}(W_{p-1,i-1}, W_{p,i-1}) \otimes \mathcal{B}(W_{p,i-1}, W_{p,i}) \xrightarrow{\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{B}}} \mathcal{B}(W_{p-1,i-1}, W_{p,i})\big],$$

which we are going to prove now.

Introduce an A_∞-functor of two variables

$$g = f|^{(Z_j)_{j<p},(Y_j)_{p<j<i},(X_j)_{j>i}}_{\{p,i\}} : \mathcal{A}_p, \mathcal{A}_i \to \mathcal{B}.$$

Recall that $p < i$. In terms of g the above equation in $\mathcal{K}$ can be rewritten as follows:

$$\big[\mathcal{A}_i(X_i,Y_i) \otimes \mathcal{A}_p(Y_p,Z_p) \xrightarrow{sg_{01}s^{-1} \otimes sg_{10}s^{-1}} \\ \mathcal{B}((Y_p,X_i)g, (Y_p,Y_i)g) \otimes \mathcal{B}((Y_p,Y_i)g, (Z_p,Y_i)g) \xrightarrow{\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{B}}} \mathcal{B}((Y_p,X_i)g, (Z_p,Y_i)g)\big] \\ = \big[\mathcal{A}_i(X_i,Y_i) \otimes \mathcal{A}_p(Y_p,Z_p) \xrightarrow{(12)_c} \mathcal{A}_p(Y_p,Z_p) \otimes \mathcal{A}_i(X_i,Y_i) \xrightarrow{sg_{10}s^{-1} \otimes sg_{01}s^{-1}} \\ \mathcal{B}((Y_p,X_i)g, (Z_p,X_i)g) \otimes \mathcal{B}((Z_p,X_i)g, (Z_p,Y_i)g) \xrightarrow{\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{B}}} \mathcal{B}((Y_p,X_i)g, (Z_p,Y_i)g)\big]. \quad (3.4.7)$$

In order to prove it we recall that $g\check{b} = (1 \boxtimes b + b \boxtimes 1)\check{g}$ by (3.2.17). The restriction of this equation to $s\mathcal{A}_i \boxtimes s\mathcal{A}_p$ gives

$$(g_{10} \otimes g_{01})b_2 + (12)_c(g_{01} \otimes g_{10})b_2 + g_{11}b_1 \\ = (1 \otimes b_1 + b_1 \otimes 1)g_{11} : s\mathcal{A}_i(X_i,Y_i) \otimes s\mathcal{A}_p(Y_p,Z_p) \to s\mathcal{B}((Y_p,X_i)g, (Z_p,Y_i)g).$$

Thus, $(g_{10} \otimes g_{01})b_2 + (12)_c(g_{01} \otimes g_{10})b_2$ is a boundary. Therefore,

$$(s \otimes s)(g_{01} \otimes g_{10})b_2s^{-1} = (12)_c(s \otimes s)(g_{10} \otimes g_{01})b_2s^{-1}$$

in $\mathcal{K}$. This implies equation (3.4.7).

Since any two staircases S' and S'' can be connected by a finite sequence of elementary transformations as above, it follows that $\varkappa(f,S') = \varkappa(f,S'')$. In particular, equation (3.4.4) holds true, so that $\mathsf{k}f$ is a (non-unital) $\mathcal{K}$-functor. □

3.4.3. Proposition. *The maps*

$$\operatorname{Ob}\mathsf{k} : \operatorname{Ob}\mathsf{A}_\infty \to \operatorname{Ob}\widehat{\mathcal{K}\text{-}\mathbf{Cat}}^{nu}, \qquad \mathcal{C} \mapsto \mathsf{k}\mathcal{C}$$

$$\mathsf{k} : \mathsf{A}_\infty((\mathcal{A}_j)_{j\in J}; \mathcal{B}) \to \widehat{\mathcal{K}\text{-}\mathbf{Cat}}^{nu}(\boxtimes^{j\in J}\mathsf{k}\mathcal{A}_j, \mathsf{k}\mathcal{B}), \quad f \mapsto \mathsf{k}f,$$

(given by (3.4.1) *for non-empty J only!) are compatible with composition and identities, and define a kind of (non-symmetric) multifunctor $\mathsf{k} : \mathsf{A}_\infty \to \widehat{\mathcal{K}\text{-}\mathbf{Cat}}^{nu}$.*

Proof. Let $\phi : I \to J$ be an order-preserving surjection with non-empty J. Suppose $f^j : (\mathcal{A}_i)_{i\in\phi^{-1}j} \to \mathcal{B}_j$, $j \in J$, $g : (\mathcal{B}_j)_{j\in J} \to \mathcal{C}$ are A_∞-functors. They are taken by k to

$$\begin{aligned}
\mathsf{k}f^j = \bigl[\otimes^{i\in\phi^{-1}j}\mathcal{A}_i(X_i,Y_i) &\xrightarrow{\otimes^{i\in\phi^{-1}j}(sf^j_{e_i^{\phi^{-1}j}}s^{-1})} \\
&\otimes^{i\in\phi^{-1}j}\,\mathcal{B}_j\bigl(((Y_k)^{\phi k=j}_{k<i},(X_k)^{\phi k=j}_{k\geqslant i})f^j,((Y_k)^{\phi k=j}_{k\leqslant i},(X_k)^{\phi k=j}_{k>i})f^j\bigr) \\
&\xrightarrow{\mu^{\phi^{-1}j}_{\mathsf{k}\mathcal{B}_j}} \mathcal{B}_j\bigl((X_i)_{i\in\phi^{-1}j}f^j,(Y_i)_{i\in\phi^{-1}j}f^j\bigr)\bigr], \\
\mathsf{k}g = \bigl[\otimes^{j\in J}\mathcal{B}_j(U_j,W_j) &\xrightarrow{\otimes^{j\in J}(sg_{e_j^J}s^{-1})} \otimes^{j\in J}\mathcal{C}\bigl(((W_l)_{l<j},(U_l)_{l\geqslant j})g,((W_l)_{l\leqslant j},(U_l)_{l>j})g\bigr) \\
&\xrightarrow{\mu^J_{\mathsf{k}\mathcal{C}}} \mathcal{C}\bigl((U_j)_{j\in J}g,(W_j)_{j\in J}g\bigr)\bigr],
\end{aligned}$$

where $e_i^{\phi^{-1}j} \in \mathbb{N}^{\phi^{-1}j}$ and $e_j^J \in \mathbb{N}^J$ are standard basis elements. The composite of these morphisms in $\widehat{\mathcal{K}\text{-}\mathbf{Cat}}^{nu}$ is

$$\begin{aligned}
(\mathsf{k}f^j)_{j\in J}\cdot_\phi \mathsf{k}g &= \bigl[\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i) \xrightarrow{\lambda^\phi} \otimes^{j\in J}\otimes^{i\in\phi^{-1}j}\,\mathcal{A}_i(X_i,Y_i) \\
&\xrightarrow{\otimes^{j\in J}\mathsf{k}f^j} \otimes^{j\in J}\mathcal{B}_j\bigl((X_i)_{i\in\phi^{-1}j}f^j,(Y_i)_{i\in\phi^{-1}j}f^j\bigr) \\
&\xrightarrow{\mathsf{k}g} \mathcal{C}\bigl(((X_i)_{i\in\phi^{-1}j}f^j)_{j\in J}g,((Y_i)_{i\in\phi^{-1}j}f^j)_{j\in J}g\bigr)\bigr] \\
= \bigl[\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i) &\xrightarrow{\lambda^\phi} \otimes^{j\in J}\otimes^{i\in\phi^{-1}j}\,\mathcal{A}_i(X_i,Y_i) \\
&\xrightarrow{\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}(sf^j_{e_i^{\phi^{-1}j}}s^{-1})} \otimes^{j\in J}\otimes^{i\in\phi^{-1}j}\,\mathcal{B}_j\bigl(((Y_k)^{\phi k=j}_{k<i},(X_k)^{\phi k=j}_{k\geqslant i})f^j, \\
&\qquad\qquad ((Y_k)^{\phi k=j}_{k\leqslant i},(X_k)^{\phi k=j}_{k>i})f^j\bigr) \\
&\xrightarrow{\otimes^{j\in J}\mu^{\phi^{-1}j}_{\mathsf{k}\mathcal{B}_j}} \otimes^{j\in J}\mathcal{B}_j\bigl((X_i)_{i\in\phi^{-1}j}f^j,(Y_i)_{i\in\phi^{-1}j}f^j\bigr) \\
&\xrightarrow{\otimes^{j\in J}(sg_{e_j^J}s^{-1})} \otimes^{j\in J}\mathcal{C}\bigl((((Y_i)_{i\in\phi^{-1}l}f^l)_{l<j},((X_i)_{i\in\phi^{-1}l}f^l)_{l\geqslant j})g, \\
&\qquad\qquad (((Y_i)_{i\in\phi^{-1}l}f^l)_{l\leqslant j},((X_i)_{i\in\phi^{-1}l}f^l)_{l>j})g\bigr) \\
&\xrightarrow{\mu^J_{\mathsf{k}\mathcal{C}}} \mathcal{C}\bigl(((X_i)_{i\in\phi^{-1}j}f^j)_{j\in J}g,((Y_i)_{i\in\phi^{-1}j}f^j)_{j\in J}g\bigr)\bigr]. \qquad (3.4.8)
\end{aligned}$$

Since $\mathsf{k}g\big|_j^{(W_l)_{l<j},(U_l)_{l>j}} : \mathcal{B}_j \to \mathcal{C}$, $V \mapsto ((W_l)_{l<j},V,(U_l)_{l>j})g$, given on morphisms by $sg_{e_j^J}s^{-1} : \mathcal{B}_j(U_j,W_j) \to \mathcal{C}\bigl(((W_l)_{l<j},(U_l)_{l\geqslant j})g,((W_l)_{l\leqslant j},(U_l)_{l>j})g\bigr)$ is a $\mathcal{K}$-functor, we have an equation in $\mathcal{K}$:

$$\begin{aligned}
&\bigl[\otimes^{i\in P}\mathcal{B}_j(V^j_{i-1},V^j_i) \xrightarrow{\mu^P_{\mathcal{B}_j}} \mathcal{B}_j(U_j,W_j) \xrightarrow{sg_{e_j^J}s^{-1}} \mathcal{C}\bigl(((W_l)_{l<j},(U_l)_{l\geqslant j})g,((W_l)_{l\leqslant j},(U_l)_{l>j})g\bigr)\bigr] \\
&= \bigl[\otimes^{i\in P}\mathcal{B}_j(V^j_{i-1},V^j_i) \xrightarrow{\otimes^{i\in P}sg_{e_j^J}s^{-1}} \otimes^{i\in P}\mathcal{C}\bigl(((W_l)_{l<j},V^j_{i-1},(U_l)_{l>j})g, \\
&\qquad\qquad ((W_l)_{l<j},V^j_i,(U_l)_{l>j})g\bigr) \\
&\qquad \xrightarrow{\mu^P_{\mathcal{C}}} \mathcal{C}\bigl(((W_l)_{l<j},(U_l)_{l\geqslant j})g,((W_l)_{l\leqslant j},(U_l)_{l>j})g\bigr)\bigr], \qquad (3.4.9)
\end{aligned}$$

where $P = \mathbf{p}$, $U_j = V_0^j$, $W_j = V_p^j$. Applying this identity to $P = \phi^{-1}j$, we turn (3.4.8) into:

$$
\begin{aligned}
\big[\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i) &\xrightarrow{\lambda^\phi} \otimes^{j\in J}\otimes^{i\in\phi^{-1}j}\mathcal{A}_i(X_i,Y_i) \xrightarrow{\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}(sf^j_{e_i^{\phi^{-1}j}}s^{-1})} \\
&\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}\mathcal{B}_j\big(((Y_k)^{\phi k=j}_{k<i},(X_k)^{\phi k=j}_{k\geqslant i})f^j,((Y_k)^{\phi k=j}_{k\leqslant i},(X_k)^{\phi k=j}_{k>i})f^j\big) \xrightarrow{\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}(sg_{e_j^J}s^{-1})} \\
&\quad \otimes^{j\in J}\otimes^{i\in\phi^{-1}j}\mathcal{C}\big\{\big(((Y_k)_{\phi k=l}f^l)_{l<j},((Y_k)^{\phi k=j}_{k<i},(X_k)^{\phi k=j}_{k\geqslant i})f^j,((X_k)_{\phi k=l}f^l)_{l>j}\big)g, \\
&\qquad \big(((Y_k)_{\phi k=l}f^l)_{l<j},((Y_k)^{\phi k=j}_{k\leqslant i},(X_k)^{\phi k=j}_{k>i})f^j,((X_k)_{\phi k=l}f^l)_{l>j}\big)g\big\} \\
&\xrightarrow{\otimes^{j\in J}\mu^{\phi^{-1}j}_{\mathsf{k}\mathcal{C}}} \otimes^{j\in J}\mathcal{C}\big((((Y_k)_{\phi k=l}f^l)_{l<j},((X_k)_{\phi k=l}f^l)_{l\geqslant j})g,(((Y_k)_{\phi k=l}f^l)_{l\leqslant j},((X_k)_{\phi k=l}f^l)_{l>j})g\big) \\
&\qquad \xrightarrow{\mu^J_{\mathsf{k}\mathcal{C}}} \mathcal{C}\big(((X_i)_{i\in\phi^{-1}j}f^j)_{j\in J}g,((Y_i)_{i\in\phi^{-1}j}f^j)_{j\in J}g\big)\big]. \qquad (3.4.10)
\end{aligned}
$$

Denote by h the composition $(f^j)_{j\in J}\cdot_\phi g : (\mathcal{A}_i)_{i\in I} \to \mathcal{C}$ in A_∞ given by (3.2.18). We have $f^j_{e_i^{\phi^{-1}j}}\cdot g_{e_j^J} = h_{e_i^I}$ if $\phi i = j$. Furthermore, $\lambda^\phi\cdot\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}h_{e_i^I} = \otimes^{i\in I}h_{e_i^I}\cdot\lambda^\phi$ by the naturality of λ^ϕ. Thus, (3.4.10) equals

$$
\begin{aligned}
\big[\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i) &\xrightarrow{\otimes^{i\in I}(sh_{e_i^I}s^{-1})} \otimes^{i\in I}\mathcal{C}\big(((Y_k)_{k<i},(X_k)_{k\geqslant i})h,((Y_k)_{k\leqslant i},(X_k)_{k>i})h\big) \\
&= \otimes^{i\in I}\mathcal{C}\big\{\big(((Y_k)_{\phi k=l}f^l)_{l<\phi i},((Y_k)^{\phi k=\phi i}_{k<i},(X_k)^{\phi k=\phi i}_{k\geqslant i})f^{\phi i},((X_k)_{\phi k=l}f^l)_{l>\phi i}\big)g, \\
&\qquad \big(((Y_k)_{\phi k=l}f^l)_{l<\phi i},((Y_k)^{\phi k=\phi i}_{k\leqslant i},(X_k)^{\phi k=\phi i}_{k>i})f^{\phi i},((X_k)_{\phi k=l}f^l)_{l>\phi i}\big)g\big\} \\
&\xrightarrow{\lambda^\phi} \otimes^{j\in J}\otimes^{i\in\phi^{-1}j}\mathcal{C}\big\{\big(((Y_k)_{\phi k=l}f^l)_{l<j},((Y_k)^{\phi k=j}_{k<i},(X_k)^{\phi k=j}_{k\geqslant i})f^j,((X_k)_{\phi k=l}f^l)_{l>j}\big)g, \\
&\qquad \big(((Y_k)_{\phi k=l}f^l)_{l<j},((Y_k)^{\phi k=j}_{k\leqslant i},(X_k)^{\phi k=j}_{k>i})f^j,((X_k)_{\phi k=l}f^l)_{l>j}\big)g\big\} \\
&\xrightarrow{\otimes^{j\in J}\mu^{\phi^{-1}j}_{\mathsf{k}\mathcal{C}}} \otimes^{j\in J}\mathcal{C}\big((((Y_k)_{\phi k=l}f^l)_{l<j},((X_k)_{\phi k=l}f^l)_{l\geqslant j})g, \\
&\qquad\qquad (((Y_k)_{\phi k=l}f^l)_{l\leqslant j},((X_k)_{\phi k=l}f^l)_{l>j})g\big) \\
&\qquad \xrightarrow{\mu^J_{\mathsf{k}\mathcal{C}}} \mathcal{C}\big(((X_i)_{i\in\phi^{-1}j}f^j)_{j\in J}g,((Y_i)_{i\in\phi^{-1}j}f^j)_{j\in J}g\big)\big]. \qquad (3.4.11)
\end{aligned}
$$

By the associativity of composition in $\mathsf{k}\mathcal{C}$, we may replace the last three arrows in (3.4.11) with $\mu^I_{\mathsf{k}\mathcal{C}}$ and get

$$
\begin{aligned}
\big[\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i) &\xrightarrow{\otimes^{i\in I}(sh_{e_i^I}s^{-1})} \otimes^{i\in I}\mathcal{C}\big(((Y_k)_{k<i},(X_k)_{k\geqslant i})h,((Y_k)_{k\leqslant i},(X_k)_{k>i})h\big) \\
&\qquad \xrightarrow{\mu^I_{\mathsf{k}\mathcal{C}}} \mathcal{C}\big((X_i)_{i\in I}h,(Y_i)_{i\in I}h\big)\big] = \mathsf{k}h.
\end{aligned}
$$

This proves the compatibility of k with composition. Compatibility with identities is obvious. □

3.4.4. Definition. An A_∞-category $\mathcal{C}$ is called *unital* if $\mathsf{k}\mathcal{C}$ is unital, that is, if for each object X of $\mathcal{C}$, there is an identity $1_X : \mathbb{k} \to \mathcal{C}(X,X) \in \mathcal{K}$ such that equations

$$(\mathrm{id}\otimes 1_Y)m_2 = 1 = (1_X\otimes\mathrm{id})m_2 : \mathcal{C}(X,Y) \to \mathcal{C}(X,Y)$$

hold true in $\mathcal{K}$. In other terms, for each object X of $\mathcal{C}$, there is a cycle defined up to a boundary, a *unit element*, ${}_X\mathbf{i}_0^{\mathcal{C}} : \mathbb{k} \to (s\mathcal{C})^{-1}(X,X)$ such that the chain maps $(1\otimes{}_Y\mathbf{i}_0^{\mathcal{C}})b_2$,

$-({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 : s\mathcal{C}(X,Y) \to s\mathcal{C}(X,Y)$ are homotopic to the identity map. This definition is equivalent to unitality in the sense of [**38**, Definition 7.3], see also [*ibid*, Lemma 7.4].

If $\mathcal{B}$ is a unital A_∞-category and $f : () \to \mathcal{B}$ is an A_∞-functor identified with an object $X = ()f$ of $\mathcal{B}$ by Remark 3.2.24, then we define $\mathsf{k}f : \boxtimes^{\mathbf{0}}() \to \mathsf{k}\mathcal{B}$, $()_{i\in\mathbf{0}} \mapsto ()f = X$, on morphisms via (3.4.1) for $n = 0$. That is,

$$\mathsf{k}f = \big[\otimes^{\mathbf{0}}() = \Bbbk \xrightarrow[1_X]{\mu^{\mathbf{0}}_{\mathsf{k}\mathcal{B}}} \mathsf{k}\mathcal{B}(X,X)\big], \qquad 1 \mapsto 1_X. \tag{3.4.12}$$

3.4.5. Definition. Suppose $\mathcal{A}_i$, $\mathcal{B}$, $i \in I$, are unital A_∞-categories. An A_∞-functor $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ is called *unital* if the $\mathcal{K}$-functor $\mathsf{k}f : \boxtimes^{i\in I}\mathsf{k}\mathcal{A}_i \to \mathsf{k}\mathcal{B}$ is unital. The set of unital A_∞-functors is denoted $\mathsf{A}^{\mathrm{u}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B}) \subset \mathsf{A}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})$.

Due to (3.4.12), for $I = \varnothing$ any A_∞-functor $f : () \to \mathcal{B}$ is unital (if $\mathcal{B}$ is unital). For one-element I an A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ between unital A_∞-categories is unital if and only if ${}_X\mathbf{i}_0^{\mathcal{A}} f_1 - {}_{Xf}\mathbf{i}_0^{\mathcal{B}} \in \operatorname{Im} b_1$ for all objects X of $\mathcal{A}$. This criterion coincides with [**38**, Definition 8.1]. Since $\widehat{\mathcal{K}\text{-}\mathbf{Cat}}$ is a submulticategory of $\widehat{\mathcal{K}\text{-}\mathbf{Cat}}^{nu}$, the subsets $\mathsf{A}^{\mathrm{u}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B}) \subset \mathsf{A}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})$ form a submulticategory $\mathsf{A}^{\mathrm{u}}_\infty \subset \mathsf{A}_\infty$ with unital A_∞-categories as objects and unital A_∞-functors as morphisms.

3.4.6. Remark. Note that an isomorphism of unital $\mathcal{K}$-categories is necessarily unital, therefore an isomorphism of unital A_∞-categories is necessarily unital.

In the unital case the statement of Proposition 3.4.3 extends to the empty set J of arguments, and to composites for arbitrary order-preserving maps $\phi : I \to J$. The proof repeats the proof of Proposition 3.4.3 word by word. As a summary, we obtain a (non-symmetric) multifunctor $\mathsf{k} : \mathsf{A}^{\mathrm{u}}_\infty \to \widehat{\mathcal{K}\text{-}\mathbf{Cat}}$. We are going to prove that it is in fact symmetric.

3.4.7. Proposition. *The multifunctor* $\mathsf{k} : \mathsf{A}^{\mathrm{u}}_\infty \to \widehat{\mathcal{K}\text{-}\mathbf{Cat}}$ *is symmetric.*

Proof. By Proposition 1.2.7, it suffices to show that, for each bijection $\sigma : I \to K$ and A_∞-categories $\mathcal{A}_i$, $\mathcal{C}_k$, $\mathcal{B}$, $i \in I$, $k \in K$, such that $\mathcal{A}_i = \mathcal{C}_{\sigma(i)}$, for each $i \in I$, the diagram

$$\begin{tikzcd}
\mathsf{A}^{\mathrm{u}}_\infty((\mathcal{C}_k)_{k\in K};\mathcal{B}) \arrow[r, "\mathsf{k}"] \arrow[d, "\mathsf{A}^{\mathrm{u}}_\infty(\sigma;\mathcal{B})"'] & \widehat{\mathcal{K}\text{-}\mathbf{Cat}}((\mathsf{k}\mathcal{C}_k)_{k\in K};\mathsf{k}\mathcal{B}) \arrow[d, "\widehat{\mathcal{K}\text{-}\mathbf{Cat}}(\sigma;\mathsf{k}\mathcal{B})"] \\
\mathsf{A}^{\mathrm{u}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B}) \arrow[r, "\mathsf{k}"] & \widehat{\mathcal{K}\text{-}\mathbf{Cat}}((\mathsf{k}\mathcal{A}_i)_{i\in I};\mathsf{k}\mathcal{B})
\end{tikzcd}$$

commutes. Without loss of generality, we may assume that $I = K = \mathbf{n}$. Furthermore, since bijections $\mathbf{n} \to \mathbf{n}$ are generated by elementary transpositions, we may assume that $\sigma = (i, i+1)$ for some $1 \leqslant i < n$. Suppose $f : \boxtimes^{k\in\mathbf{n}} Ts\mathcal{A}_k \to Ts\mathcal{B}$ is a unital A_∞-functor. Then $g = \mathsf{A}^{\mathrm{u}}_\infty(\sigma;\mathcal{B})(f)$ is given by the composite

$$\big[\boxtimes^{\mathbf{n}}[(Ts\mathcal{A}_k)_{k<i}, Ts\mathcal{A}_{i+1}, Ts\mathcal{A}_i, (Ts\mathcal{A}_k)_{k>i+1}] \xrightarrow[\sim]{\lambda^\sigma} \boxtimes^{\mathbf{n}} Ts\mathcal{A}_i \xrightarrow{f} Ts\mathcal{B}\big].$$

We must prove that

$$\mathsf{k}g = \big[\boxtimes^{\mathbf{n}}[(\mathsf{k}\mathcal{A}_k)_{k<i}, \mathsf{k}\mathcal{A}_{i+1}, \mathsf{k}\mathcal{A}_i, (\mathsf{k}\mathcal{A}_k)_{k>i+1}] \xrightarrow[\sim]{\Lambda^\sigma_{\mathcal{K}\text{-}\mathbf{Cat}}} \boxtimes^{i\in\mathbf{n}} \mathsf{k}\mathcal{A}_i \xrightarrow{\mathsf{k}f} \mathsf{k}\mathcal{B}\big].$$

Clearly, $g_{e_k} = f_{e_k}$, for $k \neq i, i+1$, $g_{e_i} = f_{e_{i+1}}$, and $g_{e_{i+1}} = f_{e_i}$. Therefore the $\mathcal{K}$-functor $\mathsf{k}g$ is given by the composite

$$\begin{tikzcd}
\otimes^{\mathbf{n}}\bigl[\bigl(\mathcal{A}_k(X_k,Y_k)\bigr)_{k<i}, \mathcal{A}_{i+1}(X_{i+1},Y_{i+1}), \mathcal{A}_i(X_i,Y_i), \bigl(\mathcal{A}_k(X_k,Y_k)\bigr)_{k>i+1}\bigr] \arrow[d, "\otimes^{\mathbf{n}}[(sf_{e_k}s^{-1})_{k<i}, sf_{e_{i+1}}s^{-1}, sf_{e_i}s^{-1}, (sf_{e_k}s^{-1})_{k>i+1}]"] \\
\begin{array}{l} \otimes^{\mathbf{n}}\bigl[\bigl(\mathcal{B}(((Y_j)_{j<k},(X_j)_{j\geqslant k})f, ((Y_j)_{j\leqslant k},(X_j)_{j>k})f)\bigr)_{k<i}, \\ \qquad \mathcal{B}(((Y_j)_{j<i},(X_j)_{j\geqslant i})f, ((Y_j)_{j<i}, X_i, Y_{i+1}, (X_j)_{j>i+1})f), \\ \qquad \mathcal{B}(((Y_j)_{j<i}, X_i, Y_{i+1}, (X_j)_{j>i+1})f, ((Y_j)_{j\leqslant i+1},(X_j)_{j>i+1})f), \\ \qquad\qquad \bigl(\mathcal{B}(((Y_j)_{j<k},(X_j)_{j\geqslant k})f, ((Y_j)_{j\leqslant k},(X_j)_{j>k})f)\bigr)_{k>i+1}\bigr] \end{array} \arrow[d, "\mu^{\mathbf{n}}_{\mathsf{k}\mathcal{B}}"] \\
\mathcal{B}((X_1,\dots,X_n)f, (Y_1,\dots,Y_n)f),
\end{tikzcd}$$

while $\Lambda^\sigma_{\mathcal{K}\text{-}\mathbf{Cat}} \cdot \mathsf{k}f$ is given by

$$\begin{aligned}
\bigl[\otimes^{\mathbf{n}}&[(\mathcal{A}_k(X_k,Y_k))_{k<i}, \mathcal{A}_{i+1}(X_{i+1},Y_{i+1}), \mathcal{A}_i(X_i,Y_i), (\mathcal{A}_k(X_k,Y_k))_{k>i+1}] \xrightarrow{\lambda^\sigma} \\
&\otimes^{k\in\mathbf{n}} \mathcal{A}_k(X_k,Y_k) \xrightarrow{\otimes^{k\in\mathbf{n}} sf_{e_k}s^{-1}} \otimes^{k\in\mathbf{n}}\mathcal{B}(((Y_j)_{j<k},(X_j)_{j\geqslant k})f, ((Y_j)_{j\leqslant k},(X_j)_{j>k})f) \\
&\qquad\qquad \xrightarrow{\mu^{\mathbf{n}}_{\mathsf{k}\mathcal{B}}} \mathcal{B}((X_1,\dots,X_n)f, (Y_1,\dots,Y_n)f)\bigr].
\end{aligned}$$

By the associativity of composition in $\mathsf{k}\mathcal{B}$, it suffices to prove the equation

$$\begin{aligned}
\bigl[&\mathcal{A}_{i+1}(X_{i+1},Y_{i+1}) \otimes \mathcal{A}_i(X_i,Y_i) \\
&\xrightarrow{sf_{e_{i+1}}s^{-1}\otimes sf_{e_i}s^{-1}} \mathcal{B}(((Y_j)_{j<i},(X_j)_{j\geqslant i})f, ((Y_j)_{j<i}, X_i, Y_{i+1}, (X_j)_{j>i+1})f) \\
&\qquad\qquad \otimes \mathcal{B}(((Y_j)_{j<i}, X_i, Y_{i+1}, (X_j)_{j>i+1})f, ((Y_j)_{j\leqslant i+1},(X_j)_{j>i+1})f) \\
&\qquad\qquad\qquad \xrightarrow{\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{B}}} \mathcal{B}(((Y_j)_{j<i},(X_j)_{j\geqslant i})f, ((Y_j)_{j\leqslant i+1},(X_j)_{j>i+1})f)\bigr] \\
= \bigl[&\mathcal{A}_{i+1}(X_{i+1},Y_{i+1}) \otimes \mathcal{A}_i(X_i,Y_i) \xrightarrow{(12)_c} \mathcal{A}_i(X_i,Y_i) \otimes \mathcal{A}_{i+1}(X_{i+1},Y_{i+1}) \\
&\xrightarrow{sf_{e_i}s^{-1}\otimes sf_{e_{i+1}}s^{-1}} \mathcal{B}(((Y_j)_{j<i},(X_j)_{j\geqslant i})f, ((Y_j)_{j\leqslant i},(X_j)_{j>i})f) \\
&\qquad\qquad \otimes \mathcal{B}(((Y_j)_{j<i+1},(X_j)_{j>i+1})f, ((Y_j)_{\leqslant i+1},(X_j)_{j>i+1})f) \\
&\qquad\qquad\qquad \xrightarrow{\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{B}}} \mathcal{B}(((Y_j)_{j<i},(X_j)_{j\geqslant i})f, ((Y_j)_{j\leqslant i+1},(X_j)_{j>i+1})f)\bigr].
\end{aligned}$$

Introduce an A_∞-functor $h = f|^{(Y_k)_{k<i},(X_k)_{k>i}}_{\{i,i+1\}} : \mathcal{A}_i, \mathcal{A}_{i+1} \to \mathcal{B}$. In terms of h the above equation in $\mathcal{K}$ can be written as follows:

$$\begin{aligned}
\bigl[&\mathcal{A}_{i+1}(X_{i+1},Y_{i+1}) \otimes \mathcal{A}_i(X_i,Y_i) \\
&\xrightarrow{sh_{01}s^{-1}\otimes sh_{10}s^{-1}} \mathcal{B}((X_i,X_{i+1})h, (Y_i,X_{i+1})h) \otimes \mathcal{B}((Y_i,X_{i+1})h, (Y_i,Y_{i+1})h) \\
&\qquad\qquad\qquad \xrightarrow{\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{B}}} \mathcal{B}((X_i,X_{i+1})h, (Y_i,Y_{i+1})h)\bigr] \\
= \bigl[&\mathcal{A}_{i+1}(X_{i+1},Y_{i+1}) \otimes \mathcal{A}_i(X_i,Y_i) \xrightarrow{(12)_c} \mathcal{A}_i(X_i,Y_i) \otimes \mathcal{A}_{i+1}(X_{i+1},Y_{i+1}) \\
&\xrightarrow{sh_{10}s^{-1}\otimes sh_{01}s^{-1}} \mathcal{B}((X_i,X_{i+1})h, (Y_i,X_{i+1})h) \otimes \mathcal{B}((Y_i,X_{i+1})h, (Y_i,X_{i+1})h)
\end{aligned}$$

$$\xrightarrow{\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{B}}} \mathcal{B}((X_i,X_{i+1})h,(Y_i,Y_{i+1})h)\big].$$

It is proven in the same way as equation (3.4.7). Namely, the restriction of the equation $h\check{b} = (1\boxtimes b + b\boxtimes 1)\check{h}$ to $s\mathcal{A}_i\boxtimes s\mathcal{A}_{i+1}$ produces

$$\begin{aligned}(h_{10}\otimes h_{01})b_2 + (12)_c(h_{01}\otimes h_{10})b_2 + h_{11}b_1 = (1\otimes b_1 + b_1\otimes 1)h_{11} :\\ s\mathcal{A}_i(X_i,Y_i)\otimes s\mathcal{A}_{i+1}(X_{i+1},Y_{i+1}) \to s\mathcal{B}((X_i,X_{i+1})h,(Y_i,Y_{i+1})h).\end{aligned}$$

Therefore $(h_{10}\otimes h_{01})b_2 + (12)_c(h_{01}\otimes h_{10})b_2$ is a boundary and

$$(s\otimes s)(h_{01}\otimes h_{10})b_2s^{-1} = (12)_c(s\otimes s)(h_{10}\otimes h_{01})b_2s^{-1}$$

in $\mathcal{K}$. The proposition is proven. □

Consider a change of symmetric Monoidal base category given by the lax symmetric Monoidal functor

$$H^0 : \mathcal{K}\to \Bbbk\text{-}\mathbf{Mod},\quad C\mapsto \mathcal{K}(\Bbbk,C) = \operatorname{Ker}(d : C^0\to C^1)/\operatorname{Im}(d : C^{-1}\to C^0) = H^0(C).$$

Here $\Bbbk = \mathbb{1}_{\mathcal{K}}$ is the graded $\Bbbk$-module $\Bbbk$ concentrated in degree 0. According to Section 1.1.14, H^0 provides a lax symmetric Monoidal **Cat**-functor $H^0_* : \mathcal{K}\text{-}\mathbf{Cat}\to\Bbbk\text{-}\mathbf{Cat}$, which we have denoted also $\mathcal{B}\mapsto\overline{\mathcal{B}}$ in Example 1.1.16. Thus, $\operatorname{Ob}\overline{\mathcal{B}} = \operatorname{Ob}\mathcal{B}$ and $\overline{\mathcal{B}}(X,Y) = H^0(\mathcal{B}(X,Y))$. By Proposition 1.2.18, there is a symmetric multifunctor $\widehat{H^0_*} : \widehat{\mathcal{K}\text{-}\mathbf{Cat}}\to\widehat{\Bbbk\text{-}\mathbf{Cat}}$. Composing it with $\mathsf{k} : \mathsf{A}^{\mathrm{u}}_\infty\to\widehat{\mathcal{K}\text{-}\mathbf{Cat}}$ we get a symmetric multifunctor, denoted

$$H^0 = \widehat{H^0_*}\circ\mathsf{k} : \mathsf{A}^{\mathrm{u}}_\infty\to\widehat{\Bbbk\text{-}\mathbf{Cat}}$$

by abuse of notation. It assigns to a unital A_∞-category $\mathcal{C}$ the $\Bbbk$-linear category $H^0(\mathcal{C})$, the *homotopy category* of $\mathcal{C}$, with the same set of objects $\operatorname{Ob}H^0(\mathcal{C}) = \operatorname{Ob}\mathcal{C}$, with $\Bbbk$-modules of morphisms $H^0(\mathcal{C})(X,Y) = H^0\big(\mathcal{C}(X,Y),m_1\big)$, and with composition induced by m_2. Objects X and Y of $\mathcal{C}$ are called *isomorphic* if they are isomorphic in $H^0(\mathcal{C})$. In other words, if there exist cycles $\alpha\in s\mathcal{C}(X,Y)$ and $\beta\in s\mathcal{C}(Y,X)$ of degree -1 such that $(\alpha\otimes\beta)b_2 - {}_X\mathbf{i}^{\mathcal{C}}_0\in\operatorname{Im}b_1$ and $(\beta\otimes\alpha)b_2 - {}_Y\mathbf{i}^{\mathcal{C}}_0\in\operatorname{Im}b_1$.

Assume that $\mathcal{C}$ is an A_∞-category and $\Bbbk$ is a field. Then the unitality of the graded $\Bbbk$-linear category $H^\bullet(\mathcal{C})$ (*cohomological unitality* [**26, 34, 48**]) is equivalent to unitality of the A_∞-category $\mathcal{C}$ itself. Indeed, any chain complex of $\Bbbk$-vector spaces is homotopy isomorphic to its cohomology, the graded $\Bbbk$-vector space equipped with zero differential. Therefore, any two chain maps inducing the same map in cohomology are homotopic. Certainly, this does not hold for arbitrary complexes of modules over an arbitrary commutative ring $\Bbbk$.

3.4.8. Proposition. *Let $(\mathcal{A}_i)_{i\in I}$, $\mathcal{B}$ be unital A_∞-categories, and let $f : (\mathcal{A}_i)_{i\in I}\to\mathcal{B}$ be an A_∞-functor. It is unital if and only if the A_∞-functor $f|_j^{(X_i)_{i\neq j}} : \mathcal{A}_j\to\mathcal{B}$ is unital, for each $j\in I$.*

Proof. If $I=\varnothing$, the statement holds true. Let $j\in I$, and let $X_i\in\operatorname{Ob}\mathcal{A}_i$ for $i\in I$, $i\neq j$. We have seen that A_∞-functors $X_i : ()\to\mathcal{A}_i$ and $\mathrm{id}_{\mathcal{A}_j} : \mathcal{A}_j\to\mathcal{A}_j$ are unital. If f is unital, then $f|_j^{(X_i)_{i\neq j}} = (\mathrm{id}_{\mathcal{A}_j},(X_i)_{i\neq j})\cdot_{\{j\}\hookrightarrow I} f$ is unital due to $\mathsf{A}^{\mathrm{u}}_\infty$ being a submulticategory.

Conversely, assume that A_∞-functors $f|_j^{(X_i)_{i\neq j}} : \mathcal{A}_j \to \mathcal{B}$ are unital for all $j \in I$. Then for each $X_i \in \operatorname{Ob}\mathcal{A}_i$, $i \in I$, we have

$$\begin{aligned} 1_{(X_j)_{j\in I}}.\mathsf{k}f &= \bigl[\otimes^{j\in I} 1_{X_j} s f_{e_j} s^{-1}\bigr]\mu^I_{\mathsf{k}\mathcal{B}} = \bigl[\otimes^{j\in I} 1_{X_j} s(f|_j^{(X_i)_{i\neq j}})_1 s^{-1}\bigr]\mu^I_{\mathsf{k}\mathcal{B}} \\ &= \bigl[\otimes^{j\in I}(1_{X_j}.(\mathsf{k}f|_j^{(X_i)_{i\neq j}}))\bigr]\mu^I_{\mathsf{k}\mathcal{B}} = \bigl[\otimes^{j\in I} 1_{(X_i)_{i\in I}f}\bigr]\mu^I_{\mathsf{k}\mathcal{B}} = 1_{(X_i)_{i\in I}\mathsf{k}f}. \end{aligned}$$

Thus $\mathsf{k}f$ is unital, hence, f is unital. □

3.4.9. Corollary. *If $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ is a unital A_∞-functor, and $J \subset I$ is a subset, then $f|_J^{(X_i)_{i\in I\smallsetminus J}} : (\mathcal{A}_j)_{j\in J} \to \mathcal{B}$ is unital, for each family of objects $(X_i)_{i\in I\smallsetminus J} \in \prod_{i\in I\smallsetminus J}\operatorname{Ob}\mathcal{A}_i$.*

Proof. Indeed, restrictions of $f|_J^{(X_i)_{i\in I\smallsetminus J}}$ to j^{th} argument coincide with those of f. □

We are going to prove that the A_∞-category $\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ is unital if $\mathcal{B}$ is unital. Recall from [**38**, Proposition 7.5] that unital elements ${}_X\mathbf{i}_0^{\mathcal{B}}$, $X \in \operatorname{Ob}\mathcal{B}$, of the A_∞-category $\mathcal{B}$ extend to a natural A_∞-transformation $\mathbf{i}^{\mathcal{B}} : \mathrm{id}_{\mathcal{B}} \to \mathrm{id}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$, determined uniquely up to equivalence, such that $(\mathbf{i}^{\mathcal{B}} \otimes \mathbf{i}^{\mathcal{B}})B_2 \equiv \mathbf{i}^{\mathcal{B}}$. It is called a *unit transformation* of $\mathcal{B}$.

For $I = \varnothing$, the unitality of $\underline{\mathsf{A}_\infty}(\,;\mathcal{B})$ follows from the isomorphism $\underline{\mathsf{A}_\infty}(\,;\mathcal{B}) \cong \mathcal{B}$. Suppose I is nonempty. The proof of the following proposition repeats the proof of [**38**, Proposition 7.7] mutatis mutandis. We provide it for the sake of completeness.

3.4.10. Proposition. *Let $\mathcal{A}_i$, $\mathcal{B}$, $i \in I$, be A_∞-categories. Suppose that $\mathcal{B}$ is unital. Then the A_∞-category $\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ is unital with unit elements*

$${}_f\mathbf{i}_0^{\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})} = f\mathbf{i}^{\mathcal{B}} \in s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,f),$$

for each A_∞-functor $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$.

Proof. Clearly, $f\mathbf{i}^{\mathcal{B}}$ is an A_∞-transformation $f \to f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$. Moreover, it is natural. Indeed, since f commutes with the differential, it follows that $(f\mathbf{i}^{\mathcal{B}})B_1 = f(\mathbf{i}^{\mathcal{B}}B_1) = 0$. It remains to prove that for each pair of A_∞-functors $f, g : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ the maps

$$\begin{aligned} s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g) &\to s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g), \quad r \mapsto (r \otimes g\mathbf{i}^{\mathcal{B}})B_2, \\ s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g) &\to s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g), \quad r \mapsto r(f\mathbf{i}^{\mathcal{B}} \otimes 1)B_2, \end{aligned}$$

are homotopy invertible. We are going to give a proof for the first map, the other map is treated similarly.

Let us define a decreasing filtration of the complex $(s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g), B_1)$. For $n \in \mathbb{N}$, we set

$$\Phi_n = \{r \in s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g) \mid \forall (n_i)_{i\in I} \in \mathbb{N}^I, \textstyle\sum_{i\in I} n_i < n \colon r_{(n_i)_{i\in I}} = 0\}.$$

Clearly, Φ_n is stable under B_1 and

$$s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g) = \Phi_0 \supset \Phi_1 \supset \dots \supset \Phi_n \supset \Phi_{n+1} \supset \dots \quad .$$

Due to (3.3.5) and (3.3.1), the chain map $a = (1 \otimes g\mathbf{i}^{\mathcal{B}})B_2$ preserves the subcomplex Φ_n. By definition $s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g) = \prod_{(n_i)_{i\in I}\in\mathbb{N}^I} V_{(n_i)_{i\in I}}$, where $V_{(n_i)_{i\in I}}$ is the graded $\Bbbk$-module of components $r_{(n_i)_{i\in I}} : \boxtimes^{i\in I} T^{n_i} s\mathcal{A}_i \to s\mathcal{B}$ of (f,g)-coderivations r, and the product is taken in the category of graded $\Bbbk$-modules. Let V_n denote the product of $V_{(n_i)_{i\in I}}$ over $(n_i)_{i\in I} \in \mathbb{N}^I$ such that $\sum_{i\in I} n_i = n$. Then $s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g) = \prod_{n=0}^{\infty} V_n$, and the filtration consists of graded $\Bbbk$-submodules $\Phi_n = 0 \times \dots \times 0 \times \prod_{m=n}^{\infty} V_m$.

The graded complex associated with this filtration is $\bigoplus_{n=0}^\infty V_n$, and the differential $d : V_n \to V_n$ induced by B_1 is given by the formula

$$r_{(n_i)_{i\in I}} d = r_{(n_i)_{i\in I}} b_1 - (-)^{r_{(n_i)_{i\in I}}} \sum_{\alpha_k+1+\beta_k=n_k}^{k\in I} \boxtimes^{i\in I}((1)_{i<k}, 1^{\otimes\alpha_k} \otimes b_1 \otimes 1^{\otimes\beta_k}, (1)_{i>k}) r_{(n_i)_{i\in I}}.$$

The associated endomorphism $\operatorname{gr} a$ of $\bigoplus_{n=0}^\infty V_n$ is given by the formula

$$(r_{(n_i)_{i\in I}}) \operatorname{gr} a = \prod_{X_i^0,\dots,X_i^{n_i}\in\operatorname{Ob}\mathcal{A}_i}^{i\in I} \bigl({}_{(X_i^0,\dots,X_i^{n_i})_{i\in I}} r_{(n_i)_{i\in I}} \otimes {}_{(X_i^{n_i})_{i\in I} g}\mathbf{i}_0^{\mathcal{B}}\bigr) b_2$$

for each $r_{(n_i)_{i\in I}} \in V_{(n_i)_{i\in I}}$, as formulas (3.3.1), (3.3.5) show. Here

$${}_{(X_i^0,\dots,X_i^{n_i})_{i\in I}} r_{(n_i)_{i\in I}} : \otimes^{i\in I} \bar{T}^{n_i} s\mathcal{A}(X_i^0, \dots, X_i^{n_i}) \to s\mathcal{B}((X_i^0)_{i\in I} f, (X_i^{n_i})_{i\in I} g),$$

where $\bar{T}^{n_i} s\mathcal{A}(X_i^0, \dots, X_i^{n_i})$ is given by (3.2.1).

Since $\mathcal{B}$ is unital, for each pair X, Y of objects of $\mathcal{B}$, the chain map $(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{B}})b_2$ is homotopic to the identity map, that is, $(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{B}})b_2 = 1 + hb_1 + b_1h$ for some $\Bbbk$-linear map $h : s\mathcal{B}(X, Y) \to s\mathcal{B}(X, Y)$ of degree -1. Let us choose such homotopies

$${}_{(X_i^0)_{i\in I},(X_i^{n_i})_{i\in I}} h : s\mathcal{B}((X_i^0)_{i\in I} f, (X_i^{n_i})_{i\in I} g) \to s\mathcal{B}((X_i^0)_{i\in I} f, (X_i^{n_i})_{i\in I} g)$$

for each pair $(X_i^0)_{i\in I}$, $(X_i^{n_i})_{i\in I} \in \prod_{i\in I} \operatorname{Ob}\mathcal{A}_i$. Denote by $H : \prod_{n=0}^\infty V_n \to \prod_{n=0}^\infty V_n$ the diagonal map

$${}_{(X_i^0,\dots,X_i^{n_i})_{i\in I}} r_{(n_i)_{i\in I}} \mapsto {}_{(X_i^0,\dots,X_i^{n_i})_{i\in I}} r_{(n_i)_{i\in I}} \cdot {}_{(X_i^0)_{i\in I},(X_i^{n_i})_{i\in I}} h.$$

Then $\operatorname{gr} a = 1 + Hd + dH$. The chain map $a - HB_1 - B_1H$ being restricted to a map $\bigoplus_{m=0}^\infty V_m \to \prod_{m=0}^\infty V_m$ gives an upper triangular $\mathbb{N} \times \mathbb{N}$ matrix which, in turn, determines the whole map. Thus, $a - HB_1 - B_1H = 1 + N$, where the $\mathbb{N} \times \mathbb{N}$ matrix N is strictly upper triangular. Therefore, $1 + N$ is invertible (since its inverse map $\sum_{i=0}^\infty (-N)^i$ makes sense). Hence, $a = (1 \otimes {}_g\mathbf{i}^{\mathcal{B}})B_2$ is homotopy invertible. □

For $I = \mathbf{n}$, $n > 1$, A_∞-categories $\mathcal{A}_1, \dots, \mathcal{A}_n$, and a unital A_∞-category $\mathcal{B}$, there is isomorphism (3.3.9)

$$\underline{\varphi}^{\mathsf{A}_\infty} : \underline{\mathsf{A}_\infty}\bigl(\mathcal{A}_2, \dots, \mathcal{A}_n; \underline{\mathsf{A}_\infty}(\mathcal{A}_1; \mathcal{B})\bigr) \to \underline{\mathsf{A}_\infty}(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{B}). \tag{3.4.13}$$

It is unital by Remark 3.4.6. The following lemma is a straightforward generalization of [**38**, Proposition 7.15]. It is proven by induction using isomorphism (3.4.13).

3.4.11. Lemma. *Let $r : f \to g : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ be a natural A_∞-transformation. Assume that for all families $(X_i)_{i\in I}$ of objects $X_i \in \operatorname{Ob}\mathcal{A}_i$ there are elements ${}_{(X_i)_{i\in I}}p_0 : \Bbbk \to (s\mathcal{B})^{-1}\bigl(((X_i)_{i\in I})g, ((X_i)_{i\in I})f\bigr)$ such that ${}_{(X_i)_{i\in I}}p_0 b_1 = 0$ and*

$$\bigl({}_{(X_i)_{i\in I}}r_0 \otimes {}_{(X_i)_{i\in I}}p_0\bigr)b_2 - {}_{(X_i)_{i\in I} f}\mathbf{i}_0^{\mathcal{B}} \in \operatorname{Im} b_1,$$
$$\bigl({}_{(X_i)_{i\in I}}p_0 \otimes {}_{(X_i)_{i\in I}}r_0\bigr)b_2 - {}_{(X_i)_{i\in I} g}\mathbf{i}_0^{\mathcal{B}} \in \operatorname{Im} b_1.$$

Then given ${}_{(X_i)_{i\in I}}p_0$ extend to a natural A_∞-transformation $p : g \to f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ inverse to r, that is,

$$(r \otimes p)B_2 - {}_f\mathbf{i}_0^{\mathsf{A}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})} \in \operatorname{Im} B_1, \qquad (p \otimes r)B_2 - {}_g\mathbf{i}_0^{\mathsf{A}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})} \in \operatorname{Im} B_1.$$

Let us show that multicategory $\mathsf{A}^{\mathrm{u}}_\infty$ is closed. If $(\mathcal{A}_i)_{i\in I}$, $\mathcal{B}$ are unital A_∞-categories, we set $\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \subset \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ to be the full A_∞-subcategory, whose objects are unital A_∞-functors $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$. Since $\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ is unital, so is $\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})$. The evaluation A_∞-functor $\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty} : (\mathcal{A}_i)_{i\in I}, \underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \to \mathcal{B}$ is taken to be the restriction of $\mathrm{ev}^{\mathsf{A}_\infty} : (\mathcal{A}_i)_{i\in I}, \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \to \mathcal{B}$. We must show that the A_∞-functor $\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}$ is unital.

If $I = \varnothing$, then $\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty} : \underline{\mathsf{A}^{\mathrm{u}}_\infty}(\,;\mathcal{B}) \to \mathcal{B}$ is the natural isomorphism, hence, it is unital by Remark 3.4.6. If $j \in I = \mathbf{n}$, $X_i \in \operatorname{Ob}\mathcal{A}_i$ for $i \neq j$, and $g \in \mathsf{A}^{\mathrm{u}}_\infty((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})$, then due to (3.3.6)

$$\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}\big|_j^{(X_i)_{i\neq j},g} = g\big|_j^{(X_i)_{i\neq j}} : Ts\mathcal{A}_j \cong \big(\boxtimes^{i\in\mathbf{n}\smallsetminus\{j\}}T^0s\mathcal{A}_i\big) \boxtimes Ts\mathcal{A}_j \boxtimes T^0s\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B}) \to Ts\mathcal{B}$$

is unital, since $g\big|_j^{(X_i)_{i\neq j}}$ is unital by Proposition 3.4.8. If $X_i \in \operatorname{Ob}\mathcal{A}_i$ for $i \in \mathbf{n}$, the first component $\big(\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}\big|_{n+1}^{(X_i)_{i\in\mathbf{n}}}\big)_1 : T^1s\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B}) \to s\mathcal{B}$ of the A_∞-functor $\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}\big|_{n+1}^{(X_i)_{i\in\mathbf{n}}}$ takes, due to (3.3.7), an A_∞-transformation $r : g \to h : (\mathcal{A}_i)_{i\in\mathbf{n}} \to \mathcal{B}$ to its 0th component $r_{0\dots0} \in s\mathcal{B}((X_1,\dots,X_n)g,(X_1,\dots,X_n)h)$. In particular, the unit element $g\mathbf{i}^{\mathcal{B}}$ of g goes to the unit element ${}_{(X_1,\dots,X_n)g}\mathbf{i}_0^{\mathcal{B}} \in s\mathcal{B}((X_1,\dots,X_n)g,(X_1,\dots,X_n)g)$. By Proposition 3.4.8, we conclude that $\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}$ is unital.

3.4.12. Proposition. *So defined $\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})$, $\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}$ turn $\mathsf{A}^{\mathrm{u}}_\infty$ into a closed multicategory.*

Proof. Let $(\mathcal{A}_i)_{i\in I}$, $(\mathcal{B}_j)_{j\in J}$, $\mathcal{C}$ be unital A_∞-categories. Denote by

$$e : \underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C}) \hookrightarrow \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})$$

the full embedding. The functions $\varphi^{\mathsf{A}^{\mathrm{u}}_\infty}$ and $\varphi^{\mathsf{A}_\infty}$ are related by embeddings:

$$\begin{tikzcd}
\mathsf{A}^{\mathrm{u}}_\infty\big((\mathcal{B}_j)_{j\in J};\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})\big) \arrow[r, "\varphi^{\mathsf{A}^{\mathrm{u}}_\infty}"] \arrow[d, hook, "e"'] & \mathsf{A}^{\mathrm{u}}_\infty\big((\mathcal{A}_i)_{i\in I},(\mathcal{B}_j)_{j\in J};\mathcal{C}\big) \arrow[dd, hook, "e"] \\
\mathsf{A}_\infty\big((\mathcal{B}_j)_{j\in J};\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})\big) \arrow[d, hook, "\mathsf{A}_\infty((\mathcal{B}_j)_{j\in J};e)"'] & \\
\mathsf{A}_\infty\big((\mathcal{B}_j)_{j\in J};\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})\big) \arrow[r, "\varphi^{\mathsf{A}_\infty}"] & \mathsf{A}_\infty\big((\mathcal{A}_i)_{i\in I},(\mathcal{B}_j)_{j\in J};\mathcal{C}\big)
\end{tikzcd}$$

Therefore, $\varphi^{\mathsf{A}^{\mathrm{u}}_\infty}$ is injective. Let us prove its surjectivity.

Suppose $g : (\mathcal{B}_j)_{j\in J} \to \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})$ is an A_∞-functor such that the A_∞-functor $f = \varphi^{\mathsf{A}_\infty}(g) : (\mathcal{A}_i)_{i\in I}, (\mathcal{B}_j)_{j\in J} \to \mathcal{C}$ is unital. Let $Y_j \in \operatorname{Ob}\mathcal{B}_j$, $j \in J$, be a family of objects. Then A_∞-functor $\big((Y_j)_{j\in J}\big)g = f\big|_I^{(Y_j)_{j\in J}} : (\mathcal{A}_i)_{i\in I} \to \mathcal{C}$ is unital by Corollary 3.4.9. Therefore, $g = he$ for an A_∞-functor $h : (\mathcal{B}_j)_{j\in J} \to \underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})$.

Let us prove that h is unital. This is obvious if $J = \varnothing$. Assume that $J \neq \varnothing$ and consider an arbitrary $k \in J$, a family $(X_i)_{i\in I} \in \prod_{i\in I}\operatorname{Ob}\mathcal{A}_i$, and a family $(Y_j)_{j\neq k} \in \prod_{j\neq k}\operatorname{Ob}\mathcal{B}_j$. The restriction of

$$f = \big[(\boxtimes^{i\in I}Ts\mathcal{A}_i) \boxtimes (\boxtimes^{j\in J}Ts\mathcal{B}_j) \xrightarrow{1\boxtimes h} (\boxtimes^{i\in I}Ts\mathcal{A}_i) \boxtimes Ts\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C}) \xrightarrow{\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}} Ts\mathcal{C}\big]$$

to the k^{th} argument

$$f\big|_k^{(X_i)_{i\in I},(Y_j)_{j\neq k}} = \big[Ts\mathcal{B}_k \xrightarrow{h|_k^{(Y_j)_{j\neq k}}} Ts\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C}) \xrightarrow{\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}|_{|I|+1}^{(X_i)_{i\in I}}} Ts\mathcal{C}\big]$$

is unital by Proposition 3.4.8. The first component

$$\big(h\big|_k^{(Y_j)_{j\neq k}}\big)_1 : s\mathcal{B}_k(Y_k,Y_k) \to s\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})\big(((Y_j)_{j\in J})h,((Y_j)_{j\in J})h\big)$$

takes the unit element ${}_{Y_k}\mathbf{i}_0^{\mathcal{B}_k}$ to some element r^k. Since $\big(h\big|_k^{(Y_j)_{j\neq k}}\big)_1$ is a chain map that preserves composition up to homotopy, we find that r^k is a cycle idempotent modulo boundary:

$$(r^k)B_1 = 0, \qquad (r^k \otimes r^k)B_2 - r^k \in \operatorname{Im} B_1.$$

The first component

$$\begin{aligned}\big(f\big|_k^{(X_i)_{i\in I},(Y_j)_{j\neq k}}\big)_1 = \big[s\mathcal{B}_k(Y_k,Y_k) &\xrightarrow{(h|_k^{(Y_j)_{j\neq k}})_1} s\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})\big(((Y_j)_{j\in J})h,((Y_j)_{j\in J})h\big) \\ &\xrightarrow{(\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}|_{|I|+1}^{(X_i)_{i\in I}})_1} s\mathcal{C}\big(((X_i)_{i\in I})((Y_j)_{j\in J})h,((X_i)_{i\in I})((Y_j)_{j\in J})h\big)\big]\end{aligned}$$

takes ${}_{Y_k}\mathbf{i}_0^{\mathcal{B}_k}$ to ${}_{(X_i)_{i\in I}}r^k_{0\dots0} \in s\mathcal{C}\big(((X_i)_{i\in I},(Y_j)_{j\in J})f,((X_i)_{i\in I},(Y_j)_{j\in J})f\big)$ due to (3.3.7). The unitality of $f\big|_k^{(X_i)_{i\in I},(Y_j)_{j\neq k}}$ implies that

$${}_{(X_i)_{i\in I}}r^k_{0\dots0} - {}_{((X_i)_{i\in I},(Y_j)_{j\in J})f}\mathbf{i}_0^{\mathcal{C}} \in \operatorname{Im} b_1.$$

By Lemma 3.4.11 this implies invertibility of r^k. Being also idempotent, r^k is equal to the unit transformation $((Y_j)_{j\in J})h\mathbf{i}^{\mathcal{C}}$ of the A_∞-functor $((Y_j)_{j\in J})h : (\mathcal{A}_i)_{i\in I} \to \mathcal{C}$ modulo boundary. Thus, $h\big|_k^{(Y_j)_{j\neq k}}$ is unital. By Proposition 3.4.8, h is a unital A_∞-functor $(\mathcal{B}_j)_{j\in J} \to \underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})$, hence, $\varphi^{\mathsf{A}^{\mathrm{u}}_\infty}$ is surjective, and, moreover, bijective. Therefore, the multicategory $\mathsf{A}^{\mathrm{u}}_\infty$ is closed. □

We are going to construct an extension of the multifunctor $\mathsf{k} : \mathsf{A}^{\mathrm{u}}_\infty \to \widehat{\mathcal{K}\text{-}\mathbf{Cat}}$ to natural A_∞-transformations as follows. Suppose $f, g : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ are unital A_∞-functors, $r : f \to g : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ is a natural A_∞-transformation. It gives rise to a $\mathcal{K}$-natural transformation of $\mathcal{K}$-functors $\mathsf{k}r : \mathsf{k}f \to \mathsf{k}g : \boxtimes^{i\in I}\mathsf{k}\mathcal{A}_i \to \mathsf{k}\mathcal{B}$. Components of $\mathsf{k}r$ are given by

$${}_{(X_i)_{i\in I}}\mathsf{k}r = {}_{(X_i)_{i\in I}}r_0 s^{-1} : \Bbbk \to \mathcal{B}((X_i)_{i\in I}f,(X_i)_{i\in I}g), \quad X_i \in \operatorname{Ob}\mathcal{A}_i, \quad i \in I.$$

Since $r_0 b_1 = 0$, $\mathsf{k}r$ is a chain map. The $\mathcal{K}$-naturality is expressed by the following equation in $\mathcal{K}$:

$$\begin{tikzcd}
\boxtimes^{i\in I}\mathsf{k}\mathcal{A}_i(X_i,Y_i) \arrow[r, "\mathsf{k}f"] \arrow[d, "\mathsf{k}g"'] \arrow[dr, phantom, "="] & \mathsf{k}\mathcal{B}((X_i)_{i\in I}f,(Y_i)_{i\in I}f) \arrow[d, "(1\otimes{}_{(Y_i)_{i\in I}}\mathsf{k}r)\mu_{\mathsf{k}\mathcal{B}}"] \\
\mathsf{k}\mathcal{B}((X_i)_{i\in I}g,(Y_i)_{i\in I}g) \arrow[r, "({}_{(X_i)_{i\in I}}\mathsf{k}r\otimes1)\mu_{\mathsf{k}\mathcal{B}}"] & \mathsf{k}\mathcal{B}((X_i)_{i\in I}f,(Y_i)_{i\in I}g)
\end{tikzcd}$$

The associativity of $\mu_{\mathsf{k}\mathcal{B}}$ allows to write it as follows:

$$\begin{aligned}
&\bigl[\otimes^{i\in I}\mathsf{k}\mathcal{A}_i(X_i,Y_i) \xrightarrow{\otimes^{i\in I}sf_{e_i}s^{-1}\otimes{}_{(Y_i)_{i\in I}}r_0s^{-1}} \\
&\qquad \otimes^{i\in I}\,\mathsf{k}\mathcal{B}(((Y_j)_{j<i},(X_j)_{j\geqslant i})f,((Y_j)_{j\leqslant i},(X_j)_{j>i})f)\otimes\mathsf{k}\mathcal{B}((Y_i)_{i\in I}f,(Y_i)_{i\in I}g) \\
&\qquad\qquad \xrightarrow{\mu^{I\sqcup\mathbf{1}}_{\mathsf{k}\mathcal{B}}} \mathsf{k}\mathcal{B}((X_i)_{i\in I}f,(Y_i)_{i\in I}g)\bigr] \\
&= \bigl[\otimes^{i\in I}\mathsf{k}\mathcal{A}_i(X_i,Y_i) \xrightarrow{{}_{(X_i)_{i\in I}}r_0s^{-1}\otimes\otimes^{i\in I}sg_{e_i}s^{-1}} \\
&\qquad \mathsf{k}\mathcal{B}((X_i)_{i\in I}f,(X_i)_{i\in I}g)\otimes\otimes^{i\in I}\mathsf{k}\mathcal{B}(((Y_j)_{j<i},(X_j)_{j\geqslant i})g,((Y_j)_{j\leqslant i},(X_j)_{j>i})g) \\
&\qquad\qquad \xrightarrow{\mu^{\mathbf{1}\sqcup I}_{\mathsf{k}\mathcal{B}}} \mathsf{k}\mathcal{B}((X_i)_{i\in I}f,(Y_i)_{i\in I}g)\bigr].
\end{aligned}$$

This equation is a consequence of the following equation in $\mathcal{K}$:

$$\begin{aligned}
&(s(f|_i^{(Y_j)_{j<i},(X_j)_{j>i}})_1s^{-1}\otimes{}_{(Y_j)_{j\leqslant i},(X_j)_{j>i}}r_0s^{-1})\mu_{\mathsf{k}\mathcal{B}} \\
&\qquad = ({}_{(Y_j)_{j<i},(X_j)_{j\geqslant i}}r_0s^{-1}\otimes s(g|_i^{(Y_j)_{j<i},(X_j)_{j>i}})_1s^{-1})\mu_{\mathsf{k}\mathcal{B}}: \\
&\qquad\qquad \mathcal{A}(X_i,Y_i)\to\mathcal{B}(((Y_j)_{j<i},(X_j)_{j\geqslant i})f,((Y_j)_{j\leqslant i},(X_j)_{j>i})g),
\end{aligned}$$

which in turn follows from the equation $(rB_1)_{e_i}=0$:

$$\begin{aligned}
&(sf_{e_i}s^{-1}\otimes r_0s^{-1})m_2-(r_0s^{-1}\otimes sg_{e_i}s^{-1})m_2+sr_{e_i}s^{-1}m_1+m_1sr_{e_i}s^{-1} \\
&\qquad = s[(f_{e_i}\otimes r_0)b_2+(r_0\otimes g_{e_i})b_2+r_{e_i}b_1+b_1r_{e_i}]s^{-1}=0: \\
&\qquad\qquad \mathcal{A}(X_i,Y_i)\to\mathcal{B}(((Y_j)_{j<i},(X_j)_{j\geqslant i})f,((Y_j)_{j\leqslant i},(X_j)_{j>i})g).
\end{aligned}$$

The 2-category $\mathcal{K}$-**Cat** is naturally a symmetric Monoidal $\Bbbk$-**Cat**-category, therefore $\widehat{\mathcal{K}\text{-}\mathbf{Cat}}$ is a symmetric $\Bbbk$-**Cat**-multicategory by Proposition 1.2.26. According to it, for each map $\phi: I\to J$, the composition in $\widehat{\mathcal{K}\text{-}\mathbf{Cat}}$ is given by the $\Bbbk$-linear functor

$$\begin{aligned}
\mu_\phi^{\widehat{\mathcal{K}\text{-}\mathbf{Cat}}} &= \bigl[\boxtimes^{J\sqcup\mathbf{1}}[(\mathcal{K}\text{-}\mathbf{Cat}(\boxtimes^{i\in\phi^{-1}j}\mathcal{A}_i,\mathcal{B}_j))_{j\in J},\mathcal{K}\text{-}\mathbf{Cat}(\boxtimes^{j\in J}\mathcal{B}_j,\mathcal{C})] \\
&\qquad \xrightarrow[\sim]{\Lambda^\gamma_{\Bbbk\text{-}\mathbf{Cat}}} \boxtimes^{j\in J}\mathcal{K}\text{-}\mathbf{Cat}(\boxtimes^{i\in\phi^{-1}j}\mathcal{A}_i,\mathcal{B}_j)\boxtimes\mathcal{K}\text{-}\mathbf{Cat}(\boxtimes^{j\in J}\mathcal{B}_j,\mathcal{C}) \\
&\qquad \xrightarrow{\boxtimes^J\boxtimes 1} \mathcal{K}\text{-}\mathbf{Cat}(\boxtimes^{j\in J}\boxtimes^{i\in\phi^{-1}j}\mathcal{A}_i,\boxtimes^{j\in J}\mathcal{B}_j)\boxtimes\mathcal{K}\text{-}\mathbf{Cat}(\boxtimes^{j\in J}\mathcal{B}_j,\mathcal{C}) \\
&\qquad\qquad \xrightarrow{\lambda^{\phi\cdot -,-}} \mathcal{K}\text{-}\mathbf{Cat}(\boxtimes^{i\in I}\mathcal{A}_i,\mathcal{C})\bigr],
\end{aligned}$$

where $\gamma: J\sqcup\mathbf{1}\to\mathbf{2}$ is given by $\gamma(j)=1$, $j\in J$, $\gamma(1)=2$, $1\in\mathbf{1}$. In particular, the action on $\mathcal{K}$-natural transformations is given by the map

$$\begin{aligned}
&\otimes^{j\in J}\,\mathcal{K}\text{-}\mathbf{Cat}(\boxtimes^{i\in\phi^{-1}j}\mathcal{A}_i,\mathcal{B}_j)(f_j,g_j)\otimes\mathcal{K}\text{-}\mathbf{Cat}(\boxtimes^{j\in J}\mathcal{B}_j,\mathcal{C})(h,k) \\
&\qquad \to\mathcal{K}\text{-}\mathbf{Cat}(\boxtimes^{i\in I}\mathcal{A}_i,\mathcal{C})((f_j)_{j\in J}\cdot h,(g_j)_{j\in J}\cdot k),\quad \otimes^{j\in J}r^j\otimes p\mapsto(r^j)_{j\in J}\cdot p,
\end{aligned}$$

where for each collection of objects $X_i\in\mathcal{A}_i$, $i\in I$,

$$\begin{aligned}
&{}_{(X_i)_{i\in I}}[(r^j)_{j\in J}\cdot p]=\bigl[\Bbbk\xrightarrow{\sim}\otimes^J\Bbbk\otimes\Bbbk\xrightarrow{\otimes^{j\in J}{}_{(X_i)_{i\in\phi^{-1}j}}r^j\otimes{}_{((X_i)_{i\in\phi^{-1}j}g_j)_{j\in J}}p} \\
&\qquad \otimes^{j\in J}\,\mathcal{B}((X_i)_{i\in\phi^{-1}j}f_j,(X_i)_{i\in\phi^{-1}j}g_j)\otimes\mathcal{C}(((X_i)_{i\in\phi^{-1}j}g_j)_{j\in J}h,((X_i)_{i\in\phi^{-1}j}g_j)_{j\in J}k) \\
&\qquad \xrightarrow{h\otimes 1}\mathcal{C}((X_i)_{i\in I}(f_j)_{j\in J}h,(X_i)_{i\in I}(g_j)_{j\in J}h)\otimes\mathcal{C}((X_i)_{i\in I}(g_j)_{j\in J}h,(X_i)_{i\in I}(g_j)_{j\in J}k) \\
&\qquad\qquad \xrightarrow{\mu_{\mathcal{C}}}\mathcal{C}((X_i)_{i\in I}(f_j)_{j\in J}h,(X_i)_{i\in I}(g_j)_{j\in J}k)\bigr].
\end{aligned}$$

The base change multifunctor $H^0 : \mathsf{A}^{\mathrm{u}}_\infty \to \widehat{\Bbbk\text{-}\mathbf{Cat}}$ turns the symmetric $\mathsf{A}^{\mathrm{u}}_\infty$-multicategory $\underline{\mathsf{A}^{\mathrm{u}}_\infty}$ into a symmetric $\Bbbk$-$\mathbf{Cat}$-multicategory, which we denote by A^u_∞, see Section 1.3.9. Note that it is essential that the multifunctor H^0 is symmetric. Thus, objects of A^u_∞ are unital A_∞-categories, and for each collection $(\mathcal{A}_i)_{i\in I}$, $\mathcal{B}$ of unital A_∞-categories, there is a $\Bbbk$-linear category $A^u_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B}) = H^0\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})$, whose objects are unital A_∞-functors, and whose morphisms are equivalence classes of natural A_∞-transformations. If $r : f \to g : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ and $p : g \to h : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ are natural A_∞-transformations, the 'vertical' composite of 2-morphisms rs^{-1} and ps^{-1} is represented by the natural A_∞-transformation $(r\otimes p)B_2s^{-1}$. Note that k preserves vertical composition:

$$\begin{aligned} {}_{(X_i)_{i\in I}}\mathsf{k}((r\otimes p)B_2) = {}_{(X_i)_{i\in I}}[(r\otimes p)B_2]_0s^{-1} &= ({}_{(X_i)_{i\in I}}r_0 \otimes {}_{(X_i)_{i\in I}}p_0)b_2s^{-1} \\ &= ({}_{(X_i)_{i\in I}}r_0s^{-1} \otimes {}_{(X_i)_{i\in I}}p_0s^{-1})m_2 = ({}_{(X_i)_{i\in I}}\mathsf{k}r \otimes {}_{(X_i)_{i\in I}}\mathsf{k}p)\mu_{\mathsf{k}\mathcal{B}}, \end{aligned}$$

by formulas (3.3.1) and (3.3.5). Unit transformations $\mathbf{i}^{\mathcal{B}} : \mathrm{id} \to \mathrm{id} : \mathcal{B} \to \mathcal{B}$ (unit elements in $s\underline{\mathsf{A}_\infty}(\mathcal{B},\mathcal{B})(\mathrm{id}_{\mathcal{B}},\mathrm{id}_{\mathcal{B}})$) provide for any unital A_∞-functor $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ the natural A_∞-transformations $f\mathbf{i}^{\mathcal{B}} : f \to f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$, representing the identity 2-morphism 1_f. Since ${}_{(X_i)_{i\in I}}(f\mathbf{i}^{\mathcal{B}})_0 = {}_{(X_i)_{i\in I}f}\mathbf{i}^{\mathcal{C}}_0$, it follows that k preserves identities. Thus, for each $I \in \operatorname{Ob}\mathcal{O}$ and A_∞-categories $\mathcal{A}_i$, $\mathcal{B}$, $i\in I$, there is a $\Bbbk$-linear functor

$$\mathsf{k} : A^u_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B}) \to \widehat{\mathcal{K}\text{-}\mathbf{Cat}}((\mathsf{k}\mathcal{A}_i)_{i\in I};\mathsf{k}\mathcal{B}).$$

We claim that these $\Bbbk$-linear functors constitute a $\Bbbk$-$\mathbf{Cat}$-multifunctor $\mathsf{k} : A^u_\infty \to \widehat{\mathcal{K}\text{-}\mathbf{Cat}}$. The composition in A^u_∞ is given by the $\Bbbk$-linear functor $\mu^{A^u_\infty}_\phi = H^0(\mu^{\underline{\mathsf{A}^{\mathrm{u}}_\infty}}_\phi) = H^0(\mathsf{k}\mu^{\underline{\mathsf{A}^{\mathrm{u}}_\infty}}_\phi)$, where according to (3.4.1) the $\mathcal{K}$-functor $\mathsf{k}\mu^{\underline{\mathsf{A}^{\mathrm{u}}_\infty}}_\phi$ is given by the composite

$$\begin{tikzcd}[row sep=large] \otimes^{j\in J}\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in\phi^{-1}j};\mathcal{B}_j)(f_j,g_j) \otimes \underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{B}_j)_{j\in J};\mathcal{C})(h,k) \arrow[d, "\otimes^{j\in J}sM_{e_j0}s^{-1}\otimes sM_{0\dots01}s^{-1}"] \\ \begin{array}{c}\otimes^{j\in J}\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})(((g_l)_{l<j},(f_l)_{l\geqslant j})h,((g_l)_{l\leqslant j},(f_l)_{l>j})h) \\ \otimes\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})((g_j)_{j\in J}h,(g_j)_{j\in J}k)\end{array} \arrow[d, "\mu^{J\sqcup\mathbf{1}}_{\mathsf{k}\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})}"] \\ \underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})((f_j)_{j\in J}h,(g_j)_{j\in J}k) \end{tikzcd}$$

3.4.13. Proposition. *There is a symmetric* $\Bbbk$-$\mathbf{Cat}$*-multifunctor* $\mathsf{k} : A^u_\infty \to \widehat{\mathcal{K}\text{-}\mathbf{Cat}}$.

Proof. It remains to prove that k is compatible with the composition $\mu^{A^u_\infty}_\phi$ on the level of 2-morphisms. Let $r^j \in s\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in\phi^{-1}j};\mathcal{B}_j)(f_j,g_j)$, $j\in J$, $p \in s\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{B}_j)_{j\in J};\mathcal{C})(h,k)$ be natural A_∞-transformations. Then $((r^js^{-1})_{j\in J},ps^{-1})\mu^{A^u_\infty}_\phi$ is the equivalence class of the following natural A_∞-transformation:

$$[\otimes^{j\in J}((g_l)_{l<j},r^j,(f_l)_{l>j},h)M_{e_j0}s^{-1} \otimes ((g_j)_{j\in J},p)M_{0\dots01}s^{-1}]\mu^{J\sqcup\mathbf{1}}_{\mathsf{k}\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})}.$$

In order to find $\mathsf{k}[((r^js^{-1})_{j\in J},ps^{-1})\mu^{A^u_\infty}_\phi]$ we need the 0th components of the above expression. Since $[(t\otimes q)B_2]_0 = (t_0\otimes q_0)b_2$, for arbitrary composable A_∞-transformations t and

q, it follows that

$$
\begin{aligned}
&{}_{(X_i)_{i\in I}}\mathsf{k}[((r^j s^{-1})_{j\in J}, ps^{-1})\mu_\phi^{A^u_\infty}] \\
&= (\otimes^{j\in J}{}_{(X_i)_{i\in\phi^{-1}j}}[((g_l)_{l<j}, r^j, (f_l)_{l>j}, h)M_{e_j0}]_0 s^{-1} \otimes {}_{(X_i)_{i\in I}}[((g_j)_{j\in J}, p)M_{0\dots01}]_0 s^{-1})\mu_{\mathsf{k}\mathcal{C}}^{J\sqcup\mathbf{1}} \\
&\qquad\qquad = (\otimes^{j\in J}{}_{(X_i)_{i\in\phi^{-1}j}}r_0^j s^{-1}\cdot sh_{e_j}s^{-1} \otimes {}_{((X_i)_{i\in\phi^{-1}j}g_j)_{j\in J}}p_0 s^{-1})\mu_{\mathsf{k}\mathcal{C}}^{J\sqcup\mathbf{1}},
\end{aligned}
$$

see Examples 3.3.4 and 3.3.5. Here

$$
\begin{aligned}
h_{e_j} = (h\big|_j^{((X_i)_{i\in\phi^{-1}l}g_l)_{l<j},((X_i)_{i\in\phi^{-1}l}f_l)_{l>j}})_1 :& \\
s\mathcal{B}_j((X_i)_{i\in\phi^{-1}j}f_j, (X_i)_{i\in\phi^{-1}j}g_j) &\to s\mathcal{C}(((X_i)_{i\in\phi^{-1}l}f_l)h, ((X_i)_{i\in\phi^{-1}l}g_l)h).
\end{aligned}
$$

By the associativity of $\mu_{\mathsf{k}\mathcal{C}}$, it follows that ${}_{(X_i)_{i\in I}}\mathsf{k}[((r^j s^{-1})_{j\in J}, ps^{-1})\mu_\phi^{A^u_\infty}]$ equals

$$
\begin{aligned}
&(\otimes^{j\in J}{}_{(X_i)_{i\in\phi^{-1}j}}r_0^j s^{-1}\cdot \otimes^{j\in J} sh_{e_j}s^{-1}\mu_{\mathsf{k}\mathcal{C}}^J \otimes {}_{((X_i)_{i\in\phi^{-1}j}g_j)_{j\in J}}p_0 s^{-1})\mu_{\mathsf{k}\mathcal{C}} \\
&\qquad = (\otimes^{j\in J}{}_{(X_i)_{i\in\phi^{-1}j}}\mathsf{k}r^j\cdot\mathsf{k}h \otimes {}_{((X_i)_{i\in\phi^{-1}j}g_j)_{j\in J}}\mathsf{k}p)\mu_{\mathsf{k}\mathcal{C}} = {}_{(X_i)_{i\in I}}[(\mathsf{k}r^j)_{j\in J}\cdot_\phi \mathsf{k}p].
\end{aligned}
$$

Therefore, $\mathsf{k}[((r^j s^{-1})_{j\in J}, ps^{-1})\mu_\phi^{A^u_\infty}] = ((\mathsf{k}r^j)_{j\in J}, \mathsf{k}p)\mu_\phi^{\widehat{\mathcal{K}\text{-}\mathbf{Cat}}}$, hence k is a $\Bbbk$-**Cat**-multifunctor. □

3.4.14. 2-category approach to unital A_∞-categories. It turns out that the 2-category structure on unital A_∞-categories studied in [**38**], as well as somewhat unnatural and obscure notions of A_∞-2-functor and A_∞-2-transformation introduced in [**40**] fit naturally into the framework of closed multicategories, and even become obvious. Let us supply the details.

The $\Bbbk$-**Cat**-multicategory A^u_∞ has an underlying $\Bbbk$-**Cat**-category, i.e., a 2-category, whose sets of 2-morphism are $\Bbbk$-modules, and vertical and horizontal compositions of 2-morphisms are (multi)linear. In particular, the notions of 2-category theory, such as isomorphisms between 1-morphisms, equivalences etc. make sense. These notions are worked out in [**38**, Sections 7,8]. For example, a natural A_∞-transformation $p : f \to g : \mathcal{A} \to \mathcal{B}$ is *invertible* if there exists a natural A_∞-transformation $q : g \to f : \mathcal{A} \to \mathcal{B}$ such that $(p \otimes q)B_2 \equiv f\mathbf{i}^{\mathcal{B}}$ and $(q \otimes p)B_2 \equiv g\mathbf{i}^{\mathcal{B}}$. Proposition 7.15 from [**38**] asserts that a natural A_∞-transformation $p : f \to g : \mathcal{A} \to \mathcal{B}$ is invertible if and only if so is $\mathsf{k}p$. Equivalently, p is invertible if and only if the cycle ${}_Xp_0 : \Bbbk \to s\mathcal{B}(Xf, Xg)$ is invertible modulo boundaries, for each $X \in \operatorname{Ob}\mathcal{A}$. A unital A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ is an A_∞-*equivalence* if there exists a unital A_∞-functor $g : \mathcal{B} \to \mathcal{A}$ and invertible natural A_∞-transformations $fg \to \mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ and $gf \to \mathrm{id}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$. Proposition 8.8 of [**38**] gives a particularly useful criterion that simplifies immensely the task of checking that a given unital A_∞-functor is an A_∞-equivalence: a unital A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ is an A_∞-equivalence if and only if $\mathsf{k}f : \mathsf{k}\mathcal{A} \to \mathsf{k}\mathcal{B}$ is an equivalence of $\mathcal{K}$-categories. Equivalently, f is an A_∞-equivalence if and only if $H^0f : H^0\mathcal{A} \to H^0\mathcal{B}$ is essentially surjective on objects, and the first component $f_1 : s\mathcal{A}(X, Y) \to s\mathcal{B}(Xf, Yf)$ is homotopy invertible, for each $X, Y \in \operatorname{Ob}\mathcal{A}$. We can say that the symmetric **Cat**-multifunctor k reflects isomorphisms between A_∞-functors and A_∞-equivalences.

The symmetric $\mathsf{A}^{\mathrm{u}}_\infty$-multicategory $\underline{\mathsf{A}^{\mathrm{u}}_\infty}$ and its underlying $\mathsf{A}^{\mathrm{u}}_\infty$-category are denoted by the same symbol. Then $\mathsf{A}^{\mathrm{u}}_\infty$-functors $F : \underline{\mathsf{A}^{\mathrm{u}}_\infty} \to \underline{\mathsf{A}_\infty}$ are the same things as strict A^u_∞-2-functors introduced in [**40**].

3.4.15. Example. From the general theory of closed multicategories it follows that an arbitrary unital A_∞-category $\mathcal{C}$ gives rise to an $\mathsf{A}^{\mathrm{u}}_\infty$-functor $\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{C}; -) : \underline{\mathsf{A}^{\mathrm{u}}_\infty} \to \underline{\mathsf{A}^{\mathrm{u}}_\infty}$, see

Example 1.3.18. The A_∞-functor

$$\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{C};-):\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A};\mathcal{B})\to\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{C};\mathcal{A}),\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{C};\mathcal{B}))$$

is computed in [**38**, Proposition 6.2]. From Remark 1.3.24 we know that it takes a unital A_∞-functor $f:\mathcal{A}\to\mathcal{B}$ to the unital A_∞-functor $\underline{\mathsf{A}^{\mathrm{u}}_\infty}(1;f):\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{C};\mathcal{A})\to\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{C};\mathcal{B})$ defined by equation (1.3.9) appropriately interpreted in the closed symmetric multicategory $\mathsf{A}^{\mathrm{u}}_\infty$.

3.4.16. Example. Similarly, each unital A_∞-category $\mathcal{C}$ gives rise to an $\mathsf{A}^{\mathrm{u}}_\infty$-functor $\underline{\mathsf{A}^{\mathrm{u}}_\infty}(-;\mathcal{C}):(\underline{\mathsf{A}^{\mathrm{u}}_\infty})^{\mathrm{op}}\to\underline{\mathsf{A}^{\mathrm{u}}_\infty}$, see Example 1.3.19. The A_∞-functor

$$\underline{\mathsf{A}^{\mathrm{u}}_\infty}(-;\mathcal{C}):\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{B};\mathcal{A})\to\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A};\mathcal{C}),\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{B};\mathcal{C}))$$

is computed in [**39**, Appendix B]. From Remark 1.3.26, it follows that it takes a unital A_∞-functor $f:\mathcal{B}\to\mathcal{A}$ to the unital A_∞-functor $\underline{\mathsf{A}^{\mathrm{u}}_\infty}(f;1):\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A};\mathcal{C})\to\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{B};\mathcal{C})$ defined by equation (1.3.9) interpreted in the closed symmetric multicategory $\mathsf{A}^{\mathrm{u}}_\infty$.

For $\mathsf{A}^{\mathrm{u}}_\infty$-functors $F,G:\underline{\mathsf{A}^{\mathrm{u}}_\infty}\to\underline{\mathsf{A}^{\mathrm{u}}_\infty}$, an $\mathsf{A}^{\mathrm{u}}_\infty$-natural transformation $F\to G:\underline{\mathsf{A}^{\mathrm{u}}_\infty}\to\underline{\mathsf{A}^{\mathrm{u}}_\infty}$ is the same thing as a strict A^u_∞-2-transformation as defined in [**40**, Definition 3.2].

Recall that we have the symmetric multifunctor $H^0:\mathsf{A}^{\mathrm{u}}_\infty\to\widehat{\Bbbk\text{-}\mathbf{Cat}}$. It gives rise to a base change $\mathbf{Cat}$-functor $H^0_*:\mathsf{A}^{\mathrm{u}}_\infty\text{-}\mathbf{Cat}\to(\Bbbk\text{-}\mathbf{Cat})\text{-}\mathbf{Cat}$. Let $F:\underline{\mathsf{A}^{\mathrm{u}}_\infty}\to\underline{\mathsf{A}^{\mathrm{u}}_\infty}$ be an $\mathsf{A}^{\mathrm{u}}_\infty$-functor. It is mapped by H^0_* to the $\Bbbk$-$\mathbf{Cat}$-functor $H^0_*F:A^u_\infty\to A^u_\infty$. A $\Bbbk$-$\mathbf{Cat}$-functor is just a strict 2-functor subject to an additional requirement that it must be $\Bbbk$-linear with respect to 2-morphisms. The $\Bbbk$-$\mathbf{Cat}$-functor H^0_*F takes a unital A_∞-category $\mathcal{A}$ to the unital A_∞-category $F\mathcal{A}$, and a unital A_∞-functor $f:\mathcal{A}\to\mathcal{B}$ to the unital A_∞-functor $Ff:F\mathcal{A}\to F\mathcal{B}$. A 2-morphism $f\to g:\mathcal{A}\to\mathcal{B}$ represented by a natural A_∞-transformation $r\in(s\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A};\mathcal{B}))^{-1}(f,g)$ is taken to the 2-morphism represented by the natural A_∞-transformation $(F_{\mathcal{A},\mathcal{B}})_1r$, where

$$(F_{\mathcal{A},\mathcal{B}})_1:s\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A};\mathcal{B})(f,g)\to\underline{\mathsf{A}^{\mathrm{u}}_\infty}(F\mathcal{A};F\mathcal{B})(Ff,Fg)$$

is the first component of the A_∞-functor $F_{\mathcal{A},\mathcal{B}}:\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A};\mathcal{B})\to\underline{\mathsf{A}^{\mathrm{u}}_\infty}(F\mathcal{A};F\mathcal{B})$. In particular, H^0_*F acts on objects and on 1-morphisms simply as F. Being a 2-functor, H^0_*F preserves equivalences in the 2-category A^u_∞, which are precisely A_∞-equivalences. We conclude that an arbitrary $\mathsf{A}^{\mathrm{u}}_\infty$-functor $F:\underline{\mathsf{A}^{\mathrm{u}}_\infty}\to\underline{\mathsf{A}^{\mathrm{u}}_\infty}$ preserves A_∞-equivalences. Similarly, an arbitrary $\mathsf{A}^{\mathrm{u}}_\infty$-functor $(\underline{\mathsf{A}^{\mathrm{u}}_\infty})^{\mathrm{op}}\to\underline{\mathsf{A}^{\mathrm{u}}_\infty}$ preserves A_∞-equivalences.

3.4.17. Example. The $\mathsf{A}^{\mathrm{u}}_\infty$-functors $\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{C};-)$ and $\underline{\mathsf{A}^{\mathrm{u}}_\infty}(-;\mathcal{C})$ from Examples 3.4.15 and 3.4.16 preserve A_∞-equivalences. Therefore, for each A_∞-equivalence $f:\mathcal{A}\to\mathcal{B}$, the A_∞-functors $\underline{\mathsf{A}^{\mathrm{u}}_\infty}(1;f)$ and $\underline{\mathsf{A}^{\mathrm{u}}_\infty}(f;1)$ are A_∞-equivalences as well.

Let $\mathcal{A}$, $\mathcal{B}$ be unital A_∞-categories. A unital A_∞-functor $f:\mathcal{A}\to\mathcal{B}$ is called *homotopy fully faithful* if the corresponding $\mathcal{K}$-functor $\mathsf{k}f:\mathsf{k}\mathcal{A}\to\mathsf{k}\mathcal{B}$ is fully faithful. That is, f is homotopy fully faithful if and only if its first component is homotopy invertible. Equivalently, f is homotopy fully faithful if and only if it admits a factorization

$$\mathcal{A}\xrightarrow{g}\mathcal{I}\xrightarrow{e}\mathcal{B}, \tag{3.4.14}$$

where $\mathcal{I}$ is a full A_∞-subcategory of $\mathcal{B}$, $e:\mathcal{I}\hookrightarrow\mathcal{B}$ is the embedding, and $g:\mathcal{A}\to\mathcal{I}$ is an A_∞-equivalence.

The following lemma will be useful when we come to A_∞-bimodules. It is a minor generalization of Example 3.4.17.

3.4.18. Lemma. *Suppose $f : \mathcal{A} \to \mathcal{B}$ is a homotopy fully faithful A_∞-functor. Then for an arbitrary unital A_∞-category $\mathcal{C}$ the A_∞-functor $\underline{\mathsf{A}}^{\mathrm{u}}_\infty(1;f) : \underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{C};\mathcal{A}) \to \underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{C};\mathcal{B})$ is homotopy fully faithful.*

Proof. Factorize f as in (3.4.14). Then by Lemma 1.3.22 the A_∞-functor $\underline{\mathsf{A}}^{\mathrm{u}}_\infty(1;f)$ factorizes as

$$\underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{C};\mathcal{A}) \xrightarrow{\underline{\mathsf{A}}^{\mathrm{u}}_\infty(1;g)} \underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{C};\mathcal{I}) \xrightarrow{\underline{\mathsf{A}}^{\mathrm{u}}_\infty(1;e)} \underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{C};\mathcal{B}).$$

The A_∞-functor $\underline{\mathsf{A}}^{\mathrm{u}}_\infty(1;g)$ is an A_∞-equivalence by Example 3.4.17, so it suffices to show that $\underline{\mathsf{A}}^{\mathrm{u}}_\infty(1;e)$ is a full embedding. It is determined unambiguously by the equation

$$\begin{tikzcd}
\mathcal{C}, \underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{C};\mathcal{I}) \arrow[r, "{1,\underline{\mathsf{A}}^{\mathrm{u}}_\infty(1;e)}"] \arrow[d, "\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}"'] & \mathcal{C}, \underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{C};\mathcal{B}) \arrow[d, "\mathrm{ev}^{\mathsf{A}_\infty}"] \\
\mathcal{I} \arrow[r, "e"] & \mathcal{B}
\end{tikzcd}$$

The left-bottom composite is a strict A_∞-functor, therefore so it the top-right composite. From formulas (3.3.6) and (3.3.6) for components of $\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}$, it follows that $\underline{\mathsf{A}}^{\mathrm{u}}_\infty(1;e)$ is a strict A_∞-functor, and that its first component is given by

$$s\underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{C};\mathcal{I})(\phi,\psi) = \prod_{n\geqslant 0}^{X,Y\in\operatorname{Ob}\mathcal{C}} \underline{\mathsf{C}}_{\Bbbk}(T^n s\mathcal{C}(X,Y), s\mathcal{I}(X\phi,Y\psi)) \xrightarrow{\prod \underline{\mathbf{gr}}(1,e_1)}$$
$$\prod_{n\geqslant 0}^{X,Y\in\operatorname{Ob}\mathcal{C}} \underline{\mathsf{C}}_{\Bbbk}(T^n s\mathcal{C}(X,Y), s\mathcal{B}(X\phi,Y\psi)) = s\underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{C};\mathcal{B})(\phi e,\psi e),$$

that is, $r = (r_n) \mapsto re = (r_n e_1)$. Since $s\mathcal{I}(X\phi,Y\psi) = s\mathcal{B}(X\phi,Y\psi)$ and e_1 is the identity morphism, the above map is the identity morphism, and the proof is complete. ☐

3.5. Opposite A_∞-categories

Recall the following definitions from [**39**, Appendix A]. Let $\mathcal{A}$ be a graded quiver. Then its *opposite quiver* $\mathcal{A}^{\mathrm{op}}$ is defined as the quiver with the same class of objects $\operatorname{Ob}\mathcal{A}^{\mathrm{op}} = \operatorname{Ob}\mathcal{A}$, and with graded $\Bbbk$-modules of morphisms $\mathcal{A}^{\mathrm{op}}(X,Y) = \mathcal{A}(Y,X)$.

Let $\gamma : Ts\mathcal{A}^{\mathrm{op}} \to Ts\mathcal{A}$ denote the following anti-isomorphism of coalgebras and algebras (free categories):

$$\gamma = (-1)^k \omega^0_c : s\mathcal{A}^{\mathrm{op}}(X_0,X_1) \otimes \dots \otimes s\mathcal{A}^{\mathrm{op}}(X_{k-1},X_k)$$
$$\to s\mathcal{A}(X_k,X_{k-1}) \otimes \dots \otimes s\mathcal{A}(X_1,X_0), \quad (3.5.1)$$

where $\omega^0 = \left(\begin{smallmatrix} 1 & 2 & \dots & k-1 & k \\ k & k-1 & \dots & 2 & 1 \end{smallmatrix}\right) \in \mathfrak{S}_k$. Clearly,

$$\gamma\Delta_0 = \Delta_0(\gamma\otimes\gamma)c = \Delta_0 c(\gamma\otimes\gamma) \quad (3.5.2)$$

which is the anti-isomorphism property. Notice also that $(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{A}$ and $\gamma^2 = \mathrm{id}$.

When $\mathcal{A}$ is an A_∞-category with the codifferential $b : Ts\mathcal{A} \to Ts\mathcal{A}$, then $\gamma b\gamma : Ts\mathcal{A}^{\mathrm{op}} \to Ts\mathcal{A}^{\mathrm{op}}$ is also a codifferential. Indeed,

$$\gamma b\gamma\Delta_0 = \gamma b\Delta_0 c(\gamma\otimes\gamma) = \gamma\Delta_0(1\otimes b + b\otimes 1)c(\gamma\otimes\gamma)$$
$$= \Delta_0(\gamma\otimes\gamma)c(1\otimes b + b\otimes 1)c(\gamma\otimes\gamma) = \Delta_0(\gamma b\gamma\otimes 1 + 1\otimes\gamma b\gamma).$$

The *opposite* A_∞*-category* $\mathcal{A}^{\mathrm{op}}$ to an A_∞-category $\mathcal{A}$ is the opposite quiver equipped with the codifferential $b^{\mathrm{op}} = \gamma b \gamma : Ts\mathcal{A}^{\mathrm{op}} \to Ts\mathcal{A}^{\mathrm{op}}$. Components of b^{op} are computed as follows:

$$\begin{aligned} b^{\mathrm{op}}_k = (-)^{k+1}\bigl[s\mathcal{A}^{\mathrm{op}}(X_0,X_1)\otimes\dots\otimes s\mathcal{A}^{\mathrm{op}}(X_{k-1},X_k) \xrightarrow{\omega^0_c} \\ s\mathcal{A}(X_k,X_{k-1})\otimes\dots\otimes s\mathcal{A}(X_1,X_0) \xrightarrow{b_k} s\mathcal{A}(X_k,X_0) = s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\bigr]. \quad (3.5.3) \end{aligned}$$

The sign $(-1)^k$ in (3.5.1) ensures that the above definition agrees with the definition of the opposite $\mathcal{K}$-category, meaning that, for an arbitrary A_∞-category $\mathcal{A}$, $\mathsf{k}(\mathcal{A}^{\mathrm{op}}) = (\mathsf{k}\mathcal{A})^{\mathrm{op}}$. Indeed, clearly, both $\mathcal{K}$-categories have $\operatorname{Ob}\mathcal{A}$ as the set of objects. Furthermore, for each pair of objects $X, Y \in \operatorname{Ob}\mathcal{A}$,

$$m^{\mathrm{op}}_1 = sb^{\mathrm{op}}_1 s^{-1} = sb_1 s^{-1} = m_1 : \mathcal{A}^{\mathrm{op}}(X,Y) = \mathcal{A}(Y,X) \to \mathcal{A}(Y,X) = \mathcal{A}^{\mathrm{op}}(X,Y),$$

therefore $(\mathsf{k}\mathcal{A}^{\mathrm{op}})(X,Y) = (\mathcal{A}^{\mathrm{op}}(X,Y), m^{\mathrm{op}}_1) = (\mathcal{A}(Y,X), m_1) = (\mathsf{k}\mathcal{A})^{\mathrm{op}}(X,Y)$. Finally, compositions in both categories coincide:

$$\mu_{\mathsf{k}\mathcal{A}^{\mathrm{op}}} = m^{\mathrm{op}}_2 = (s\otimes s)b^{\mathrm{op}}_2 s^{-1} = -(s\otimes s)cb_2 s^{-1} = c(s\otimes s)b_2 s^{-1} = cm_2 = \mu_{(\mathsf{k}\mathcal{A})^{\mathrm{op}}}.$$

In particular, it follows that $\mathcal{A}^{\mathrm{op}}$ is unital if so is $\mathcal{A}$, with the same unit elements.

For an arbitrary A_∞-functor $f : \boxtimes^{i\in\mathbf{n}} Ts\mathcal{A}_i \to Ts\mathcal{B}$ there is another A_∞-functor f^{op} defined by the commutative square

$$\begin{tikzcd} \boxtimes^{i\in\mathbf{n}} Ts\mathcal{A}_i \arrow[r, "f"] \arrow[d, "\boxtimes^{\mathbf{n}}\gamma"'] & Ts\mathcal{B} \arrow[d, "\gamma"] \\ \boxtimes^{i\in\mathbf{n}} Ts\mathcal{A}^{\mathrm{op}}_i \arrow[r, "f^{\mathrm{op}}"] & Ts\mathcal{B}^{\mathrm{op}} \end{tikzcd}$$

Since $\gamma^2 = \mathrm{id}$, the A_∞-functor f^{op} is found as the composite

$$f^{\mathrm{op}} = \bigl[\boxtimes^{i\in\mathbf{n}} Ts\mathcal{A}^{\mathrm{op}} \xrightarrow{\boxtimes^{\mathbf{n}}\gamma} \boxtimes^{i\in\mathbf{n}} Ts\mathcal{A}_i \xrightarrow{f} Ts\mathcal{B} \xrightarrow{\gamma} Ts\mathcal{B}^{\mathrm{op}}\bigr].$$

3.5.1. Lemma. *For an arbitrary* A_∞*-functor* $f : \boxtimes^{i\in\mathbf{n}} Ts\mathcal{A}_i \to Ts\mathcal{B}$*, the* $\mathcal{K}$*-functors* $\mathsf{k}f^{\mathrm{op}}, (\mathsf{k}f)^{\mathrm{op}} : \boxtimes^{i\in\mathbf{n}} \mathsf{k}\mathcal{A}^{\mathrm{op}}_i \to \mathsf{k}\mathcal{B}^{\mathrm{op}}$ *coincide.*

Proof. The case $n = 1$ is straightforward. We provide a proof in the case $n = 2$, which we are going to use later.

Let $f : Ts\mathcal{A}\boxtimes Ts\mathcal{B} \to Ts\mathcal{C}$ be an A_∞-functor. The components of f^{op} are given by

$$\begin{aligned} f^{\mathrm{op}}_{kn} = (-)^{k+n-1}\bigl[s\mathcal{A}^{\mathrm{op}}(X_0,X_1)\otimes\dots\otimes s\mathcal{A}^{\mathrm{op}}(X_{k-1},X_k)\otimes \\ \otimes s\mathcal{B}^{\mathrm{op}}(U_0,U_1)\otimes\dots\otimes s\mathcal{B}^{\mathrm{op}}(U_{n-1},U_n) \\ = s\mathcal{A}(X_1,X_0)\otimes\dots\otimes s\mathcal{A}(X_k,X_{k-1})\otimes s\mathcal{B}(U_1,U_0)\otimes\dots\otimes s\mathcal{B}(U_n,U_{n-1}) \\ \xrightarrow{\pi^{kn}_c} s\mathcal{A}(X_k,X_{k-1})\otimes\dots\otimes s\mathcal{A}(X_1,X_0)\otimes s\mathcal{B}(U_n,U_{n-1})\otimes\dots\otimes s\mathcal{B}(U_1,U_0) \\ \xrightarrow{f_{kn}} s\mathcal{C}((X_k,U_n)f,(X_0,U_0)f) = s\mathcal{C}^{\mathrm{op}}((X_0,U_0)f,(X_k,U_n)f)\bigr], \quad (3.5.4) \end{aligned}$$

where $\pi^{kn} = \left(\begin{smallmatrix} 1 & 2 & \dots & k & k+1 & k+2 & \dots & k+n \\ k & k-1 & \dots & 1 & k+n & k+n-1 & \dots & k+1 \end{smallmatrix}\right) \in \mathfrak{S}_{k+n}$, and π^{kn}_c is the corresponding permutation isomorphism.

Clearly, both $\mathsf{k}f^{\mathrm{op}}$ and $(\mathsf{k}f)^{\mathrm{op}}$ act as $\operatorname{Ob} f$ on objects. Let $X, Y \in \operatorname{Ob} \mathcal{A}$, $U, V \in \operatorname{Ob} \mathcal{B}$. Then

$$\mathsf{k}f^{\mathrm{op}} = \bigl[\mathcal{A}^{\mathrm{op}}(X,Y) \otimes \mathcal{B}^{\mathrm{op}}(U,V) \xrightarrow{sf^{\mathrm{op}}_{10}s^{-1} \otimes sf^{\mathrm{op}}_{01}s^{-1}} \\ \mathcal{C}^{\mathrm{op}}((X,U)f,(Y,U)f) \otimes \mathcal{C}^{\mathrm{op}}((Y,U)f,(Y,V)f) \xrightarrow{\mu_{\mathsf{k}\mathcal{C}^{\mathrm{op}}}} \mathcal{C}^{\mathrm{op}}((X,U)f,(Y,V)f)\bigr].$$

By (3.5.4),

$$f^{\mathrm{op}}_{10} = f_{10} : s\mathcal{A}^{\mathrm{op}}(X,Y) \to s\mathcal{C}((Y,U)f,(X,U)f) = s\mathcal{C}^{\mathrm{op}}((X,U)f,(Y,U)f),$$
$$f^{\mathrm{op}}_{01} = f_{01} : s\mathcal{B}^{\mathrm{op}}(U,V) \to s\mathcal{C}((Y,V)f,(Y,U)f) = s\mathcal{C}^{\mathrm{op}}((Y,U)f,(Y,V)f),$$

therefore

$$\mathsf{k}f^{\mathrm{op}} = \bigl[\mathcal{A}(Y,X) \otimes \mathcal{B}(V,U) \xrightarrow{sf_{10}s^{-1} \otimes sf_{01}s^{-1}} \mathcal{C}((Y,U)f,(X,U)f) \otimes \mathcal{C}((Y,V)f,(Y,U)f) \\ \xrightarrow{c} \mathcal{C}((Y,V)f,(Y,U)f) \otimes \mathcal{C}((Y,U)f,(X,U)f) \xrightarrow{\mu_{\mathsf{k}\mathcal{C}}} \mathcal{C}((Y,V)f,(X,U)f)\bigr] \\ = \bigl[\mathcal{A}(Y,X) \otimes \mathcal{B}(V,U) \xrightarrow{c} \mathcal{B}(V,U) \otimes \mathcal{A}(Y,X) \xrightarrow{sf_{01}s^{-1} \otimes sf_{10}s^{-1}} \\ \mathcal{C}((Y,V)f,(Y,U)f) \otimes \mathcal{C}((Y,U)f,(X,U)f) \xrightarrow{\mu_{\mathsf{k}\mathcal{C}}} \mathcal{C}((Y,V)f,(X,U)f)\bigr].$$

Further,

$$(\mathsf{k}f)^{\mathrm{op}} = \bigl[\mathcal{A}(Y,X) \otimes \mathcal{B}(V,U) \xrightarrow{sf_{10}s^{-1} \otimes sf_{01}s^{-1}} \\ \mathcal{C}((Y,V)f,(X,V)f) \otimes \mathcal{C}((X,V)f,(X,U)f) \xrightarrow{\mu_{\mathsf{k}\mathcal{C}}} \mathcal{C}((Y,V)f,(X,U)f)\bigr].$$

We must therefore prove the following equation in $\mathcal{K}$:

$$\bigl[\mathcal{A}(Y,X) \otimes \mathcal{B}(V,U) \xrightarrow{c} \mathcal{B}(V,U) \otimes \mathcal{A}(Y,X) \xrightarrow{sf_{01}s^{-1} \otimes sf_{10}s^{-1}} \\ \mathcal{C}((Y,V)f,(Y,U)f) \otimes \mathcal{C}((Y,U)f,(X,U)f) \xrightarrow{\mu_{\mathsf{k}\mathcal{C}}} \mathcal{C}((Y,V)f,(X,U)f)\bigr] \\ = \bigl[\mathcal{A}(Y,X) \otimes \mathcal{B}(V,U) \xrightarrow{sf_{10}s^{-1} \otimes sf_{01}s^{-1}} \mathcal{C}((Y,V)f,(X,V)f) \otimes \mathcal{C}((X,V)f,(X,U)f) \\ \xrightarrow{\mu_{\mathsf{k}\mathcal{C}}} \mathcal{C}((Y,V)f,(X,U)f)\bigr]. \quad (3.5.5)$$

Since f is an A_∞-functor, the equation $fb = (b \boxtimes 1 + 1 \boxtimes b)f : Ts\mathcal{A} \boxtimes Ts\mathcal{B} \to Ts\mathcal{C}$ holds true. Restricting it to $s\mathcal{A} \boxtimes s\mathcal{B}$ and composing with the projection $\mathrm{pr}_1 : Ts\mathcal{C} \to s\mathcal{C}$, we obtain

$$(f_{10} \otimes f_{01})b_2 + c(f_{01} \otimes f_{10})b_2 + f_{11}b_1 \\ = (1 \otimes b_1 + b_1 \otimes 1)f_{11} : s\mathcal{A}(Y,X) \otimes s\mathcal{B}(V,U) \to s\mathcal{C}((Y,V)f,(X,U)f).$$

Thus, $(f_{10} \otimes f_{01})b_2 + c(f_{01} \otimes f_{10})b_2$ is a boundary. Therefore,

$$(s \otimes s)(f_{10} \otimes f_{01})b_2 = c(s \otimes s)(f_{01} \otimes f_{10})b_2$$

in $\mathcal{K}$. This implies equation (3.5.5). □

In particular, f^{op} is a unital A_∞-functor if f is unital.

3.5.2. Proposition. *The correspondences $\mathcal{A} \mapsto \mathcal{A}^{\mathrm{op}}$, $f \mapsto f^{\mathrm{op}}$ define a symmetric multifunctor $-^{\mathrm{op}} : \mathsf{A}_\infty \to \mathsf{A}_\infty$ which restricts to a symmetric multifunctor $-^{\mathrm{op}} : \mathsf{A}^{\mathrm{u}}_\infty \to \mathsf{A}^{\mathrm{u}}_\infty$.*

Proof. Straightforward. □

As an arbitrary symmetric multifunctor between closed multicategories, $-^{\mathrm{op}}$ possesses a closing transformation $\underline{\mathrm{op}} : \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathrm{op}} \to \underline{\mathsf{A}_\infty}((\mathcal{A}_i^{\mathrm{op}})_{i\in I};\mathcal{B}^{\mathrm{op}})$ uniquely determined by the following equation in A_∞:

$$\bigl[(\mathcal{A}_i^{\mathrm{op}})_{i\in I}, \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathrm{op}} \xrightarrow{1,\underline{\mathrm{op}}} (\mathcal{A}_i^{\mathrm{op}})_{i\in I}, \underline{\mathsf{A}_\infty}((\mathcal{A}_i^{\mathrm{op}})_{i\in I};\mathcal{B}^{\mathrm{op}}) \xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty}} \mathcal{B}^{\mathrm{op}}\bigr] = (\mathrm{ev}^{\mathsf{A}_\infty})^{\mathrm{op}}. \tag{3.5.6}$$

The A_∞-functor $(\mathrm{ev}^{\mathsf{A}_\infty})^{\mathrm{op}}$ acts on objects in the same way as $\mathrm{ev}^{\mathsf{A}_\infty}$. It follows that $(X_i)_{i\in I}(f)\underline{\mathrm{op}} = (X_i)_{i\in I}f$ for an arbitrary A_∞-functor $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ and a family of objects $X_i \in \operatorname{Ob}\mathcal{A}_i$, $i \in I$. The components

$$\begin{aligned}(\mathrm{ev}^{\mathsf{A}_\infty})^{\mathrm{op}}_{(m_i),m} = -\bigl[\boxtimes^{i\in I} T^{m_i} s\mathcal{A}_i^{\mathrm{op}} \boxtimes T^m s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathrm{op}} &\xrightarrow{\boxtimes^I(\gamma)_I\boxtimes\gamma} \\ \boxtimes^{i\in I} T^{m_i} s\mathcal{A}_i \boxtimes T^m s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B}) &\xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty}_{(m_i),m}} s\mathcal{B}\bigr]\end{aligned} \tag{3.5.7}$$

vanish unless $m = 0$ or $m = 1$ since the same holds for $\mathrm{ev}^{\mathsf{A}_\infty}_{(m_i),m}$. From equations (3.5.6) and (3.5.7) we infer that

$$\begin{aligned}&(\otimes^{i\in I} 1^{\otimes m_i} \otimes \operatorname{Ob}\underline{\mathrm{op}})\,\mathrm{ev}^{\mathsf{A}_\infty}_{(m_i),0} \\ &= -\bigl[\otimes^{i\in I} \otimes^{p_i\in\mathbf{m}_i} s\mathcal{A}_i^{\mathrm{op}}(X^i_{p_i-1}, X^i_{p_i}) \otimes T^0 s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathrm{op}}(f,f) \\ &\qquad \cong \otimes^{i\in I} \otimes^{p_i\in\mathbf{m}_i} s\mathcal{A}_i^{\mathrm{op}}(X^i_{p_i-1}, X^i_{p_i}) \xrightarrow{\otimes^{i\in I}(-)^{m}_{i}\omega^0_c} \\ &\qquad\qquad \otimes^{i\in I} \otimes^{p_i\in\mathbf{m}_i} s\mathcal{A}_i(X^i_{m_i-p_i}, X^i_{m_i-p_i+1}) \xrightarrow{f_{(m_i)}} s\mathcal{B}((X^i_0)_{i\in I}f, (X^i_{m_i})_{i\in I}f)\bigr],\end{aligned}$$

therefore $(f)\underline{\mathrm{op}} = f^{\mathrm{op}} : (\mathcal{A}_i^{\mathrm{op}})_{i\in I} \to \mathcal{B}^{\mathrm{op}}$. Similarly,

$$\begin{aligned}&(\otimes^{i\in I} 1^{\otimes m_i} \otimes \underline{\mathrm{op}}_1)\,\mathrm{ev}^{\mathsf{A}_\infty}_{(m_i),1} \\ &= \bigl[\otimes^{i\in I} \otimes^{p_i\in\mathbf{m}_i} s\mathcal{A}_i^{\mathrm{op}}(X^i_{p_i-1}, X^i_{p_i}) \otimes s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathrm{op}}(f,g) \\ &\quad \xrightarrow{\otimes^{i\in I}(-)^{m_i}\omega^0_c\otimes 1} \otimes^{i\in I} \otimes^{p_i\in\mathbf{m}_i} s\mathcal{A}_i(X^i_{m_i-p_i}, X^i_{m_i-p_i+1}) \otimes s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(g,f) \\ &\quad \xrightarrow{\otimes^{i\in I}\otimes^{p_i\in\mathbf{m}_i}1\otimes\mathrm{pr}} \otimes^{i\in I} \otimes^{p_i\in\mathbf{m}_i} s\mathcal{A}_i(X^i_{m_i-p_i}, X^i_{m_i-p_i+1}) \\ &\qquad\qquad \otimes \underline{\mathbf{gr}}(\otimes^{i\in I} \otimes^{p_i\in\mathbf{m}_i} s\mathcal{A}_i(X^i_{m_i-p_i}, X^i_{m_i-p_i+1}), s\mathcal{B}((X^i_{m_i})_{i\in I}g, (X^i_0)_{i\in I}f)) \\ &\qquad\qquad \xrightarrow{\mathrm{ev}^{\mathbf{gr}}} s\mathcal{B}((X^i_{m_i})_{i\in I}g, (X^i_0)_{i\in I}f) = s\mathcal{B}^{\mathrm{op}}((X^i_0)_{i\in I}f^{\mathrm{op}}, (X^i_{m_i})_{i\in I}g^{\mathrm{op}})\bigr].\end{aligned}$$

It follows that the map $\underline{\mathrm{op}}_1 : s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathrm{op}}(f,g) \to s\underline{\mathsf{A}_\infty}((\mathcal{A}_i^{\mathrm{op}})_{i\in I};\mathcal{B}^{\mathrm{op}})(f^{\mathrm{op}},g^{\mathrm{op}})$ takes an A_∞-transformation $r : g \to f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ to the opposite A_∞-transformation $r^{\mathrm{op}} \overset{\mathrm{def}}{=} (r)\underline{\mathrm{op}}_1 : f^{\mathrm{op}} \to g^{\mathrm{op}} : (\mathcal{A}_i^{\mathrm{op}})_{i\in I} \to \mathcal{B}^{\mathrm{op}}$ with the components

$$\begin{aligned}[(r)\underline{\mathrm{op}}_1]_{(m_i)} = (-)^{m_1+\dots+m_n}\bigl[\otimes^{i\in I} \otimes^{p_i\in\mathbf{m}_i} s\mathcal{A}_i^{\mathrm{op}}(X^i_{p_i-1}, X^i_{p_i}) &\xrightarrow{\otimes^{i\in I}\omega^0_c} \\ \otimes^{i\in I} \otimes^{p_i\in\mathbf{m}_i} s\mathcal{A}_i(X^i_{m_i-p_i}, X^i_{m_i-p_i+1}) &\xrightarrow{r_{(m_i)}} s\mathcal{B}((X^i_{m_i})_{i\in I}g, (X^i_0)_{i\in I}f)\bigr].\end{aligned}$$

The higher components of $\underline{\mathrm{op}}$ vanish. Similar computations can be performed in the multicategory $\mathsf{A}^{\mathrm{u}}_\infty$. They lead to the same formulas for $\underline{\mathrm{op}}$, which means that the A_∞-functor $\underline{\mathrm{op}}$ restricts to a unital A_∞-functor $\underline{\mathrm{op}} : \underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathrm{op}} \to \underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i^{\mathrm{op}})_{i\in I};\mathcal{B}^{\mathrm{op}})$ if the A_∞-categories $\mathcal{A}_i$, $i \in I$, $\mathcal{B}$ are unital.

3.5.3. Remark. Suppose that $\mathcal{B}$ is a unital A_∞-category, and $r : f \to g : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ is an isomorphism of A_∞-functors. Then $r^{\mathrm{op}} : g^{\mathrm{op}} \to f^{\mathrm{op}} : (\mathcal{A}_i^{\mathrm{op}})_{i\in I} \to \mathcal{B}^{\mathrm{op}}$ is an isomorphism as well. Indeed, since $\underline{\mathrm{op}}_1$ is a chain map, it follows that $r^{\mathrm{op}} = (r)\underline{\mathrm{op}}_1$ is a natural A_∞-transformation. Furthermore, the 0^{th} components of the transformations r and r^{op} coincide. The claim follows from Lemma 3.4.11. Note that in the case when $\mathcal{A}_i$, $i \in I$, are unital A_∞-categories, we may just argue that $\underline{\mathrm{op}}$ is a unital A_∞-functor and therefore takes cycles invertible modulo boundaries to cycles invertible modulo boundaries.

CHAPTER 4

A_∞-bimodules

The definition of A_∞-bimodule over A_∞-algebras has been given by Tradler [**59, 60**]. The notion of bimodule over some kind of A_∞-categories was introduced by Lefèvre-Hasegawa under the name of bipolydule [**34**]. Here we extend Tradler's definition of A_∞-bimodules improved in [**55**] from graded $\Bbbk$-modules to graded quivers. We will see that A_∞-bimodules over A_∞-categories $\mathcal{A}$ and $\mathcal{B}$ form a differential graded category. It turns out to be isomorphic to the differential graded category $\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}}, \mathcal{B}; \underline{\mathsf{C}}_\Bbbk)$. In particular, there is a bijection between $\mathcal{A}$-$\mathcal{B}$-bimodules and A_∞-functors $\mathcal{A}^{\mathrm{op}}, \mathcal{B} \to \underline{\mathsf{C}}_\Bbbk$. This observation is very helpful, although obvious. On the one hand, it allows to apply general results about A_∞-functors to the study of A_∞-bimodules. On the other hand, A_∞-bimodules are often more suited for computations than A_∞-functors. For example, the Yoneda Lemma proven in Appendix A is formulated using A_∞-functors, but the proof uses A_∞-bimodules.

The chapter is organized as follows. In Section 4.1, we introduce the notion of differential graded bicomodule over a pair of graded coalgebras. Specializing to coalgebras of the form $Ts\mathcal{A}$, where $\mathcal{A}$ is an A_∞-category, we obtain the definition of A_∞-bimodule. We establish an isomorphism between the differential graded category of A_∞-bimodules over A_∞-categories $\mathcal{A}$ and $\mathcal{B}$ and the differential graded category $\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}}, \mathcal{B}; \underline{\mathsf{C}}_\Bbbk)$. There is nothing surprising in this statement. In fact, it is in complete analogy with the ordinary category theory: on the one hand, a bimodule over $\Bbbk$-linear categories $\mathcal{A}$ and $\mathcal{B}$ can be defined as a collection of $\Bbbk$-modules $\mathcal{P}(X, Y)$, $X \in \operatorname{Ob} \mathcal{A}$, $Y \in \operatorname{Ob} \mathcal{B}$, together with $\Bbbk$-linear action maps $\mathcal{A}(U, X) \otimes \mathcal{P}(X, Y) \otimes \mathcal{B}(Y, V) \to \mathcal{P}(U, V)$ compatible with compositions and identities; on the other hand, an $\mathcal{A}$-$\mathcal{B}$-bimodule can be defined as a $\Bbbk$-linear bifunctor $\mathcal{A}^{\mathrm{op}} \boxtimes \mathcal{B} \to \Bbbk$-**Mod**. That these definitions are equivalent is a straightforward exercise. In the case of A_∞-bimodules it is still straightforward, however computations become cumbersome.

A basic example of A_∞-bimodule is provided by regular A_∞-bimodule. It is discussed in Section 4.2. For an A_∞-category $\mathcal{A}$, we introduce an A_∞-functor $\operatorname{Hom}_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}}, \mathcal{A} \to \underline{\mathsf{C}}_\Bbbk$ as a unique A_∞-functor that corresponds to the regular $\mathcal{A}$-$\mathcal{A}$-bimodule. It is an ingredient of the definition of Serre A_∞-functor. Furthermore, by the closedness of the multicategory A_∞, the A_∞-functor $\operatorname{Hom}_{\mathcal{A}}$ gives rise to an A_∞-functor $\mathscr{Y} : \mathcal{A} \to \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}}; \underline{\mathsf{C}}_\Bbbk)$, called the Yoneda A_∞-functor.

We study some operations on A_∞-bimodules (restriction of scalars, taking opposite and dual bimodule) in Section 4.3. These are necessary for the definition of Serre A_∞-functor.

Section 4.4 is devoted to unital A_∞-bimodules, which are defined simply as A_∞-bimodules that correspond to unital A_∞-functors. A unital A_∞-functor $g : \mathcal{C} \to \mathcal{A}$ gives rise to a unital $\mathcal{A}$-$\mathcal{C}$-bimodule $\mathcal{A}^g$ defined as the bimodule corresponding to the A_∞-functor $(1, g) \operatorname{Hom}_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}}, \mathcal{C} \to \underline{\mathsf{C}}_\Bbbk$. We prove that the mapping $g \mapsto \mathcal{A}^g$ extends to a homotopy fully faithful A_∞-functor $\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{C}; \mathcal{A}) \to \underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A}^{\mathrm{op}}, \mathcal{C}; \underline{\mathsf{C}}_\Bbbk)$ and characterize it image. Here we use some of the properties of the Yoneda A_∞-functor proven in Appendix A. Finally, Section 4.5 is an expository section. Here we introduce A_∞-modules as a particular case of

A_∞-bimodules. The results obtained for A_∞-bimodules remain true also for A_∞-modules. These results are stated for the record, since we are going to use them in Appendix A.

The definitions of bicomodule and A_∞-bimodule are due to Volodymyr Lyubashenko. He also suggested Propositions 4.1.2 and 4.1.3. The proofs of these statement have been given by the author. The definition of the Yoneda A_∞-functor via regular bimodules is Prof. Lyubashenko's finding. Section 4.3 contains authors results only. The definition of unital A_∞-bimodules was suggested by Prof. Lyubashenko. He also proved Propositions 4.4.2 and 4.4.4.

4.1. Definitions

Consider the monoidal category $(\mathscr{Q}/S,\otimes)$ of graded quivers with a fixed set of objects S, see Section 3.2.1. When S is a 1-element set, the category $(\mathscr{Q}/S,\otimes)$ reduces to the category of graded $\Bbbk$-modules used by Tradler.

4.1.1. Definition. Let A, C be coassociative counital coalgebras in $(\mathscr{Q}/R,\otimes_R)$ resp. $(\mathscr{Q}/S,\otimes_S)$, i.e., graded coalgebras in the terminology of Section 3.2.2. A *counital* (A,C)*-bicomodule* (P,δ^P) consists of a graded span P with $\operatorname{Ob}_s P = R$, $\operatorname{Ob}_t P = S$, and a coaction $\delta^P = (\delta',\delta'') : P \to (A\otimes_R P)\oplus(P\otimes_S C)$ of degree 0 such that the following diagram commutes

$$\begin{tikzcd}
P \arrow[r, "\delta"] \arrow[d, "\delta"'] & (A\otimes_R P)\oplus(P\otimes_S C) \arrow[d, "(\Delta\otimes1)\oplus(\delta\otimes1)"] \\
(A\otimes_R P)\oplus(P\otimes_S C) \arrow[r, "(1\otimes\delta)\oplus(1\otimes\Delta)"] & (A\otimes_R A\otimes_R P)\oplus(A\otimes_R P\otimes_S C)\oplus(P\otimes_S C\otimes_S C)
\end{tikzcd}$$

and $\delta'\cdot(\varepsilon\otimes1) = 1 = \delta''\cdot(1\otimes\varepsilon) : P \to P$.

The equation presented on the diagram consists in fact of three equations claiming that P is a left A-comodule, a right C-comodule and the coactions commute.

Let A, B, C, D be graded coalgebras; let $\phi : A \to B$, $\psi : C \to D$ be morphisms of graded coalgebras; let $\chi : A \to B$ be a (ϕ,ϕ)-coderivation and let $\xi : C \to D$ be a (ψ,ψ)-coderivation of certain degree, that is,

$$\chi\Delta = \Delta(\phi\otimes\chi+\chi\otimes\phi), \qquad \xi\Delta = \Delta(\psi\otimes\xi+\xi\otimes\psi).$$

Suppose that $\delta : P \to (A\otimes P)\oplus(P\otimes C)$ is a counital (A,C)-bicomodule and that $\delta : Q \to (B\otimes Q)\oplus(Q\otimes D)$ is a counital (B,D)-bicomodule. A morphism of graded spans $f : P \to Q$ of degree 0 with $\operatorname{Ob}_s f = \operatorname{Ob}\phi$, $\operatorname{Ob}_t f = \operatorname{Ob}\psi$ is a (ϕ,ψ)*-bicomodule homomorphism* if $f\delta' = \delta'(\phi\otimes f) : P \to B\otimes Q$ and $f\delta'' = \delta''(f\otimes\psi) : P \to Q\otimes D$. Define a (ϕ,ψ,f,χ,ξ)*-connection* as a morphism of graded spans $r : P \to Q$ of certain degree with $\operatorname{Ob}_s r = \operatorname{Ob}\phi$, $\operatorname{Ob}_t r = \operatorname{Ob}\psi$ such that

$$\begin{tikzcd}
P \arrow[r, "\delta"] \arrow[d, "r"'] & (A\otimes P)\oplus(P\otimes C) \arrow[d, "(\phi\otimes r+\chi\otimes f)\oplus(f\otimes\xi+r\otimes\psi)"] \\
Q \arrow[r, "\delta"'] & (B\otimes Q)\oplus(Q\otimes D)
\end{tikzcd}$$

commutes.

Let (A, b^A), (C, b^C) be differential graded coalgebras and let P be an (A, C)-bicomodule with an $(\mathrm{id}_A, \mathrm{id}_C, \mathrm{id}_P, b^A, b^C)$-connection $b^P : P \to P$ of degree 1, that is,

$$b^P \delta' = \delta'(1 \otimes b^P + b^A \otimes 1), \qquad b^P \delta'' = \delta''(1 \otimes b^C + b^P \otimes 1).$$

Its *curvature* $(b^P)^2 : P \to P$ is always an (A, C)-bicomodule homomorphism of degree 2. If it vanishes, b^P is called a *flat connection* (a differential) on P.

Equivalently, bicomodules with flat connections are bicomodules which live in the category of differential graded spans. The set of A-C-bicomodules becomes the set of objects of a differential graded category A-C-bicomod. For differential graded bicomodules P, Q, the k-th component of the graded $\Bbbk$-module A-C-bicomod(P, Q) consists of $(\mathrm{id}_A, \mathrm{id}_C)$-bicomodule homomorphisms $t : P \to Q$ of degree k. The differential of t is the commutator $tm_1 = tb^Q - (-)^t b^P t : P \to Q$, which is again a homomorphism of bicomodules, naturally of degree $k + 1$. Composition of homomorphisms of bicomodules is the ordinary composition of morphisms of graded spans.

The main example of a bicomodule is the following. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ be graded quivers. Let $\mathcal{P}$, $\mathcal{Q}$ be graded spans with $\operatorname{Ob}_s \mathcal{P} = \operatorname{Ob} \mathcal{A}$, $\operatorname{Ob}_t \mathcal{P} = \operatorname{Ob} \mathcal{C}$, $\operatorname{Ob}_s \mathcal{Q} = \operatorname{Ob} \mathcal{B}$, $\operatorname{Ob}_t \mathcal{Q} = \operatorname{Ob} \mathcal{D}$. Take graded coalgebras $A = Ts\mathcal{A}$, $B = Ts\mathcal{B}$, $C = Ts\mathcal{C}$, $D = Ts\mathcal{D}$ and bicomodules $P = Ts\mathcal{A} \otimes s\mathcal{P} \otimes Ts\mathcal{C}$, $Q = Ts\mathcal{B} \otimes s\mathcal{Q} \otimes Ts\mathcal{D}$ equipped with the cut comultiplications (coactions)

$$\begin{aligned}
\Delta_0(a_1, \dots, a_n) &= \sum_{i=0}^{n} (a_1, \dots, a_i) \otimes (a_{i+1}, \dots, a_n), \\
\delta(a_1, \dots, a_k, p, c_{k+1}, \dots, c_{k+l}) &= \sum_{i=0}^{k} (a_1, \dots, a_i) \otimes (a_{i+1}, \dots, p, \dots, c_{k+l}) \\
&+ \sum_{i=k}^{k+l} (a_1, \dots, p, \dots, c_i) \otimes (c_{i+1}, \dots, c_{k+l}).
\end{aligned}$$

Let $\phi : Ts\mathcal{A} \to Ts\mathcal{B}$, $\psi : Ts\mathcal{C} \to Ts\mathcal{D}$ be morphisms of augmented graded coalgebras. Let $g : P \to Q$ be a morphism of graded spans of certain degree with $\operatorname{Ob}_s g = \operatorname{Ob} \phi$, $\operatorname{Ob}_t g = \operatorname{Ob} \psi$. Define matrix coefficients of g to be

$$\begin{aligned}
g_{kl;mn} = (\mathrm{in}_k \otimes 1 \otimes \mathrm{in}_l) \cdot g \cdot (\mathrm{pr}_m \otimes 1 \otimes \mathrm{pr}_n) :& \\
T^k s\mathcal{A} \otimes s\mathcal{P} \otimes T^l s\mathcal{C} \to T^m s\mathcal{B} \otimes s\mathcal{Q} \otimes T^n s\mathcal{D},& \quad k, l, m, n \geqslant 0.
\end{aligned}$$

Coefficients $g_{kl;00} : T^k s\mathcal{A} \otimes s\mathcal{P} \otimes T^l s\mathcal{C} \to s\mathcal{Q}$ are abbreviated to g_{kl} and called components of g. Denote by $\check{g}$ the composite $g \cdot (\mathrm{pr}_0 \otimes 1 \otimes \mathrm{pr}_0) : Ts\mathcal{A} \otimes s\mathcal{P} \otimes Ts\mathcal{C} \to s\mathcal{Q}$. The restriction of $\check{g}$ to the summand $T^k s\mathcal{A} \otimes s\mathcal{P} \otimes T^l s\mathcal{C}$ is precisely the component g_{kl}.

Let $f : P \to Q$ be a (ϕ, ψ)-bicomodule homomorphism. It is uniquely recovered from its components similarly to Tradler [**59**, Lemma 4.2]. Let us supply the details. The coaction δ^P has two components,

$$\begin{aligned}
\delta' = \Delta_0 \otimes 1 \otimes 1 &: Ts\mathcal{A} \otimes s\mathcal{P} \otimes Ts\mathcal{C} \to Ts\mathcal{A} \otimes Ts\mathcal{A} \otimes s\mathcal{P} \otimes Ts\mathcal{C}, \\
\delta'' = 1 \otimes 1 \otimes \Delta_0 &: Ts\mathcal{A} \otimes s\mathcal{P} \otimes Ts\mathcal{C} \to Ts\mathcal{A} \otimes s\mathcal{P} \otimes Ts\mathcal{C} \otimes Ts\mathcal{C},
\end{aligned}$$

and similarly for δ^Q. As f is a (ϕ, ψ)-bicomodule homomorphism, it satisfies the equations

$$\begin{aligned}
f(\Delta_0 \otimes 1 \otimes 1) = (\Delta_0 \otimes 1 \otimes 1)(\phi \otimes f) &: Ts\mathcal{A} \otimes s\mathcal{P} \otimes Ts\mathcal{C} \to Ts\mathcal{B} \otimes Ts\mathcal{B} \otimes s\mathcal{Q} \otimes Ts\mathcal{D}, \\
f(1 \otimes 1 \otimes \Delta_0) = (1 \otimes 1 \otimes \Delta_0)(f \otimes \psi) &: Ts\mathcal{A} \otimes s\mathcal{P} \otimes Ts\mathcal{C} \to Ts\mathcal{B} \otimes s\mathcal{Q} \otimes Ts\mathcal{D} \otimes Ts\mathcal{D}.
\end{aligned}$$

It follows that

$$f(\Delta_0 \otimes 1 \otimes \Delta_0) = (\Delta_0 \otimes 1 \otimes \Delta_0)(\phi \otimes f \otimes \psi) : \\ Ts\mathcal{A} \otimes s\mathcal{P} \otimes Ts\mathcal{C} \to Ts\mathcal{B} \otimes Ts\mathcal{B} \otimes s\mathcal{Q} \otimes Ts\mathcal{D} \otimes Ts\mathcal{D}.$$

Composing both sides with the morphism

$$1 \otimes \operatorname{pr}_0 \otimes 1 \otimes \operatorname{pr}_0 \otimes 1 : Ts\mathcal{B} \otimes Ts\mathcal{B} \otimes s\mathcal{Q} \otimes Ts\mathcal{D} \otimes Ts\mathcal{D} \to Ts\mathcal{B} \otimes s\mathcal{Q} \otimes Ts\mathcal{D}, \qquad (4.1.1)$$

and taking into account the identities $\Delta_0(1 \otimes \operatorname{pr}_0) = 1$, $\Delta_0(\operatorname{pr}_0 \otimes 1) = 1$, we obtain

$$f = (\Delta_0 \otimes 1 \otimes \Delta_0)(\phi \otimes \check{f} \otimes \psi). \qquad (4.1.2)$$

This equation implies the following formulas for matrix coefficients of f:

$$f_{kl;mn} = \sum_{\substack{i_1+\dots+i_m+p=k \\ j_1+\dots+j_n+q=l}} (\phi_{i_1} \otimes \dots \otimes \phi_{i_m} \otimes f_{pq} \otimes \psi_{j_1} \otimes \dots \otimes \psi_{j_n}) : \\ T^k s\mathcal{A} \otimes s\mathcal{P} \otimes T^l s\mathcal{C} \to T^m s\mathcal{B} \otimes s\mathcal{Q} \otimes T^n s\mathcal{D}, \quad k, l, m, n \geqslant 0, \qquad (4.1.3)$$

see also (3.2.9) for the formula of matrix coefficients of augmented graded coalgebra morphisms ϕ and ψ. In particular, if $k < m$ or $l < n$, the matrix coefficient $f_{kl;mn}$ vanishes.

Let $r : P \to Q$ be a $(\phi, \psi, f, \chi, \xi)$-connection. It satisfies the following equations:

$$r(\Delta_0 \otimes 1 \otimes 1) = (\Delta_0 \otimes 1 \otimes 1)(\phi \otimes r + \chi \otimes f) : \\ Ts\mathcal{A} \otimes s\mathcal{P} \otimes Ts\mathcal{C} \to Ts\mathcal{B} \otimes Ts\mathcal{B} \otimes s\mathcal{Q} \otimes Ts\mathcal{D},$$

$$r(1 \otimes 1 \otimes \Delta_0) = (1 \otimes 1 \otimes \Delta_0)(f \otimes \xi + r \otimes \psi) : \\ Ts\mathcal{A} \otimes s\mathcal{P} \otimes Ts\mathcal{C} \to Ts\mathcal{B} \otimes s\mathcal{Q} \otimes Ts\mathcal{D} \otimes Ts\mathcal{D}.$$

They imply that

$$r(\Delta_0 \otimes 1 \otimes \Delta_0) = (\Delta_0 \otimes 1 \otimes \Delta_0)(\phi \otimes f \otimes \xi + \phi \otimes r \otimes \psi + \chi \otimes f \otimes \psi) : \\ Ts\mathcal{A} \otimes s\mathcal{P} \otimes Ts\mathcal{C} \to Ts\mathcal{B} \otimes Ts\mathcal{B} \otimes s\mathcal{Q} \otimes Ts\mathcal{D} \otimes Ts\mathcal{D}.$$

Composing both side with morphism (4.1.1) we obtain

$$r = (\Delta_0 \otimes 1 \otimes \Delta_0)(\phi \otimes \check{f} \otimes \xi + \phi \otimes \check{r} \otimes \psi + \chi \otimes \check{f} \otimes \psi). \qquad (4.1.4)$$

From this equation we find the following expression for the matrix coefficient $r_{kl;mn}$:

$$\sum_{\substack{i_1+\dots+i_m+i=k \\ j+j_1+\dots+j_p+t+j_{p+1}+\dots+j_{p+q}=l}}^{p+1+q=n} \phi_{i_1} \otimes \dots \otimes \phi_{i_m} \otimes f_{ij} \otimes \psi_{j_1} \otimes \dots \otimes \psi_{j_p} \otimes \xi_t \otimes \psi_{j_{p+1}} \otimes \dots \otimes \psi_{j_{p+q}} \\ + \sum_{\substack{i_1+\dots+i_m+i=k \\ j+j_1+\dots+j_n=l}} \phi_{i_1} \otimes \dots \otimes \phi_{i_m} \otimes r_{ij} \otimes \psi_{j_1} \otimes \dots \otimes \psi_{j_n} \\ + \sum_{\substack{i_1+\dots+i_a+u+i_{a+1}+\dots+i_{a+c}+i=k \\ j+j_1+\dots+j_n=l}}^{a+1+c=m} \phi_{i_1} \otimes \dots \otimes \phi_{i_a} \otimes \chi_u \otimes \phi_{i_{a+1}} \otimes \dots \otimes \phi_{i_{a+c}} \otimes f_{ij} \otimes \psi_{j_1} \otimes \dots \otimes \psi_{j_n} : \\ T^k s\mathcal{A} \otimes s\mathcal{P} \otimes T^l s\mathcal{C} \to T^m s\mathcal{B} \otimes s\mathcal{Q} \otimes T^n s\mathcal{D}, \quad k, l, m, n \geqslant 0, \qquad (4.1.5)$$

see also (3.2.12) for the formula of matrix coefficients of coderivations ξ and χ.

Let $\mathcal{A}$, $\mathcal{C}$ be A_∞-categories and let $\mathcal{P}$ be a graded span with $\operatorname{Ob}_s \mathcal{P} = \operatorname{Ob}\mathcal{A}$ and $\operatorname{Ob}_t \mathcal{P} = \operatorname{Ob}\mathcal{C}$. Let $A = Ts\mathcal{A}$, $C = Ts\mathcal{C}$, and consider the bicomodule $P = Ts\mathcal{A}\otimes s\mathcal{P}\otimes Ts\mathcal{C}$. The set of $(1,1,1,b^{\mathcal{A}},b^{\mathcal{C}})$-connections $b^{\mathcal{P}} : P \to P$ of degree 1 with $(b^{\mathcal{P}}_{00})^2 = 0$ is in bijection with the set of morphisms of augmented graded coalgebras $\phi^{\mathcal{P}} : Ts\mathcal{A}^{\mathrm{op}} \boxtimes Ts\mathcal{C} \to Ts\underline{\mathsf{C}}_{\Bbbk}$. Indeed, collections of complexes $(\phi^{\mathcal{P}}(X,Y),d)^{X\in\operatorname{Ob}\mathcal{A}}_{Y\in\operatorname{Ob}\mathcal{C}}$ are identified with the differential graded spans $(\mathcal{P}, sb^{\mathcal{P}}_{00}s^{-1})$. In particular, for each pair of objects $X \in \operatorname{Ob}\mathcal{A}$, $Y \in \operatorname{Ob}\mathcal{C}$ holds $(\phi^{\mathcal{P}}(X,Y))[1] = (s\mathcal{P}(X,Y), -b^{\mathcal{P}}_{00})$. The components $b^{\mathcal{P}}_{kn}$ and $\phi^{\mathcal{P}}_{kn}$ are related for $(k,n) \neq (0,0)$ by the formula

$$
\begin{aligned}
b^{\mathcal{P}}_{kn} = \bigl[& s\mathcal{A}(X_k,X_{k-1}) \otimes \dots \otimes s\mathcal{A}(X_1,X_0) \otimes s\mathcal{P}(X_0,Y_0) \otimes s\mathcal{C}(Y_0,Y_1) \otimes \dots \otimes s\mathcal{C}(Y_{n-1},Y_n) \\
& \xrightarrow{\tilde{\gamma}\otimes 1^{\otimes n}} s\mathcal{P}(X_0,Y_0) \otimes s\mathcal{A}^{\mathrm{op}}(X_0,X_1) \otimes \dots \otimes s\mathcal{A}^{\mathrm{op}}(X_{k-1},X_k) \\
& \qquad\qquad\qquad\qquad\qquad\qquad \otimes s\mathcal{C}(Y_0,Y_1) \otimes \dots \otimes s\mathcal{C}(Y_{n-1},Y_n) \\
& \xrightarrow{1\otimes\phi^{\mathcal{P}}_{kn}} s\mathcal{P}(X_0,Y_0) \otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(X_0,Y_0),\mathcal{P}(X_k,Y_n)) \\
& \xrightarrow{1\otimes s^{-1}} s\mathcal{P}(X_0,Y_0) \otimes \underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(X_0,Y_0),\mathcal{P}(X_k,Y_n)) \\
& \qquad \xrightarrow{1\otimes[1]} s\mathcal{P}(X_0,Y_0) \otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(X_0,Y_0),s\mathcal{P}(X_k,Y_n)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_{\Bbbk}}} s\mathcal{P}(X_k,Y_n) \bigr],
\end{aligned}
$$

where $\tilde{\gamma} = (12\dots k+1)\cdot\gamma$, the anti-isomorphism γ is defined by (3.5.1), and the shift differential graded functor $[1] : \underline{\mathsf{C}}_{\Bbbk} \to \underline{\mathsf{C}}_{\Bbbk}$ is defined in Example 3.2.22.

Components of $b^{\mathcal{P}}$ can be written in a more concise form. Given objects $X, Y \in \operatorname{Ob}\mathcal{A}$, $Z, W \in \operatorname{Ob}\mathcal{C}$, define

$$
\begin{aligned}
\check{b}^{\mathcal{P}}_{+} = \bigl[& Ts\mathcal{A}(Y,X) \otimes s\mathcal{P}(X,Z) \otimes Ts\mathcal{C}(Z,W) \\
& \xrightarrow{c\otimes 1} s\mathcal{P}(X,Z) \otimes Ts\mathcal{A}(Y,X) \otimes Ts\mathcal{C}(Z,W) \\
& \xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}(X,Z) \otimes Ts\mathcal{A}^{\mathrm{op}}(X,Y) \otimes Ts\mathcal{C}(Z,W) \\
& \xrightarrow{1\otimes\check{\phi}^{\mathcal{P}}} s\mathcal{P}(X,Z) \otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(X,Z),\mathcal{P}(Y,W)) \\
& \qquad \xrightarrow{(s\otimes s)^{-1}} \mathcal{P}(X,Z) \otimes \underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(X,Z),\mathcal{P}(Y,W)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_{\Bbbk}}} \mathcal{P}(Y,W) \xrightarrow{s} s\mathcal{P}(Y,W) \bigr] \\
= \bigl[& Ts\mathcal{A}(Y,X) \otimes s\mathcal{P}(X,Z) \otimes Ts\mathcal{C}(Z,W) \\
& \xrightarrow{c\otimes 1} s\mathcal{P}(X,Z) \otimes Ts\mathcal{A}(Y,X) \otimes Ts\mathcal{C}(Z,W) \\
& \xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}(X,Z) \otimes Ts\mathcal{A}^{\mathrm{op}}(X,Y) \otimes Ts\mathcal{C}(Z,W) \\
& \xrightarrow{1\otimes\check{\phi}^{\mathcal{P}}} s\mathcal{P}(X,Z) \otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(X,Z),\mathcal{P}(Y,W)) \\
& \qquad \xrightarrow{1\otimes s^{-1}[1]} s\mathcal{P}(X,Z) \otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(X,Z),s\mathcal{P}(Y,W)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_{\Bbbk}}} s\mathcal{P}(Y,W) \bigr], \quad (4.1.6)
\end{aligned}
$$

where $\gamma : Ts\mathcal{A} \to Ts\mathcal{A}^{\mathrm{op}}$ is anti-isomorphism (3.5.1), and $\check{\phi}^{\mathcal{P}}$ denotes, as usual, the composite $\phi^{\mathcal{P}}\operatorname{pr}_1 : Ts\mathcal{A}^{\mathrm{op}} \boxtimes Ts\mathcal{C} \to s\underline{\mathsf{C}}_{\Bbbk}$. Conversely, components of the A_∞-functor $\phi^{\mathcal{P}}$ can be found as

$$
\begin{aligned}
\check{\phi}^{\mathcal{P}} = \bigl[& Ts\mathcal{A}^{\mathrm{op}}(X,Y) \otimes Ts\mathcal{C}(Z,W) \xrightarrow{\gamma\otimes 1} Ts\mathcal{A}(Y,X) \otimes Ts\mathcal{C}(Z,W) \\
& \xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(X,Z), s\mathcal{P}(X,Z) \otimes Ts\mathcal{A}(Y,X) \otimes Ts\mathcal{C}(Z,W)) \\
& \qquad \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,(c\otimes 1)\check{b}^{\mathcal{P}}_{+})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(X,Z), s\mathcal{P}(Y,W)) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(X,Z),\mathcal{P}(Y,W)) \bigr]. \quad (4.1.7)
\end{aligned}
$$

Define also

$$\check{b}^{\mathcal{P}}_0 = \bigl[Ts\mathcal{A}(Y,X) \otimes s\mathcal{P}(X,Z) \otimes Ts\mathcal{C}(Z,W) \xrightarrow{\mathrm{pr}_0 \otimes 1 \otimes \mathrm{pr}_0} s\mathcal{P}(X,Z) \xrightarrow{b^{\mathcal{P}}_{00}} s\mathcal{P}(X,Z)\bigr]. \qquad (4.1.8)$$

Note that $\check{b}^{\mathcal{P}}_+$ vanishes on $T^0s\mathcal{A}(Y,X) \otimes s\mathcal{P}(X,Z) \otimes T^0s\mathcal{C}(Z,W)$ since $\check{\phi}^{\mathcal{P}}$ vanishes on $T^0s\mathcal{A}^{\mathrm{op}}(X,Y) \otimes T^0s\mathcal{C}(Z,W)$. It follows that $\check{b}^{\mathcal{P}} = \check{b}^{\mathcal{P}}_+ + \check{b}^{\mathcal{P}}_0$.

The following statement was proven by Lefèvre-Hasegawa in assumption that the ground ring is a field [**34**, Lemme 5.3.0.1].

4.1.2. Proposition. *The connection $b^{\mathcal{P}}$ is flat, that is, $(Ts\mathcal{A} \otimes s\mathcal{P} \otimes Ts\mathcal{C}, b^{\mathcal{P}})$ is a bicomodule in ${}^d\mathscr{Q}$, if and only if the corresponding augmented coalgebra homomorphism $\phi^{\mathcal{P}} : Ts\mathcal{A}^{\mathrm{op}} \boxtimes Ts\mathcal{C} \to Ts\underline{\mathsf{C}}_\Bbbk$ is an A_∞-functor.*

The reader is advised to skip the proof on the first reading.

Proof. According to (4.1.4),

$$\begin{aligned} b^{\mathcal{P}} &= (\Delta_0 \otimes 1 \otimes \Delta_0)(1 \otimes \mathrm{pr}_0 \otimes 1 \otimes \mathrm{pr}_0 \otimes b^{\mathcal{C}} + 1 \otimes \check{b}^{\mathcal{P}} \otimes 1 + b^{\mathcal{A}} \otimes \mathrm{pr}_0 \otimes 1 \otimes \mathrm{pr}_0 \otimes 1) \\ &= 1 \otimes 1 \otimes b^{\mathcal{C}} + (\Delta_0 \otimes 1 \otimes \Delta_0)(1 \otimes \check{b}^{\mathcal{P}} \otimes 1) + b^{\mathcal{A}} \otimes 1 \otimes 1. \end{aligned}$$

The morphism of graded spans $(b^{\mathcal{P}})^2 : P \to P$ is a $(1,1,1,0,0)$-connection of degree 2, therefore the equation $(b^{\mathcal{P}})^2 = 0$ is equivalent to its particular case $(b^{\mathcal{P}})^2(\mathrm{pr}_0 \otimes 1 \otimes \mathrm{pr}_0) = 0 : Ts\mathcal{A} \otimes s\mathcal{P} \otimes Ts\mathcal{C} \to s\mathcal{P}$. In terms of $\check{b}^{\mathcal{P}}$, the latter reads as follows:

$$(b^{\mathcal{A}} \otimes 1 \otimes 1 + 1 \otimes 1 \otimes b^{\mathcal{C}})\check{b}^{\mathcal{P}} + (\Delta_0 \otimes 1 \otimes \Delta_0)(1 \otimes \check{b}^{\mathcal{P}} \otimes 1)\check{b}^{\mathcal{P}} = 0. \qquad (4.1.9)$$

Note that $(b^{\mathcal{A}} \otimes 1 \otimes 1 + 1 \otimes 1 \otimes b^{\mathcal{C}})\check{b}^{\mathcal{P}} = (b^{\mathcal{A}} \otimes 1 \otimes 1 + 1 \otimes 1 \otimes b^{\mathcal{C}})\check{b}^{\mathcal{P}}_+$ since $b^{\mathcal{A}}\mathrm{pr}_0 = 0$, $b^{\mathcal{C}}\mathrm{pr}_0 = 0$. The second term in the above equation splits into a sum of four summands, which we are going to compute separately. First of all,

$$\begin{aligned} (\Delta_0 \otimes 1 \otimes \Delta_0)(1 \otimes \check{b}^{\mathcal{P}}_0 \otimes 1)\check{b}^{\mathcal{P}}_0 &= (\Delta_0(\mathrm{pr}_0 \otimes \mathrm{pr}_0) \otimes 1 \otimes \Delta(\mathrm{pr}_0 \otimes \mathrm{pr}_0))(b^{\mathcal{P}}_{00})^2 \\ &= (\mathrm{pr}_0 \otimes 1 \otimes \mathrm{pr}_0)s^{-1}d^2s = 0 : Ts\mathcal{A} \otimes s\mathcal{P} \otimes Ts\mathcal{C} \to s\mathcal{P}. \end{aligned}$$

Secondly,

$$\begin{aligned} &(\Delta_0 \otimes 1 \otimes \Delta_0)(1 \otimes \check{b}^{\mathcal{P}}_+ \otimes 1)\check{b}^{\mathcal{P}}_0 + (\Delta_0 \otimes 1 \otimes \Delta_0)(1 \otimes \check{b}^{\mathcal{P}}_0 \otimes 1)\check{b}^{\mathcal{P}}_+ \\ &= (\Delta_0(\mathrm{pr}_0 \otimes 1) \otimes 1 \otimes \Delta_0(1 \otimes \mathrm{pr}_0))\check{b}^{\mathcal{P}}_+ b^{\mathcal{P}}_{00} + (\Delta_0(1 \otimes \mathrm{pr}_0) \otimes b^{\mathcal{P}}_{00} \otimes \Delta(\mathrm{pr}_0 \otimes 1))\check{b}^{\mathcal{P}}_+ \\ &= \check{b}^{\mathcal{P}}_+ b^{\mathcal{P}}_{00} + (1 \otimes b^{\mathcal{P}}_{00} \otimes 1)\check{b}^{\mathcal{P}}_+ \\ &= \bigl[Ts\mathcal{A}(X,Y) \otimes s\mathcal{P}(Y,Z) \otimes Ts\mathcal{C}(Z,W) \\ &\qquad \xrightarrow{c \otimes 1} s\mathcal{P}(Y,Z) \otimes Ts\mathcal{A}(X,Y) \otimes Ts\mathcal{C}(Z,W) \\ &\qquad \xrightarrow{1 \otimes \gamma \otimes 1} s\mathcal{P}(Y,Z) \otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X) \otimes Ts\mathcal{C}(Z,W) \\ &\qquad \xrightarrow{1 \otimes \check{\phi}^{\mathcal{P}}} s\mathcal{P}(Y,Z) \otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(Y,Z), \mathcal{P}(X,W)) \\ &\qquad \xrightarrow{1 \otimes s^{-1}} s\mathcal{P}(Y,Z) \otimes \underline{\mathsf{C}}_\Bbbk(\mathcal{P}(Y,Z), \mathcal{P}(X,W)) \\ &\qquad\quad \xrightarrow{1 \otimes [1]} s\mathcal{P}(Y,Z) \otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{P}(Y,Z), s\mathcal{P}(X,W)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}} s\mathcal{P}(X,W) \xrightarrow{b^{\mathcal{P}}_{00}} s\mathcal{P}(X,W)\bigr] \\ &+ \bigl[Ts\mathcal{A}(X,Y) \otimes s\mathcal{P}(Y,Z) \otimes Ts\mathcal{C}(Z,W) \\ &\qquad \xrightarrow{1 \otimes b^{\mathcal{P}}_{00} \otimes 1} Ts\mathcal{A}(X,Y) \otimes s\mathcal{P}(Y,Z) \otimes Ts\mathcal{C}(Z,W) \\ &\qquad \xrightarrow{c \otimes 1} s\mathcal{P}(Y,Z) \otimes Ts\mathcal{A}(X,Y) \otimes Ts\mathcal{C}(Z,W) \end{aligned}$$

$$
\begin{aligned}
&\xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\check{\phi}^{\mathcal{P}}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{P}(X,W))\\
&\xrightarrow{1\otimes s^{-1}} s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{P}(X,W))\\
&\qquad\qquad\xrightarrow{1\otimes[1]} s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{P}(X,W))\xrightarrow{\mathrm{ev}^{\underline{\mathsf{C}}_{\Bbbk}}} s\mathcal{P}(X,W)\big]\\
={}&\big[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{c\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\check{\phi}^{\mathcal{P}}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{P}(X,W))\\
&\xrightarrow{1\otimes s^{-1}} s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{P}(X,W))\\
&\quad\xrightarrow{1\otimes[1]} s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{P}(X,W))\xrightarrow{\mathrm{ev}^{\underline{\mathsf{C}}_{\Bbbk}}\,b_{00}^{\mathcal{P}}-(b_{00}^{\mathcal{P}}\otimes 1)\,\mathrm{ev}^{\underline{\mathsf{C}}_{\Bbbk}}} s\mathcal{P}(X,W)\big].
\end{aligned}
$$

The complexes $s\mathcal{P}(Y,Z)$ and $s\mathcal{P}(X,W)$ carry the differential $-b_{00}^{\mathcal{P}}$. Since $\mathrm{ev}^{\underline{\mathsf{C}}_{\Bbbk}}$ is a chain map, it follows that $\mathrm{ev}^{\underline{\mathsf{C}}_{\Bbbk}}\,b_{00}^{\mathcal{P}}-(b_{00}^{\mathcal{P}}\otimes 1)\,\mathrm{ev}^{\underline{\mathsf{C}}_{\Bbbk}}=-(1\otimes m_1^{\underline{\mathsf{C}}_{\Bbbk}})\,\mathrm{ev}^{\underline{\mathsf{C}}_{\Bbbk}}$, therefore

$$
\begin{aligned}
&(\Delta_0\otimes 1\otimes\Delta_0)(1\otimes\check{b}_+^{\mathcal{P}}\otimes 1)\check{b}_0^{\mathcal{P}}+(\Delta_0\otimes 1\otimes\Delta_0)(1\otimes\check{b}_0^{\mathcal{P}}\otimes 1)\check{b}_+^{\mathcal{P}}\\
&=-\big[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)\\
&\qquad\xrightarrow{c\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)\\
&\qquad\xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\\
&\qquad\xrightarrow{1\otimes\check{\phi}^{\mathcal{P}}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{P}(X,W))\\
&\qquad\xrightarrow{1\otimes s^{-1}} s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{P}(X,W))\\
&\qquad\qquad\qquad\xrightarrow{1\otimes[1]} s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{P}(X,W))\xrightarrow{(1\otimes m_1^{\underline{\mathsf{C}}_{\Bbbk}})\,\mathrm{ev}^{\underline{\mathsf{C}}_{\Bbbk}}} s\mathcal{P}(X,W)\big].
\end{aligned}
$$

Since $[1]$ is a differential graded functor, it follows that $[1]m_1^{\underline{\mathsf{C}}_{\Bbbk}}=m_1^{\underline{\mathsf{C}}_{\Bbbk}}[1]$. Together with the relation $b_1^{\underline{\mathsf{C}}_{\Bbbk}}s^{-1}=s^{-1}m_1^{\underline{\mathsf{C}}_{\Bbbk}}$ this implies that

$$
\begin{aligned}
&(\Delta_0\otimes 1\otimes\Delta_0)(1\otimes\check{b}_+^{\mathcal{P}}\otimes 1)\check{b}_0^{\mathcal{P}}+(\Delta_0\otimes 1\otimes\Delta_0)(1\otimes\check{b}_0^{\mathcal{P}}\otimes 1)\check{b}_+^{\mathcal{P}}\\
&=-\big[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)\\
&\qquad\xrightarrow{c\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)\\
&\qquad\xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\\
&\qquad\xrightarrow{1\otimes\check{\phi}^{\mathcal{P}}\,b_1^{\underline{\mathsf{C}}_{\Bbbk}}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{P}(X,W))\\
&\qquad\xrightarrow{1\otimes s^{-1}} s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{P}(X,W))\\
&\qquad\qquad\qquad\qquad\xrightarrow{1\otimes[1]} s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{P}(X,W))\xrightarrow{\mathrm{ev}^{\underline{\mathsf{C}}_{\Bbbk}}} s\mathcal{P}(X,W)\big].
\end{aligned}
$$

Next, let us compute

$$
\begin{aligned}
&(\Delta_0\otimes 1\otimes\Delta_0)(1\otimes\check{b}^{\mathcal{P}}_+\otimes 1)\check{b}^{\mathcal{P}}_+ = \bigl[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{\Delta_0\otimes 1\otimes\Delta_0} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes Ts\mathcal{A}(U,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes c\otimes 1\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(U,Y)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes 1\otimes\gamma\otimes 1\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes 1\otimes\check{\phi}^{\mathcal{P}}\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(Y,Z),\mathcal{P}(U,V))\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes 1\otimes s^{-1}\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_\Bbbk(\mathcal{P}(Y,Z),\mathcal{P}(U,V))\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes 1\otimes[1]\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{P}(Y,Z),s\mathcal{P}(U,V))\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes\mathrm{ev}^{\underline{\mathsf{C}}_\Bbbk}\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(U,V)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum c\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(U,V)\otimes Ts\mathcal{A}(X,U)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes\gamma\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(U,V)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes\check{\phi}^{\mathcal{P}}} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(U,V)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(U,V),\mathcal{P}(X,W))\\
&\xrightarrow{\sum 1\otimes s^{-1}} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(U,V)\otimes \underline{\mathsf{C}}_\Bbbk(\mathcal{P}(U,V),\mathcal{P}(X,W))\\
&\qquad\xrightarrow{\sum 1\otimes[1]} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(U,V)\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{P}(U,V),s\mathcal{P}(X,W)) \xrightarrow{\sum\mathrm{ev}^{\underline{\mathsf{C}}_\Bbbk}} s\mathcal{P}(X,W)\bigr].
\end{aligned}
$$

The latter can be written as

$$
\begin{aligned}
&\bigl[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W) \xrightarrow{c\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\Delta_0 c(\gamma\otimes\gamma)\otimes\Delta_0} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\\
&\hphantom{\xrightarrow{1\otimes\Delta_0 c(\gamma\otimes\gamma)\otimes\Delta_0} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)}\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes 1\otimes c\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes\check{\phi}^{\mathcal{P}}\otimes\check{\phi}^{\mathcal{P}}} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(Y,Z),\mathcal{P}(U,V))\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(U,V),\mathcal{P}(X,W))
\end{aligned}
$$

$$\begin{aligned}
&\xrightarrow{\sum 1\otimes s^{-1}\otimes s^{-1}} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{P}(U,V))\otimes \underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(U,V),\mathcal{P}(X,W))\\
&\xrightarrow{\sum 1\otimes[1]\otimes[1]} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{P}(U,V))\otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(U,V),s\mathcal{P}(X,W))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\xrightarrow{\sum(\operatorname{ev}^{\mathsf{C}_\Bbbk}\otimes 1)\operatorname{ev}^{\mathsf{C}_\Bbbk}} s\mathcal{P}(X,W)\big].
\end{aligned}$$

Using the identities $\Delta_0 c(\gamma\otimes\gamma)=\gamma\Delta_0$ (see (3.5.2)), $(\operatorname{ev}^{\mathsf{C}_\Bbbk}\otimes 1)\operatorname{ev}^{\mathsf{C}_\Bbbk}=(1\otimes m_2^{\mathsf{C}_\Bbbk})\operatorname{ev}^{\mathsf{C}_\Bbbk}$, and $(s^{-1}\otimes s^{-1})m_2^{\mathsf{C}_\Bbbk}=-b_2^{\mathsf{C}_\Bbbk}s^{-1}$, we transform the above expression as follows:

$$\begin{aligned}
&(\Delta_0\otimes 1\otimes\Delta_0)(1\otimes\check{b}^{\mathcal{P}}_{+}\otimes 1)\check{b}^{\mathcal{P}}_{+}=-\big[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{c\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\Delta_0\otimes\Delta_0} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes 1\otimes c\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes\check{\phi}^{\mathcal{P}}\otimes\check{\phi}^{\mathcal{P}}} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{P}(U,V))\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(U,V),\mathcal{P}(X,W))\\
&\xrightarrow{\sum 1\otimes b_2^{\mathsf{C}_\Bbbk}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{P}(X,W))\\
&\xrightarrow{1\otimes s^{-1}} s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{P}(X,W))\\
&\qquad\qquad\qquad\qquad\xrightarrow{1\otimes[1]} s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{P}(X,W))\xrightarrow{\operatorname{ev}^{\mathsf{C}_\Bbbk}} s\mathcal{P}(X,W)\big].
\end{aligned}$$

Finally,

$$\begin{aligned}
&(b^{\mathcal{A}}\otimes 1\otimes 1+1\otimes 1\otimes b^{\mathcal{C}})\check{b}^{\mathcal{P}}_{+}=\big[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)\\
&\quad\xrightarrow{c\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)\\
&\quad\xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\\
&\quad\xrightarrow{1\otimes(b^{\mathcal{A}^{\mathrm{op}}}\otimes 1+1\otimes b^{\mathcal{C}})\check{\phi}^{\mathcal{P}}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{P}(X,W))\\
&\qquad\qquad\quad\xrightarrow{(s\otimes s)^{-1}} \mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{P}(X,W))\xrightarrow{\operatorname{ev}^{\mathsf{C}_\Bbbk}} \mathcal{P}(X,W)\xrightarrow{s} s\mathcal{P}(X,W)\big]
\end{aligned}$$

since $b^{\mathcal{A}^{\mathrm{op}}}=\gamma b^{\mathcal{A}}\gamma : Ts\mathcal{A}^{\mathrm{op}}(Y,X)\to Ts\mathcal{A}^{\mathrm{op}}(Y,X)$. We conclude that the left hand side of (4.1.9) equals $(c\otimes 1)(1\otimes\gamma\otimes 1)(1\otimes R)(s\otimes s)^{-1}\operatorname{ev}^{\mathsf{C}_\Bbbk}s^{-1}$, where

$$\begin{aligned}
R=\big[&Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\xrightarrow{b^{\mathcal{A}^{\mathrm{op}}}\otimes 1+1\otimes b^{\mathcal{C}}}\\
&\qquad\qquad\qquad Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\xrightarrow{\check{\phi}^{\mathcal{P}}} s\underline{\mathsf{C}}_{\Bbbk}(\phi^{\mathcal{P}}(Y,Z),\phi^{\mathcal{P}}(X,W))\big]\\
&-\check{\phi}^{\mathcal{P}}b_1^{\mathsf{C}_\Bbbk}\\
&-\big[Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)
\end{aligned}$$

$$\xrightarrow{\Delta_0\otimes\Delta_0} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)$$

$$\xrightarrow{\sum 1\otimes c\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(V,W)$$

$$\xrightarrow{\sum\check{\phi}^{\mathcal{P}}\otimes\check{\phi}^{\mathcal{P}}} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\underline{\mathsf{C}}_\Bbbk(\phi^{\mathcal{P}}(Y,Z),\phi^{\mathcal{P}}(U,V))\otimes s\underline{\mathsf{C}}_\Bbbk(\phi^{\mathcal{P}}(U,V),\phi^{\mathcal{P}}(X,W))$$

$$\xrightarrow{\sum b_2^{\underline{\mathsf{C}}_\Bbbk}} s\underline{\mathsf{C}}_\Bbbk(\phi^{\mathcal{P}}(Y,Z),\phi^{\mathcal{P}}(X,W))\big].$$

By (3.2.6) and (3.2.17), it follows that R equals

$$(b^{\mathcal{A}^{\mathrm{op}}}\boxtimes 1+1\boxtimes b^{\mathcal{C}})\check{\phi}^{\mathcal{P}}-\phi^{\mathcal{P}}\check{b}^{\underline{\mathsf{C}}_\Bbbk} : Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\to s\underline{\mathsf{C}}_\Bbbk(\phi^{\mathcal{P}}(Y,Z),\phi^{\mathcal{P}}(X,W)).$$

By closedness, $b^{\mathcal{P}}$ is a flat connection if and only if $R=0$, for all objects $X,Y\in\operatorname{Ob}\mathcal{A}$, $Z,W\in\operatorname{Ob}\mathcal{C}$, that is, if $\phi^{\mathcal{P}}$ is an A_∞-functor. □

Let $\mathcal{A}$, $\mathcal{C}$ be A_∞-categories. The full subcategory of the differential graded category $Ts\mathcal{A}$-$Ts\mathcal{C}$-bicomod consisting of **dg**-bicomodules whose underlying graded bicomodule has the form $Ts\mathcal{A}\otimes s\mathcal{P}\otimes Ts\mathcal{C}$ is denoted by $\mathcal{A}$-$\mathcal{C}$-bimod. Its objects are called *A_∞-bimodules*, extending the terminology of Tradler [**59**].

4.1.3. Proposition. *The differential graded categories* $\mathcal{A}$-$\mathcal{C}$-bimod *and* $\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_\Bbbk)$ *are isomorphic.*

Proof. Proposition 4.1.2 establishes a bijection between the sets of objects of the differential graded categories $\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_\Bbbk)$ and $\mathcal{A}$-$\mathcal{C}$-bimod. Let us extend it to an isomorphism of differential graded categories. Let $\phi,\psi:\mathcal{A}^{\mathrm{op}},\mathcal{C}\to\underline{\mathsf{C}}_\Bbbk$ be A_∞-functors, $\mathcal{P}$, $\mathcal{Q}$ the corresponding $\mathcal{A}$-$\mathcal{C}$-bimodules. Define a $\Bbbk$-linear map

$$\Phi:\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi)\to\mathcal{A}\text{-}\mathcal{C}\text{-bimod}(\mathcal{P},\mathcal{Q})$$

of degree 0 as follows. With an element $rs^{-1}\in\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi)$ an $(\mathrm{id}_{Ts\mathcal{A}},\mathrm{id}_{Ts\mathcal{C}})$-bicomodule homomorphism $t=(rs^{-1})\Phi:Ts\mathcal{A}\otimes s\mathcal{P}\otimes Ts\mathcal{C}\to Ts\mathcal{A}\otimes s\mathcal{Q}\otimes Ts\mathcal{C}$ is associated given by its components

$$t_{kn}=(-)^{r+1}\big[s\mathcal{A}(X_k,X_{k-1})\otimes\cdots\otimes s\mathcal{A}(X_1,X_0)\otimes s\mathcal{P}(X_0,Y_0)\otimes s\mathcal{C}(Y_0,Y_1)\otimes\cdots\otimes s\mathcal{C}(Y_{n-1},Y_n)$$

$$\xrightarrow{\tilde{\gamma}\otimes 1^{\otimes n}} s\mathcal{P}(X_0,Y_0)\otimes s\mathcal{A}^{\mathrm{op}}(X_0,X_1)\otimes\cdots\otimes s\mathcal{A}^{\mathrm{op}}(X_{k-1},X_k)\otimes s\mathcal{C}(Y_0,Y_1)\otimes\cdots\otimes s\mathcal{C}(Y_{n-1},Y_n)$$

$$\xrightarrow{1\otimes r_{kn}} s\mathcal{P}(X_0,Y_0)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(X_0,Y_0),\mathcal{Q}(X_k,Y_n))$$

$$\xrightarrow{1\otimes s^{-1}[1]} s\mathcal{P}(X_0,Y_0)\otimes\underline{\mathsf{C}}_\Bbbk(s\mathcal{P}(X_0,Y_0),s\mathcal{Q}(X_k,Y_n))$$

$$\xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}} s\mathcal{Q}(X_k,Y_n)\big],\quad k,n\geqslant 0,$$

or more concisely,

$$\begin{aligned}
\check{t} = (-)^{r+1}\big[&Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W) \\
&\xrightarrow{c\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W) \\
&\xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W) \\
&\xrightarrow{1\otimes\check{r}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{Q}(X,W)) \\
&\qquad\qquad \xrightarrow{1\otimes s^{-1}[1]} s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{Q}(X,W)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_{\Bbbk}}} s\mathcal{Q}(X,W)\big],
\end{aligned}$$

where $\check{r} = r\cdot\mathrm{pr}_1 : Ts\mathcal{A}^{\mathrm{op}}\boxtimes Ts\mathcal{C}\to s\underline{\mathsf{C}}_{\Bbbk}$ is the morphism of graded spans that collects components of r, $\mathrm{Ob}_s\,\check{r} = \mathrm{Ob}\,\phi$, $\mathrm{Ob}_t\,\check{r} = \mathrm{Ob}\,\psi$. The closedness of **gr** implies that the map $\Phi : \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_{\Bbbk})(\phi,\psi)\to\mathcal{A}\text{-}\mathcal{C}\text{-bimod}(\mathcal{P},\mathcal{Q})$ is an isomorphism. Let us prove that it also commutes with the differential. We must prove that

$$((rs^{-1})\Phi)d = (rs^{-1}m_1^{\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_{\Bbbk})})\Phi = ((rB_1)s^{-1})\Phi,$$

for each element $r\in s\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_{\Bbbk})(\phi,\psi)$. Both sides of the equation are $(\mathrm{id}_{Ts\mathcal{A}},\mathrm{id}_{Ts\mathcal{C}})$-bimodule homomorphisms of degree $\deg r+1$, therefore it suffices to prove the equation $[((rs^{-1})\Phi)d]^\vee = [((rB_1)s^{-1})\Phi]^\vee$. Using (4.1.4), we obtain:

$$\begin{aligned}
[((rs^{-1})\Phi)d]^\vee = (td)^\vee &= (t\cdot b^{\mathcal{Q}})^\vee - (-)^t(b^{\mathcal{P}}\cdot t)^\vee = t\cdot\check{b}^{\mathcal{Q}} - (-)^t b^{\mathcal{P}}\cdot\check{t} \\
&= t\cdot\check{b}^{\mathcal{Q}}_+ && (4.1.10)\\
&- (-)^t(\Delta_0\otimes 1\otimes\Delta_0)(1\otimes\check{b}^{\mathcal{P}}_+\otimes 1)\check{t} && (4.1.11)\\
&+ t\cdot\check{b}^{\mathcal{Q}}_0 - (-)^t(\Delta_0\otimes 1\otimes\Delta_0)(1\otimes\check{b}^{\mathcal{P}}_0\otimes 1)\check{t} && (4.1.12)\\
&- (-)^t(b^{\mathcal{A}}\otimes 1\otimes 1 + 1\otimes 1\otimes b^{\mathcal{C}})\check{t}. && (4.1.13)
\end{aligned}$$

Let us compute summands (4.1.10)–(4.1.13) separately. According to (4.1.2), expression (4.1.10) equals

$$\begin{aligned}
t\cdot\check{b}^{\mathcal{Q}}_+ &= (\Delta_0\otimes 1\otimes\Delta_0)(1\otimes\check{t}\otimes 1)\check{b}^{\mathcal{Q}}_+ \\
&= (-)^{r+1}\big[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W) \\
&\xrightarrow{\Delta_0\otimes 1\otimes\Delta_0} \bigoplus_{U\in\mathrm{Ob}\,\mathcal{A},V\in\mathrm{Ob}\,\mathcal{C}} Ts\mathcal{A}(X,U)\otimes Ts\mathcal{A}(U,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W) \\
&\xrightarrow{\sum 1\otimes c\otimes 1\otimes 1} \bigoplus_{U\in\mathrm{Ob}\,\mathcal{A},V\in\mathrm{Ob}\,\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(U,Y)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W) \\
&\xrightarrow{\sum 1\otimes 1\otimes\gamma\otimes 1\otimes 1} \bigoplus_{U\in\mathrm{Ob}\,\mathcal{A},V\in\mathrm{Ob}\,\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W) \\
&\xrightarrow{\sum 1\otimes 1\otimes\check{r}\otimes 1} \bigoplus_{U\in\mathrm{Ob}\,\mathcal{A},V\in\mathrm{Ob}\,\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{Q}(U,V))\otimes Ts\mathcal{C}(V,W) \\
&\xrightarrow{\sum 1\otimes 1\otimes s^{-1}[1]\otimes 1} \bigoplus_{U\in\mathrm{Ob}\,\mathcal{A},V\in\mathrm{Ob}\,\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(Y,Z)\otimes\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{Q}(U,V))\otimes Ts\mathcal{C}(V,W) \\
&\xrightarrow{\sum 1\otimes\mathrm{ev}^{\mathsf{C}_{\Bbbk}}\otimes 1} \bigoplus_{U\in\mathrm{Ob}\,\mathcal{A},V\in\mathrm{Ob}\,\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{Q}(U,V)\otimes Ts\mathcal{C}(V,W)
\end{aligned}$$

$$\xrightarrow{\sum c\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{Q}(U,V)\otimes Ts\mathcal{A}(X,U)\otimes Ts\mathcal{C}(V,W)$$

$$\xrightarrow{\sum 1\otimes\gamma\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{Q}(U,V)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(V,W)$$

$$\xrightarrow{\sum 1\otimes\check{\psi}} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{Q}(U,V)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{Q}(U,V),\mathcal{Q}(X,W))$$

$$\xrightarrow{\sum 1\otimes s^{-1}[1]} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{Q}(U,V)\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{Q}(U,V),s\mathcal{Q}(X,W)) \xrightarrow{\sum \mathrm{ev}^{\mathsf{C}_\Bbbk}} s\mathcal{Q}(X,W)\big].$$

As in the proof of Proposition 4.1.2, the above composite can be transformed as follows:

$$(-)^{r+1}\big[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)$$

$$\xrightarrow{c\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)$$

$$\xrightarrow{1\otimes\Delta_0 c(\gamma\otimes\gamma)\otimes\Delta_0} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)$$

$$\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)$$

$$\xrightarrow{\sum 1\otimes 1\otimes c\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(V,W)$$

$$\xrightarrow{\sum 1\otimes\check{r}\otimes\check{\psi}} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(Y,Z),\mathcal{Q}(U,V))\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{Q}(U,V),\mathcal{Q}(X,W))$$

$$\xrightarrow{\sum 1\otimes s^{-1}[1]\otimes s^{-1}[1]} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{P}(Y,Z),s\mathcal{Q}(U,V))\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{Q}(U,V),s\mathcal{Q}(X,W))$$

$$\xrightarrow{\sum(\mathrm{ev}^{\mathsf{C}_\Bbbk}\otimes 1)\,\mathrm{ev}^{\mathsf{C}_\Bbbk}} s\mathcal{Q}(X,W)\big].$$

Applying the already mentioned identities $\Delta_0 c(\gamma\otimes\gamma)=\gamma\Delta_0$, $(\mathrm{ev}^{\mathsf{C}_\Bbbk}\otimes 1)\,\mathrm{ev}^{\mathsf{C}_\Bbbk}=(1\otimes m_2^{\mathsf{C}_\Bbbk})\,\mathrm{ev}^{\mathsf{C}_\Bbbk}$, and $(s^{-1}\otimes s^{-1})m_2^{\mathsf{C}_\Bbbk}=-b_2^{\mathsf{C}_\Bbbk}s^{-1}$, we find:

$$t\cdot\check{b}_+^{\mathcal{Q}}=(-)^r\big[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)$$

$$\xrightarrow{c\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)$$

$$\xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)$$

$$\xrightarrow{1\otimes\Delta_0\otimes\Delta_0} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)$$

$$\xrightarrow{\sum 1\otimes 1\otimes c\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(V,W)$$

$$\xrightarrow{\sum 1\otimes\check{r}\otimes\check{\psi}} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(Y,Z),\mathcal{Q}(U,V))\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{Q}(U,V),\mathcal{Q}(X,W))$$

$$\xrightarrow{\sum 1\otimes b_2^{\mathsf{C}_\Bbbk}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(Y,Z),\mathcal{Q}(X,W))$$

$$\xrightarrow{1\otimes s^{-1}[1]} s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{P}(Y,Z),s\mathcal{Q}(X,W)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}} s\mathcal{Q}(X,W)\big].$$

Similarly, composite (4.1.11) equals

$$-(-)^t(\Delta_0\otimes 1\otimes\Delta_0)(1\otimes\check{b}^{\mathcal{P}}_+\otimes 1)\check{t} = -\bigl[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)$$

$$\xrightarrow{\Delta_0\otimes1\otimes\Delta_0}\bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes Ts\mathcal{A}(U,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)$$

$$\xrightarrow{\sum 1\otimes c\otimes1\otimes1}\bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(U,Y)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)$$

$$\xrightarrow{\sum 1\otimes1\otimes\gamma\otimes1\otimes1}\bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)$$

$$\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)$$

$$\xrightarrow{\sum 1\otimes1\otimes\check{\phi}\otimes1}\bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{P}(U,V))\otimes Ts\mathcal{C}(V,W)$$

$$\xrightarrow{\sum 1\otimes1\otimes s^{-1}[1]\otimes1}\bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{P}(U,V))\otimes Ts\mathcal{C}(V,W)$$

$$\xrightarrow{\sum 1\otimes\mathrm{ev}^{\underline{\mathsf{C}}_{\Bbbk}}\otimes1}\bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(U,V)\otimes Ts\mathcal{C}(V,W)$$

$$\xrightarrow{\sum c\otimes1}\bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(U,V)\otimes Ts\mathcal{A}(X,U)\otimes Ts\mathcal{C}(V,W)$$

$$\xrightarrow{\sum 1\otimes\gamma\otimes1}\bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(U,V)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(V,W)$$

$$\xrightarrow{\sum 1\otimes\check{r}}\bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(U,V)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(U,V),\mathcal{Q}(X,W))$$

$$\xrightarrow{\sum 1\otimes s^{-1}[1]}\bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(U,V)\otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(U,V),s\mathcal{Q}(X,W))\xrightarrow{\sum\mathrm{ev}^{\underline{\mathsf{C}}_{\Bbbk}}} s\mathcal{Q}(X,W)\bigr]$$

$$=(-)^{r+1}\bigl[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)$$

$$\xrightarrow{c\otimes1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)$$

$$\xrightarrow{1\otimes\Delta_0c(\gamma\otimes\gamma)\otimes\Delta_0}\bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)$$

$$\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)$$

$$\xrightarrow{\sum 1\otimes1\otimes c\otimes1}\bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(V,W)$$

$$\xrightarrow{\sum 1\otimes\check{\phi}\otimes\check{r}}\bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{P}(U,V))\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(U,V),\mathcal{Q}(X,W))$$

$$\xrightarrow{\sum 1\otimes s^{-1}[1]\otimes s^{-1}[1]}\bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{P}(U,V))\otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(U,V),s\mathcal{Q}(X,W))$$

$$\xrightarrow{\sum(\mathrm{ev}^{\underline{\mathsf{C}}_{\Bbbk}}\otimes1)\,\mathrm{ev}^{\underline{\mathsf{C}}_{\Bbbk}}} s\mathcal{Q}(X,W)\bigr]$$

$$=(-)^{r}\bigl[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)$$

$$
\begin{aligned}
&\xrightarrow{c\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\Delta_0\otimes\Delta_0} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes 1\otimes c\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes\check{\phi}\otimes\check{r}} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{P}(U,V))\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(U,V),\mathcal{Q}(X,W))\\
&\xrightarrow{\sum 1\otimes b_2^{\underline{\mathsf{C}}_{\Bbbk}}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{Q}(X,W))\\
&\qquad\qquad\xrightarrow{1\otimes s^{-1}[1]} s\mathcal{P}(Y,Z)\otimes\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{Q}(X,W))\xrightarrow{\mathrm{ev}^{\mathsf{C}_{\Bbbk}}} s\mathcal{Q}(X,W)\big].
\end{aligned}
$$

By (4.1.2), expression (4.1.12) can be written as follows:

$$
\begin{aligned}
t\cdot\check{b}_0^{\mathcal{Q}}-&(-)^t(\Delta_0\otimes 1\otimes\Delta_0)(1\otimes\check{b}_0^{\mathcal{P}}\otimes 1)\check{t}\\
&=(\Delta_0\otimes 1\otimes\Delta_0)(1\otimes\check{t}\otimes 1)(\mathrm{pr}_0\otimes 1\otimes\mathrm{pr}_0)b_{00}^{\mathcal{Q}}\\
&\quad-(-)^t(\Delta_0\otimes 1\otimes\Delta_0)(1\otimes\mathrm{pr}_0\otimes 1\otimes\mathrm{pr}_0\otimes 1)(1\otimes b_{00}^{\mathcal{P}}\otimes 1)\check{t}\\
&=\check{t}\cdot b_{00}^{\mathcal{Q}}-(-)^t(1\otimes b_{00}^{\mathcal{P}}\otimes 1)\check{t},
\end{aligned}
$$

therefore

$$
\begin{aligned}
t\cdot\check{b}_0^{\mathcal{Q}}-(-)^t(\Delta_0\otimes 1\otimes\Delta_0)(1\otimes\check{b}_0^{\mathcal{P}}\otimes 1)\check{t}&=(-)^{r+1}\big[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{c\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\check{r}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{Q}(X,W))\\
&\qquad\xrightarrow{1\otimes s^{-1}[1]} s\mathcal{P}(Y,Z)\otimes\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{Q}(X,W))\xrightarrow{\mathrm{ev}^{\mathsf{C}_{\Bbbk}}} s\mathcal{Q}(X,W)\xrightarrow{b_{00}^{\mathcal{Q}}} s\mathcal{Q}(X,W)\big]\\
&-\big[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes b_{00}^{\mathcal{P}}\otimes 1} Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{c\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\check{r}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{Q}(X,W))\\
&\qquad\qquad\xrightarrow{1\otimes s^{-1}[1]} s\mathcal{P}(Y,Z)\otimes\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{Q}(X,W))\xrightarrow{\mathrm{ev}^{\mathsf{C}_{\Bbbk}}} s\mathcal{Q}(X,W)\big]\\
&=(-)^{r+1}\big[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{c\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\check{r}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{Q}(X,W))
\end{aligned}
$$

$$\xrightarrow{1\otimes s^{-1}[1]} s\mathcal{P}(Y,Z)\otimes\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{Q}(X,W)) \xrightarrow{\operatorname{ev}^{\mathsf{C}_{\Bbbk}}\cdot b^{\mathcal{Q}}_{00}-(b^{\mathcal{P}}_{00}\otimes 1)\operatorname{ev}^{\mathsf{C}_{\Bbbk}}} s\mathcal{Q}(X,W)\bigr].$$

The complexes $s\mathcal{P}(Y,Z)$ and $s\mathcal{Q}(X,W)$ carry the differential $-b^{\mathcal{P}}_{00}$. Since $\operatorname{ev}^{\mathsf{C}_{\Bbbk}}$ is a chain map, it follows that $\operatorname{ev}^{\mathsf{C}_{\Bbbk}} b^{\mathcal{P}}_{00}-(b^{\mathcal{P}}_{00}\otimes 1)\operatorname{ev}^{\mathsf{C}_{\Bbbk}}=-(1\otimes m^{\underline{\mathsf{C}}_{\Bbbk}}_1)\operatorname{ev}^{\mathsf{C}_{\Bbbk}}$. Together with the relation $b^{\underline{\mathsf{C}}_{\Bbbk}}_1 s^{-1}[1]=s^{-1}[1]m^{\underline{\mathsf{C}}_{\Bbbk}}_1$ this implies that

$$\begin{aligned}
t\cdot\check{b}^{\mathcal{Q}}_0-(-)^t(\Delta_0\otimes 1\otimes\Delta_0)(1\otimes\check{b}^{\mathcal{P}}_0\otimes 1)\check{t} &= (-)^r\bigl[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{c\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\check{r}b^{\underline{\mathsf{C}}_{\Bbbk}}_1} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{Q}(X,W))\\
&\qquad\xrightarrow{1\otimes s^{-1}[1]} s\mathcal{P}(Y,Z)\otimes\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{Q}(X,W))\xrightarrow{\operatorname{ev}^{\mathsf{C}_{\Bbbk}}} s\mathcal{Q}(X,W)\bigr].
\end{aligned}$$

Finally, notice that

$$\begin{aligned}
-(-)^t(b^{\mathcal{A}}\otimes 1\otimes 1+1\otimes 1\otimes b^{\mathcal{C}})\check{t} &= -\bigl[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{c\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes(b^{\mathcal{A}^{\mathrm{op}}}\otimes 1+1\otimes b^{\mathcal{C}})\check{r}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{Q}(X,W))\\
&\qquad\xrightarrow{1\otimes s^{-1}[1]} s\mathcal{P}(Y,Z)\otimes\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{Q}(X,W))\xrightarrow{\operatorname{ev}^{\mathsf{C}_{\Bbbk}}} s\mathcal{Q}(X,W)\bigr].
\end{aligned}$$

Summing up, we conclude that $(td)^\vee=(-)^r(c\otimes 1)(1\otimes\gamma\otimes 1)(1\otimes R)(1\otimes s^{-1}[1])\operatorname{ev}^{\mathsf{C}_{\Bbbk}}$, where

$$\begin{aligned}
R=\bigl[&Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{\Delta_0\otimes\Delta_0}\bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes c\otimes 1}\bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(V,W)\\
&\qquad\qquad\xrightarrow{\sum(\check{\phi}\otimes\check{r}+\check{r}\otimes\check{\psi})b^{\underline{\mathsf{C}}_{\Bbbk}}_2} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{Q}(X,W))\bigr]\\
&+\check{r}b^{\underline{\mathsf{C}}_{\Bbbk}}_1-(-)^r(b^{\mathcal{A}^{\mathrm{op}}}\otimes 1+1\otimes b^{\mathcal{C}})\check{r}=[rb^{\underline{\mathsf{C}}_{\Bbbk}}-(-)^r(b^{\mathcal{A}^{\mathrm{op}}}\otimes 1+1\otimes b^{\mathcal{C}})r]^\vee=[rB_1]^\vee,
\end{aligned}$$

by the Proposition 3.2.15, which says how to recover the (ϕ,ψ)-coderivation r from its components, and by formula (3.3.3) for the component B_1. The claim follows.

Let us prove that the constructed chain maps are compatible with composition. Let $\phi,\psi,\chi:\mathcal{A}^{\mathrm{op}},\mathcal{C}\to\underline{\mathsf{C}}_{\Bbbk}$ be A_∞-functors, and let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{T}$ be the corresponding $\mathcal{A}$-$\mathcal{C}$-bimodules. Pick arbitrary $r\in s\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_{\Bbbk})(\phi,\psi)$ and $q\in s\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_{\Bbbk})(\psi,\chi)$, and denote by $t=(rs^{-1})\Phi$ and $u=(qs^{-1})\Phi$ the corresponding bicomodule homomorphisms. We must show that

$$t\cdot u=((rs^{-1}\otimes qs^{-1})m_2^{\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_{\Bbbk})})\Phi=(-)^{q+1}((r\otimes q)B_2s^{-1})\Phi.$$

Again, it suffices to prove the equation $(t\cdot u)^\vee = (-)^{q+1}[((r\otimes q)B_2s^{-1})\Phi]^\vee$. We have:

$$\begin{aligned}
&(t\cdot u)^\vee = t\cdot\check{u} = (\Delta_0\otimes 1\otimes\Delta_0)(1\otimes\check{t}\otimes 1)\check{u}\\
&= (-)^{r+q}\big[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{\Delta_0\otimes 1\otimes\Delta_0} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes Ts\mathcal{A}(U,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes c\otimes 1\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(U,Y)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes 1\otimes\gamma\otimes 1\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes 1\otimes\check{r}\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(Y,Z),\mathcal{Q}(U,V))\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes 1\otimes s^{-1}[1]\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{P}(Y,Z)\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{P}(Y,Z),s\mathcal{Q}(U,V))\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes\operatorname{ev}^{\mathsf{C}_\Bbbk}\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}(X,U)\otimes s\mathcal{Q}(U,V)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum c\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{Q}(U,V)\otimes Ts\mathcal{A}(X,U)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes\gamma\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{Q}(U,V)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes\check{q}} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{Q}(U,V)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{Q}(U,V),\mathcal{T}(X,W))\\
&\qquad\qquad\xrightarrow{\sum 1\otimes s^{-1}[1]} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{Q}(U,V)\otimes\underline{\mathsf{C}}_\Bbbk(s\mathcal{Q}(U,V),s\mathcal{T}(X,W)) \xrightarrow{\sum\operatorname{ev}^{\mathsf{C}_\Bbbk}} s\mathcal{T}(X,W)\big]\\
&= (-)^r\big[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{c\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\Delta_0c(\gamma\otimes\gamma)\otimes\Delta_0} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes 1\otimes c\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes\check{r}\otimes\check{q}} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(Y,Z),\mathcal{Q}(U,V))\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{Q}(U,V),\mathcal{T}(X,W))\\
&\xrightarrow{\sum 1\otimes s^{-1}[1]\otimes s^{-1}[1]} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes\underline{\mathsf{C}}_\Bbbk(s\mathcal{P}(Y,Z),s\mathcal{Q}(U,V))\otimes\underline{\mathsf{C}}_\Bbbk(s\mathcal{Q}(U,V),s\mathcal{T}(X,W))\\
&\qquad\qquad\qquad\qquad\xrightarrow{\sum(\operatorname{ev}^{\mathsf{C}_\Bbbk}\otimes 1)\operatorname{ev}^{\mathsf{C}_\Bbbk}} s\mathcal{T}(X,W)\big]\\
&= (-)^{r+1}\big[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W)
\end{aligned}$$

$$
\begin{aligned}
&\xrightarrow{c\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{1\otimes\Delta_0\otimes\Delta_0} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes 1\otimes c\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes\check{r}\otimes\check{q}} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{Q}(U,V))\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{Q}(U,V),\mathcal{T}(X,W))\\
&\xrightarrow{\sum 1\otimes b_2^{\mathsf{C}_{\Bbbk}}} s\mathcal{P}(Y,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{T}(X,W))\\
&\qquad\qquad\xrightarrow{1\otimes s^{-1}[1]} s\mathcal{P}(Y,Z)\otimes\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{T}(X,W))\xrightarrow{\mathrm{ev}^{\mathsf{C}_{\Bbbk}}} s\mathcal{T}(X,W)\big].
\end{aligned}
$$

It remains to note that by (3.3.2) and (3.3.4)

$$
\begin{aligned}
[(r\otimes q)B_2]^{\vee} = \big[&Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{\Delta_0\otimes\Delta_0} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum 1\otimes c\otimes 1} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} Ts\mathcal{A}^{\mathrm{op}}(Y,U)\otimes Ts\mathcal{C}(Z,V)\otimes Ts\mathcal{A}^{\mathrm{op}}(U,X)\otimes Ts\mathcal{C}(V,W)\\
&\xrightarrow{\sum\check{r}\otimes\check{q}} \bigoplus_{U\in\operatorname{Ob}\mathcal{A},V\in\operatorname{Ob}\mathcal{C}} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{Q}(U,V))\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{Q}(U,V),\mathcal{T}(X,W))\\
&\qquad\qquad\xrightarrow{\sum b_2^{\mathsf{C}_{\Bbbk}}} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{T}(X,W))\big],
\end{aligned}
$$

and $(-)^{r+1}=(-)^{q+1}(-)^{(r+q+1)+1}=(-)^{q+1}(-)^{\deg[(r\otimes q)B_2]+1}$. The claim follows from the definition of Φ.

Both **dg**-categories $\mathcal{A}$-$\mathcal{C}$-bimod and $\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_{\Bbbk})$ are unital. The units are the identity morphisms in the ordinary categories $Z^0(\mathcal{A}$-$\mathcal{C}$-bimod) and $Z^0(\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_{\Bbbk}))$. The **dg**-functor Φ induces an isomorphism $Z^0\Phi$ of these categories. Hence, $Z^0\Phi$ is unital. In other words, Φ is unital. The proposition is proven. $\square$

Let us write explicitly the inverse map $\Phi^{-1}:\mathcal{A}\text{-}\mathcal{C}\text{-bimod}(\mathcal{P},\mathcal{Q})\to\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_{\Bbbk})(\phi,\psi)$. It takes a bicomodule homomorphism $t:Ts\mathcal{A}\otimes s\mathcal{P}\otimes Ts\mathcal{C}\to Ts\mathcal{A}\otimes s\mathcal{Q}\otimes Ts\mathcal{C}$ to an A_∞-transformation $rs^{-1}\in\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_{\Bbbk})(\phi,\psi)$ given by its components

$$
\begin{aligned}
\check{r}=(-)^t\big[&Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\xrightarrow{\gamma\otimes 1} Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)\\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{P}(Y,Z)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,(c\otimes 1)\check{t})}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z),s\mathcal{Q}(X,W))\xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z),\mathcal{Q}(X,W))\big].
\end{aligned}
\tag{4.1.14}
$$

The isomorphisms Φ and Φ^{-1} yield a tool that allows to move back and forth between the language of A_∞-functors and A_∞-bimodules.

4.2. Regular A_∞-bimodule

Let $\mathcal{A}$ be an A_∞-category. Extending the notion of regular A_∞-bimodule given by Tradler [**59**, Lemma 5.1(a)] from the case of A_∞-algebras to A_∞-categories, define the *regular* $\mathcal{A}$-$\mathcal{A}$-*bimodule* $\mathcal{R} = \mathcal{R}_\mathcal{A}$ as follows. Its underlying graded quiver coincides with $\mathcal{A}$. Components of the codifferential $b^\mathcal{R}$ are given by

$$\check{b}^\mathcal{R} = \bigl[Ts\mathcal{A} \otimes s\mathcal{A} \otimes Ts\mathcal{A} \xrightarrow{\mu_{Ts\mathcal{A}}} Ts\mathcal{A} \xrightarrow{\check{b}^\mathcal{A}} s\mathcal{A}\bigr],$$

where $\mu_{Ts\mathcal{A}}$ is the multiplication in the tensor quiver $Ts\mathcal{A}$, see Remark 3.2.11. Equivalently, $b^\mathcal{R}_{kn} = b^\mathcal{A}_{k+1+n}$, $k, n \geqslant 0$. Flatness of $b^\mathcal{R}$ in form (4.1.9) is equivalent to the A_∞-identity $b^\mathcal{A} \cdot \check{b}^\mathcal{A} = 0$. Indeed, the three summands of the left hand side of (4.1.9) correspond to three kinds of subintervals of the interval $[1, k+1+n] \cap \mathbb{N}$. Subintervals of the first two types miss the point $k+1$ and those of the third type contain it.

4.2.1. Definition. Define an A_∞-functor $\operatorname{Hom}_\mathcal{A} : \mathcal{A}^{\mathrm{op}}, \mathcal{A} \to \underline{\mathsf{C}}_\Bbbk$ as the A_∞-functor $\phi^\mathcal{R}$ that corresponds to the regular $\mathcal{A}$-$\mathcal{A}$-bimodule $\mathcal{R} = \mathcal{R}_\mathcal{A}$.

The A_∞-functor $\operatorname{Hom}_\mathcal{A}$ takes a pair of objects $X, Z \in \operatorname{Ob}\mathcal{A}$ to the chain complex $(\mathcal{A}(X,Z), m_1)$. Components of $\operatorname{Hom}_\mathcal{A}$ are found from equation (4.1.7):

$$\begin{aligned}(\operatorname{Hom}_\mathcal{A})_{kn} = \bigl[T^ks\mathcal{A}^{\mathrm{op}}(X,Y) \otimes T^ns\mathcal{A}(Z,W) &\xrightarrow{\gamma\otimes 1} T^ks\mathcal{A}(Y,X) \otimes T^ns\mathcal{A}(Z,W) \\ &\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z), s\mathcal{A}(X,Z) \otimes T^ks\mathcal{A}(Y,X) \otimes T^ns\mathcal{A}(Z,W)) \\ &\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,(c\otimes 1)b^\mathcal{A}_{k+1+n})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z), s\mathcal{A}(Y,W)) \\ &\qquad\qquad \xrightarrow{[-1]s} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z), \mathcal{A}(Y,W))\bigr]. \end{aligned} \tag{4.2.1}$$

4.2.2. Proposition. *For an arbitrary A_∞-category $\mathcal{A}$,*

$$\mathsf{k}\operatorname{Hom}_\mathcal{A} = \operatorname{Hom}_{\mathsf{k}\mathcal{A}} : \mathsf{k}\mathcal{A}^{\mathrm{op}} \boxtimes \mathsf{k}\mathcal{A} \to \underline{\mathcal{K}}.$$

Proof. Let $X, Y, U, V \in \operatorname{Ob}\mathcal{A}$. Then

$$\begin{aligned}\mathsf{k}\operatorname{Hom}_\mathcal{A} = \bigl[\mathcal{A}^{\mathrm{op}}(X,Y) \otimes \mathcal{A}(U,V) &\xrightarrow{s(\operatorname{Hom}_\mathcal{A})_{10}s^{-1}\otimes s(\operatorname{Hom}_\mathcal{A})_{01}s^{-1}} \\ \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,U), \mathcal{A}(Y,U)) \otimes \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(Y,U), \mathcal{A}(Y,V)) &\xrightarrow{m_2^{\underline{\mathsf{C}}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,U), \mathcal{A}(Y,V))\bigr].\end{aligned}$$

According to (4.2.3) and to the identity $[-1] = \underline{\mathsf{C}}_\Bbbk(s,1) \cdot \underline{\mathsf{C}}_\Bbbk(1,s^{-1})$,

$$\begin{aligned}s(\operatorname{Hom}_\mathcal{A})_{10}s^{-1} &= -\bigl[\mathcal{A}(Y,X) \xrightarrow{s} s\mathcal{A}(Y,X) \xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,U), s\mathcal{A}(X,U) \otimes s\mathcal{A}(Y,X)) \\ &\qquad \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,cb_2)} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,U), s\mathcal{A}(Y,U)) \xrightarrow{\underline{\mathsf{C}}_\Bbbk(s,1)\cdot\underline{\mathsf{C}}_\Bbbk(1,s^{-1})} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,U), \mathcal{A}(Y,U))\bigr] \\ &= -\bigl[\mathcal{A}(Y,X) \xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,U), \mathcal{A}(X,U) \otimes \mathcal{A}(Y,X)) \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,s\otimes s)} \\ &\qquad \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,U), s\mathcal{A}(X,U) \otimes s\mathcal{A}(Y,X)) \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,cb_2s^{-1})} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,U), \mathcal{A}(Y,U))\bigr] \\ &= \bigl[\mathcal{A}(Y,X) \xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,U), \mathcal{A}(X,U) \otimes \mathcal{A}(Y,X)) \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,c)} \\ &\qquad \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,U), \mathcal{A}(Y,X) \otimes \mathcal{A}(X,U)) \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,m_2)} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,U), \mathcal{A}(Y,U))\bigr]. \end{aligned} \tag{4.2.2}$$

Similarly we obtain from equation (4.2.1)

$$\begin{aligned}
s(\mathrm{Hom}_{\mathcal{A}})_{01}s^{-1} = \bigl[\mathcal{A}(U,V) &\xrightarrow{s} s\mathcal{A}(U,V) \\
&\xrightarrow{\mathrm{coev}^{\underline{\mathsf{C}}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(Y,U), s\mathcal{A}(Y,U)\otimes s\mathcal{A}(U,V)) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,b_2)} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(Y,U), s\mathcal{A}(Y,V)) \xrightarrow{[-1]} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(Y,U),\mathcal{A}(Y,V))\bigr] \\
= \bigl[\mathcal{A}(U,V) &\xrightarrow{\mathrm{coev}^{\underline{\mathsf{C}}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(Y,U),\mathcal{A}(Y,U)\otimes\mathcal{A}(U,V)) \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,m_2)} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(Y,U),\mathcal{A}(Y,V))\bigr].
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathsf{k}\,\mathrm{Hom}_{\mathcal{A}} = \bigl[&\mathcal{A}(Y,X)\otimes\mathcal{A}(U,V) \\
&\xrightarrow{\mathrm{coev}^{\underline{\mathsf{C}}_\Bbbk}\otimes\mathrm{coev}^{\underline{\mathsf{C}}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,U),\mathcal{A}(X,U)\otimes\mathcal{A}(Y,X))\otimes\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(Y,U),\mathcal{A}(Y,U)\otimes\mathcal{A}(U,V)) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,cm_2)\otimes\underline{\mathsf{C}}_\Bbbk(1,m_2)} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,U),\mathcal{A}(Y,U))\otimes\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(Y,U),\mathcal{A}(Y,V)) \\
&\qquad\qquad\qquad\qquad\qquad\xrightarrow{m_2^{\underline{\mathsf{C}}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,U),\mathcal{A}(Y,V))\bigr].
\end{aligned}$$

Equation (3.1.1) allows to write the above expression as follows:

$$\begin{aligned}
\mathsf{k}\,\mathrm{Hom}_{\mathcal{A}} = \bigl[&\mathcal{A}(Y,X)\otimes\mathcal{A}(U,V) \\
&\xrightarrow{\mathrm{coev}^{\underline{\mathsf{C}}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,U),\mathcal{A}(X,U)\otimes\mathcal{A}(Y,X)\otimes\mathcal{A}(U,V)) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,cm_2\otimes1)} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,U),\mathcal{A}(Y,U)\otimes\mathcal{A}(U,V)) \\
&\qquad\qquad\qquad\qquad\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,m_2)} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,U),\mathcal{A}(Y,V))\bigr] \\
= \bigl[&\mathcal{A}(Y,X)\otimes\mathcal{A}(U,V) \\
&\xrightarrow{\mathrm{coev}^{\underline{\mathsf{C}}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,U),\mathcal{A}(X,U)\otimes\mathcal{A}(Y,X)\otimes\mathcal{A}(U,V)) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,c\otimes1)} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,U),\mathcal{A}(Y,X)\otimes\mathcal{A}(X,U)\otimes\mathcal{A}(U,V)) \\
&\qquad\qquad\qquad\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,(m_2\otimes1)m_2)} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,U),\mathcal{A}(Y,V))\bigr] \\
= \mathrm{Hom}&_{\mathsf{k}\mathcal{A}},
\end{aligned}$$

by (2.1.2). The proposition is proven. □

4.2.3. The Yoneda A_∞-functor. The closedness of the multicategory A_∞ implies that there exists a unique A_∞-functor $\mathscr{Y}:\mathcal{A}\to\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_\Bbbk)$ (called the *Yoneda A_∞-functor*) such that

$$\mathrm{Hom}_{\mathcal{A}} = \bigl[\mathcal{A}^{\mathrm{op}},\mathcal{A} \xrightarrow{1,\mathscr{Y}} \mathcal{A}^{\mathrm{op}},\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_\Bbbk) \xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty}} \underline{\mathsf{C}}_\Bbbk\bigr].$$

Explicit formula (3.3.6) for evaluation component $\mathrm{ev}^{\mathsf{A}_\infty}_{k0}$ shows that the value of $\mathscr{Y}$ on an object Z of $\mathcal{A}$ is given by the restriction A_∞-functor

$$Z\mathscr{Y} = H^Z = H^Z_{\mathcal{A}} = \mathrm{Hom}_{\mathcal{A}}\big|_1^Z : \mathcal{A}^{\mathrm{op}}\to\underline{\mathsf{C}}_\Bbbk, \quad X\mapsto(\mathcal{A}(X,Z),m_1) = \mathrm{Hom}_{\mathcal{A}}(X,Z)$$

with the components

$$\begin{aligned} H^Z_k &= (\mathrm{Hom}_{\mathcal{A}})_{k0} \\ &= (-1)^k \big[T^k s\mathcal{A}^{\mathrm{op}}(X,Y) \xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z), s\mathcal{A}(X,Z) \otimes T^k s\mathcal{A}^{\mathrm{op}}(X,Y)) \\ &\qquad \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\omega^0_c b^{\mathcal{A}}_{k+1})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z), s\mathcal{A}(Y,Z)) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z), \mathcal{A}(Y,Z))\big], \qquad (4.2.3) \end{aligned}$$

where $\omega^0 = \left(\begin{smallmatrix} 0 & 1 & \dots & k-1 & k \\ k & k-1 & \dots & 1 & 0 \end{smallmatrix}\right) \in \mathfrak{S}_{k+1}$, and ω^0_c is the corresponding permutation isomorphism. By (3.2.19), the k^{th} component of $\mathrm{Hom}_{\mathcal{A}}\big|^Z_1$ equals $(1, \mathrm{Ob}\,\mathscr{Y})\,\mathrm{ev}^{\mathsf{A}_\infty}_{k0}$. Equivalently, components of the A_∞-functor $H^Z : \mathcal{A}^{\mathrm{op}} \to \underline{\mathsf{C}}_\Bbbk$ are determined by the equation

$$\begin{aligned} s^{\otimes k} H^Z_k s^{-1} = (-1)^{k(k+1)/2+1} \big[T^k s\mathcal{A}^{\mathrm{op}}(X,Y) \xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \\ \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z), \mathcal{A}(X,Z) \otimes T^k \mathcal{A}^{\mathrm{op}}(X,Y)) \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\omega^0_c m^{\mathcal{A}}_{k+1})} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z), \mathcal{A}(Y,Z))\big]. \end{aligned}$$

An A_∞-functor of the form $H^Z : \mathcal{A}^{\mathrm{op}} \to \underline{\mathsf{C}}_\Bbbk$, for some $Z \in \mathrm{Ob}\,\mathcal{A}$, is called a *representable A_∞-functor.*

Since $\mathrm{ev}^{\mathsf{A}_\infty}_{km}$ vanishes unless $m \leqslant 1$, formula (3.3.7) for the component $\mathrm{ev}^{\mathsf{A}_\infty}_{k1}$ implies that the component $(\mathrm{Hom}_{\mathcal{A}})_{kn}$ is determined for $n \geqslant 1$, $k \geqslant 0$, by $\mathscr{Y}_{nk}$ which is the composition of $\mathscr{Y}_n$ with

$$\mathrm{pr}_k : s\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}}; \underline{\mathsf{C}}_\Bbbk)(H^Z, H^W) \to \underline{\mathsf{C}}_\Bbbk(T^k s\mathcal{A}^{\mathrm{op}}(X,Y), s\underline{\mathsf{C}}_\Bbbk(XH^Z, YH^W)),$$

as follows:

$$\begin{aligned} (\mathrm{Hom}_{\mathcal{A}})_{kn} = \big[&T^k s\mathcal{A}^{\mathrm{op}}(X,Y) \otimes T^n s\mathcal{A}(Z,W) \\ &\xrightarrow{1\otimes\mathscr{Y}_{nk}} T^k s\mathcal{A}^{\mathrm{op}}(X,Y) \otimes \underline{\mathsf{C}}_\Bbbk\big(T^k s\mathcal{A}^{\mathrm{op}}(X,Y), s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z), \mathcal{A}(Y,W))\big) \\ &\qquad\qquad \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z), \mathcal{A}(Y,W))\big]. \end{aligned}$$

Conversely, the component $\mathscr{Y}_n$ is determined by the components $(\mathrm{Hom}_{\mathcal{A}})_{kn}$ for all $k \geqslant 0$ via the formula

$$\begin{aligned} \mathscr{Y}_{nk} = \big[T^n s\mathcal{A}(Z,W) &\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(T^k s\mathcal{A}^{\mathrm{op}}(X,Y), T^k s\mathcal{A}^{\mathrm{op}}(X,Y) \otimes T^n s\mathcal{A}(Z,W)) \\ &\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,(\mathrm{Hom}_{\mathcal{A}})_{kn})} \underline{\mathsf{C}}_\Bbbk\big(T^k s\mathcal{A}^{\mathrm{op}}(X,Y), s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z), \mathcal{A}(Y,W))\big)\big]. \end{aligned}$$

Plugging in expression (4.2.1) we get

$$\begin{aligned} \mathscr{Y}_{nk} = \big[&T^n s\mathcal{A}(Z,W) \xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(T^k s\mathcal{A}^{\mathrm{op}}(X,Y), T^k s\mathcal{A}^{\mathrm{op}}(X,Y) \otimes T^n s\mathcal{A}(Z,W)) \\ &\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{coev}^{\mathsf{C}_\Bbbk})} \underline{\mathsf{C}}_\Bbbk\big(T^k s\mathcal{A}^{\mathrm{op}}(X,Y), \\ &\qquad\qquad \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z), s\mathcal{A}(X,Z) \otimes T^k s\mathcal{A}^{\mathrm{op}}(X,Y) \otimes T^n s\mathcal{A}(Z,W))\big) \\ &\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\underline{\mathsf{C}}_\Bbbk(1,(1\otimes\gamma\otimes1)(c\otimes1)b^{\mathcal{A}}_{k+1+n}))} \underline{\mathsf{C}}_\Bbbk\big(T^k s\mathcal{A}^{\mathrm{op}}(X,Y), \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z), s\mathcal{A}(Y,W))\big) \\ &\qquad \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,[-1]s)} \underline{\mathsf{C}}_\Bbbk\big(T^k s\mathcal{A}^{\mathrm{op}}(X,Y), s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z), \mathcal{A}(Y,W))\big)\big]. \end{aligned}$$

We will see in Section 4.4 that $\mathscr{Y}$ is a unital A_∞-functor if $\mathcal{A}$ is a unital A_∞-category. In this case, it is homotopy fully faithful as we prove in Appendix A.

4.3. Operations on A_∞-bimodules

4.3.1. Restriction of scalars. Let $f:\mathcal{A}\to\mathcal{B}$, $g:\mathcal{C}\to\mathcal{D}$ be A_∞-functors. Let $\mathcal{P}$ be a $\mathcal{B}$-$\mathcal{D}$-bimodule, $\phi:\mathcal{B}^{\mathrm{op}},\mathcal{D}\to\underline{\mathsf{C}}_\Bbbk$ the corresponding A_∞-functor. Define an $\mathcal{A}$-$\mathcal{C}$-bimodule ${}_f\mathcal{P}_g$ as the bimodule corresponding to the composite

$$\mathcal{A}^{\mathrm{op}},\mathcal{C}\xrightarrow{f^{\mathrm{op}},g}\mathcal{B}^{\mathrm{op}},\mathcal{D}\xrightarrow{\phi}\underline{\mathsf{C}}_\Bbbk.$$

Its underlying graded span is given by ${}_f\mathcal{P}_g(X,Y)=\mathcal{P}(Xf,Yg)$, $X\in\operatorname{Ob}\mathcal{A}$, $Y\in\operatorname{Ob}\mathcal{C}$. Components of the codifferential $b^{{}_f\mathcal{P}_g}$ are found using formulas (4.1.6) and (4.1.8):

$$\begin{aligned}
\check{b}_+^{{}_f\mathcal{P}_g}&=\bigl[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Yf,Zg)\otimes Ts\mathcal{C}(Z,W)\\
&\qquad\xrightarrow{c\otimes1}s\mathcal{P}(Yf,Zg)\otimes Ts\mathcal{A}(X,Y)\otimes Ts\mathcal{C}(Z,W)\\
&\qquad\xrightarrow{1\otimes\gamma\otimes1}s\mathcal{P}(Yf,Zg)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes Ts\mathcal{C}(Z,W)\\
&\qquad\xrightarrow{1\otimes[(f^{\mathrm{op}},g)\phi]^\vee}s\mathcal{P}(Yf,Zg)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(Yf,Zg),\mathcal{P}(Xf,Wg))\\
&\qquad\quad\xrightarrow{1\otimes s^{-1}[1]}s\mathcal{P}(Yf,Zg)\otimes\underline{\mathsf{C}}_\Bbbk(s\mathcal{P}(Yf,Zg),s\mathcal{P}(Xf,Wg))\xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}}s\mathcal{P}(Xf,Wg)\bigr]\\
&=\bigl[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Yf,Zg)\otimes Ts\mathcal{C}(Z,W)\\
&\qquad\xrightarrow{f\otimes1\otimes g}Ts\mathcal{B}(Xf,Yf)\otimes s\mathcal{P}(Yf,Zg)\otimes Ts\mathcal{D}(Zg,Wg)\\
&\qquad\xrightarrow{c\otimes1}s\mathcal{P}(Yf,Zg)\otimes Ts\mathcal{B}(Xf,Yf)\otimes Ts\mathcal{D}(Zg,Wg)\\
&\qquad\xrightarrow{1\otimes\gamma\otimes1}s\mathcal{P}(Yf,Zg)\otimes Ts\mathcal{B}^{\mathrm{op}}(Yf,Xf)\otimes Ts\mathcal{D}(Zg,Wg)\\
&\qquad\xrightarrow{1\otimes\check{\phi}}s\mathcal{P}(Yf,Zg)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(Yf,Zg),\mathcal{P}(Xf,Wg))\\
&\qquad\quad\xrightarrow{1\otimes s^{-1}[1]}s\mathcal{P}(Yf,Zg)\otimes\underline{\mathsf{C}}_\Bbbk(s\mathcal{P}(Yf,Zg),s\mathcal{P}(Xf,Wg))\xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}}s\mathcal{P}(Xf,Wg)\bigr],\\
\check{b}_0^{{}_f\mathcal{P}_g}&=\bigl[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Yf,Zg)\otimes Ts\mathcal{C}(Z,W)\xrightarrow{\mathrm{pr}_0\otimes1\otimes\mathrm{pr}_0}\\
&\qquad\qquad s\mathcal{P}(Yf,Zg)\xrightarrow{b^{\mathcal{P}}_{00}}s\mathcal{P}(Yf,Zg)\bigr].
\end{aligned}$$

These equations can be combined into a single formula

$$\begin{aligned}
\check{b}^{{}_f\mathcal{P}_g}&=\bigl[Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Yf,Zg)\otimes Ts\mathcal{C}(Z,W)\xrightarrow{f\otimes1\otimes g}\\
&\qquad Ts\mathcal{B}(Xf,Yf)\otimes s\mathcal{P}(Yf,Zg)\otimes Ts\mathcal{D}(Zg,Wg)\xrightarrow{\check{b}^{\mathcal{P}}}s\mathcal{P}(Xf,Wg)\bigr].
\end{aligned}\tag{4.3.1}$$

Let $f:\mathcal{A}\to\mathcal{B}$ be an A_∞-functor. Define an $(\mathrm{id}_{Ts\mathcal{A}},\mathrm{id}_{Ts\mathcal{A}})$-bicomodule homomorphism $t^f:\mathcal{R}_{\mathcal{A}}=\mathcal{A}\to{}_f\mathcal{B}_f={}_f(\mathcal{R}_{\mathcal{B}})_f$ of degree 0 by its components

$$\check{t}^f=\bigl[Ts\mathcal{A}(X,Y)\otimes s\mathcal{A}(Y,Z)\otimes Ts\mathcal{A}(Z,W)\xrightarrow{\mu_{Ts\mathcal{A}}}Ts\mathcal{A}(X,W)\xrightarrow{\check{f}}s\mathcal{B}(Xf,Wf)\bigr],$$

or in extended form,

$$\begin{aligned}
t^f_{kn}&=\bigl[s\mathcal{A}(X_k,X_{k-1})\otimes\dots\otimes s\mathcal{A}(X_1,X_0)\otimes s\mathcal{A}(X_0,Z_0)\\
&\qquad\otimes s\mathcal{A}(Z_0,Z_1)\otimes\dots\otimes s\mathcal{A}(Z_{n-1},Z_n)\xrightarrow{f_{k+1+n}}s\mathcal{B}(X_kf,Z_nf)\bigr].
\end{aligned}\tag{4.3.2}$$

We claim that $t^f d = 0$. As usual, it suffices to show that $(t^f d)^\vee = 0$. From the identity

$$(t^f d)^\vee = t^f \cdot \check{b}^{f(\mathcal{R}_\mathcal{B})_f} - b^{\mathcal{R}_\mathcal{A}} \cdot \check{t}^f = (\Delta_0 \otimes 1 \otimes \Delta_0)(1 \otimes \check{t}^f \otimes 1)\check{b}^{f(\mathcal{R}_\mathcal{B})_f} \\ - (b^\mathcal{A} \otimes 1 \otimes 1 + 1 \otimes 1 \otimes b^\mathcal{A})\check{t}^f - (\Delta_0 \otimes 1 \otimes \Delta_0)(1 \otimes \check{b}^{\mathcal{R}_\mathcal{A}} \otimes 1)\check{t}^f$$

it follows that

$$\begin{aligned} (t^f d)^\vee = \bigl[& Ts\mathcal{A}(X,Y) \otimes s\mathcal{A}(Y,Z) \otimes Ts\mathcal{A}(Z,W) \\ & \xrightarrow{\Delta_0 \otimes 1 \otimes \Delta_0} \bigoplus_{U,V \in \operatorname{Ob}\mathcal{A}} Ts\mathcal{A}(X,U) \otimes Ts\mathcal{A}(U,Y) \otimes s\mathcal{A}(Y,Z) \\ & \qquad \otimes Ts\mathcal{A}(Z,V) \otimes Ts\mathcal{A}(V,W) \\ & \xrightarrow{\sum 1 \otimes \mu_{Ts\mathcal{A}} \otimes 1} \bigoplus_{U,V \in \operatorname{Ob}\mathcal{A}} Ts\mathcal{A}(X,U) \otimes Ts\mathcal{A}(U,V) \otimes Ts\mathcal{A}(V,W) \\ & \xrightarrow{\sum 1 \otimes \check{f} \otimes 1} \bigoplus_{U,V \in \operatorname{Ob}\mathcal{A}} Ts\mathcal{A}(X,U) \otimes s\mathcal{B}(Uf,Vf) \otimes Ts\mathcal{A}(V,W) \\ & \xrightarrow{\sum f \otimes 1 \otimes f} \bigoplus_{U,V \in \operatorname{Ob}\mathcal{A}} Ts\mathcal{B}(Xf,Uf) \otimes s\mathcal{B}(Uf,Vf) \otimes Ts\mathcal{B}(Vf,Wf) \\ & \qquad \xrightarrow{\mu_{Ts\mathcal{B}}} Ts\mathcal{B}(Xf,Wf) \xrightarrow{\check{b}^\mathcal{B}} s\mathcal{B}(Xf,Wf)\bigr] \\ - \bigl[& Ts\mathcal{A}(X,Y) \otimes s\mathcal{A}(Y,Z) \otimes Ts\mathcal{A}(Z,W) \\ & \xrightarrow{b^\mathcal{A} \otimes 1 \otimes 1 + 1 \otimes 1 \otimes b^\mathcal{A}} Ts\mathcal{A}(X,Y) \otimes s\mathcal{A}(Y,Z) \otimes Ts\mathcal{A}(Z,W) \\ & \qquad \xrightarrow{\mu_{Ts\mathcal{A}}} Ts\mathcal{A}(X,W) \xrightarrow{\check{f}} s\mathcal{B}(Xf,Wf)\bigr] \\ - \bigl[& Ts\mathcal{A}(X,Y) \otimes s\mathcal{A}(Y,Z) \otimes Ts\mathcal{A}(Z,W) \\ & \xrightarrow{\Delta_0 \otimes 1 \otimes \Delta_0} \bigoplus_{U,V \in \operatorname{Ob}\mathcal{A}} Ts\mathcal{A}(X,U) \otimes Ts\mathcal{A}(U,Y) \otimes s\mathcal{A}(Y,Z) \\ & \qquad \otimes Ts\mathcal{A}(Z,V) \otimes Ts\mathcal{A}(V,W) \\ & \xrightarrow{\sum 1 \otimes \mu_{Ts\mathcal{A}} \otimes 1} \bigoplus_{U,V \in \operatorname{Ob}\mathcal{A}} Ts\mathcal{A}(X,U) \otimes Ts\mathcal{A}(U,V) \otimes Ts\mathcal{A}(V,W) \\ & \xrightarrow{\sum 1 \otimes \check{b}^\mathcal{A} \otimes 1} \bigoplus_{U,V \in \operatorname{Ob}\mathcal{A}} Ts\mathcal{A}(X,U) \otimes s\mathcal{A}(U,V) \otimes Ts\mathcal{A}(V,W) \\ & \qquad \xrightarrow{\sum \mu_{Ts\mathcal{A}}} Ts\mathcal{A}(X,W) \xrightarrow{\check{f}} s\mathcal{B}(Xf,Wf)\bigr]. \end{aligned}$$

Likewise Section 4.2 we see that the equation $(t^f d)^\vee = 0$ is equivalent to $f \cdot \check{b}^\mathcal{B} = b^\mathcal{A} \cdot \check{f}$.

4.3.2. Corollary. *Let $f : \mathcal{A} \to \mathcal{B}$ be an A_∞-functor. There is a natural A_∞-transformation $r^f : \operatorname{Hom}_\mathcal{A} \to (f^{\mathrm{op}}, f) \cdot \operatorname{Hom}_\mathcal{B} : \mathcal{A}^{\mathrm{op}}, \mathcal{A} \to \underline{\mathsf{C}}_\Bbbk$ depicted as follows:*

$$\begin{tikzcd} \mathcal{A}^{\mathrm{op}},\mathcal{A} \arrow[rr, "\operatorname{Hom}_\mathcal{A}"] \arrow[dr, "f^{\mathrm{op}},f"'] & {} \arrow[d, Rightarrow, "r^f"] & \underline{\mathsf{C}}_\Bbbk \\ & \mathcal{B}^{\mathrm{op}},\mathcal{B} \arrow[ur, "\operatorname{Hom}_\mathcal{B}"'] & \end{tikzcd}$$

It is invertible if f is homotopy full and faithful.

Proof. Define $r^f = (t^f)\Phi^{-1}s \in s\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}}, \mathcal{A}; \underline{\mathsf{C}}_{\Bbbk})(\mathrm{Hom}_{\mathcal{A}}, (f^{\mathrm{op}}, f)\,\mathrm{Hom}_{\mathcal{B}})$, where

$$\Phi : \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}}, \mathcal{A}; \underline{\mathsf{C}}_{\Bbbk}) \to \mathcal{A}\text{-}\mathcal{A}\text{-bimod}$$

is the isomorphism of **dg**-categories from Proposition 4.1.3, and $t^f : \mathcal{A} \to {}_f\mathcal{B}_f$ is the closed bicomodule homomorphism defined above. Since Φ is an invertible chain map, it follows that r^f is a natural A_∞-transformation. Suppose f is homotopy full and faithful. That is, its first component f_1 is homotopy invertible. This implies that the $(0,0)$-component

$$ {}_{X,Z}r^f_{00} = \bigl[\Bbbk \xrightarrow{\mathrm{coev}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z), s\mathcal{A}(X,Z)) \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,f_1)} \\ \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z), s\mathcal{B}(Xf,Zf)) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,Z), \mathcal{B}(Xf,Zf))\bigr],$$

found from (4.1.14) and (4.3.2), is invertible modulo boundaries, thus r^f is invertible by Lemma 3.4.11. The corollary is proven. □

4.3.3. Opposite bimodule. Let $\mathcal{P}$ be an $\mathcal{A}$-$\mathcal{C}$-bimodule, $\phi : \mathcal{A}^{\mathrm{op}}, \mathcal{C} \to \underline{\mathsf{C}}_{\Bbbk}$ the corresponding A_∞-functor. Define an *opposite bimodule* $\mathcal{P}^{\mathrm{op}}$ as the $\mathcal{C}^{\mathrm{op}}$-$\mathcal{A}^{\mathrm{op}}$-bimodule corresponding to the A_∞-functor

$$\underline{\mathsf{A}_\infty}(\mathsf{X}; \underline{\mathsf{C}}_{\Bbbk})(\phi) = (\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{A}^{\mathrm{op}}}) \cdot_{\mathsf{X}:\mathbf{2}\to\mathbf{2}} \phi = \bigl[Ts\mathcal{C} \boxtimes Ts\mathcal{A}^{\mathrm{op}} \xrightarrow{c} Ts\mathcal{A}^{\mathrm{op}} \boxtimes Ts\mathcal{C} \xrightarrow{\phi} Ts\underline{\mathsf{C}}_{\Bbbk}\bigr].$$

Its underlying graded span is given by $\mathcal{P}^{\mathrm{op}}(Y,X) = \mathcal{P}(X,Y)$, $X \in \mathrm{Ob}\,\mathcal{A}$, $Y \in \mathrm{Ob}\,\mathcal{C}$. Components of the differential $b^{\mathcal{P}^{\mathrm{op}}}$ are found from equations (4.1.6) and (4.1.8):

$$\begin{aligned}
\check{b}^{\mathcal{P}^{\mathrm{op}}}_{+} &= \bigl[Ts\mathcal{C}^{\mathrm{op}}(W,Z) \otimes s\mathcal{P}^{\mathrm{op}}(Z,Y) \otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X) \\
&\qquad \xrightarrow{c\otimes 1} s\mathcal{P}(Y,Z) \otimes Ts\mathcal{C}^{\mathrm{op}}(W,Z) \otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X) \\
&\qquad \xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}(Y,Z) \otimes Ts\mathcal{C}(Z,W) \otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X) \\
&\qquad \xrightarrow{1\otimes c} s\mathcal{P}(Y,Z) \otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X) \otimes Ts\mathcal{C}(Z,W) \\
&\qquad \xrightarrow{1\otimes\check{\phi}} s\mathcal{P}(Y,Z) \otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(Y,Z), \mathcal{P}(X,W)) \\
&\qquad \xrightarrow{1\otimes s^{-1}[1]} s\mathcal{P}(Y,Z) \otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{P}(Y,Z), s\mathcal{P}(X,W)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_{\Bbbk}}} s\mathcal{P}(X,W) = s\mathcal{P}^{\mathrm{op}}(W,X)\bigr] \\
&= \bigl[Ts\mathcal{C}^{\mathrm{op}}(W,Z) \otimes s\mathcal{P}^{\mathrm{op}}(Z,Y) \otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X) \\
&\qquad \xrightarrow{(13)_c} Ts\mathcal{A}^{\mathrm{op}}(Y,X) \otimes s\mathcal{P}(Y,Z) \otimes Ts\mathcal{C}^{\mathrm{op}}(W,Z) \\
&\qquad \xrightarrow{\gamma\otimes 1\otimes\gamma} Ts\mathcal{A}(X,Y) \otimes s\mathcal{P}(Y,Z) \otimes Ts\mathcal{C}(Z,W) \xrightarrow{\check{b}^{\mathcal{P}}_{+}} s\mathcal{P}(X,W) = s\mathcal{P}^{\mathrm{op}}(W,X)\bigr], \\
\check{b}^{\mathcal{P}^{\mathrm{op}}}_{0} &= \bigl[Ts\mathcal{C}^{\mathrm{op}}(W,X) \otimes s\mathcal{P}^{\mathrm{op}}(Z,Y) \otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X) \xrightarrow{\mathrm{pr}_0\otimes 1\otimes\mathrm{pr}_0} s\mathcal{P}(Y,Z) \xrightarrow{b^{\mathcal{P}}_{00}} s\mathcal{P}(Y,Z)\bigr].
\end{aligned}$$

These equations are particular cases of a single formula

$$\begin{aligned}
\check{b}^{\mathcal{P}^{\mathrm{op}}} &= \bigl[Ts\mathcal{C}^{\mathrm{op}}(W,Z)\otimes s\mathcal{P}^{\mathrm{op}}(Z,Y)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X) \\
&\qquad \xrightarrow{(13)_c} Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}^{\mathrm{op}}(W,Z) \\
&\qquad \xrightarrow{\gamma\otimes 1\otimes\gamma} Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W) \xrightarrow{\check{b}^{\mathcal{P}}} s\mathcal{P}(X,W) = s\mathcal{P}^{\mathrm{op}}(W,X)\bigr] \\
&= -\bigl[Ts\mathcal{C}^{\mathrm{op}}(W,Z)\otimes s\mathcal{P}^{\mathrm{op}}(Z,Y)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X) \\
&\qquad \xrightarrow{(13)_c} Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes s\mathcal{P}^{\mathrm{op}}(Z,Y)\otimes Ts\mathcal{C}^{\mathrm{op}}(W,Z) \\
&\qquad \xrightarrow{\gamma\otimes\gamma\otimes\gamma} Ts\mathcal{A}(X,Y)\otimes s\mathcal{P}(Y,Z)\otimes Ts\mathcal{C}(Z,W) \\
&\qquad\qquad \xrightarrow{\check{b}^{\mathcal{P}}} s\mathcal{P}(X,W) = s\mathcal{P}^{\mathrm{op}}(W,X)\bigr]. \qquad (4.3.3)
\end{aligned}$$

4.3.4. Proposition. *Let $\mathcal{A}$ be an A_∞-category. Then $\mathcal{R}^{\mathrm{op}}_{\mathcal{A}} = \mathcal{R}_{\mathcal{A}^{\mathrm{op}}}$ as $\mathcal{A}^{\mathrm{op}}$-$\mathcal{A}^{\mathrm{op}}$-bimodules.*

Proof. Clearly, the underlying graded spans of both bimodules coincide. Computing $\check{b}^{\mathcal{R}^{\mathrm{op}}_{\mathcal{A}}}$ by formula (4.3.3) yields

$$\begin{aligned}
\check{b}^{\mathcal{R}^{\mathrm{op}}_{\mathcal{A}}} &= -\bigl[Ts\mathcal{A}^{\mathrm{op}}(W,Z)\otimes s\mathcal{A}^{\mathrm{op}}(Z,Y)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X) \\
&\qquad \xrightarrow{(13)_c} Ts\mathcal{A}^{\mathrm{op}}(Y,X)\otimes s\mathcal{A}^{\mathrm{op}}(Z,Y)\otimes Ts\mathcal{A}^{\mathrm{op}}(W,Z) \\
&\qquad \xrightarrow{\gamma\otimes\gamma\otimes\gamma} Ts\mathcal{A}(X,Y)\otimes s\mathcal{A}(Y,Z)\otimes Ts\mathcal{A}(Z,W) \xrightarrow{\mu_{Ts\mathcal{A}}} Ts\mathcal{A}(X,W) \\
&\qquad\qquad \xrightarrow{\check{b}^{\mathcal{A}}} s\mathcal{A}(X,W)\bigr] \\
&= -\bigl[Ts\mathcal{A}^{\mathrm{op}}(W,Z)\otimes s\mathcal{A}^{\mathrm{op}}(Z,Y)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X) \\
&\qquad \xrightarrow{\mu_{Ts\mathcal{A}^{\mathrm{op}}}} Ts\mathcal{A}^{\mathrm{op}}(W,X) \xrightarrow{\gamma} Ts\mathcal{A}(X,W) \xrightarrow{\check{b}^{\mathcal{A}}} s\mathcal{A}(X,W)\bigr]
\end{aligned}$$

since $\gamma : Ts\mathcal{A}^{\mathrm{op}} \to Ts\mathcal{A}$ is a category anti-isomorphism. Since $b^{\mathcal{A}^{\mathrm{op}}} = \gamma b^{\mathcal{A}}\gamma$, it follows that $\check{b}^{\mathcal{A}^{\mathrm{op}}} = -\gamma\check{b}^{\mathcal{A}} : Ts\mathcal{A}^{\mathrm{op}}(W,X) \to s\mathcal{A}(W,X)$, therefore

$$\begin{aligned}
\check{b}^{\mathcal{R}^{\mathrm{op}}_{\mathcal{A}}} &= \bigl[Ts\mathcal{A}^{\mathrm{op}}(W,Z)\otimes s\mathcal{A}^{\mathrm{op}}(Z,Y)\otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X) \xrightarrow{\mu_{Ts\mathcal{A}^{\mathrm{op}}}} \\
&\qquad\qquad Ts\mathcal{A}^{\mathrm{op}}(W,X) \xrightarrow{\check{b}^{\mathcal{A}^{\mathrm{op}}}} s\mathcal{A}^{\mathrm{op}}(W,X)\bigr] = \check{b}^{\mathcal{R}_{\mathcal{A}^{\mathrm{op}}}}.
\end{aligned}$$

The proposition is proven. □

4.3.5. Corollary. *Let $\mathcal{A}$ be an A_∞-category. Then*

$$\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}} = \bigl[Ts\mathcal{A}\boxtimes Ts\mathcal{A}^{\mathrm{op}} \xrightarrow{c} Ts\mathcal{A}^{\mathrm{op}}\boxtimes Ts\mathcal{A} \xrightarrow{\mathrm{Hom}_{\mathcal{A}}} Ts\underline{\mathsf{C}}_{\Bbbk}\bigr].$$

4.3.6. Duality A_∞-functor. The regular module $\Bbbk$, viewed as a complex concentrated in degree 0, determines the *duality A_∞-functor* $D = H^{\Bbbk} : \underline{\mathsf{C}}_{\Bbbk}{}^{\mathrm{op}} \to \underline{\mathsf{C}}_{\Bbbk}$. It maps a complex M to its dual $(\underline{\mathsf{C}}_{\Bbbk}(M,\Bbbk), m_1) = (\underline{\mathsf{C}}_{\Bbbk}(M,\Bbbk), -\underline{\mathsf{C}}_{\Bbbk}(d,1))$. Since $\underline{\mathsf{C}}_{\Bbbk}$ is a differential graded category, components D_k vanish if $k > 1$, due to (4.2.3). The component D_1 is given by

$$\begin{aligned}
D_1 = -\bigl[&s\underline{\mathsf{C}}_{\Bbbk}{}^{\mathrm{op}}(M,N) = s\underline{\mathsf{C}}_{\Bbbk}(N,M) \xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\underline{\mathsf{C}}_{\Bbbk}(M,\Bbbk), s\underline{\mathsf{C}}_{\Bbbk}(M,\Bbbk)\otimes s\underline{\mathsf{C}}_{\Bbbk}(N,M)) \\
&\quad \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,cb_2^{\underline{\mathsf{C}}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\underline{\mathsf{C}}_{\Bbbk}(M,\Bbbk), s\underline{\mathsf{C}}_{\Bbbk}(N,\Bbbk)) \\
&\qquad\qquad \xrightarrow{[-1]} \underline{\mathsf{C}}_{\Bbbk}(\underline{\mathsf{C}}_{\Bbbk}(M,\Bbbk), \underline{\mathsf{C}}_{\Bbbk}(N,\Bbbk)) \xrightarrow{s} s\underline{\mathsf{C}}_{\Bbbk}(\underline{\mathsf{C}}_{\Bbbk}(M,\Bbbk), \underline{\mathsf{C}}_{\Bbbk}(N,\Bbbk))\bigr].
\end{aligned}$$

It follows that

$$sD_1s^{-1} = \bigl[\underline{\mathsf{C}}_\Bbbk{}^{\mathrm{op}}(M,N) = \underline{\mathsf{C}}_\Bbbk(N,M) \xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \\ \underline{\mathsf{C}}_\Bbbk(\underline{\mathsf{C}}_\Bbbk(M,\Bbbk), \underline{\mathsf{C}}_\Bbbk(M,\Bbbk) \otimes \underline{\mathsf{C}}_\Bbbk(N,M)) \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,cm_2^{\underline{\mathsf{C}}_\Bbbk})} \underline{\mathsf{C}}_\Bbbk(\underline{\mathsf{C}}_\Bbbk(M,\Bbbk), \underline{\mathsf{C}}_\Bbbk(N,\Bbbk))\bigr],$$

cf. (4.2.2). Equation (2.1.6) implies that $\mathsf{k}D = \underline{\mathcal{K}}(-,\Bbbk) : \mathsf{k}\underline{\mathsf{C}}_\Bbbk{}^{\mathrm{op}} = \underline{\mathcal{K}}^{\mathrm{op}} \to \mathsf{k}\underline{\mathsf{C}}_\Bbbk = \underline{\mathcal{K}}$.

4.3.7. Dual A_∞-bimodule. Let $\mathcal{A}$, $\mathcal{C}$ be A_∞-categories, and let $\mathcal{P}$ be an $\mathcal{A}$-$\mathcal{C}$-bimodule with a flat $(1,1,1,b^{\mathcal{A}},b^{\mathcal{C}})$-connection $b^{\mathcal{P}} : Ts\mathcal{A} \otimes s\mathcal{P} \otimes Ts\mathcal{C} \to Ts\mathcal{A} \otimes s\mathcal{P} \otimes Ts\mathcal{C}$, and let $\phi^{\mathcal{P}} : Ts\mathcal{A}^{\mathrm{op}} \boxtimes Ts\mathcal{C} \to Ts\underline{\mathsf{C}}_\Bbbk$ be the corresponding A_∞-functor. Define a *dual* $\mathcal{C}$-$\mathcal{A}$-*bimodule* $\mathcal{P}^*$ as the bimodule that corresponds to the following A_∞-functor:

$$\phi^{\mathcal{P}^*} = \mathsf{A}_\infty(\mathsf{X};\underline{\mathsf{C}}_\Bbbk)((\phi^{\mathcal{P}})^{\mathrm{op}} \cdot D) : \mathcal{C}^{\mathrm{op}}, \mathcal{A} \to \underline{\mathsf{C}}_\Bbbk.$$

Equivalently,

$$\phi^{\mathcal{P}^*} = \bigl[Ts\mathcal{C}^{\mathrm{op}} \boxtimes Ts\mathcal{A} \xrightarrow{c} Ts\mathcal{A} \boxtimes Ts\mathcal{C}^{\mathrm{op}} \xrightarrow{\gamma\boxtimes\gamma} \\ Ts\mathcal{A}^{\mathrm{op}} \boxtimes Ts\mathcal{C} \xrightarrow{\phi^{\mathcal{P}}} Ts\underline{\mathsf{C}}_\Bbbk \xrightarrow{\gamma} Ts\underline{\mathsf{C}}_\Bbbk{}^{\mathrm{op}} \xrightarrow{D} Ts\underline{\mathsf{C}}_\Bbbk\bigr]. \qquad (4.3.4)$$

The underlying graded span of $\mathcal{P}^*$ is given by $\mathrm{Ob}_s\,\mathcal{P}^* = \mathrm{Ob}_t\,\mathcal{P} = \mathrm{Ob}\,\mathcal{C}$, $\mathrm{Ob}_t\,\mathcal{P}^* = \mathrm{Ob}_s\,\mathcal{P} = \mathrm{Ob}\,\mathcal{A}$, $\mathcal{P}^*(X,Y) = \underline{\mathsf{C}}_\Bbbk(\mathcal{P}(Y,X),\Bbbk)$, $X \in \mathrm{Ob}\,\mathcal{C}$, $Y \in \mathrm{Ob}\,\mathcal{A}$. Moreover,

$$\begin{aligned}\check{\phi}^{\mathcal{P}^*} &= \phi^{\mathcal{P}^*}\,\mathrm{pr}_1 \\ &= \bigl[Ts\mathcal{C}^{\mathrm{op}}(Y,Z) \otimes Ts\mathcal{A}(X,W) \\ &\quad \xrightarrow{c(\gamma\otimes\gamma)} Ts\mathcal{A}^{\mathrm{op}}(W,X) \otimes Ts\mathcal{C}(Z,Y) \\ &\quad \xrightarrow{\check{\phi}^{\mathcal{P}}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(W,Z),\mathcal{P}(X,Y)) \\ &\quad \xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(X,Y),\Bbbk), s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(X,Y),\Bbbk) \otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(W,Z),\mathcal{P}(X,Y))) \\ &\quad \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,cb_2^{\underline{\mathsf{C}}_\Bbbk})} \underline{\mathsf{C}}_\Bbbk(s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(X,Y),\Bbbk), s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(W,Z),\Bbbk)) \\ &\quad \xrightarrow{[-1]} \underline{\mathsf{C}}_\Bbbk(\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(X,Y),\Bbbk), \underline{\mathsf{C}}_\Bbbk(\mathcal{P}(W,Z),\Bbbk)) \\ &\qquad\qquad \xrightarrow{s} s\underline{\mathsf{C}}_\Bbbk(\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(X,Y),\Bbbk), \underline{\mathsf{C}}_\Bbbk(\mathcal{P}(W,Z),\Bbbk))\bigr]\end{aligned}$$

(the minus sign present in $\gamma : s\underline{\mathsf{C}}_\Bbbk{}^{\mathrm{op}} \to s\underline{\mathsf{C}}_\Bbbk$ cancels that present in D_1). According to (4.1.6),

$$\begin{aligned}\check{b}_+^{\mathcal{P}^*} = \bigl[&Ts\mathcal{C}(X,Y) \otimes s\mathcal{P}^*(Y,Z) \otimes Ts\mathcal{A}(Z,W) \\ &\xrightarrow{c\otimes 1} s\mathcal{P}^*(Y,Z) \otimes Ts\mathcal{C}(X,Y) \otimes Ts\mathcal{A}(Z,W) \\ &\xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{P}^*(Y,Z) \otimes Ts\mathcal{C}^{\mathrm{op}}(Y,X) \otimes Ts\mathcal{A}(Z,W) \\ &\xrightarrow{1\otimes c} s\mathcal{P}^*(Y,Z) \otimes Ts\mathcal{A}(Z,W) \otimes Ts\mathcal{C}^{\mathrm{op}}(Y,X) \\ &\xrightarrow{1\otimes\gamma\otimes\gamma} s\mathcal{P}^*(Y,Z) \otimes Ts\mathcal{A}^{\mathrm{op}}(W,Z) \otimes Ts\mathcal{C}(X,Y) \\ &\xrightarrow{1\otimes\check{\phi}^{\mathcal{P}}} s\mathcal{P}^*(Y,Z) \otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(W,X),\mathcal{P}(Z,Y)) \\ &\xrightarrow{1\otimes\mathrm{coev}^{\mathsf{C}_\Bbbk}} s\mathcal{P}^*(Y,Z) \otimes \underline{\mathsf{C}}_\Bbbk(s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(Z,Y),\Bbbk), \\ &\qquad\qquad s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(Z,Y),\Bbbk) \otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(W,X),\mathcal{P}(Z,Y)))\end{aligned}$$

$$\xrightarrow{1\otimes\underline{\mathsf{C}}_\Bbbk(1,cb_2^{\mathsf{C}_\Bbbk})} s\mathcal{P}^*(Y,Z)\otimes\underline{\mathsf{C}}_\Bbbk(s\mathcal{P}^*(Y,Z),s\mathcal{P}^*(X,W)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}} s\mathcal{P}^*(X,W)\big]$$

$$= \big[Ts\mathcal{C}(X,Y)\otimes s\mathcal{P}^*(Y,Z)\otimes Ts\mathcal{A}(Z,W)$$

$$\xrightarrow{(123)_c} Ts\mathcal{A}(Z,W)\otimes Ts\mathcal{C}(X,Y)\otimes s\mathcal{P}^*(Y,Z)$$

$$\xrightarrow{\gamma\otimes1\otimes1} Ts\mathcal{A}^{\mathrm{op}}(W,Z)\otimes Ts\mathcal{C}(X,Y)\otimes s\mathcal{P}^*(Y,Z)$$

$$\xrightarrow{\check{\phi}^{\mathcal{P}}\otimes1} s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(W,X),\mathcal{P}(Z,Y))\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(Z,Y),\Bbbk)$$

$$\xrightarrow{b_2^{\mathsf{C}_\Bbbk}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{P}(W,X),\Bbbk) = s\mathcal{P}^*(X,W)\big],$$

by properties of closed monoidal categories. Similarly, by (4.1.8)

$$\check{b}_0^{\mathcal{P}^*} = \big[Ts\mathcal{C}(X,Y)\otimes s\mathcal{P}^*(Y,Z)\otimes Ts\mathcal{A}(Z,W) \xrightarrow{\mathrm{pr}_0\otimes1\otimes\mathrm{pr}_0}$$

$$s\mathcal{P}^*(Y,Z) \xrightarrow{-s^{-1}\underline{\mathsf{C}}_\Bbbk(d,1)s} s\mathcal{P}^*(Y,Z)\big],$$

where d is the differential in the complex $\phi^{\mathcal{P}}(Z,Y)=\mathcal{P}(Z,Y)$.

4.3.8. Proposition. *Let $\mathcal{A}$ be an A_∞-category. Denote by $\mathcal{R}$ the regular $\mathcal{A}$-bimodule. Then*

$$\phi^{\mathcal{R}^*} = \mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}^{\mathrm{op}}\cdot D : \mathcal{A}^{\mathrm{op}},\mathcal{A}\to\underline{\mathsf{C}}_\Bbbk.$$

Proof. Formula (4.3.4) implies that

$$\phi^{\mathcal{R}^*} = \big[Ts\mathcal{A}^{\mathrm{op}}\boxtimes Ts\mathcal{A} \xrightarrow{c} Ts\mathcal{A}\boxtimes Ts\mathcal{A}^{\mathrm{op}} \xrightarrow{\gamma\boxtimes\gamma}$$

$$Ts\mathcal{A}^{\mathrm{op}}\boxtimes Ts\mathcal{A} \xrightarrow{\mathrm{Hom}_{\mathcal{A}}} Ts\underline{\mathsf{C}}_\Bbbk \xrightarrow{\gamma} Ts\underline{\mathsf{C}}_\Bbbk{}^{\mathrm{op}} \xrightarrow{D} Ts\underline{\mathsf{C}}_\Bbbk\big]$$

$$= \big[Ts\mathcal{A}^{\mathrm{op}}\boxtimes Ts\mathcal{A} \xrightarrow{\gamma\boxtimes\gamma} Ts\mathcal{A}\boxtimes Ts\mathcal{A}^{\mathrm{op}} \xrightarrow{c}$$

$$Ts\mathcal{A}^{\mathrm{op}}\boxtimes Ts\mathcal{A} \xrightarrow{\mathrm{Hom}_{\mathcal{A}}} Ts\underline{\mathsf{C}}_\Bbbk \xrightarrow{\gamma} Ts\underline{\mathsf{C}}_\Bbbk{}^{\mathrm{op}} \xrightarrow{D} Ts\underline{\mathsf{C}}_\Bbbk\big].$$

By Corollary 4.3.5, $\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}} = [Ts\mathcal{A}\boxtimes Ts\mathcal{A}^{\mathrm{op}} \xrightarrow{c} Ts\mathcal{A}^{\mathrm{op}}\boxtimes Ts\mathcal{A} \xrightarrow{\mathrm{Hom}_{\mathcal{A}}} Ts\underline{\mathsf{C}}_\Bbbk]$, therefore

$$\phi^{\mathcal{R}^*} = \big[Ts\mathcal{A}^{\mathrm{op}}\boxtimes Ts\mathcal{A} \xrightarrow{\gamma\boxtimes\gamma} Ts\mathcal{A}\boxtimes Ts\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}} Ts\underline{\mathsf{C}}_\Bbbk \xrightarrow{\gamma} Ts\underline{\mathsf{C}}_\Bbbk{}^{\mathrm{op}} \xrightarrow{D} Ts\underline{\mathsf{C}}_\Bbbk\big]$$

$$= \big[Ts\mathcal{A}^{\mathrm{op}}\boxtimes Ts\mathcal{A} \xrightarrow{\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}^{\mathrm{op}}} Ts\underline{\mathsf{C}}_\Bbbk{}^{\mathrm{op}} \xrightarrow{D} Ts\underline{\mathsf{C}}_\Bbbk\big].$$

The proposition is proven. □

4.4. Unital A_∞-bimodules

4.4.1. Definition. An $\mathcal{A}$-$\mathcal{C}$-bimodule $\mathcal{P}$ corresponding to an A_∞-functor $\phi:\mathcal{A}^{\mathrm{op}},\mathcal{C}\to\underline{\mathsf{C}}_\Bbbk$ is called *unital* if the A_∞-functor ϕ is unital.

4.4.2. Proposition. *An $\mathcal{A}$-$\mathcal{C}$-bimodule $\mathcal{P}$ is unital if and only if, for each $X\in\mathrm{Ob}\,\mathcal{A}$, $Y\in\mathrm{Ob}\,\mathcal{C}$, the composites*

$$\big[s\mathcal{P}(X,Y)\cong s\mathcal{P}(X,Y)\otimes\Bbbk \xrightarrow{1\otimes_Y\mathbf{i}_0^{\mathcal{C}}} s\mathcal{P}(X,Y)\otimes s\mathcal{C}(Y,Y) \xrightarrow{b_{01}^{\mathcal{P}}} s\mathcal{P}(X,Y)\big],$$

$$-\big[s\mathcal{P}(X,Y)\cong\Bbbk\otimes s\mathcal{P}(X,Y) \xrightarrow{{}_X\mathbf{i}_0^{\mathcal{A}}\otimes1} s\mathcal{A}(X,X)\otimes s\mathcal{P}(X,Y) \xrightarrow{b_{10}^{\mathcal{P}}} s\mathcal{P}(X,Y)\big]$$

are homotopic to the identity map.

Proof. The second statement expands to the property that

$$\big[s\mathcal{P}(X,Y) \xrightarrow{s^{-1}\otimes_Y \mathbf{i}_0^{\mathcal{C}}\phi_{01}s^{-1}} \mathcal{P}(X,Y)\otimes \underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(X,Y),\mathcal{P}(X,Y)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}s} s\mathcal{P}(X,Y)\big],$$

$$\big[s\mathcal{P}(X,Y) \xrightarrow{s^{-1}\otimes_X \mathbf{i}_0^{\mathcal{A}}\phi_{10}s^{-1}} \mathcal{P}(X,Y)\otimes \underline{\mathsf{C}}_{\Bbbk}(\mathcal{P}(X,Y),\mathcal{P}(X,Y)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}s} s\mathcal{P}(X,Y)\big]$$

are homotopic to identity. That is,

$$_Y\mathbf{i}_0^{\mathcal{C}}\big(\phi_{01}\big|_{\mathcal{C}}^{X}\big) - 1_{s\mathcal{P}(X,Y)}s \in \operatorname{Im} b_1, \qquad {}_X\mathbf{i}_0^{\mathcal{A}}\big(\phi_{10}\big|_{\mathcal{A}^{\mathrm{op}}}^{Y}\big) - 1_{s\mathcal{P}(X,Y)}s \in \operatorname{Im} b_1$$

for all $X\in \operatorname{Ob}\mathcal{A}$, $Y\in\operatorname{Ob}\mathcal{C}$. By Proposition 3.4.8, the A_∞-functor ϕ is unital. □

4.4.3. Remark. Suppose $\mathcal{A}$ is a unital A_∞-category. By the above criterion, the regular $\mathcal{A}$-bimodule $\mathcal{R}_{\mathcal{A}}$ is unital, and therefore the A_∞-functor $\operatorname{Hom}_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}},\mathcal{A} \to \underline{\mathsf{C}}_{\Bbbk}$ is unital. In particular, for each object Z of $\mathcal{A}$, the representable A_∞-functor

$$H^Z = \operatorname{Hom}_{\mathcal{A}}\big|_1^Z : \mathcal{A}^{\mathrm{op}} \to \underline{\mathsf{C}}_{\Bbbk}$$

is unital, by Corollary 3.4.9. Thus, the Yoneda A_∞-functor $\mathscr{Y} : \mathcal{A} \to \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})$ takes values in the full A_∞-subcategory $\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})$ of $\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})$. Furthermore, by the closedness of the multicategory $\mathsf{A}^{\mathrm{u}}_\infty$, the A_∞-functor $\mathscr{Y}$ is unital.

Let $g:\mathcal{C}\to\mathcal{A}$ be an A_∞-functor. Then an $\mathcal{A}$-$\mathcal{C}$-bimodule $\mathcal{A}^g$ is associated with it via the A_∞-functor

$$\begin{aligned}\mathcal{A}^g &= \big[\mathcal{A}^{\mathrm{op}},\mathcal{C} \xrightarrow{1,g} \mathcal{A}^{\mathrm{op}},\mathcal{A} \xrightarrow{1,\mathscr{Y}} \mathcal{A}^{\mathrm{op}},\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}},\underline{\mathsf{C}}_{\Bbbk}) \xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty}} \underline{\mathsf{C}}_{\Bbbk}\big]\\ &= \big[\mathcal{A}^{\mathrm{op}},\mathcal{C} \xrightarrow{1,g} \mathcal{A}^{\mathrm{op}},\mathcal{A} \xrightarrow{\operatorname{Hom}_{\mathcal{A}}} \underline{\mathsf{C}}_{\Bbbk}\big]. \qquad (4.4.1)\end{aligned}$$

4.4.4. Proposition. *Suppose that the A_∞-categories $\mathcal{A}$ and $\mathcal{C}$ are unital. Then the $\mathcal{A}$-$\mathcal{C}$-bimodule $\mathcal{A}^g$ is unital if and only if g is a unital A_∞-functor.*

Proof. The "if" part is obvious. For the proof of "only if" part, suppose that A_∞-functor (4.4.1) is unital. Let us prove that $g:\mathcal{C}\to\mathcal{A}$ is unital. Denote by f the composite $g\cdot\mathscr{Y}:\mathcal{C}\to\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})$. The A_∞-functor

$$f' = \big[\mathcal{A}^{\mathrm{op}},\mathcal{C} \xrightarrow{1,f} \mathcal{A}^{\mathrm{op}},\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A}^{\mathrm{op}},\underline{\mathsf{C}}_{\Bbbk}) \xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty}} \underline{\mathsf{C}}_{\Bbbk}\big]$$

is unital by assumption. The bijection

$$\varphi^{\mathsf{A}_\infty} : \mathsf{A}_\infty(\mathcal{C};\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})) \to \mathsf{A}_\infty(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_{\Bbbk})$$

shows that given f' can be obtained from a unique f. The bijection

$$\varphi^{\mathsf{A}^{\mathrm{u}}_\infty} : \mathsf{A}^{\mathrm{u}}_\infty(\mathcal{C};\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})) \to \mathsf{A}^{\mathrm{u}}_\infty(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_{\Bbbk})$$

shows that such A_∞-functor f is unital.

Thus, the composition of $g:\mathcal{C}\to\mathcal{A}$ with the unital homotopy fully faithful A_∞-functor $\mathscr{Y}:\mathcal{A}\to\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})$ is unital. Applying the multifunctor k, we find that the composite

$$\mathsf{k}\mathcal{C} \xrightarrow{\mathsf{k}g} \mathsf{k}\mathcal{A} \xrightarrow{\mathsf{k}\mathscr{Y}} \mathsf{k}\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})$$

is a unital $\mathcal{K}$-functor, and the second $\mathcal{K}$-functor is unital and fully faithful. It is an elementary category theory fact that $\mathsf{k}g$ is a unital $\mathcal{K}$-functor. Indeed, since the above composite is unital, for each object X of $\mathcal{C}$,

$$\big[\Bbbk \xrightarrow{1_X^{\mathsf{k}\mathcal{C}}} \mathsf{k}\mathcal{C}(X,X) \xrightarrow{\mathsf{k}g} \mathsf{k}\mathcal{A}(Xg,Xg) \xrightarrow{\mathsf{k}\mathscr{Y}} \underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})(H^{Xg},H^{Xg})\big] = 1_{H^{Xg}}^{\mathsf{k}\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})}.$$

On the other hand, since $\mathsf{k}\mathscr{Y}$ is unital, it follows that

$$\bigl[\Bbbk \xrightarrow{1^{\mathsf{k}\mathcal{A}}_{Xg}} \mathsf{k}\mathcal{A}(Xg,Xg) \xrightarrow{\mathsf{k}\mathscr{Y}} \underline{\mathsf{A}}^{\mathrm{u}}_{\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})(H^{Xg},H^{Xg})\bigr] = 1^{\mathsf{k}\underline{\mathsf{A}}^{\mathrm{u}}_{\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})}_{H^{Xg}}.$$

Since $\mathsf{k}\mathscr{Y}$ is fully faithful, we conclude that $(1^{\mathsf{k}\mathcal{C}}_X)\mathsf{k}g = 1^{\mathsf{k}\mathcal{A}}_{Xg}$, so that $\mathsf{k}g$ is a unital $\mathcal{K}$-functor, and therefore g is a unital A_∞-functor. □

As we already noticed, $\mathcal{A}$-$\mathcal{C}$-bimodules are objects of the **dg**-category $\mathcal{A}$-$\mathcal{C}$-bimod isomorphic to the **dg**-category $\underline{\mathsf{A}}_{\infty}(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_{\Bbbk})$. By Proposition 4.4.2, the full differential graded subcategory of $\mathcal{A}$-$\mathcal{C}$-bimod consisting of unital $\mathcal{A}$-$\mathcal{C}$-bimodules is isomorphic to the **dg**-category $\underline{\mathsf{A}}^{\mathrm{u}}_{\infty}(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_{\Bbbk})$.

4.4.5. Proposition. *Suppose $\mathcal{A}$ and $\mathcal{C}$ are unital A_∞-categories. There is a homotopy fully faithful A_∞-functor $\underline{\mathsf{A}}^{\mathrm{u}}_{\infty}(\mathcal{C};\mathcal{A}) \to \underline{\mathsf{A}}^{\mathrm{u}}_{\infty}(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_{\Bbbk})$, $g \mapsto \mathcal{A}^g$. In particular, A_∞-functors $g,h:\mathcal{C}\to\mathcal{A}$ are isomorphic if and only if the bimodules $\mathcal{A}^g$ and $\mathcal{A}^h$ are isomorphic.*

Proof. The functor in question is the composite

$$\underline{\mathsf{A}}^{\mathrm{u}}_{\infty}(\mathcal{C};\mathcal{A}) \xrightarrow{\underline{\mathsf{A}}^{\mathrm{u}}_{\infty}(1;\mathscr{Y})} \underline{\mathsf{A}}^{\mathrm{u}}_{\infty}(\mathcal{C};\underline{\mathsf{A}}^{\mathrm{u}}_{\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})) \xrightarrow[\sim]{\underline{\varphi}^{\mathsf{A}^{\mathrm{u}}_{\infty}}} \underline{\mathsf{A}}^{\mathrm{u}}_{\infty}(\mathcal{A}^{\mathrm{op}},\mathcal{C};\underline{\mathsf{C}}_{\Bbbk}).$$

Since $\mathcal{A}$ is unital, $\mathscr{Y}:\mathcal{A}\to\underline{\mathsf{A}}^{\mathrm{u}}_{\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})$ is homotopy fully faithful by Proposition A.8. Thus, the claim follows from Lemma 3.4.18. □

4.4.6. Proposition. *Let $\mathcal{A},\mathcal{C}$ be A_∞-categories, and suppose $\mathcal{A}$ is unital. Let $\mathcal{P}$ be an $\mathcal{A}$-$\mathcal{C}$-bimodule, $\phi^{\mathcal{P}}:\mathcal{A}^{\mathrm{op}},\mathcal{C}\to\underline{\mathsf{C}}_{\Bbbk}$ the corresponding A_∞-functor. The $\mathcal{A}$-$\mathcal{C}$-bimodule $\mathcal{P}$ is isomorphic to $\mathcal{A}^g$ for some A_∞-functor $g:\mathcal{C}\to\mathcal{A}$ if and only if for each object $Y\in\operatorname{Ob}\mathcal{C}$ the A_∞-functor $\phi^{\mathcal{P}}|^{Y}_{1}:\mathcal{A}^{\mathrm{op}}\to\underline{\mathsf{C}}_{\Bbbk}$ is representable.*

If conditions of the proposition are satisfied, the $\mathcal{A}$-$\mathcal{C}$-bimodule $\mathcal{P}$ is called *representable*, and an A_∞-functor $g:\mathcal{C}\to\mathcal{A}$ such that $\mathcal{P}\cong\mathcal{A}^g$ is said to *represent* $\mathcal{P}$.

Proof. The "only if" part is obvious. For the proof of "if", consider the A_∞-functor $f=(\varphi^{\mathsf{A}_\infty})^{-1}(\phi^{\mathcal{P}}):\mathcal{C}\to\underline{\mathsf{A}}_{\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})$. It acts on objects by $Y\mapsto\phi^{\mathcal{P}}|^{Y}_{1}$, $Y\in\operatorname{Ob}\mathcal{C}$, therefore it takes values in the **dg**-subcategory $\operatorname{Rep}(\mathcal{A}^{\mathrm{op}},\underline{\mathsf{C}}_{\Bbbk})$ of representable A_∞-functors. By Corollary A.9, the A_∞-functor $\mathscr{Y}:\mathcal{A}\to\operatorname{Rep}(\mathcal{A}^{\mathrm{op}},\underline{\mathsf{C}}_{\Bbbk})$ is an A_∞-equivalence. Denote by $\Psi:\operatorname{Rep}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})\to\mathcal{A}$ a quasi-inverse A_∞-functor to $\mathscr{Y}$. Let g denote the A_∞-functor $f\cdot\Psi:\mathcal{C}\to\mathcal{A}$. Then the composite $g\cdot\mathscr{Y}=f\cdot\Psi\cdot\mathscr{Y}:\mathcal{C}\to\underline{\mathsf{A}}_{\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})$ is isomorphic to f, therefore the A_∞-functor

$$\varphi^{\mathsf{A}_\infty}(g\cdot\mathscr{Y}) = \bigl[\mathcal{A}^{\mathrm{op}},\mathcal{C} \xrightarrow{1,g\cdot\mathscr{Y}} \mathcal{A}^{\mathrm{op}},\underline{\mathsf{A}}_{\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk}) \xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty}} \underline{\mathsf{C}}_{\Bbbk}\bigr],$$

corresponding to the bimodule $\mathcal{A}^g$, is isomorphic to $\varphi^{\mathsf{A}_\infty}(f)=\phi^{\mathcal{P}}$. Thus, $\mathcal{A}^g$ is isomorphic to $\mathcal{P}$. □

4.4.7. Lemma. *If $\mathcal{A}$-$\mathcal{C}$-bimodule $\mathcal{P}$ is unital, then the dual $\mathcal{C}$-$\mathcal{A}$-bimodule $\mathcal{P}^*$ is unital as well.*

Proof. The A_∞-functor $\phi^{\mathcal{P}^*}$ is the composite of two A_∞-functors, $(\phi^{\mathcal{P}})^{\mathrm{op}}:\mathcal{A},\mathcal{C}^{\mathrm{op}}\to\underline{\mathsf{C}}_{\Bbbk}{}^{\mathrm{op}}$ and $D=H^{\Bbbk}:\underline{\mathsf{C}}_{\Bbbk}{}^{\mathrm{op}}\to\underline{\mathsf{C}}_{\Bbbk}$. The latter is unital by Remark 4.4.3. The former is unital if and only if $\phi^{\mathcal{P}}$ is unital. □

4.5. A_∞-modules

A_∞-modules are particular cases of A_∞-bimodules discussed above. That is why statements about A_∞-modules are only formulated for the record.

Consider the monoidal category $(\mathscr{Q}/S,\otimes)$ of graded quivers with a fixed set of objects S. When S is a 1-element set, the category $(\mathscr{Q}/S,\otimes)$ reduces to the category of graded $\Bbbk$-modules used by Keller [**26**] in his definition of A_∞-modules over A_∞-algebras. Let C, D be graded coalgebras; let $\psi: C\to D$ be a morphism of graded coalgebras; let $\delta: M\to M\otimes C$ and $\delta: N\to N\otimes D$ be counital comodules; let $f: M\to N$ be a ψ-comodule homomorphism, $f\delta=\delta(f\otimes\psi)$; let $\xi: C\to D$ be a (ψ,ψ)-coderivation, $\xi\Delta_0=\Delta_0(\psi\otimes\xi+\xi\otimes\psi)$. Define a (ψ,f,ξ)-*connection* as a morphism $r: M\to N$ of certain degree such that the diagram

$$\begin{tikzcd}
M \arrow[r, "\delta"] \arrow[d, "r"'] & M\otimes C \arrow[d, "f\otimes\xi+r\otimes\psi"] \\
N \arrow[r, "\delta"] & N\otimes D
\end{tikzcd}$$

commutes, compare with Tradler [**59**]. Let (C,b^C) be a differential graded coalgebra. Let a counital comodule M have a $(1,1,b^C)$-connection $b^M: M\to M$ of degree 1, that is, $b^M\delta=\delta(1\otimes b^C+b^M\otimes 1)$. Its *curvature* $(b^M)^2: M\to M$ is always a C-comodule homomorphism of degree 2. If it vanishes, b^M is called a flat connection (a differential) on M.

Equivalently, we may consider the category $({}^d\mathscr{Q}/S,\otimes)$ of differential graded quivers, and coalgebras and comodules therein. For A_∞-applications it suffices to consider coalgebras (resp. comodules) whose underlying graded coalgebra (resp. comodule) has the form $Ts\mathcal{A}$ (resp. $s\mathcal{M}\otimes Ts\mathcal{C}$).

A C-comodule with a $(1,1,b^C)$-connection is the same as an (A,C)-bicomodule P with an $(\mathrm{id}_A,\mathrm{id}_C,\mathrm{id}_P,b^A,b^C)$-connection $b^P: P\to P$ of degree 1, where (A,b^A) is the trivial differential graded coalgebra $\Bbbk$ with the trivial coactions.

Let $\mathcal{M}\in\mathrm{Ob}\,\mathscr{Q}/S$ be a graded quiver such that $\mathcal{M}(X,Y)=\mathcal{M}(Y)$ depends only on $Y\in S$. For each quiver $\mathcal{C}\in\mathrm{Ob}\,\mathscr{Q}/S$, the tensor quiver $C=(Ts\mathcal{C},\Delta_0)$ is a graded coalgebra. The comodule $\delta=1\otimes\Delta_0: M=s\mathcal{M}\otimes Ts\mathcal{C}\to s\mathcal{M}\otimes Ts\mathcal{C}\otimes Ts\mathcal{C}$ is counital. Let $(\mathcal{C},b^{\mathcal{C}})$ be an A_∞-category. Equivalently, we consider augmented differential graded coalgebras of the form $(Ts\mathcal{C},\Delta_0,b^{\mathcal{C}})$. Let $b^{\mathcal{M}}: s\mathcal{M}\otimes Ts\mathcal{C}\to s\mathcal{M}\otimes Ts\mathcal{C}$ be a $(1,1,b^{\mathcal{C}})$-connection. Define matrix coefficients of $b^{\mathcal{M}}$ to be

$$b^{\mathcal{M}}_{mn}=(1\otimes\mathrm{in}_m)\cdot b^{\mathcal{M}}\cdot(1\otimes\mathrm{pr}_n): s\mathcal{M}\otimes T^m s\mathcal{C}\to s\mathcal{M}\otimes T^n s\mathcal{C},\quad m,n\geqslant 0.$$

Coefficients $b^{\mathcal{M}}_{m0}: s\mathcal{M}\otimes T^m s\mathcal{C}\to s\mathcal{M}$ are abbreviated to $b^{\mathcal{M}}_m$ and called components of $b^{\mathcal{M}}$.

A version of the following statement occurs in [**34**, Lemme 2.1.2.1].

4.5.1. Lemma. *Any $(1,1,b^{\mathcal{C}})$-connection $b^{\mathcal{M}}: s\mathcal{M}\otimes Ts\mathcal{C}\to s\mathcal{M}\otimes Ts\mathcal{C}$ is determined in a unique way by its components $b^{\mathcal{M}}_n: s\mathcal{M}\otimes T^n s\mathcal{C}\to s\mathcal{M}$, $n\geqslant 0$. Matrix coefficients of $b^{\mathcal{M}}$ are expressed via components of $b^{\mathcal{M}}$ and components of the codifferential $b^{\mathcal{C}}$ as follows:*

$$b^{\mathcal{M}}_{mn}=b^{\mathcal{M}}_{m-n}\otimes 1^{\otimes n}+\sum_{\substack{p+k+q=m\\ p+1+q=n}}1^{\otimes 1+p}\otimes b^{\mathcal{C}}_k\otimes 1^{\otimes q}: s\mathcal{M}\otimes T^m s\mathcal{C}\to s\mathcal{M}\otimes T^n s\mathcal{C}$$

for $m\geqslant n$. If $m<n$, the matrix coefficient $b^{\mathcal{M}}_{mn}$ vanishes.

The morphism $(b^{\mathcal{M}})^2 : s\mathcal{M} \otimes Ts\mathcal{C} \to s\mathcal{M} \otimes Ts\mathcal{C}$ is a $(1,1,0)$-connection of degree 2, therefore equation $(b^{\mathcal{M}})^2 = 0$ is equivalent to its particular case $(b^{\mathcal{M}})^2(1 \otimes \mathrm{pr}_0) = 0 : s\mathcal{M} \otimes Ts\mathcal{C} \to s\mathcal{M}$. Thus $b^{\mathcal{M}}$ is a flat connection if for each $m \geqslant 0$ the following equation holds:

$$\sum_{n=0}^{m}(b^{\mathcal{M}}_{m-n} \otimes 1^{\otimes n})b^{\mathcal{M}}_n + \sum_{p+k+q=m}(1^{\otimes 1+p} \otimes b^{\mathcal{C}}_k \otimes 1^{\otimes q})b^{\mathcal{M}}_{p+1+q} = 0 : s\mathcal{M} \otimes T^m s\mathcal{C} \to s\mathcal{M}. \quad (4.5.1)$$

Equivalently, a $Ts\mathcal{C}$-comodule with a flat connection is the $Ts\mathcal{C}$-comodule $(s\mathcal{M} \otimes Ts\mathcal{C}, b^{\mathcal{M}})$ in the category $({}^d\mathcal{Q}/S, \otimes)$. It consists of the following data: a graded $\Bbbk$-module $\mathcal{M}(X)$ for each object X of $\mathcal{C}$; a family of $\Bbbk$-linear maps of degree 1

$$b^{\mathcal{M}}_n : s\mathcal{M}(X_0) \otimes s\mathcal{C}(X_0, X_1) \otimes \cdots \otimes s\mathcal{C}(X_{n-1}, X_n) \to s\mathcal{M}(X_n), \quad n \geqslant 0,$$

subject to equations (4.5.1). Equation (4.5.1) for $m = 0$ implies $(b^{\mathcal{M}}_0)^2 = 0$, that is, $(s\mathcal{M}(X), b^{\mathcal{M}}_0)$ is a chain complex, for each object $X \in \operatorname{Ob}\mathcal{C}$. We call a $Ts\mathcal{C}$-comodule with a flat connection $(s\mathcal{M} \otimes Ts\mathcal{C}, b^{\mathcal{M}})$, $\mathcal{M}(*, Y) = \mathcal{M}(Y)$, a $\mathcal{C}$-*module* (an A_∞-*module over* $\mathcal{C}$). $\mathcal{C}$-modules form a differential graded category $\mathcal{C}$-mod. The notion of a module over some kind of A_∞-category was introduced by Lefèvre-Hasegawa under the name of polydule [**34**].

4.5.2. Proposition. *An arbitrary A_∞-functor $\phi : \mathcal{C} \to \underline{\mathsf{C}}_\Bbbk$ determines a $Ts\mathcal{C}$-comodule $s\mathcal{M} \otimes Ts\mathcal{C}$ with a flat connection $b^{\mathcal{M}}$ by the formulas: $\mathcal{M}(X) = X\phi$, for each object X of $\mathcal{C}$, $b^{\mathcal{M}}_0 = s^{-1}ds : s\mathcal{M}(X) \to s\mathcal{M}(X)$, where d is the differential in the complex $X\phi$, and for $n > 0$*

$$\begin{aligned} b^{\mathcal{M}}_n = \big[&s\mathcal{M}(X_0) \otimes s\mathcal{C}(X_0, X_1) \otimes \cdots \otimes s\mathcal{C}(X_{n-1}, X_n) \\ &\xrightarrow{1\otimes\phi_n} s\mathcal{M}(X_0) \otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{M}(X_0), \mathcal{M}(X_n)) \\ &\xrightarrow{(s\otimes s)^{-1}} \mathcal{M}(X_0) \otimes \underline{\mathsf{C}}_\Bbbk(\mathcal{M}(X_0), \mathcal{M}(X_n)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}} \mathcal{M}(X_n) \xrightarrow{s} s\mathcal{M}(X_n)\big]. \end{aligned} \quad (4.5.2)$$

This mapping from A_∞-functors to $\mathcal{C}$-modules is bijective. Moreover, the differential graded categories $\underline{\mathsf{A}_\infty}(\mathcal{C}; \underline{\mathsf{C}}_\Bbbk)$ and $\mathcal{C}$-mod are isomorphic.

An A_∞-module over $\mathcal{C}$ is defined as an A_∞-functor $\phi : \mathcal{C} \to \underline{\mathsf{C}}_\Bbbk$ by Seidel [**48**, Section 1j]. The above proposition shows that both definitions of $\mathcal{C}$-modules are equivalent. In the differential graded case $\mathcal{C}$-modules are actively used by Drinfeld [**13**].

Notice that a graded quiver $\mathcal{M} \in \operatorname{Ob}\mathcal{Q}/S$ such that $\mathcal{M}(X, Y) = \mathcal{M}(Y)$ depends only on $Y \in S$ is nothing else but a graded span $\mathcal{M}$ with $\operatorname{Ob}_s\mathcal{M} = \{*\}$, $\operatorname{Ob}_t\mathcal{M} = S$. Thus, $Ts\mathcal{C}$-comodules of the form $s\mathcal{M} \otimes Ts\mathcal{C}$ are nothing else but $Ts\mathcal{A}$-$Ts\mathcal{C}$-bicomodules $Ts\mathcal{A} \otimes s\mathcal{M} \otimes Ts\mathcal{C}$ for the graded quiver $\mathcal{A}$ with one object $*$ and with $\mathcal{A}(*, *) = 0$. Furthermore, A_∞-modules $\mathcal{M}$ over an A_∞-category $\mathcal{C}$ are the same as $\mathcal{A}$-$\mathcal{C}$-bicomodules.

4.5.3. Definition. Let $\mathcal{C}$ be a unital A_∞-category. A $\mathcal{C}$-module $\mathcal{M}$ determined by an A_∞-functor $\phi : \mathcal{C} \to \underline{\mathsf{C}}_\Bbbk$ is called *unital* if ϕ is unital.

4.5.4. Proposition. *A $\mathcal{C}$-module $\mathcal{M}$ is unital if and only if, for each $X \in \operatorname{Ob}\mathcal{C}$, the composition*

$$\big[s\mathcal{M}(X) \cong s\mathcal{M}(X) \otimes \Bbbk \xrightarrow{1\otimes {}_X\mathbf{i}^{\mathcal{C}}_0} s\mathcal{M}(X) \otimes s\mathcal{C}(X, X) \xrightarrow{b^{\mathcal{M}}_1} s\mathcal{M}(X)\big]$$

is homotopic to the identity map.

Proof. The second statement expands to the property that

$$\bigl[s\mathcal{M}(X) \cong s\mathcal{M}(X) \otimes \Bbbk \xrightarrow{s^{-1}\otimes {}_X\mathbf{i}_0^{\mathcal{C}}} \mathcal{M}(X) \otimes s\mathcal{C}(X,X) \\ \xrightarrow{1\otimes\phi_1 s^{-1}} \mathcal{M}(X) \otimes \underline{\mathsf{C}}_\Bbbk(\mathcal{M}(X),\mathcal{M}(X)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}} \mathcal{M}(X) \xrightarrow{s} s\mathcal{M}(X)\bigr]$$

is homotopic to the identity map. That is,

$$ {}_X\mathbf{i}_0^{\mathcal{C}}\phi_1 s^{-1} = 1_{s\mathcal{M}(X)} + v m_1^{\underline{\mathsf{C}}_\Bbbk}, \quad \text{or,} \quad {}_X\mathbf{i}_0^{\mathcal{C}}\phi_1 = 1_{\mathcal{M}(X)} s + v s b_1^{\underline{\mathsf{C}}_\Bbbk}.$$

In other words, the A_∞-functor ϕ is unital. □

CHAPTER 5

Serre A_∞-functors

Having worked out A_∞-bimodules, it is now easy to define Serre A_∞-functors. Namely, an A_∞-functor S from a unital A_∞-category $\mathcal{A}$ to itself is a (right) Serre functor if it represents the dual of the regular $\mathcal{A}$-$\mathcal{A}$-bimodule. Assembling results of Chapters 2, 3, and 4, we prove that a Serre A_∞-functor $S : \mathcal{A} \to \mathcal{A}$ gives rise to a Serre $\mathcal{K}$-functor $\mathsf{k}S : \mathsf{k}\mathcal{A} \to \mathsf{k}\mathcal{A}$. If $\Bbbk$ is a field, then the induced functor in the homotopy category $H^0(\mathcal{A})$ is an ordinary Serre functor. Moreover, it turns out that the $\Bbbk$-**Cat**-multifunctor k reflects Serre functors, meaning that a unital A_∞-category $\mathcal{A}$ admits a Serre A_∞-functor if and only if $\mathsf{k}\mathcal{A}$ admits a Serre $\mathcal{K}$-functor. If $\Bbbk$ is a field and the cohomology of $\mathcal{A}$ is finite dimensional, then a Serre A_∞-functor $S : \mathcal{A} \to \mathcal{A}$, if it exists, is homotopy fully faithful. Furthermore, if $\Bbbk$ is a field and $\mathcal{A}$ is closed under shifts, then the existence of a Serre A_∞-functor is equivalent to the existence of an ordinary Serre functor $H^0(\mathcal{A}) \to H^0(\mathcal{A})$. In particular, if a triangulated category $\mathcal{C}$ is the homotopy category of a pretriangulated A_∞-category $\mathcal{A}$ in the sense of [**3**], and $\mathcal{C}$ admits a Serre functor, then $\mathcal{A}$ admits a Serre A_∞-functor.

The definition of Serre A_∞-functor via A_∞-bimodules was suggested to the author by Volodymyr Lyubashenko. It was also his idea to consider as an example the strict case of Serre A_∞-functors. Proposition 5.2.1 is mainly his contribution. The results of this chapter have been published in [**41**].

5.1. Basic properties of Serre A_∞-functors

Let us make formal the definition of Serre A_∞-functor given in the introduction to the chapter.

5.1.1. Definition (cf. Soibelman [**51**], Kontsevich and Soibelman, sequel to [**31**])**.** A *right Serre A_∞-functor* $S : \mathcal{A} \to \mathcal{A}$ in a unital A_∞-category $\mathcal{A}$ is an A_∞-functor for which the $\mathcal{A}$-bimodules $\mathcal{A}^S = \bigl[\mathcal{A}^{\mathrm{op}}, \mathcal{A} \xrightarrow{1,S} \mathcal{A}^{\mathrm{op}}, \mathcal{A} \xrightarrow{\mathrm{Hom}_{\mathcal{A}}} \underline{\mathsf{C}}_\Bbbk\bigr]$ and $\mathcal{A}^*$ are isomorphic. If, moreover, S is an A_∞-equivalence, it is called a *Serre A_∞-functor.*

By Lemma 4.4.7 and Proposition 4.4.4, if a right Serre A_∞-functor exists, then it is unital. By Proposition 4.4.5, it is unique up to isomorphism.

5.1.2. Proposition. *If $S : \mathcal{A} \to \mathcal{A}$ is a (right) Serre A_∞-functor, then $\mathsf{k}S : \mathsf{k}\mathcal{A} \to \mathsf{k}\mathcal{A}$ is a (right) Serre $\mathcal{K}$-functor.*

Proof. Let $p : \mathcal{A}^S \to \mathcal{A}^*$ be an isomorphism. More precisely, p is an isomorphism

$$(1, S) \cdot \mathrm{Hom}_{\mathcal{A}} \to \mathrm{Hom}^{\mathrm{op}}_{\mathcal{A}^{\mathrm{op}}} \cdot D : \mathcal{A}^{\mathrm{op}}, \mathcal{A} \to \underline{\mathsf{C}}_\Bbbk.$$

We visualize this by the following diagram:

$$\begin{tikzcd}
\mathcal{A}^{\mathrm{op}},\mathcal{A} \arrow[r, "1{,}S"] \arrow[d, "\mathrm{Hom}^{\mathrm{op}}_{\mathcal{A}^{\mathrm{op}}}"'] & \mathcal{A}^{\mathrm{op}},\mathcal{A} \arrow[d, "\mathrm{Hom}_{\mathcal{A}}"] \arrow[dl, Rightarrow, "p"'] \\
\underline{\mathsf{C}}_{\Bbbk}{}^{\mathrm{op}} \arrow[r, "D"] & \underline{\mathsf{C}}_{\Bbbk}
\end{tikzcd}$$

Applying the $\Bbbk$-**Cat**-multifunctor k, and using Lemma 3.5.1, Proposition 4.2.2, and results of Section 4.3.6, we get a similar diagram in $\mathcal{K}$-**Cat**:

$$\begin{tikzcd}
\mathsf{k}\mathcal{A}^{\mathrm{op}}\boxtimes\mathsf{k}\mathcal{A} \arrow[r, "1\boxtimes\mathsf{k}S"] \arrow[d, "\mathrm{Hom}^{\mathrm{op}}_{\mathsf{k}\mathcal{A}^{\mathrm{op}}}"'] & \mathsf{k}\mathcal{A}^{\mathrm{op}}\boxtimes\mathsf{k}\mathcal{A} \arrow[d, "\mathrm{Hom}_{\mathsf{k}\mathcal{A}}"] \arrow[dl, Rightarrow, "\mathsf{k}p"'] \\
\underline{\mathcal{K}}^{\mathrm{op}} \arrow[r, "\underline{\mathcal{K}}(-{,}\Bbbk)"] & \underline{\mathcal{K}}
\end{tikzcd}$$

Since $\mathsf{k}p$ is an isomorphism, it follows that $\mathsf{k}S$ is a right Serre $\mathcal{K}$-functor.

The A_∞-functor S is an equivalence if and only if $\mathsf{k}S$ is a $\mathcal{K}$-equivalence. $\square$

When $\mathcal{A}$ is an A_∞-algebra and S is its identity endomorphism, the natural transformation $p:\mathcal{A}\to\mathcal{A}^*$ identifies with an ∞-inner-product on $\mathcal{A}$, as defined by Tradler [**59**, Definition 5.3].

5.1.3. Corollary. *Let $\mathcal{A}$ be a unital A_∞-category. Then $\mathcal{A}$ admits a (right) Serre A_∞-functor if and only if $\mathsf{k}\mathcal{A}$ admits a (right) Serre $\mathcal{K}$-functor.*

Proof. The "only if" part is proven above. Suppose $\mathsf{k}\mathcal{A}$ admits a Serre $\mathcal{K}$-functor. By Proposition 2.2.5 this implies representability of the $\mathcal{K}$-functor

$$\mathrm{Hom}_{\mathsf{k}\mathcal{A}}(Y,-)^{\mathrm{op}}\cdot\underline{\mathcal{K}}(-,\Bbbk)=\mathsf{k}[\mathrm{Hom}_{\mathcal{A}}(Y,-)^{\mathrm{op}}\cdot D]:\mathsf{k}\mathcal{A}^{\mathrm{op}}\to\underline{\mathcal{K}}=\mathsf{k}\underline{\mathsf{C}}_{\Bbbk},$$

for each object $Y\in\operatorname{Ob}\mathcal{A}$. By Corollary A.6 the A_∞-functor

$$\mathrm{Hom}_{\mathcal{A}}(Y,-)^{\mathrm{op}}\cdot D=(\mathrm{Hom}^{\mathrm{op}}_{\mathcal{A}^{\mathrm{op}}}\cdot D)\big|_1^Y:\mathcal{A}^{\mathrm{op}}\to\underline{\mathsf{C}}_{\Bbbk}$$

is representable, for each $Y\in\operatorname{Ob}\mathcal{A}$. By Proposition 4.4.6 the bimodule $\mathcal{A}^*$ corresponding to the A_∞-functor $\mathrm{Hom}^{\mathrm{op}}_{\mathcal{A}^{\mathrm{op}}}\cdot D$ is isomorphic to $\mathcal{A}^S$ for some A_∞-functor $S:\mathcal{A}\to\mathcal{A}$. $\square$

5.1.4. Corollary. *Suppose $\mathcal{A}$ is a Hom-reflexive A_∞-category,* i.e., *the complex $\mathcal{A}(X,Y)$ is reflexive in $\mathcal{K}$ for each pair of objects $X,Y\in\operatorname{Ob}\mathcal{A}$. If $S:\mathcal{A}\to\mathcal{A}$ is a right Serre A_∞-functor, then S is homotopy fully faithful.*

Proof. The $\mathcal{K}$-functor $\mathsf{k}S$ is fully faithful by Proposition 2.2.4. $\square$

5.1.5. Definition. A unital A_∞-category $\mathcal{A}$ is called *closed under shifts* if the $\mathcal{K}$-category $\mathsf{k}\mathcal{A}$ is closed under shifts.

In [**3**] a different, but equivalent, definition of A_∞-categories closed under shifts is given. In particular, a pretriangulated A_∞-category in the sense of [**3**] is closed under shifts. We are not going to pursue the subject here.

The above corollary applies, in particular, if $\Bbbk$ is a field and all homology spaces $H^n(\mathcal{A}(X,Y))$ are finite dimensional. If $\mathcal{A}$ is closed under shifts, the latter condition is equivalent to requiring that $H^0(\mathcal{A}(X,Y))$ be finite dimensional for each pair $X,Y\in\operatorname{Ob}\mathcal{A}$. Indeed, $H^n(\mathcal{A}(X,Y))=H^n(\mathsf{k}\mathcal{A}(X,Y))=H^0(\mathsf{k}\mathcal{A}(X,Y)[n])=H^0((\mathsf{k}\mathcal{A})^{[]}((X,0),(Y,n)))$.

The $\mathcal{K}$-category $\mathsf{k}\mathcal{A}$ is closed under shifts by definition, therefore there exists an isomorphism $\alpha : (Y, n) \to (Z, 0)$ in $(\mathsf{k}\mathcal{A})^{[]}$, for some $Z \in \operatorname{Ob}\mathcal{A}$. It induces an isomorphism

$$(\mathsf{k}\mathcal{A})^{[]}(1,\alpha) : (\mathsf{k}\mathcal{A})^{[]}((X,0),(Y,n)) \xrightarrow{\sim} (\mathsf{k}\mathcal{A})^{[]}((X,0),(Z,0)) = \mathsf{k}\mathcal{A}(X,Z)$$

in $\mathcal{K}$, thus an isomorphism in cohomology

$$H^n(\mathcal{A}(X,Y)) = H^0((\mathsf{k}\mathcal{A})^{[]}((X,0),(Y,n))) \cong H^0(\mathsf{k}\mathcal{A}(X,Z)) = H^0(\mathcal{A}(X,Z)).$$

The latter space is finite dimensional by assumption.

5.1.6. Theorem. *Suppose $\Bbbk$ is a field, $\mathcal{A}$ is a unital A_∞-category closed under shifts. Then the following conditions are equivalent:*

(a) *$\mathcal{A}$ admits a (right) Serre A_∞-functor;*
(b) *$\mathsf{k}\mathcal{A}$ admits a (right) Serre $\mathcal{K}$-functor;*
(c) *$H^\bullet\mathcal{A} \overset{\mathrm{def}}{=} H^\bullet_*(\mathsf{k}\mathcal{A})$ admits a (right) Serre $\mathbf{gr}$-functor;*
(d) *$H^0(\mathcal{A})$ admits a (right) Serre $\Bbbk$-linear functor.*

Proof. Equivalence of (a) and (b) is proven in Corollary 5.1.3. Conditions (b) and (c) are equivalent due to Proposition 2.3.3, because $H^\bullet : \mathcal{K} \to \mathbf{gr}$ is an equivalence of symmetric monoidal categories. Condition (c) implies (d) for arbitrary $\mathbf{gr}$-category by Corollary 2.3.5, in particular, for $H^\bullet\mathcal{A}$. Note that $H^\bullet\mathcal{A}$ is closed under shifts by the discussion preceding Proposition 2.3.8. Therefore, (d) implies (c) due to Proposition 2.3.8. □

An application of this theorem is the following. Let $\Bbbk$ be a field. Drinfeld's construction of quotients of pretriangulated $\mathbf{dg}$-categories [**13**] allows to find a pretriangulated $\mathbf{dg}$-category $\mathcal{A}$ such that $H^0(\mathcal{A})$ is some familiar derived category (e.g. the bounded derived category $D^b(X)$ of coherent sheaves on a projective variety X). If a right Serre functor exists for $H^0(\mathcal{A})$, then $\mathcal{A}$ admits a right Serre A_∞-functor S by the above theorem. That is the case of $H^0(\mathcal{A}) \simeq D^b(X)$, where X is a smooth projective variety [**4**, Example 3.2]. Notice that $S : \mathcal{A} \to \mathcal{A}$ does not have to be a $\mathbf{dg}$-functor.

5.1.7. Proposition. *Let $S : \mathcal{A} \to \mathcal{A}$, $S' : \mathcal{B} \to \mathcal{B}$ be right Serre A_∞-functors. Let $g : \mathcal{B} \to \mathcal{A}$ be an A_∞-equivalence. Then the A_∞-functors $S'g : \mathcal{B} \to \mathcal{A}$ and $gS : \mathcal{B} \to \mathcal{A}$ are isomorphic.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
\mathcal{B}^{\mathrm{op}},\mathcal{B} \arrow[rrr, "1{,}S'"] \arrow[dr, "g^{\mathrm{op}}{,}g"] \arrow[dd, "\mathrm{Hom}^{\mathrm{op}}_{\mathcal{B}^{\mathrm{op}}}"'] & & & \mathcal{B}^{\mathrm{op}},\mathcal{B} \arrow[dl, "g^{\mathrm{op}}{,}g"'] \arrow[dd, "\mathrm{Hom}_{\mathcal{B}}"] \\
{} & \mathcal{A}^{\mathrm{op}},\mathcal{A} \arrow[r, "1{,}S"] \arrow[l, Rightarrow, "(r^g)^{\mathrm{op}}"', "\sim"] \arrow[dl, "\mathrm{Hom}^{\mathrm{op}}_{\mathcal{A}^{\mathrm{op}}}"] & \mathcal{A}^{\mathrm{op}},\mathcal{A} \arrow[dr, "\mathrm{Hom}_{\mathcal{A}}"'] & {} \arrow[l, Rightarrow, "r^g"', "\sim"] \\
\underline{\mathsf{C}}_{\Bbbk}{}^{\mathrm{op}} \arrow[rrr, "D"] & & & \underline{\mathsf{C}}_{\Bbbk}
\end{tikzcd}$$

Here the natural A_∞-isomorphism r^g is that constructed in Corollary 4.3.2. The exterior and the lower trapezoid commute up to natural A_∞-isomorphisms by definition of right Serre functor. It follows that the A_∞-functors $(g^{\mathrm{op}}, S'g) \cdot \mathrm{Hom}_{\mathcal{A}} : \mathcal{B}^{\mathrm{op}}, \mathcal{B} \to \underline{\mathsf{C}}_{\Bbbk}$ and $(g^{\mathrm{op}}, gS) \cdot \mathrm{Hom}_{\mathcal{A}} : \mathcal{B}^{\mathrm{op}}, \mathcal{B} \to \underline{\mathsf{C}}_{\Bbbk}$ are isomorphic. Consider the A_∞-functors

$$\mathcal{B} \xrightarrow{S'g} \mathcal{A} \xrightarrow{\mathscr{Y}} \underline{\mathsf{A}}^{\mathrm{u}}_{\infty}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk}) \xrightarrow{\underline{\mathsf{A}}^{\mathrm{u}}_{\infty}(g^{\mathrm{op}};1)} \underline{\mathsf{A}}^{\mathrm{u}}_{\infty}(\mathcal{B}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})$$

and

$$\mathcal{B} \xrightarrow{gS} \mathcal{A} \xrightarrow{\mathscr{Y}} \underline{\mathsf{A}^{\mathrm{u}}_{\infty}}(\mathcal{A}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk}) \xrightarrow{\mathsf{A}^{\mathrm{u}}_{\infty}(g^{\mathrm{op}};1)} \underline{\mathsf{A}^{\mathrm{u}}_{\infty}}(\mathcal{B}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})$$

that correspond to $(g^{\mathrm{op}}, S'g)\cdot \mathrm{Hom}_{\mathcal{A}} : \mathcal{B}^{\mathrm{op}}, \mathcal{B} \to \underline{\mathsf{C}}_{\Bbbk}$ and $(g^{\mathrm{op}}, gS)\cdot \mathrm{Hom}_{\mathcal{A}} : \mathcal{B}^{\mathrm{op}}, \mathcal{B} \to \underline{\mathsf{C}}_{\Bbbk}$ by closedness. More precisely, the upper line is equal to $(\underline{\varphi}^{\mathsf{A}^{\mathrm{u}}_{\infty}})^{-1}((g^{\mathrm{op}}, S'g)\cdot \mathrm{Hom}_{\mathcal{A}})$ and the bottom line is equal to $(\underline{\varphi}^{\mathsf{A}^{\mathrm{u}}_{\infty}})^{-1}((g^{\mathrm{op}}, gS)\cdot \mathrm{Hom}_{\mathcal{A}})$, where

$$\underline{\varphi}^{\mathsf{A}^{\mathrm{u}}_{\infty}} : \underline{\mathsf{A}^{\mathrm{u}}_{\infty}}(\mathcal{B};\underline{\mathsf{A}^{\mathrm{u}}_{\infty}}(\mathcal{B}^{\mathrm{op}};\underline{\mathsf{C}}_{\Bbbk})) \to \underline{\mathsf{A}^{\mathrm{u}}_{\infty}}(\mathcal{B}^{\mathrm{op}},\mathcal{B};\underline{\mathsf{C}}_{\Bbbk})$$

is the natural isomorphism of A_∞-categories coming from the closed structure. It follows that the A_∞-functors $S'\cdot g\cdot \mathscr{Y}\cdot \underline{\mathsf{A}^{\mathrm{u}}_{\infty}}(g^{\mathrm{op}};1)$ and $g\cdot S\cdot \mathscr{Y}\cdot \underline{\mathsf{A}^{\mathrm{u}}_{\infty}}(g^{\mathrm{op}};1)$ are isomorphic. Obviously, the A_∞-functor g^{op} is an equivalence, therefore so is the A_∞-functor $\underline{\mathsf{A}^{\mathrm{u}}_{\infty}}(g^{\mathrm{op}};1)$ since $\underline{\mathsf{A}^{\mathrm{u}}_{\infty}}(-;\underline{\mathsf{C}}_{\Bbbk})$ is an $\underline{\mathsf{A}^{\mathrm{u}}_{\infty}}$-functor, see Example 3.4.17 and the discussion preceding it. Therefore, the A_∞-functors $S'\cdot g\cdot \mathscr{Y}$ and $g\cdot S\cdot \mathscr{Y}$ are isomorphic. However, this implies that the A_∞-functors $(1,S'g)\cdot \mathrm{Hom}_{\mathcal{A}} = \underline{\varphi}^{\mathsf{A}^{\mathrm{u}}_{\infty}}(S'\cdot g\cdot \mathscr{Y})$ and $(1,gS)\cdot \mathrm{Hom}_{\mathcal{A}} = \underline{\varphi}^{\mathsf{A}^{\mathrm{u}}_{\infty}}(g\cdot S\cdot \mathscr{Y})$ are isomorphic as well. These A_∞-functors correspond to $(\mathcal{A},\mathcal{B})$-bimodules $\mathcal{A}^{S'g}$ and $\mathcal{A}^{gS}$ respectively. Proposition 4.4.5 implies an isomorphism between the A_∞-functors $S'g$ and gS. $\square$

5.2. The strict case of Serre A_∞-functors

Let $\mathcal{A}$ be an A_∞-category, let $S : \mathcal{A} \to \mathcal{A}$ be an A_∞-functor. The $(0,0)$-component of a cycle $p \in \underline{\mathsf{A}_{\infty}}(\mathcal{A}^{\mathrm{op}},\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(\mathcal{A}^S,\mathcal{A}^*)[1]^{-1}$ determines for all objects X, Y of $\mathcal{A}$ a degree 0 map

$$\begin{aligned}\Bbbk \cong T^0 s\mathcal{A}^{\mathrm{op}}(X,X)\otimes T^0 s\mathcal{A}(Y,Y) &\xrightarrow{p_{00}} \\ s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,YS),\mathcal{A}^*(X,Y)) &\xrightarrow{s^{-1}} \underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,YS),\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(Y,X),\Bbbk)).\end{aligned}$$

The obtained mapping $\mathcal{A}(X,YS) \to \underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(Y,X),\Bbbk)$ is a chain map, since $p_{00}s^{-1}m_1 = 0$. Its homotopy class gives $\psi_{X,Y}$ from (2.2.1) when the pair (S,p) is projected to $(\mathsf{k}S, \psi = \mathsf{k}p)$ via the multifunctor k.

Let us consider a particularly simple case of an A_∞-category $\mathcal{A}$ with a right Serre functor $S : \mathcal{A} \to \mathcal{A}$ which is a strict A_∞-functor (only the first component does not vanish) and with an invertible natural A_∞-transformation $p : \mathcal{A}^S \to \mathcal{A}^* : \mathcal{A}^{\mathrm{op}},\mathcal{A} \to \underline{\mathsf{C}}_{\Bbbk}$ whose only non-vanishing component is

$$p_{00} : T^0 s\mathcal{A}^{\mathrm{op}}(X,X)\otimes T^0 s\mathcal{A}(Y,Y) \to s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,YS),\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(Y,X),\Bbbk)).$$

The invertibility of p, equivalent to the invertibility of p_{00}, means that the induced chain maps $r_{00} : \mathcal{A}(X,YS) \to \underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(Y,X),\Bbbk)$ are homotopy invertible, for all objects X, Y of $\mathcal{A}$. General formula (3.3.3) for pB_1 gives the components $(pB_1)_{00} = p_{00}b_1$ and

$$(pB_1)_{kn} = ([(1,S)\,\mathrm{Hom}_{\mathcal{A}}]_{kn}\otimes p_{00})b_2^{\underline{\mathsf{C}}_{\Bbbk}} + (p_{00}\otimes [\mathrm{Hom}^{\mathrm{op}}_{\mathcal{A}^{\mathrm{op}}}\cdot D]_{kn})b_2^{\underline{\mathsf{C}}_{\Bbbk}} \qquad (5.2.1)$$

for $k+n>0$. Since p is natural, $pB_1=0$, thus the right hand side of (5.2.1) must vanish. Expanding out the first summand we get

$$\begin{aligned}
&(-)^k\big[T^ks\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes T^ns\mathcal{A}(Y_0,Y_n)\\
&\qquad\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}}\underline{\mathsf{C}}_\Bbbk\big(s\mathcal{A}(X_0,Y_0S),s\mathcal{A}(X_0,Y_0S)\otimes T^ks\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes T^ns\mathcal{A}(Y_0,Y_n)\big)\\
&\qquad\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\rho_c(1^{\otimes k+1}\otimes S_1^{\otimes n})b_{k+1+n}r_{00})}\underline{\mathsf{C}}_\Bbbk\big(s\mathcal{A}(X_0,Y_0S),s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(Y_n,X_k),\Bbbk)\big)\\
&\qquad\qquad\qquad\xrightarrow{[-1]s}s\underline{\mathsf{C}}_\Bbbk\big(\mathcal{A}(X_0,Y_0S),\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(Y_n,X_k),\Bbbk)\big)\big].
\end{aligned}$$

Expanding out the second summand we obtain

$$\begin{aligned}
&-(-)^k\big[T^ks\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes T^ns\mathcal{A}(Y_0,Y_n)\\
&\qquad\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}}\underline{\mathsf{C}}_\Bbbk\big(s\mathcal{A}(Y_n,X_k),s\mathcal{A}(Y_n,X_k)\otimes T^ks\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes T^ns\mathcal{A}(Y_0,Y_n)\big)\\
&\quad\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,(123)_c(1\otimes1\otimes\omega_c^0))}\underline{\mathsf{C}}_\Bbbk\big(s\mathcal{A}(Y_n,X_k),T^ns\mathcal{A}(Y_0,Y_n)\otimes s\mathcal{A}(Y_n,X_k)\otimes T^ks\mathcal{A}(X_k,X_0)\big)\\
&\qquad\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,b_{n+1+k})}\underline{\mathsf{C}}_\Bbbk\big(s\mathcal{A}(Y_n,X_k),s\mathcal{A}(Y_0,X_0)\big)\xrightarrow{[-1]s}s\underline{\mathsf{C}}_\Bbbk\big(\mathcal{A}(Y_n,X_k),\mathcal{A}(Y_0,X_0)\big)\\
&\qquad\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}}\underline{\mathsf{C}}_\Bbbk\big(s\mathcal{A}(X_0,Y_0S),s\mathcal{A}(X_0,Y_0S)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(Y_n,X_k),\mathcal{A}(Y_0,X_0))\big)\\
&\qquad\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,(r_{00}\otimes1)cb_2)}\underline{\mathsf{C}}_\Bbbk\big(s\mathcal{A}(X_0,Y_0S),s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(Y_n,X_k),\Bbbk)\big)\\
&\qquad\qquad\qquad\xrightarrow{[-1]s}s\underline{\mathsf{C}}_\Bbbk\big(\mathcal{A}(X_0,Y_0S),\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(Y_n,X_k),\Bbbk)\big)\big].
\end{aligned}$$

The sum of the two above expressions must vanish. The obtained equation can be simplified further by closedness of $\mathsf{C}_\Bbbk$. The homotopy isomorphism r_{00} induces the pairing

$$q_{00}=\big[\mathcal{A}(Y,X)\otimes\mathcal{A}(X,YS)\xrightarrow{1\otimes r_{00}}\mathcal{A}(Y,X)\otimes\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(Y,X),\Bbbk)\xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}}\Bbbk\big].$$

Using it we write down the naturality condition for p as follows: for all $k\geqslant0$, $n\geqslant0$,

$$\begin{aligned}
&\big[\mathcal{A}(Y_n,X_k)\otimes T^k\mathcal{A}(X_k,X_0)\otimes\mathcal{A}(X_0,Y_0S)\otimes T^n\mathcal{A}(Y_0,Y_n)\\
&\qquad\qquad\xrightarrow{(1^{\otimes3}\otimes(sS_1s^{-1})^{\otimes n})(1\otimes m_{k+1+n})}\mathcal{A}(Y_n,X_k)\otimes\mathcal{A}(X_k,Y_nS)\xrightarrow{q_{00}}\Bbbk\big]\\
&=(-)^{(k+1)(n+1)}\big[\mathcal{A}(Y_n,X_k)\otimes T^k\mathcal{A}(X_k,X_0)\otimes\mathcal{A}(X_0,Y_0S)\otimes T^n\mathcal{A}(Y_0,Y_n)\\
&\qquad\qquad\xrightarrow{(1234)_c}T^n\mathcal{A}(Y_0,Y_n)\otimes\mathcal{A}(Y_n,X_k)\otimes T^k\mathcal{A}(X_k,X_0)\otimes\mathcal{A}(X_0,Y_0S)\\
&\qquad\qquad\qquad\xrightarrow{m_{n+1+k}\otimes1}\mathcal{A}(Y_0,X_0)\otimes\mathcal{A}(X_0,Y_0S)\xrightarrow{q_{00}}\Bbbk\big]. \qquad (5.2.2)
\end{aligned}$$

Let us give a sufficient condition for this equation to hold true.

5.2.1. Proposition. *Let $\mathcal{A}$ be an A_∞-category, and let $S:\mathcal{A}\to\mathcal{A}$ be a strict A_∞-functor. Suppose given a pairing $q_{00}:\mathcal{A}(Y,X)\otimes\mathcal{A}(X,YS)\to\Bbbk$, for all objects X, Y of $\mathcal{A}$. Assume that for all $X,Y\in\operatorname{Ob}\mathcal{A}$*

(a) *q_{00} is a chain map;*
(b) *the induced chain map*

$$r_{00}=\big[\mathcal{A}(X,YS)\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}}\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(Y,X),\mathcal{A}(Y,X)\otimes\mathcal{A}(X,YS))\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,q_{00})}\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(Y,X),\Bbbk)\big]$$

is homotopy invertible;

(c) *the pairing q_{00} is symmetric in a sense similar to diagram (2.2.11), namely, the following diagram of chain maps commutes:*

$$\begin{tikzcd}
\mathcal{A}(X,YS)\otimes\mathcal{A}(Y,X) \arrow[r, "1\otimes sS_1s^{-1}"] \arrow[d, "c"'] \arrow[dr, phantom, "="] & \mathcal{A}(X,YS)\otimes\mathcal{A}(YS,XS) \arrow[d, "q_{00}"] \\
\mathcal{A}(Y,X)\otimes\mathcal{A}(X,YS) \arrow[r, "q_{00}"] & \Bbbk
\end{tikzcd} \tag{5.2.3}$$

(d) *the following equation holds for all $k \geqslant 0$ and all objects $X_0, \dots, X_k, Y$:*

$$\begin{aligned}
&\big[\mathcal{A}(Y,X_k)\otimes T^k\mathcal{A}(X_k,X_0)\otimes\mathcal{A}(X_0,YS) \xrightarrow{1\otimes m_{k+1}} \mathcal{A}(Y,X_k)\otimes\mathcal{A}(X_k,YS) \xrightarrow{q_{00}} \Bbbk\big] \\
&= (-)^{k+1}\big[\mathcal{A}(Y,X_k)\otimes T^k\mathcal{A}(X_k,X_0)\otimes\mathcal{A}(X_0,YS) \\
&\qquad\qquad \xrightarrow{m_{1+k}\otimes 1} \mathcal{A}(Y,X_0)\otimes\mathcal{A}(X_0,YS) \xrightarrow{q_{00}} \Bbbk\big].
\end{aligned} \tag{5.2.4}$$

Then the natural A_∞-transformation $p : \mathcal{A}^S \to \mathcal{A}^* : \mathcal{A}^{\mathrm{op}}, \mathcal{A} \to \underline{\mathsf{C}}_\Bbbk$ with the only non-vanishing component $p_{00} : 1 \mapsto r_{00}$ is invertible and $S : \mathcal{A} \to \mathcal{A}$ is a Serre A_∞-functor.

Notice that (5.2.4) is precisely the case of (5.2.2) with $n = 0$. On the other hand, diagram (2.2.11) written for the $\mathcal{K}$-category $\mathcal{C} = \mathsf{k}\mathcal{A}$ and the pairing $\phi = [q_{00}]$ says that (5.2.3) has to commute only up to homotopy. Thus, condition (c) is sufficient but not necessary.

Proof. We have to prove equation (5.2.2) for all $k \geqslant 0$, $n \geqslant 0$. The case of $n = 0$ holds by condition (d). Let us proceed by induction on n. Assume that (5.2.2) holds true for all $k \geqslant 0$, $0 \leqslant n < N$. Let us prove equation (5.2.2) for $k \geqslant 0$, $n = N$. We have

$$\begin{aligned}
&(-)^{(k+1)(n+1)}(13524)_c\cdot(m_{n+1+k}\otimes 1)\cdot q_{00} \\
&\overset{\text{(d)}}{=} (-)^{(k+1)(n+1)+k+n+1}(13524)_c\cdot(1\otimes m_{n+k+1})\cdot q_{00} \\
&= (-)^{kn}(12345)_c\cdot(m_{n+k+1}\otimes 1)\cdot c\cdot q_{00} \\
&\overset{\text{(c)}}{=} (-)^{kn}(12345)_c\cdot(m_{n+k+1}\otimes sS_1s^{-1})\cdot q_{00} \\
&= (-)^{(k+2)n}(1^{\otimes 3}\otimes sS_1s^{-1}\otimes 1)\cdot(12345)_c\cdot(m_{n+k+1}\otimes 1)\cdot q_{00} \\
&\underset{\text{for } k+1,n-1}{\overset{\text{by (5.2.2)}}{=}} (1^{\otimes 3}\otimes sS_1s^{-1}\otimes 1)\cdot(1^{\otimes 4}\otimes T^{n-1}(sS_1s^{-1}))\cdot(1\otimes m_{k+1+n})\cdot q_{00} : \\
&\mathcal{A}(Y_n,X_k)\otimes T^k\mathcal{A}(X_k,X_0)\otimes\mathcal{A}(X_0,Y_0S)\otimes\mathcal{A}(Y_0,Y_1)\otimes T^{n-1}\mathcal{A}(Y_1,Y_n)\to\Bbbk.
\end{aligned}$$

This is just equation (5.2.2) for k, n. □

Some authors like to consider a special case of the above in which $S = [d]$ is the shift functor (when it makes sense), the paring q_{00} is symmetric and cyclically symmetric with respect to n-ary compositions, cf. [**10**, Section 6.2]. Then $\mathcal{A}$ is called a d-Calabi–Yau A_∞-category. General Serre A_∞-functors cover a wider scope, although they require more data to work with. A detailed study of Calabi–Yau A_∞-categories is a possible subject for future research.

APPENDIX A

The Yoneda Lemma

A version of the classical Yoneda Lemma is presented in Mac Lane's book [**44**, Section III.2] as the following statement. For any category $\mathcal{C}$, there is an isomorphism of functors

$$\mathrm{ev}^{\mathbf{Cat}} \cong \big[\mathcal{C} \times \underline{\mathbf{Cat}}(\mathcal{C},\mathbf{Set}) \xrightarrow{\mathscr{Y}^{\mathrm{op}}\times 1} \underline{\mathbf{Cat}}(\mathcal{C},\mathbf{Set})^{\mathrm{op}} \times \underline{\mathbf{Cat}}(\mathcal{C},\mathbf{Set}) \xrightarrow{\mathrm{Hom}_{\underline{\mathbf{Cat}}(\mathcal{C},\mathbf{Set})}} \mathbf{Set}\big],$$

where $\mathscr{Y} : \mathcal{C}^{\mathrm{op}} \to \underline{\mathbf{Cat}}(\mathcal{C},\mathbf{Set})$, $X \mapsto \mathcal{C}(X,-)$, is the Yoneda embedding. Here we generalize this to A_∞-setting. The following formulation of the Yoneda Lemma was suggested to the author by Volodymyr Lyubashenko.

A.1. Theorem (The Yoneda Lemma)**.** *For any A_∞-category $\mathcal{A}$ there is a natural A_∞-transformation*

$$\Omega : \mathrm{ev}^{\mathsf{A}_\infty} \to \big[\mathcal{A}, \underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk) \xrightarrow{\mathscr{Y}^{\mathrm{op}},1} \underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)^{\mathrm{op}}, \underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk) \xrightarrow{\mathrm{Hom}_{\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)}} \underline{\mathsf{C}}_\Bbbk\big].$$

If the A_∞-category $\mathcal{A}$ is unital, Ω restricts to an invertible *natural A_∞-transformation*

$$\begin{tikzcd}
\mathcal{A}, \underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk) \arrow[rr, "\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}"] \arrow[dr, "\mathscr{Y}^{\mathrm{op}},1"'] & {} \arrow[d, Rightarrow, "\Omega"] & \underline{\mathsf{C}}_\Bbbk \\
& \underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)^{\mathrm{op}}, \underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk) \arrow[ur, "\mathrm{Hom}_{\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)}"'] &
\end{tikzcd}$$

Previously published A_∞-versions of Yoneda Lemma assert that for a unital A_∞-category $\mathcal{A}$, the Yoneda A_∞-functor $\mathscr{Y} : \mathcal{A}^{\mathrm{op}} \to \underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)$ is homotopy full and faithful [**16**, Theorem 9.1], [**39**, Theorem A.11]. A more general form of the Yoneda Lemma is considered by Seidel [**48**, Lemma 2.12]. We will see that these are corollaries of the above theorem.

Proof. First of all we describe the A_∞-transformation Ω for an arbitrary A_∞-category $\mathcal{A}$. The discussion of Section 4.3.1 applied to the A_∞-functor

$$\psi = \big[\mathcal{A}, \underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk) \xrightarrow{\mathscr{Y}^{\mathrm{op}},1} \underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)^{\mathrm{op}}, \underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk) \xrightarrow{\mathrm{Hom}_{\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)}} \underline{\mathsf{C}}_\Bbbk\big]$$

presents the corresponding $\mathcal{A}^{\mathrm{op}}$-$\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)$-bimodule $\mathcal{Q} = {}_{\mathscr{Y}}\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)_1$ via the regular A_∞-bimodule. Thus,

$$(\mathcal{Q}(X,f), sb^{\mathcal{Q}}_{00}s^{-1}) = (\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^X,f), sB_1s^{-1}).$$

According to (4.2.3) $H^X = \mathcal{A}^{\mathrm{op}}(-,X) = \mathcal{A}(X,-)$ has the components

$$\begin{aligned} H^X_k = (\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}})_{k0} = \big[T^k s\mathcal{A}(Y,Z) &\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Y), s\mathcal{A}(X,Y) \otimes T^k s\mathcal{A}(Y,Z)) \\ &\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,b^{\mathcal{A}}_{1+k})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Y), s\mathcal{A}(X,Z)) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Y),\mathcal{A}(X,Z))\big]. \quad \text{(A.1.1)} \end{aligned}$$

We have $b^{\mathcal{Q}}_{00} = B_1$ and, moreover, by (4.3.1)

$$\begin{aligned}\check{b}^{\mathcal{Q}} = \bigl[&Ts\mathcal{A}^{\mathrm{op}}(Y,X) \otimes s\mathcal{Q}(X,f) \otimes Ts\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f,g) \\ &\xrightarrow{\mathscr{Y}\otimes1\otimes1} Ts\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^Y,H^X) \otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^X,f) \otimes Ts\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f,g) \\ &\qquad\qquad\xrightarrow{\check{B}} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^Y,g) = s\mathcal{Q}(Y,g)\bigr].\end{aligned}$$

Since $\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)$ is a differential graded category, $B_p = 0$ for $p > 2$. Therefore, $b^{\mathcal{Q}}_{kn} = 0$ if $n > 1$, and $b^{\mathcal{Q}}_{k1} = 0$ if $k > 0$. The non-trivial components are (for $k > 0$)

$$\begin{aligned}b^{\mathcal{Q}}_{k0} = \bigl[&T^ks\mathcal{A}^{\mathrm{op}}(Y,X) \otimes s\mathcal{Q}(X,f) \otimes T^0s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f,f) \\ &\xrightarrow{\mathscr{Y}_k\otimes1\otimes1} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^Y,H^X) \otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^X,f) \\ &\qquad\qquad\xrightarrow{B_2} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^Y,f) = s\mathcal{Q}(Y,f)\bigr], \\ b^{\mathcal{Q}}_{01} = \bigl[&T^0s\mathcal{A}^{\mathrm{op}}(X,X) \otimes s\mathcal{Q}(X,f) \otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f,g) \\ &\qquad\qquad\xrightarrow{B_2} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^X,g) = s\mathcal{Q}(X,g)\bigr]. \qquad \text{(A.1.2)}\end{aligned}$$

Denote by $\mathcal{E}$ the $\mathcal{A}^{\mathrm{op}}$-$\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)$-bimodule corresponding to the evaluation A_∞-functor $\mathrm{ev}^{\mathsf{A}_\infty} : \mathcal{A},\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk) \to \underline{\mathsf{C}}_\Bbbk$. For any object X of $\mathcal{A}$ and any A_∞-functor $f : \mathcal{A} \to \underline{\mathsf{C}}_\Bbbk$ the complex $(\mathcal{E}(X,f), sb^{\mathcal{E}}_{00}s^{-1})$ is (Xf,d). According to (4.1.6)

$$\begin{aligned}\check{b}^{\mathcal{E}}_{+} = \bigl[&Ts\mathcal{A}^{\mathrm{op}}(Y,X) \otimes s\mathcal{E}(X,f) \otimes Ts\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f,g) \\ &\xrightarrow{c\otimes1} s\mathcal{E}(X,f) \otimes Ts\mathcal{A}^{\mathrm{op}}(Y,X) \otimes Ts\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f,g) \\ &\xrightarrow{1\otimes\gamma\otimes1} s\mathcal{E}(X,f) \otimes Ts\mathcal{A}(X,Y) \otimes Ts\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f,g) \\ &\xrightarrow{1\otimes\check{\mathrm{ev}}^{\mathsf{A}_\infty}} s\mathcal{E}(X,f) \otimes s\underline{\mathsf{C}}_\Bbbk(Xf,Yg) \\ &\qquad\qquad\xrightarrow{1\otimes s^{-1}[1]} Xf[1] \otimes \underline{\mathsf{C}}_\Bbbk(Xf[1],Yg[1]) \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}} Yg[1] = s\mathcal{E}(Y,g)\bigr].\end{aligned}$$

Explicit formulas (3.3.6) and (3.3.7) for $\mathrm{ev}^{\mathsf{A}_\infty}$ show that $b^{\mathcal{E}}_{kn} = 0$ if $n > 1$. The remaining components are described as

$$\begin{aligned}b^{\mathcal{E}}_{k0} = \bigl[&T^ks\mathcal{A}^{\mathrm{op}}(Y,X) \otimes s\mathcal{E}(X,f) \otimes T^0s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f,f) \\ &\xrightarrow{c\otimes1} s\mathcal{E}(X,f) \otimes T^ks\mathcal{A}^{\mathrm{op}}(Y,X) \\ &\xrightarrow{1\otimes\gamma} s\mathcal{E}(X,f) \otimes T^ks\mathcal{A}(X,Y) \\ &\xrightarrow{1\otimes f_k} Xf[1] \otimes s\underline{\mathsf{C}}_\Bbbk(Xf,Yf) \\ &\qquad\qquad\xrightarrow{1\otimes s^{-1}[1]} Xf[1] \otimes \underline{\mathsf{C}}_\Bbbk(Xf[1],Yf[1]) \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}} Yf[1] = s\mathcal{E}(Y,f)\bigr]\end{aligned}$$

for $k > 0$, and if $k \geqslant 0$ there is

$$
\begin{aligned}
b^{\mathcal{E}}_{k1} = \bigl[& T^k s\mathcal{A}^{\mathrm{op}}(Y,X) \otimes s\mathcal{E}(X,f) \otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f,g) \\
& \xrightarrow{c\otimes 1} s\mathcal{E}(X,f) \otimes T^k s\mathcal{A}^{\mathrm{op}}(Y,X) \otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f,g) \\
& \xrightarrow{1\otimes\gamma\otimes 1} s\mathcal{E}(X,f) \otimes T^k s\mathcal{A}(X,Y) \otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f,g) \\
& \xrightarrow{1\otimes 1\otimes \mathrm{pr}_k} s\mathcal{E}(X,f) \otimes T^k s\mathcal{A}(X,Y) \otimes \underline{\mathsf{C}}_\Bbbk(T^k s\mathcal{A}(X,Y), s\underline{\mathsf{C}}_\Bbbk(Xf,Yg)) \\
& \xrightarrow{1\otimes \mathrm{ev}^{\underline{\mathsf{C}}_\Bbbk}} Xf[1] \otimes s\underline{\mathsf{C}}_\Bbbk(Xf,Yg) \\
& \qquad\qquad \xrightarrow{1\otimes s^{-1}[1]} Xf[1] \otimes \underline{\mathsf{C}}_\Bbbk(Xf[1],Yg[1]) \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}} Yg[1] = s\mathcal{E}(Y,g) \bigr].
\end{aligned}
$$

The A_∞-transformation Ω in question is constructed via a homomorphism

$$
\mho = (\Omega s^{-1})\Phi : Ts\mathcal{A}^{\mathrm{op}} \otimes s\mathcal{E} \otimes Ts\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk) \to Ts\mathcal{A}^{\mathrm{op}} \otimes s\mathcal{Q} \otimes Ts\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)
$$

of $Ts\mathcal{A}^{\mathrm{op}}$-$Ts\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)$-bicomodules thanks to Proposition 4.1.3. Its matrix coefficients are recovered from its components via formula (4.1.3) as

$$
\begin{aligned}
\mho_{kl;mn} = \sum_{\substack{m+p=k\\ q+n=l}} (1^{\otimes m} \otimes \mho_{pq} \otimes 1^{\otimes n}) : & \\
& T^k s\mathcal{A}^{\mathrm{op}} \otimes s\mathcal{E} \otimes T^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk) \to T^m s\mathcal{A}^{\mathrm{op}} \otimes s\mathcal{Q} \otimes T^n s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk).
\end{aligned}
$$

The composition of the morphism

$$
\mho_{pq} : T^p s\mathcal{A}^{\mathrm{op}}(X_0,X_p) \otimes X_p f_0[1] \otimes T^q s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_q) \to s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^{X_0},f_q)
$$

with the projection

$$
\mathrm{pr}_n : s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^{X_0},f_q) \to \underline{\mathsf{C}}_\Bbbk(T^n s\mathcal{A}(Z_0,Z_n), s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_n f_q)) \tag{A.1.3}
$$

is given by the composite

$$
\begin{aligned}
& \mho_{pq;n} \overset{\mathrm{def}}{=} \mho_{pq}\cdot \mathrm{pr}_n = (-)^{p+1}\bigl[T^p s\mathcal{A}^{\mathrm{op}}(X_0,X_p) \otimes X_p f_0[1] \otimes T^q s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_q) \\
& \xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0) \otimes T^n s\mathcal{A}(Z_0,Z_n), \\
& \quad s\mathcal{A}(X_0,Z_0) \otimes T^n s\mathcal{A}(Z_0,Z_n) \otimes T^p s\mathcal{A}^{\mathrm{op}}(X_0,X_p) \otimes X_p f_0[1] \otimes T^q s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_q)) \\
& \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{perm})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0) \otimes T^n s\mathcal{A}(Z_0,Z_n), \\
& \quad\; X_p f_0[1] \otimes T^p s\mathcal{A}(X_p,X_0) \otimes s\mathcal{A}(X_0,Z_0) \otimes T^n s\mathcal{A}(Z_0,Z_n) \otimes T^q s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_q)) \\
& \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes \mathrm{ev}^{\mathsf{A}_\infty}_{p+1+n,q})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0) \otimes T^n s\mathcal{A}(Z_0,Z_n), X_p f_0[1] \otimes s\underline{\mathsf{C}}_\Bbbk(X_p f_0, Z_n f_q)) \\
& \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0) \otimes T^n s\mathcal{A}(Z_0,Z_n), X_p f_0[1] \otimes \underline{\mathsf{C}}_\Bbbk(X_p f_0[1], Z_n f_q[1])) \\
& \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{ev}^{\mathsf{C}_\Bbbk})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0) \otimes T^n s\mathcal{A}(Z_0,Z_n), Z_n f_q[1]) \\
& \xrightarrow{(\underline{\varphi}^{\mathsf{C}_\Bbbk})^{-1}} \underline{\mathsf{C}}_\Bbbk(T^n s\mathcal{A}(Z_0,Z_n), \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), Z_n f_q[1])) \\
& \qquad\qquad\qquad \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,[-1]s)} \underline{\mathsf{C}}_\Bbbk(T^n s\mathcal{A}(Z_0,Z_n), s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0), Z_n f_q))\bigr].
\end{aligned} \tag{A.1.4}
$$

Thus, an element $x_1 \otimes \cdots \otimes x_p \otimes y \otimes r_1 \otimes \cdots \otimes r_q \in T^p s\mathcal{A}^{\mathrm{op}}(X_0,X_p) \otimes X_p f_0[1] \otimes T^q s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_q)$ is mapped to an A_∞-transformation $(x_1 \otimes \cdots \otimes x_p \otimes y \otimes r_1 \otimes \cdots \otimes$

$r_q)\mho_{pq} \in s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^{X_0}, f_q)$ with components

$$
\begin{gathered}
[(x_1 \otimes \dots \otimes x_p \otimes y \otimes r_1 \otimes \dots \otimes r_q)\mho_{pq}]_n : T^n s\mathcal{A}(Z_0, Z_n) \to s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0, Z_0), Z_n f_q), \\
z_1 \otimes \dots \otimes z_n \mapsto (z_1 \otimes \dots \otimes z_n \otimes x_1 \otimes \dots \otimes x_p \otimes y \otimes r_1 \otimes \dots \otimes r_q)\mho'_{pq;n},
\end{gathered}
$$

where $\mho'_{pq;n} \overset{\text{def}}{=} (1^{\otimes n} \otimes \mho_{pq;n})\operatorname{ev}^{\mathsf{C}_\Bbbk} = (1^{\otimes n} \otimes \mho_{pq} \cdot \operatorname{pr}_n)\operatorname{ev}^{\mathsf{C}_\Bbbk} = (1^{\otimes n} \otimes \mho_{pq})\operatorname{ev}^{\mathsf{A}_\infty}_{n1}$ is given by

$$
\begin{aligned}
\mho'_{pq;n} = (-)^{p+1}\bigl[& T^n s\mathcal{A}(Z_0, Z_n) \otimes T^p s\mathcal{A}^{\mathrm{op}}(X_0, X_p) \otimes X_p f_0[1] \otimes T^q s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0, f_q) \\
& \xrightarrow{\operatorname{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), s\mathcal{A}(X_0, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n) \\
& \qquad \otimes T^p s\mathcal{A}^{\mathrm{op}}(X_0, X_p) \otimes X_p f_0[1] \otimes T^q s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0, f_q)) \\
& \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{perm})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), X_p f_0[1] \otimes T^p s\mathcal{A}(X_p, X_0) \\
& \qquad \otimes s\mathcal{A}(X_0, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n) \otimes T^q s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0, f_q)) \\
& \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes\operatorname{ev}^{\mathsf{A}_\infty}_{p+1+n,q})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), X_p f_0[1] \otimes s\underline{\mathsf{C}}_\Bbbk(X_p f_0, Z_n f_q)) \\
& \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), X_p f_0[1] \otimes \underline{\mathsf{C}}_\Bbbk(X_p f_0[1], Z_n f_q[1])) \\
& \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\operatorname{ev}^{\mathsf{C}_\Bbbk})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), Z_n f_q[1]) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0, Z_0), Z_n f_q)\bigr].
\end{aligned}
$$

It follows that $\mho_{pq} : T^p s\mathcal{A}^{\mathrm{op}} \otimes s\mathcal{E} \otimes T^q s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk) \to s\mathcal{Q}$ vanishes if $q > 1$. The other components are given by

$$
\begin{aligned}
\mho'_{p0;n} = (-)^{p+1}\bigl[& T^n s\mathcal{A}(Z_0, Z_n) \otimes T^p s\mathcal{A}^{\mathrm{op}}(X_0, X_p) \otimes X_p f[1] \\
& \xrightarrow{\operatorname{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), s\mathcal{A}(X_0, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n) \\
& \qquad \otimes T^p s\mathcal{A}^{\mathrm{op}}(X_0, X_p) \otimes X_p f[1]) \\
& \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{perm})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), X_p f[1] \otimes T^p s\mathcal{A}(X_p, X_0) \\
& \qquad \otimes s\mathcal{A}(X_0, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n)) \\
& \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes f_{p+1+n})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), X_p f[1] \otimes s\underline{\mathsf{C}}_\Bbbk(X_p f, Z_n f)) \\
& \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), X_p f[1] \otimes \underline{\mathsf{C}}_\Bbbk(X_p f[1], Z_n f[1])) \\
& \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\operatorname{ev}^{\mathsf{C}_\Bbbk})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), Z_n f[1]) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0, Z_0), Z_n f)\bigr]
\end{aligned} \tag{A.1.5}
$$

and

$$
\begin{aligned}
\mho'_{p1;n} = (-)^{p+1}\bigl[& T^n s\mathcal{A}(Z_0, Z_n) \otimes T^p s\mathcal{A}^{\mathrm{op}}(X_0, X_p) \otimes X_p f[1] \otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f, g) \\
& \xrightarrow{\operatorname{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), s\mathcal{A}(X_0, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n) \\
& \qquad \otimes T^p s\mathcal{A}^{\mathrm{op}}(X_0, X_p) \otimes X_p f[1] \otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f, g)) \\
& \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{perm})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), X_p f[1] \otimes T^p s\mathcal{A}(X_p, X_0) \\
& \qquad \otimes s\mathcal{A}(X_0, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n) \otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f, g)) \\
& \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes 1^{\otimes p+1+n}\otimes\operatorname{pr}_{p+1+n})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), X_p f[1] \otimes T^p s\mathcal{A}(X_p, X_0) \\
& \qquad \otimes s\mathcal{A}(X_0, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n) \\
& \otimes \underline{\mathsf{C}}_\Bbbk(T^p s\mathcal{A}(X_p, X_0) \otimes s\mathcal{A}(X_0, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n), s\underline{\mathsf{C}}_\Bbbk(X_p f, Z_n g)))
\end{aligned}
$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_pf[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_pf,Z_ng))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_pf[1]\otimes \underline{\mathsf{C}}_{\Bbbk}(X_pf[1],Z_ng[1]))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_ng[1]) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_ng)\big].$$

The naturality of the A_∞-transformation Ω is implied by the following lemma.

A.2. Lemma. *The bicomodule homomorphism $\mho$ is a chain map.*

Proof. Equivalently, we have to prove the equation $\mho\check{b}^{\Omega} = b^{\mathcal{E}}\check{\mho}$. In components, the expressions

$$(\mho\check{b}^{\Omega})_{kl} = \sum_{\substack{m+p=k\\ q+n=l}} (1^{\otimes m}_{s\mathcal{A}^{\mathrm{op}}} \otimes \mho_{pq} \otimes 1^{\otimes n}_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})})b^{\Omega}_{mn}, \tag{A.2.1}$$

$$(b^{\mathcal{E}}\check{\mho})_{kl} = \sum_{p+t+q=l} (1^{\otimes k}_{s\mathcal{A}^{\mathrm{op}}} \otimes 1_{s\mathcal{E}} \otimes 1^{\otimes p}_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})} \otimes b_t^{\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})} \otimes 1^{\otimes q}_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})})\mho_{k,p+1+q} \tag{A.2.2}$$

$$+ \sum_{\substack{m+i=k\\ j+n=l}} (1^{\otimes m}_{s\mathcal{A}^{\mathrm{op}}} \otimes b^{\mathcal{E}}_{ij} \otimes 1^{\otimes n}_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})})\mho_{mn} \tag{A.2.3}$$

$$+ \sum_{a+u+c=k} (1^{\otimes a}_{s\mathcal{A}^{\mathrm{op}}} \otimes b_u^{\mathcal{A}^{\mathrm{op}}} \otimes 1^{\otimes c}_{s\mathcal{A}^{\mathrm{op}}} \otimes 1_{s\mathcal{E}} \otimes 1^{\otimes l}_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})})\mho_{a+1+c,l} \tag{A.2.4}$$

must coincide for all $k,l \geqslant 0$. Let us analyze this equation in detail. Since $b^{\Omega}_{mn} = 0$ unless $n = 0$ or $(m,n) = (0,1)$, it follows that the right hand side of (A.2.1) reduces to

$$\sum_{m=0}^{k}(1^{\otimes m}_{s\mathcal{A}^{\mathrm{op}}} \otimes \mho_{k-m,l})b^{\Omega}_{m0} + (\mho_{k,l-1} \otimes 1_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})})b^{\Omega}_{01}.$$

Since $\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})$ is a differential graded category, sum (A.2.2) reduces to

$$\sum_{p=1}^{l}(1^{\otimes k}_{s\mathcal{A}^{\mathrm{op}}} \otimes 1_{s\mathcal{E}} \otimes 1^{\otimes(p-1)}_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})} \otimes B_1 \otimes 1^{\otimes(l-p)}_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})})\mho_{kl}$$

$$+\sum_{p=1}^{l-1}(1^{\otimes k}_{s\mathcal{A}^{\mathrm{op}}} \otimes 1_{s\mathcal{E}} \otimes 1^{\otimes(p-1)}_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})} \otimes B_2 \otimes 1^{\otimes(l-p-1)}_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})})\mho_{k,l-1}.$$

Since $b^{\mathcal{E}}_{ij} = 0$ if $j > 1$, sum (A.2.3) equals

$$\sum_{m=0}^{k-1}(1^{\otimes m}_{s\mathcal{A}^{\mathrm{op}}} \otimes b^{\mathcal{E}}_{k-m,0} \otimes 1^{\otimes l}_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})})\mho_{ml} + (1^{\otimes k}_{s\mathcal{A}^{\mathrm{op}}} \otimes b^{\mathcal{E}}_{00} \otimes 1^{\otimes l}_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})})\mho_{kl}$$

$$+\sum_{m=0}^{k}(1^{\otimes m}_{s\mathcal{A}^{\mathrm{op}}} \otimes b^{\mathcal{E}}_{k-m,1} \otimes 1^{\otimes(l-1)}_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})})\mho_{m,l-1}.$$

Sum (A.2.4) does not allow further simplification. Therefore, the equation to prove is

$$
\begin{aligned}
&\sum_{m=0}^{k}(1_{s\mathcal{A}^{\mathrm{op}}}^{\otimes m}\otimes\mho_{k-m,l})b^{\mathcal{Q}}_{m0}+(\mho_{k,l-1}\otimes 1_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})})b^{\mathcal{Q}}_{01}\\
&=\sum_{p=1}^{l}(1_{s\mathcal{A}^{\mathrm{op}}}^{\otimes k}\otimes 1_{s\mathcal{E}}\otimes 1_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})}^{\otimes(p-1)}\otimes B_1\otimes 1_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})}^{\otimes(l-p)})\mho_{kl}\\
&+\sum_{p=1}^{l-1}(1_{s\mathcal{A}^{\mathrm{op}}}^{\otimes k}\otimes 1_{s\mathcal{E}}\otimes 1_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})}^{\otimes(p-1)}\otimes B_2\otimes 1_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})}^{\otimes(l-p-1)})\mho_{k,l-1}\\
&+\sum_{m=0}^{k}(1_{s\mathcal{A}^{\mathrm{op}}}^{\otimes m}\otimes b^{\mathcal{E}}_{k-m,0}\otimes 1_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})}^{\otimes l})\mho_{ml}+\sum_{m=0}^{k}(1_{s\mathcal{A}^{\mathrm{op}}}^{\otimes m}\otimes b^{\mathcal{E}}_{k-m,1}\otimes 1_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})}^{\otimes(l-1)})\mho_{m,l-1}\\
&\qquad\qquad+\sum_{a+u+c=k}(1_{s\mathcal{A}^{\mathrm{op}}}^{\otimes a}\otimes b^{\mathcal{A}^{\mathrm{op}}}_{u}\otimes 1_{s\mathcal{A}^{\mathrm{op}}}^{\otimes c}\otimes 1_{s\mathcal{E}}\otimes 1_{s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})}^{\otimes l})\mho_{a+1+c,l}.
\end{aligned}
$$

Write it in more detailed form using explicit formulas (A.1.2) for components of $b^{\mathcal{Q}}$:

$$
\begin{aligned}
S\overset{\mathrm{def}}{=}&\sum_{m=1}^{k}\bigl[\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\xrightarrow{\mathscr{Y}_m\otimes\mho_{k-m,l}}\\
&\qquad s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(H^{X_0},H^{X_m})\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(H^{X_m},f_l)\xrightarrow{B_2}s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(H^{X_0},f_l)\bigr]\\
&+\bigl[\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\xrightarrow{\mho_{kl}}\\
&\qquad s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(H^{X_0},f_l)\xrightarrow{B_1}s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(H^{X_0},f_l)\bigr]\\
&+\bigl[\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\xrightarrow{\mho_{k,l-1}\otimes1}\\
&\qquad s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(H^{X_0},f_{l-1})\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_{l-1},f_l)\xrightarrow{B_2}s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(H^{X_0},f_l)\bigr]\\
&-\sum_{p=1}^{l}\bigl[\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\xrightarrow{1^{\otimes k}\otimes1\otimes1^{\otimes p-1}\otimes B_1\otimes1^{\otimes l-p}}\\
&\qquad\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\xrightarrow{\mho_{kl}}s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(H^{X_0},f_l)\bigr]\\
&-\sum_{p=1}^{l-1}\bigl[\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\xrightarrow{1^{\otimes k}\otimes1\otimes1^{\otimes p-1}\otimes B_2\otimes1^{\otimes l-p-1}}\\
&\qquad\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^{l-1} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,\dots,f_{p-1},f_{p+1},\dots,f_l)\\
&\qquad\qquad\xrightarrow{\mho_{k,l-1}}s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(H^{X_0},f_l)\bigr]\\
&-\sum_{m=0}^{k}\bigl[\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\xrightarrow{1^{\otimes m}\otimes b^{\mathcal{E}}_{k-m,0}\otimes1^{\otimes l}}\\
&\qquad\bar{T}^m s\mathcal{A}^{\mathrm{op}}(X_0,X_m)\otimes X_mf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\xrightarrow{\mho_{ml}}s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(H^{X_0},f_l)\bigr]\\
&-\sum_{m=0}^{k}\bigl[\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\xrightarrow{1^{\otimes m}\otimes b^{\mathcal{E}}_{k-m,1}\otimes1^{\otimes l-1}}
\end{aligned}
$$

$$\bar T^m s\mathcal{A}^{\mathrm{op}}(X_0,X_m)\otimes X_m f_1[1]\otimes \bar T^{l-1} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_1,f_l) \xrightarrow{\mho_{m,l-1}} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^{X_0},f_l)\big]$$
$$-\sum_{a+u+c=k}\big[\bar T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_k f_0[1]\otimes \bar T^{l} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l) \xrightarrow{1^{\otimes a}\otimes b_u^{\mathcal{A}^{\mathrm{op}}}\otimes 1^{\otimes c}\otimes 1\otimes 1^{\otimes l}}$$
$$\bar T^{a+1+c} s\mathcal{A}^{\mathrm{op}}(X_0,\dots,X_a,X_{a+u},\dots,X_k)\otimes X_k f_0[1]\otimes \bar T^{l} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l)$$
$$\xrightarrow{\mho_{a+1+c,l}} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^{X_0},f_l)\big] = 0.$$

The above equation is equivalent to the system of equations

$$S\cdot \mathrm{pr}_n = 0 : \bar T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_k f_0[1]\otimes \bar T^{l} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l)$$
$$\to s\underline{\mathsf{C}}_\Bbbk(\bar T^n s\mathcal{A}(Z_0,Z_n), s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_n f_l)),$$

where $n\geqslant 0$, and $Z_0,\dots,Z_n\in \operatorname{Ob}\mathcal{A}$. By closedness, each of these equations is equivalent to

$$(1^{\otimes n}\otimes S\cdot\mathrm{pr}_n)\,\mathrm{ev}^{\mathsf{C}_\Bbbk} = (1^{\otimes n}\otimes S)\,\mathrm{ev}^{\mathsf{A}_\infty}_{n1} = 0 :$$
$$\bar T^n s\mathcal{A}(Z_0,Z_n)\otimes \bar T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_k f_0[1]\otimes \bar T^{l} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l)$$
$$\to s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_n f_l).$$

The fact that $\mathrm{ev}^{\mathsf{A}_\infty}$ is an A_∞-functor combined with explicit formulas (3.3.6) and (3.3.7) for components of $\mathrm{ev}^{\mathsf{A}_\infty}$ allows to derive certain identities. Specifically, restricting the identity $[\mathrm{ev}^{\mathsf{A}_\infty} b^{\underline{\mathsf{C}}_\Bbbk} - (b^{\mathcal{A}}\boxtimes 1 + 1\boxtimes B)\,\mathrm{ev}^{\mathsf{A}_\infty}]\,\mathrm{pr}_1 = 0 : Ts\mathcal{A}\boxtimes Ts\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)\to s\underline{\mathsf{C}}_\Bbbk$ to the summand $\bar T^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\psi,\chi)$ yields

$$\big[\bar T^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\psi,\chi)$$
$$\xrightarrow{1^{\otimes n}\otimes B_2} \bar T^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\chi)$$
$$\xrightarrow{1^{\otimes n}\otimes \mathrm{pr}_n} \bar T^n s\mathcal{A}(Z_0,Z_n)\otimes \underline{\mathsf{C}}_\Bbbk(\bar T^n s\mathcal{A}(Z_0,Z_n), s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_n\chi)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_n\chi)\big]$$
$$= \big[\bar T^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\psi,\chi)$$
$$\xrightarrow{1^{\otimes n}\otimes B_2} \bar T^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\chi) \xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty}_{n1}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_n\chi)\big]$$
$$= \sum_{p+q=n}\big[\bar T^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\psi,\chi)$$
$$\xrightarrow{\mathrm{perm}} \bar T^p s\mathcal{A}(Z_0,Z_p)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi)\otimes \bar T^q s\mathcal{A}(Z_p,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\psi,\chi)$$
$$\xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty}_{p1}\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{q1}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_p\psi)\otimes s\underline{\mathsf{C}}_\Bbbk(Z_p\psi,Z_n\chi) \xrightarrow{b_2^{\mathsf{C}_\Bbbk}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_n\chi)\big]. \tag{A.2.5}$$

Restricting the same identity to the summand $\bar T^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi)$ yields

$$\big[\bar T^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi) \xrightarrow{1^{\otimes n}\otimes B_1} \bar T^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi)$$
$$\xrightarrow{1^{\otimes n}\otimes \mathrm{pr}_n} \bar T^n s\mathcal{A}(Z_0,Z_n)\otimes \underline{\mathsf{C}}_\Bbbk(\bar T^n s\mathcal{A}(Z_0,Z_n), s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_n\psi)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_n\psi)\big]$$
$$= \big[\bar T^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi) \xrightarrow{1^{\otimes n}\otimes B_1}$$
$$\bar T^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi) \xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty}_{n1}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_n\psi)\big]$$
$$= \big[\bar T^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi) \xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty}_{n1}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_n\psi) \xrightarrow{b_1^{\mathsf{C}_\Bbbk}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_n\psi)\big]$$

$$\begin{aligned}
&+\sum_{p+q=n}^{q>0}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi)\xrightarrow{\text{perm}} \\
&\qquad \bar{T}^p s\mathcal{A}(Z_0,Z_p)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi)\otimes\bar{T}^q s\mathcal{A}(Z_p,Z_n)\otimes T^0 s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\psi,\psi) \\
&\qquad\qquad \xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty}_{p1}\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{q0}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_p\psi)\otimes s\underline{\mathsf{C}}_\Bbbk(Z_p\psi,Z_n\psi)\xrightarrow{b_2^{\mathsf{C}_\Bbbk}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_n\psi)\big] \\
&+\sum_{p+q=n}^{p>0}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi)\xrightarrow{\sim} \\
&\qquad \bar{T}^p s\mathcal{A}(Z_0,Z_p)\otimes T^0 s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\phi)\otimes\bar{T}^q s\mathcal{A}(Z_p,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi) \\
&\qquad\qquad \xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty}_{p0}\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{q1}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_p\phi)\otimes s\underline{\mathsf{C}}_\Bbbk(Z_p\phi,Z_n\psi)\xrightarrow{b_2^{\mathsf{C}_\Bbbk}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_n\psi)\big] \\
&-\sum_{\alpha+t+\beta=n}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi)\xrightarrow{1^{\otimes\alpha}\otimes b_t\otimes 1^{\otimes\beta}\otimes 1} \\
&\qquad\qquad \bar{T}^{\alpha+1+\beta}\mathcal{A}(Z_0,\dots,Z_\alpha,Z_{\alpha+t},\dots,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi) \\
&\qquad\qquad\qquad \xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty}_{\alpha+1+\beta,1}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_n\psi)\big] \\
&=\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi)\xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty}_{n1}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_n\psi)\xrightarrow{b_1^{\mathsf{C}_\Bbbk}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_n\psi)\big] \\
&+\sum_{p+q=n}^{q>0}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi)\xrightarrow{\text{perm}} \\
&\qquad \bar{T}^p s\mathcal{A}(Z_0,Z_p)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi)\otimes\bar{T}^q s\mathcal{A}(Z_p,Z_n)\otimes T^0 s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\psi,\psi) \\
&\qquad\qquad \xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty}_{p1}\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{q0}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_p\psi)\otimes s\underline{\mathsf{C}}_\Bbbk(Z_p\psi,Z_n\psi)\xrightarrow{b_2^{\mathsf{C}_\Bbbk}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_n\psi)\big] \\
&+\sum_{p+q=n}^{p>0}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi) \\
&\qquad\qquad \xrightarrow{\phi_p\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{q1}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_p\phi)\otimes s\underline{\mathsf{C}}_\Bbbk(Z_p\phi,Z_n\psi)\xrightarrow{b_2^{\mathsf{C}_\Bbbk}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_n\psi)\big] \\
&-\sum_{\alpha+t+\beta=n}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi)\xrightarrow{1^{\otimes\alpha}\otimes b_t\otimes 1^{\otimes\beta}\otimes 1} \\
&\qquad\qquad \bar{T}^{\alpha+1+\beta} s\mathcal{A}(Z_0,\dots,Z_\alpha,Z_{\alpha+t},\dots,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(\phi,\psi) \\
&\qquad\qquad\qquad \xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty}_{\alpha+1+\beta,1}} s\underline{\mathsf{C}}_\Bbbk(Z_0\phi,Z_n\psi)\big]. \qquad \text{(A.2.6)}
\end{aligned}$$

With identities (A.2.5) and (A.2.6) in hand, it is the matter of straightforward verification to check that $(1^{\otimes n}\otimes S\cdot\mathrm{pr}_n)\,\mathrm{ev}^{\mathsf{C}_\Bbbk}$ admits the following presentation:

$$\begin{aligned}
&\sum_{m=1}^{k}\sum_{p+q=n}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_k f_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l) \\
&\qquad\xrightarrow{\text{perm}}\bar{T}^p s\mathcal{A}(Z_0,Z_p)\otimes\bar{T}^m s\mathcal{A}^{\mathrm{op}}(X_0,X_m)\otimes\bar{T}^q s\mathcal{A}(Z_p,Z_n) \\
&\qquad\qquad\otimes\bar{T}^{k-m} s\mathcal{A}^{\mathrm{op}}(X_m,X_k)\otimes X_k f_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l) \\
&\qquad\xrightarrow{(\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}})_{pm}\otimes\mho'_{k-m,l;q}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),\mathcal{A}(X_m,Z_p))\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_m,Z_p),Z_n f_l)
\end{aligned}$$

$$\xrightarrow{b_2^{\underline{\mathsf{C}}_\Bbbk}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_nf_l)\big] \tag{A.2.7}$$

$$\begin{aligned}
+ \big[\bar{T}^n s\mathcal{A}(Z_0,Z_n) \otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k) \otimes X_kf_0[1] \otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l) \\
\xrightarrow{\mho'_{kl;n}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_nf_l) \xrightarrow{b_1^{\underline{\mathsf{C}}_\Bbbk}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_nf_l)\big]
\end{aligned} \tag{A.2.8}$$

$$\begin{aligned}
+ \sum_{p+q=n}^{q>0} \big[\bar{T}^n s\mathcal{A}(Z_0,Z_n) \otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k) \otimes X_kf_0[1] \otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l) \\
\xrightarrow{\mathrm{perm}} \bar{T}^p s\mathcal{A}(Z_0,Z_p) \otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k) \otimes X_kf_0[1] \\
\otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l) \otimes \bar{T}^q s\mathcal{A}(Z_p,Z_n) \otimes T^0 s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_l,f_l) \\
\xrightarrow{\mho'_{kl;p}\otimes\mathrm{ev}_{q0}^{\mathsf{A}_\infty}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_pf_l) \otimes s\underline{\mathsf{C}}_\Bbbk(Z_pf_l,Z_nf_l) \\
\xrightarrow{b_2^{\underline{\mathsf{C}}_\Bbbk}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_nf_l)\big]
\end{aligned} \tag{A.2.9}$$

$$\begin{aligned}
+ \sum_{p+q=n}^{p>0} \big[\bar{T}^n s\mathcal{A}(Z_0,Z_n) \otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k) \otimes X_kf_0[1] \otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l) \\
\xrightarrow{H_p^{X_0}\otimes\mho'_{kl;q}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),\mathcal{A}(X_0,Z_p)) \otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_p),Z_nf_l) \\
\xrightarrow{b_2^{\underline{\mathsf{C}}_\Bbbk}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_nf_l)\big]
\end{aligned} \tag{A.2.10}$$

$$\begin{aligned}
- \sum_{\alpha+t+\beta=n} \big[\bar{T}^n s\mathcal{A}(Z_0,Z_n) \otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k) \otimes X_kf_0[1] \otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l) \\
\xrightarrow{1^{\otimes\alpha}\otimes b_t\otimes 1^{\otimes\beta}\otimes 1^{\otimes k}\otimes 1\otimes 1^{\otimes l}} \bar{T}^{\alpha+1+\beta} s\mathcal{A}(Z_0,\dots,Z_\alpha,Z_{\alpha+t},\dots,Z_n) \\
\otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k) \otimes X_kf_0[1] \otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l) \\
\xrightarrow{\mho'_{kl;\alpha+1+\beta}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_nf_l)\big]
\end{aligned} \tag{A.2.11}$$

$$\begin{aligned}
+ \sum_{p+q=n} \big[\bar{T}^n s\mathcal{A}(Z_0,Z_n) \otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k) \otimes X_kf_0[1] \otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l) \\
\xrightarrow{\mathrm{perm}} \bar{T}^p s\mathcal{A}(Z_0,Z_p) \otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k) \otimes X_kf_0[1] \\
\otimes \bar{T}^{l-1} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_{l-1}) \otimes \bar{T}^q s\mathcal{A}(Z_p,Z_n) \otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_{l-1},f_l) \\
\xrightarrow{\mho'_{k,l-1;p}\otimes\mathrm{ev}_{q1}^{\mathsf{A}_\infty}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_pf_{l-1}) \otimes s\underline{\mathsf{C}}_\Bbbk(Z_pf_{l-1},Z_nf_l) \\
\xrightarrow{b_2^{\underline{\mathsf{C}}_\Bbbk}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_nf_l)\big]
\end{aligned} \tag{A.2.12}$$

$$\begin{aligned}
- \sum_{p=1}^{l} \big[\bar{T}^n s\mathcal{A}(Z_0,Z_n) \otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k) \otimes X_kf_0[1] \otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l) \\
\xrightarrow{1^{\otimes n}\otimes 1^{\otimes k}\otimes 1\otimes 1^{\otimes p-1}\otimes B_1\otimes 1^{\otimes l-p}} \bar{T}^n s\mathcal{A}(Z_0,Z_n) \otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k) \\
\otimes X_kf_0[1] \otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l) \\
\xrightarrow{\mho'_{kl;n}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_nf_l)\big]
\end{aligned} \tag{A.2.13}$$

$$\begin{aligned}
&-\sum_{p=1}^{l-1}\bigl[\bar T^n s\mathcal{A}(Z_0,Z_n)\otimes\bar T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar T^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l)\\
&\qquad\xrightarrow{1^{\otimes n}\otimes1^{\otimes k}\otimes1\otimes1^{\otimes p-1}\otimes B_2\otimes1^{\otimes l-p-1}}\bar T^n s\mathcal{A}(Z_0,Z_n)\otimes\bar T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\\
&\qquad\qquad\otimes X_kf_0[1]\otimes\bar T^{l-1} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,\dots,f_{p-1},f_{p+1},\dots,f_l)\\
&\qquad\qquad\xrightarrow{\mho'_{k,l-1;n}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_nf_l)\bigr] \qquad (\text{A.2.14})
\end{aligned}$$

$$\begin{aligned}
&-\sum_{m=0}^{k}\bigl[\bar T^n s\mathcal{A}(Z_0,Z_n)\otimes\bar T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar T^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l)\\
&\qquad\xrightarrow{1^{\otimes n}\otimes1^{\otimes m}\otimes b^{\mathcal{E}}_{k-m,0}\otimes1^{\otimes l}}\bar T^n s\mathcal{A}(Z_0,Z_n)\otimes\bar T^m s\mathcal{A}^{\mathrm{op}}(X_0,X_m)\\
&\qquad\qquad\otimes X_mf_0[1]\otimes\bar T^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l)\xrightarrow{\mho'_{ml;n}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_nf_l)\bigr] \qquad (\text{A.2.15})
\end{aligned}$$

$$\begin{aligned}
&-\sum_{m=0}^{k}\bigl[\bar T^n s\mathcal{A}(Z_0,Z_n)\otimes\bar T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar T^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l)\\
&\qquad\xrightarrow{1^{\otimes n}\otimes1^{\otimes m}\otimes b^{\mathcal{E}}_{k-m,1}\otimes1^{\otimes l-1}}\bar T^n s\mathcal{A}(Z_0,Z_n)\otimes\bar T^m s\mathcal{A}^{\mathrm{op}}(X_0,X_m)\\
&\qquad\qquad\otimes X_mf_1[1]\otimes\bar T^{l-1} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_1,f_l)\\
&\qquad\qquad\xrightarrow{\mho'_{m,l-1;n}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_nf_l)\bigr] \qquad (\text{A.2.16})
\end{aligned}$$

$$\begin{aligned}
&-\sum_{a+u+c=k}\bigl[\bar T^n s\mathcal{A}(Z_0,Z_n)\otimes\bar T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar T^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l)\\
&\qquad\xrightarrow{1^{\otimes n}\otimes1^{\otimes a}\otimes b_u^{\mathcal{A}^{\mathrm{op}}}\otimes1^{\otimes c}\otimes1\otimes1^{\otimes l}}\bar T^n s\mathcal{A}(Z_0,Z_n)\\
&\qquad\qquad\otimes\bar T^{a+1+c} s\mathcal{A}^{\mathrm{op}}(X_0,\dots,X_a,X_{a+u},\dots,X_k)\\
&\qquad\qquad\otimes X_kf_0[1]\otimes\bar T^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l)\\
&\qquad\qquad\xrightarrow{\mho'_{a+1+c,l;n}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_nf_l)\bigr]. \qquad (\text{A.2.17})
\end{aligned}$$

Appearance of the component $(\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}})_{pm}$ in term (A.2.7) is explained by the identity $(\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}})_{pm}=((1,\mathscr{Y})\,\mathrm{ev}^{\mathsf{A}_\infty})_{pm}=(1^{\otimes p}\otimes\mathscr{Y}_m)\,\mathrm{ev}^{\mathsf{A}_\infty}_{p1}$, which holds true by the definition of the Yoneda A_∞-functor $\mathscr{Y}:\mathcal{A}^{\mathrm{op}}\to\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)$. Expanding $(\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}})_{pm}$ according to formula (4.2.1), term (A.2.7) can be written as follows:

$$\begin{aligned}
&-(-)^{k+1}\bigl[\bar T^n s\mathcal{A}(Z_0,Z_n)\otimes\bar T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar T^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l)\\
&\xrightarrow{\mathrm{perm}}\bar T^p s\mathcal{A}(Z_0,Z_p)\otimes\bar T^m s\mathcal{A}^{\mathrm{op}}(X_0,X_m)\otimes\bar T^q s\mathcal{A}(Z_p,Z_n)\\
&\qquad\qquad\otimes\bar T^{k-m} s\mathcal{A}^{\mathrm{op}}(X_m,X_k)\otimes X_kf_0[1]\otimes\bar T^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l)\\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}\otimes\,\mathrm{coev}^{\mathsf{C}_\Bbbk}}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes\bar T^p s\mathcal{A}(Z_0,Z_p)\otimes\bar T^m s\mathcal{A}^{\mathrm{op}}(X_0,X_m))\\
&\qquad\qquad\otimes\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_m,Z_p),s\mathcal{A}(X_m,Z_p)\otimes\bar T^q s\mathcal{A}(Z_p,Z_n)\\
&\qquad\qquad\otimes\bar T^{k-m} s\mathcal{A}^{\mathrm{op}}(X_m,X_k)\otimes X_kf_0[1]\otimes\bar T^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{perm})\otimes\underline{\mathsf{C}}_\Bbbk(1,\mathrm{perm})}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0),\bar T^m s\mathcal{A}(X_m,X_0)\otimes s\mathcal{A}(X_0,Z_0)\otimes\bar T^p s\mathcal{A}(Z_0,Z_p))
\end{aligned}$$

$$
\begin{aligned}
&\otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_m,Z_p),X_kf_0[1]\otimes \bar{T}^{k-m}s\mathcal{A}(X_k,X_m)\\
&\qquad\qquad \otimes \mathcal{A}(X_m,Z_p)\otimes \bar{T}^q s\mathcal{A}(Z_p,Z_n)\otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,b_{m+1+p})\otimes\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{k-m+1+q,l})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_m,Z_p))\\
&\qquad\qquad \otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_m,Z_p),X_kf_0[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_kf_0,Z_nf_l))\\
&\xrightarrow{1\otimes\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_m,Z_p))\\
&\qquad\qquad \otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_m,Z_p),X_kf_0[1]\otimes \underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1],Z_nf_l[1]))\\
&\xrightarrow{1\otimes\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_m,Z_p))\otimes\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_m,Z_p),Z_nf_l[1])\\
&\xrightarrow{[-1]s\otimes[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),\mathcal{A}(X_m,Z_p))\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_m,Z_p),Z_nf_l)\\
&\qquad\qquad \xrightarrow{b_2^{\underline{\mathsf{C}}_{\Bbbk}}} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_nf_l)\big].
\end{aligned}
$$

Replacing the last two arrows by the composite $m_2^{\underline{\mathsf{C}}_{\Bbbk}}[-1]s$ and applying identity (3.1.1) yields:

$$
\begin{aligned}
&-(-)^{k+1}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\\
&\qquad\qquad \otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0)\\
&\qquad\qquad \otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes(1^{\otimes(k-m)}\otimes b_{m+1+p}\otimes1^{\otimes q}\otimes1^{\otimes l})\,\mathrm{ev}^{\mathsf{A}_\infty}_{k-m+1+q,l})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_kf_0,Z_nf_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1],Z_nf_l[1]))\\
&\qquad\qquad \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_nf_l[1]) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_nf_l)\big].
\end{aligned}
$$

Denote $d' = s^{-1}ds = s^{-1}d^{X_kf_0}s = b_{00}^{\mathcal{E}} : X_kf_0[1] = s\mathcal{E}(X_k,f_0) \to X_kf_0[1] = s\mathcal{E}(X_k,f_0)$. Thus, the shifted complex $X_kf_0[1]$ carries the differential $-d'$. Since $m_1^{\underline{\mathsf{C}}_{\Bbbk}} = -\underline{\mathsf{C}}_{\Bbbk}(1,d') + \underline{\mathsf{C}}_{\Bbbk}(b_1,1) : \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_nf_l[1]) \to \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_nf_l[1])$, it follows that term (A.2.8) equals

$$
\begin{aligned}
&(-)^{k+1}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\\
&\qquad\qquad \otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0)\\
&\qquad\qquad \otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+n,l})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_kf_0,Z_nf_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1],Z_nf_l[1]))
\end{aligned}
$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_nf_l[1])$$

$$\xrightarrow{m_1^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_nf_l[1]) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_nf_l)\big]$$

$$= (-)^{k+1}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)$$

$$\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)$$

$$\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0)$$

$$\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+n,l})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_kf_0,Z_nf_l))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1],Z_nf_l[1]))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,-\mathrm{ev}^{\mathsf{C}_{\Bbbk}}\cdot d')} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_nf_l[1]) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_nf_l)\big]$$

$$-(-)^{k+1}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)$$

$$\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)$$

$$\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0)$$

$$\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes(1^{\otimes k}\otimes b_1\otimes 1^{\otimes n}\otimes 1^{\otimes l})\,\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+n,l})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_kf_0,Z_nf_l))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1],Z_nf_l[1]))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_nf_l[1]) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_nf_l)\big].$$

Since $\mathrm{ev}^{\mathsf{C}_{\Bbbk}}$ is a chain map, it follows that

$$-\mathrm{ev}^{\mathsf{C}_{\Bbbk}}\cdot d' = -(d'\otimes 1)\,\mathrm{ev}^{\mathsf{C}_{\Bbbk}} + (1\otimes m_1^{\mathsf{C}_{\Bbbk}})\,\mathrm{ev}^{\mathsf{C}_{\Bbbk}} : X_kf_0[1]\otimes\underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1],Z_nf_l[1])\to Z_nf_l[1],$$

therefore term (A.2.8) equals

$$(-)^{k+1}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)$$

$$\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)$$

$$\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0)$$

$$\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+n,l}\,b_1^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_kf_0,Z_nf_l))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1],Z_nf_l[1]))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_nf_l[1]) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_nf_l)\big]$$

$$\begin{aligned}
&+ (-)^{k+1}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l)\\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\\
&\qquad\qquad\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{perm})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), X_kf_0[1]\otimes\bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0)\\
&\qquad\qquad\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,d'\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+n,l})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), X_kf_0[1]\otimes s\underline{\mathsf{C}}_\Bbbk(X_kf_0,Z_nf_l))\\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), X_kf_0[1]\otimes \underline{\mathsf{C}}_\Bbbk(X_kf_0[1],Z_nf_l[1]))\\
&\qquad\qquad\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{ev}^{\mathsf{C}_\Bbbk})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), Z_nf_l[1]) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_nf_l)\big]\\
&- (-)^{k+1}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l)\\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\\
&\qquad\qquad\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{perm})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), X_kf_0[1]\otimes\bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0)\\
&\qquad\qquad\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes(1^{\otimes k}\otimes b_1\otimes 1^{\otimes n}\otimes 1^{\otimes l})\,\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+n,l})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), X_kf_0[1]\otimes s\underline{\mathsf{C}}_\Bbbk(X_kf_0,Z_nf_l))\\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), X_kf_0[1]\otimes \underline{\mathsf{C}}_\Bbbk(X_kf_0[1],Z_nf_l[1]))\\
&\qquad\qquad\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{ev}^{\mathsf{C}_\Bbbk})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), Z_nf_l[1]) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_nf_l)\big]. \qquad \text{(A.2.18)}
\end{aligned}$$

Using the identity

$$f = \big[X \xrightarrow{\mathrm{coev}^{\underline{\mathsf{C}}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(Y, Y\otimes X) \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes f)} \underline{\mathsf{C}}_\Bbbk(Y, Y\otimes\underline{\mathsf{C}}_\Bbbk(Y,Z)) \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{ev}^{\mathsf{C}_\Bbbk})} \underline{\mathsf{C}}_\Bbbk(Y,Z)\big]$$

valid for an arbitrary $f\in\underline{\mathsf{C}}_\Bbbk(X,\underline{\mathsf{C}}_\Bbbk(Y,Z))$ by general properties of closed monoidal categories, term (A.2.9) can be written as follows:

$$\begin{aligned}
&- (-)^{k+1}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l)\\
&\xrightarrow{\mathrm{perm}} \bar{T}^p s\mathcal{A}(Z_0,Z_p)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l)\\
&\qquad\qquad\otimes\bar{T}^q s\mathcal{A}(Z_p,Z_n)\otimes T^0 s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_l,f_l)\\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}\otimes\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^p s\mathcal{A}(Z_0,Z_p)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\\
&\qquad\qquad\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l))\\
&\qquad\qquad\otimes\underline{\mathsf{C}}_\Bbbk(Z_pf_l[1], Z_pf_l[1]\otimes\bar{T}^q s\mathcal{A}(Z_p,Z_n)\otimes T^0 s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_l,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{perm})\otimes 1} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), X_kf_0[1]\otimes\bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0)\\
&\qquad\qquad\otimes\bar{T}^p s\mathcal{A}(Z_0,Z_p)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l))\\
&\qquad\qquad\otimes\underline{\mathsf{C}}_\Bbbk(Z_pf_l[1], Z_pf_l[1]\otimes\bar{T}^q s\mathcal{A}(Z_p,Z_n)\otimes T^0 s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_l,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+p,l})\otimes\underline{\mathsf{C}}_\Bbbk(1,1\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{q0})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), X_kf_0[1]\otimes s\underline{\mathsf{C}}_\Bbbk(X_kf_0,Z_pf_l))
\end{aligned}$$

$$\otimes \underline{\mathsf{C}}_{\Bbbk}(Z_pf_l[1], Z_pf_l[1] \otimes s\underline{\mathsf{C}}_{\Bbbk}(Z_pf_l, Z_nf_l))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])\otimes\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0, Z_0), X_kf_0[1] \otimes \underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1], Z_pf_l[1]))$$

$$\otimes \underline{\mathsf{C}}_{\Bbbk}(Z_pf_l[1], Z_pf_l[1] \otimes \underline{\mathsf{C}}_{\Bbbk}(Z_pf_l[1], Z_nf_l[1]))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})\otimes\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0, Z_0), Z_pf_l[1]) \otimes \underline{\mathsf{C}}_{\Bbbk}(Z_pf_l[1], Z_nf_l[1])$$

$$\xrightarrow{[-1]s\otimes[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0, Z_0), Z_pf_l) \otimes s\underline{\mathsf{C}}_{\Bbbk}(Z_pf_l, Z_nf_l)$$

$$\xrightarrow{b_2^{\underline{\mathsf{C}}_{\Bbbk}}} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0, Z_0), Z_nf_l)\big]. \quad \text{(A.2.19)}$$

Replacing the last two arrows by the composite $m_2^{\underline{\mathsf{C}}_{\Bbbk}}[-1]s$ and applying identity (3.1.1) leads to

$$-(-)^{k+1}\big[\bar{T}^n s\mathcal{A}(Z_0, Z_n) \otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0, X_k) \otimes X_kf_0[1] \otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_{\Bbbk})(f_0, f_l)$$

$$\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0, Z_0), s\mathcal{A}(X_0, Z_0) \otimes \bar{T}^n s\mathcal{A}(Z_0, Z_n)$$

$$\otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0, X_k) \otimes X_kf_0[1] \otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_{\Bbbk})(f_0, f_l))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0, Z_0), X_kf_0[1] \otimes \bar{T}^k s\mathcal{A}(X_k, X_0) \otimes s\mathcal{A}(X_0, Z_0)$$

$$\otimes \bar{T}^n s\mathcal{A}(Z_0, Z_n) \otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_{\Bbbk})(f_0, f_l))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes\mathrm{perm})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0, Z_0), X_kf_0[1] \otimes \bar{T}^k s\mathcal{A}(X_k, X_0) \otimes s\mathcal{A}(X_0, Z_0) \otimes \bar{T}^p s\mathcal{A}(Z_0, Z_p)$$

$$\otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_{\Bbbk})(f_0, f_l) \otimes \bar{T}^q s\mathcal{A}(Z_p, Z_n) \otimes T^0 s\underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_{\Bbbk})(f_l, f_l))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+p,l}\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{q0})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0, Z_0), X_kf_0[1] \otimes s\underline{\mathsf{C}}_{\Bbbk}(X_kf_0, Z_pf_l) \otimes s\underline{\mathsf{C}}_{\Bbbk}(Z_pf_l, Z_nf_l))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1]\otimes s^{-1}[1])}$$

$$\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0, Z_0), X_kf_0[1] \otimes \underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1], Z_pf_l[1]) \otimes \underline{\mathsf{C}}_{\Bbbk}(Z_pf_l[1], Z_nf_l[1]))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,(\mathrm{ev}^{\mathsf{C}_{\Bbbk}}\otimes 1)\,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0, Z_0), Z_nf_l[1]) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0, Z_0), Z_nf_l)\big]. \quad \text{(A.2.20)}$$

Since

$$(\mathrm{ev}^{\mathsf{C}_{\Bbbk}} \otimes 1)\,\mathrm{ev}^{\mathsf{C}_{\Bbbk}} = (1 \otimes m_2^{\underline{\mathsf{C}}_{\Bbbk}})\,\mathrm{ev}^{\mathsf{C}_{\Bbbk}} :$$

$$X_kf_0[1] \otimes \underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1], Z_pf_l[1]) \otimes \mathsf{C}_{\Bbbk}(Z_pf_l[1], Z_nf_l[1]) \to Z_nf_l[1],$$

and $(s^{-1}[1] \otimes s^{-1}[1])m_2^{\underline{\mathsf{C}}_{\Bbbk}} = -b_2^{\underline{\mathsf{C}}_{\Bbbk}} s^{-1}[1]$, we infer that term (A.2.9) equals

$$(-)^{k+1}\big[\bar{T}^n s\mathcal{A}(Z_0, Z_n) \otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0, X_k) \otimes X_kf_0[1] \otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_{\Bbbk})(f_0, f_l)$$

$$\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0, Z_0), s\mathcal{A}(X_0, Z_0) \otimes \bar{T}^n s\mathcal{A}(Z_0, Z_n)$$

$$\otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0, X_k) \otimes X_kf_0[1] \otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_{\Bbbk})(f_0, f_l))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0, Z_0), X_kf_0[1] \otimes \bar{T}^k s\mathcal{A}(X_k, X_0) \otimes s\mathcal{A}(X_0, Z_0)$$

$$\otimes \bar{T}^n s\mathcal{A}(Z_0, Z_n) \otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_{\Bbbk})(f_0, f_l))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes\mathrm{perm}\cdot(\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+p,l}\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{q0})b_2^{\underline{\mathsf{C}}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0, Z_0), X_kf_0[1] \otimes s\underline{\mathsf{C}}_{\Bbbk}(X_kf_0, Z_nf_l))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0, Z_0), X_kf_0[1] \otimes \underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1], Z_nf_l[1]))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_nf_l[1]) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_nf_l)\big].$$

Similarly, term (A.2.10) equals

$$\begin{aligned}
&-(-)^{k+1}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}\otimes\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes \bar{T}^p s\mathcal{A}(Z_0,Z_p))\otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_p),\\
&\quad s\mathcal{A}(X_0,Z_p)\otimes \bar{T}^q s\mathcal{A}(Z_p,Z_n)\otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{1\otimes\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes \bar{T}^p s\mathcal{A}(Z_0,Z_p))\\
&\qquad\otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_p),X_kf_0[1]\otimes \bar{T}^k s\mathcal{A}(X_k,X_0)\\
&\qquad\qquad\otimes s\mathcal{A}(X_0,Z_p)\otimes \bar{T}^q s\mathcal{A}(Z_p,Z_n)\otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,b_{p+1})\otimes\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+q,l})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_p))\\
&\qquad\qquad\otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_p),X_kf_0[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_kf_0,Z_nf_l))\\
&\xrightarrow{1\otimes\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])}\\
&\quad \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_p))\otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_p),X_kf_0[1]\otimes \underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1],Z_nf_l[1]))\\
&\xrightarrow{1\otimes\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_p))\otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_p),Z_nf_l[1])\\
&\xrightarrow{[-1]s\otimes[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),\mathcal{A}(X_0,Z_p))\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_p),Z_nf_l)\\
&\qquad\qquad\qquad\xrightarrow{b_2^{\underline{\mathsf{C}}_{\Bbbk}}} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_nf_l)\big]\\
={}&-(-)^{k+1}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes \bar{T}^n s\mathcal{A}(Z_0,Z_n)\\
&\qquad\qquad\otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes \bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0)\\
&\qquad\qquad\otimes \bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes(1^{\otimes k}\otimes b_{p+1}\otimes 1^{\otimes q}\otimes 1^{\otimes l})\,\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+q,l})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_kf_0,Z_nf_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes \underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1],Z_nf_l[1]))\\
&\qquad\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_nf_l[1]) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_nf_l)\big]
\end{aligned}$$

due to identity (3.1.1). It follows immediately from the naturality of $\mathrm{coev}^{\mathsf{C}_{\Bbbk}}$ that term (A.2.11) equals

$$\begin{aligned}
&(-)^{k+1}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes \bar{T}^n s\mathcal{A}(Z_0,Z_n)\\
&\qquad\qquad\otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes \bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0)\\
&\qquad\qquad\otimes \bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))
\end{aligned}$$

$$\begin{aligned}
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes(1^{\otimes k}\otimes 1\otimes 1^{\otimes\alpha}\otimes b_t\otimes 1^{\otimes\beta}\otimes 1^{\otimes l})\,\mathrm{ev}^{\mathsf{A}_\infty}_{k+\alpha+2+\beta,l})} \\
&\qquad\qquad \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_kf_0,Z_nf_l)) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes \underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1],Z_nf_l[1])) \\
&\qquad\qquad \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_nf_l[1]) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_nf_l)\big].
\end{aligned}$$

Term (A.2.12) can be written as follows:

$$\begin{aligned}
&-(-)^{k+1}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l) \\
&\xrightarrow{\mathrm{perm}} \bar{T}^p s\mathcal{A}(Z_0,Z_p)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^{l-1}s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_{l-1}) \\
&\qquad\qquad \otimes\bar{T}^q s\mathcal{A}(Z_p,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_{l-1},f_l) \\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}\otimes\,\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^p s\mathcal{A}(Z_0,Z_p)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k) \\
&\qquad\qquad \otimes X_kf_0[1]\otimes\bar{T}^{l-1}s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_{l-1})) \\
&\qquad\qquad \otimes\underline{\mathsf{C}}_{\Bbbk}(Z_pf_{l-1}[1],Z_pf_{l-1}[1]\otimes\bar{T}^q s\mathcal{A}(Z_p,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_{l-1},f_l)) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})\otimes 1} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0) \\
&\qquad\qquad \otimes\bar{T}^p s\mathcal{A}(Z_0,Z_p)\otimes\bar{T}^{l-1}s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_{l-1})) \\
&\qquad\qquad \otimes\underline{\mathsf{C}}_{\Bbbk}(Z_pf_{l-1}[1],Z_pf_{l-1}[1]\otimes\bar{T}^q s\mathcal{A}(Z_p,Z_n)\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_{l-1},f_l)) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+p,l-1})\otimes\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{q1})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_kf_0,Z_pf_{l-1})) \\
&\qquad\qquad \otimes\underline{\mathsf{C}}_{\Bbbk}(Z_pf_{l-1}[1],Z_pf_{l-1}[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(Z_pf_{l-1},Z_nf_l)) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])\otimes\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1],Z_pf_{l-1}[1])) \\
&\qquad\qquad \otimes\underline{\mathsf{C}}_{\Bbbk}(Z_pf_{l-1}[1],Z_pf_{l-1}[1]\otimes\underline{\mathsf{C}}_{\Bbbk}(Z_pf_{l-1}[1],Z_nf_l[1])) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})\otimes\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_pf_{l-1}[1])\otimes\underline{\mathsf{C}}_{\Bbbk}(Z_pf_{l-1}[1],Z_nf_l[1]) \\
&\xrightarrow{[-1]s\otimes[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_pf_{l-1})\otimes s\underline{\mathsf{C}}_{\Bbbk}(Z_pf_{l-1},Z_nf_l) \xrightarrow{b_2^{\mathsf{C}_{\Bbbk}}} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_nf_l)\big].
\end{aligned}$$

The obtained expression is of the same type as (A.2.19). Transform it in the same manner to conclude that term (A.2.12) equals

$$\begin{aligned}
&(-)^{k+1}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l) \\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n) \\
&\qquad\qquad \otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0) \\
&\qquad\qquad \otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes\mathrm{perm}\cdot(\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+p,l-1}\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{q1})b_2^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_kf_0,Z_nf_l)) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1],Z_nf_l[1])) \\
&\qquad\qquad \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_nf_l[1]) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_nf_l)\big].
\end{aligned}$$

Obviously, term (A.2.13) can be written as follows:

$$
\begin{aligned}
&(-)^{k+1}\bigl[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l)\\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\\
&\qquad\qquad\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{perm})}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0)\\
&\qquad\qquad\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes(1^{\otimes k}\otimes1\otimes1^{\otimes n}\otimes1^{\otimes p-1}\otimes B_1\otimes1^{\otimes l-p})\,\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+n,l})}\\
&\qquad\qquad\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes s\underline{\mathsf{C}}_\Bbbk(X_kf_0,Z_nf_l))\\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes s^{-1}[1])}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\underline{\mathsf{C}}_\Bbbk(X_kf_0[1],Z_nf_l[1]))\\
&\qquad\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{ev}^{\mathsf{C}_\Bbbk})}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0),Z_nf_l[1])\xrightarrow{[-1]s}s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_nf_l)\bigr].
\end{aligned}
$$

Similar presentation holds for term (A.2.14):

$$
\begin{aligned}
&(-)^{k+1}\bigl[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l)\\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\\
&\qquad\qquad\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{perm})}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0)\\
&\qquad\qquad\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes(1^{\otimes k}\otimes1\otimes1^{\otimes n}\otimes1^{\otimes p-1}\otimes B_2\otimes1^{\otimes l-p-1})\,\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+n,l-1})}\\
&\qquad\qquad\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes s\underline{\mathsf{C}}_\Bbbk(X_kf_0,Z_nf_l))\\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes s^{-1}[1])}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\underline{\mathsf{C}}_\Bbbk(X_kf_0[1],Z_nf_l[1]))\\
&\qquad\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{ev}^{\mathsf{C}_\Bbbk})}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0),Z_nf_l[1])\xrightarrow{[-1]s}s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_nf_l)\bigr].
\end{aligned}
$$

If $m=k$, term (A.2.15) equals

$$
\begin{aligned}
&(-)^{k+1}\bigl[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l)\\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\\
&\qquad\qquad\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{perm})}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0)\\
&\qquad\qquad\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,d'\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+n,l})}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes s\underline{\mathsf{C}}_\Bbbk(X_kf_0,Z_nf_l))\\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes s^{-1}[1])}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\underline{\mathsf{C}}_\Bbbk(X_kf_0[1],Z_nf_l[1]))\\
&\qquad\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{ev}^{\mathsf{C}_\Bbbk})}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0),Z_nf_l[1])\xrightarrow{[-1]s}s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),Z_nf_l)\bigr],
\end{aligned}
$$

it cancels one of the summands present in (A.2.18). Suppose that $0 \leqslant m \leqslant k-1$. Expressing $b^{\mathcal{E}}_{k-m,0}$ via $\mathrm{ev}^{\mathsf{A}_\infty}_{k-m,0}$, we write term (A.2.15) as follows:

$$
\begin{aligned}
&(-)^{k+1}\bigl[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\\
&\xrightarrow{\mathrm{perm}}\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^m s\mathcal{A}^{\mathrm{op}}(X_0,X_m)\otimes X_kf_0[1]\otimes\bar{T}^{k-m} s\mathcal{A}(X_k,X_m)\\
&\qquad\qquad\otimes T^0 s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_0)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\\
&\xrightarrow{1^{\otimes n}\otimes1^{\otimes m}\otimes1\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{k-m,0}\otimes1^{\otimes l}}\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^m s\mathcal{A}^{\mathrm{op}}(X_0,X_m)\otimes X_kf_0[1]\\
&\qquad\qquad\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_kf_0,X_mf_0)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\\
&\xrightarrow{1^{\otimes n}\otimes1^{\otimes m}\otimes1\otimes s^{-1}[1]\otimes1^{\otimes l}}\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^m s\mathcal{A}^{\mathrm{op}}(X_0,X_m)\otimes X_kf_0[1]\\
&\qquad\qquad\otimes\underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1],X_mf_0[1])\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\\
&\xrightarrow{1^{\otimes n}\otimes1^{\otimes m}\otimes\mathrm{ev}^{\mathsf{C}_{\Bbbk}}\otimes1^{\otimes l}}\\
&\qquad\qquad\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^m s\mathcal{A}^{\mathrm{op}}(X_0,X_m)\otimes X_mf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^m s\mathcal{A}^{\mathrm{op}}(X_0,X_m)\\
&\qquad\qquad\otimes X_mf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_mf_0[1]\otimes\bar{T}^m s\mathcal{A}(X_m,X_0)\otimes s\mathcal{A}(X_0,Z_0)\\
&\qquad\qquad\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{m+1+n,l})}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_mf_0[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_mf_0,Z_nf_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_mf_0[1]\otimes\underline{\mathsf{C}}_{\Bbbk}(X_mf_0[1],Z_nf_l[1]))\\
&\qquad\qquad\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_nf_l[1])\xrightarrow{[-1]s}s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_nf_l)\bigr]\\
&=(-)^{k+1}\bigl[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\\
&\qquad\qquad\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\bar{T}^{k-m} s\mathcal{A}(X_k,X_m)\otimes T^0 s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_0)\\
&\qquad\qquad\otimes\bar{T}^m s\mathcal{A}(X_m,X_0)\otimes s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{k-m,0}\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{m+1+n,l})}\\
&\qquad\qquad\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_kf_0,X_mf_0)\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_mf_0,Z_nf_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1]\otimes s^{-1}[1])}\\
&\qquad\qquad\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1],X_mf_0[1])\otimes\underline{\mathsf{C}}_{\Bbbk}(X_mf_0[1],Z_nf_l[1]))\\
&\qquad\qquad\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,(\mathrm{ev}^{\mathsf{C}_{\Bbbk}}\otimes1)\,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_nf_l[1])\xrightarrow{[-1]s}s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_nf_l)\bigr].
\end{aligned}
$$

The further transformations are parallel to (A.2.20). We conclude that term (A.2.15) equals

$$
\begin{aligned}
&-(-)^{k+1}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\\
&\qquad\qquad\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0)\\
&\qquad\qquad\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes(\mathrm{ev}^{\mathsf{A}_\infty}_{k-m,0}\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{m+1+n,l})b_2^{\mathsf{C}_{\Bbbk}})}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_kf_0,Z_nf_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1],Z_nf_l[1]))\\
&\qquad\qquad\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_nf_l[1])\xrightarrow{[-1]s}s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_nf_l)\big].
\end{aligned}
$$

The case of term (A.2.16) is quite similar, we only give the result:

$$
\begin{aligned}
&-(-)^{k+1}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\\
&\qquad\qquad\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0)\\
&\qquad\qquad\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes\mathrm{perm}\cdot(\mathrm{ev}^{\mathsf{A}_\infty}_{k-m,1}\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{m+1+n,l-1})b_2^{\mathsf{C}_{\Bbbk}})}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_kf_0,Z_nf_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1],Z_nf_l[1]))\\
&\qquad\qquad\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_nf_l[1])\xrightarrow{[-1]s}s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_nf_l)\big].
\end{aligned}
$$

Finally, using formula (3.5.3) for $b_u^{\mathcal{A}^{\mathrm{op}}}$, we find that term (A.2.17) equals

$$
\begin{aligned}
&(-)^{a+2+c}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)\\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\\
&\qquad\qquad\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes1^{\otimes n}\otimes1^{\otimes a}\otimes b_u^{\mathcal{A}^{\mathrm{op}}}\otimes1^{\otimes c}\otimes1\otimes1^{\otimes l})}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\\
&\qquad\otimes\bar{T}^{a+1+c} s\mathcal{A}^{\mathrm{op}}(X_0,\dots,X_a,X_{a+u},\dots,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\bar{T}^{c+1+a} s\mathcal{A}(X_k,\dots,X_{a+u},X_a,\dots,X_0)\\
&\qquad\qquad\otimes s\mathcal{A}(X_0,Z_0)\otimes\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes\mathrm{ev}^{\mathsf{A}_\infty}_{c+2+a+n,l})}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_kf_0,Z_nf_l))\\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\otimes s^{-1}[1])}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),X_kf_0[1]\otimes\underline{\mathsf{C}}_{\Bbbk}(X_kf_0[1],Z_nf_l[1]))\\
&\qquad\qquad\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\otimes\,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})}\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0),Z_nf_l[1])\xrightarrow{[-1]s}s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0),Z_nf_l)\big]\\
&=(-)^{a+u+c+1}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes\bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kf_0[1]\otimes\bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)
\end{aligned}
$$

$$
\begin{aligned}
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0), s\mathcal{A}(X_0,Z_0)\otimes \bar{T}^n s\mathcal{A}(Z_0,Z_n) \\
&\qquad\qquad \otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_k f_0[1]\otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0), X_k f_0[1]\otimes \bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0) \\
&\qquad\qquad \otimes \bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes(1^{\otimes c}\otimes b_u\otimes 1^{\otimes a}\otimes 1\otimes 1^{\otimes n}\otimes 1^{\otimes l})\,\mathrm{ev}^{\mathsf{A}_\infty}_{c+2+a+n,l})} \\
&\qquad\qquad \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0), X_k f_0[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_k f_0, Z_n f_l)) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0), X_k f_0[1]\otimes \underline{\mathsf{C}}_{\Bbbk}(X_k f_0[1], Z_n f_l[1])) \\
&\qquad \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0), Z_n f_l[1]) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0), Z_n f_l)\big].
\end{aligned}
$$

The overall sign is $(-)^{a+u+c+1} = (-)^{k+1}$.

Summing up, we conclude that

$$
\begin{aligned}
(1^{\otimes n}\otimes S\cdot \mathrm{pr}_n)\,\mathrm{ev}^{\mathsf{C}_{\Bbbk}} &= (-)^{k+1}\big[\bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_k f_0[1] \\
&\qquad\qquad \otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l) \\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0), s\mathcal{A}(X_0,Z_0)\otimes \bar{T}^n s\mathcal{A}(Z_0,Z_n) \\
&\qquad\qquad \otimes \bar{T}^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_k f_0[1]\otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{perm})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0), X_k f_0[1]\otimes \bar{T}^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0) \\
&\qquad\qquad \otimes \bar{T}^n s\mathcal{A}(Z_0,Z_n)\otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(f_0,f_l)) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes R)} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0), X_k f_0[1]\otimes s\underline{\mathsf{C}}_{\Bbbk}(X_k f_0, Z_n f_l)) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0), X_k f_0[1]\otimes \underline{\mathsf{C}}_{\Bbbk}(X_k f_0[1], Z_n f_l[1])) \\
&\qquad \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X_0,Z_0), Z_n f_l[1]) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X_0,Z_0), Z_n f_l)\big],
\end{aligned}
$$

where

$$
\begin{aligned}
R = &-\sum_{m=1}^{k}\sum_{p+q=n}(1^{\otimes k-m}\otimes b_{m+1+p}\otimes 1^{\otimes q}\otimes 1^{\otimes l})\,\mathrm{ev}^{\mathsf{A}_\infty}_{k-m+1+q,l} + \mathrm{ev}^{\mathsf{A}_\infty}_{k+1+n,l}\, b^{\underline{\mathsf{C}}_{\Bbbk}} \\
&-(1^{\otimes k}\otimes b_1\otimes 1^{\otimes n}\otimes 1^{\otimes l})\,\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+n,l} + \sum_{p+q=n}^{q>0}\mathrm{perm}\cdot(\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+p,l}\otimes \mathrm{ev}^{\mathsf{A}_\infty}_{q0})b_2^{\underline{\mathsf{C}}_{\Bbbk}} \\
&-\sum_{p+q=n}^{p>0}(1^{\otimes k}\otimes b_{p+1}\otimes 1^{\otimes q}\otimes 1^{\otimes l})\,\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+q,l} \\
&-\sum_{\alpha+t+\beta=n}(1^{\otimes k}\otimes 1\otimes 1^{\otimes \alpha}\otimes b_t\otimes 1^{\otimes \beta}\otimes 1^{\otimes l})\,\mathrm{ev}^{\mathsf{A}_\infty}_{k+\alpha+2+\beta,l} \\
&+\sum_{p+q=n}\mathrm{perm}\cdot(\mathrm{ev}^{\mathsf{A}_\infty}_{k+1+p,l-1}\otimes \mathrm{ev}^{\mathsf{A}_\infty}_{q1})b_2^{\underline{\mathsf{C}}_{\Bbbk}}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{p=1}^{l} (1^{\otimes k+1+n} \otimes 1^{\otimes p-1} \otimes B_1 \otimes 1^{\otimes l-p}) \operatorname{ev}^{\mathsf{A}_\infty}_{k+1+n,l} \\
&- \sum_{p=1}^{l-1} (1^{\otimes k+1+n} \otimes 1^{\otimes p-1} \otimes B_2 \otimes 1^{\otimes l-p-1}) \operatorname{ev}^{\mathsf{A}_\infty}_{k+1+n,l-1} \\
&+ \sum_{m=0}^{k-1} (\operatorname{ev}^{\mathsf{A}_\infty}_{k-m,0} \otimes \operatorname{ev}^{\mathsf{A}_\infty}_{m+1+n,l}) b_2^{\mathsf{C}_\Bbbk} + \sum_{m=0}^{k} \operatorname{perm} \cdot (\operatorname{ev}^{\mathsf{A}_\infty}_{k-m,1} \otimes \operatorname{ev}^{\mathsf{A}_\infty}_{m+1+n,l-1}) b_2^{\mathsf{C}_\Bbbk} \\
&- \sum_{a+u+c=k} (1^{\otimes c} \otimes b_u \otimes 1^{\otimes a} \otimes 1 \otimes 1^{\otimes n} \otimes 1^{\otimes l}) \operatorname{ev}^{\mathsf{A}_\infty}_{c+2+a+n,l} : \\
&\bar{T}^k s\mathcal{A}(X_k, X_0) \otimes s\mathcal{A}(X_0, Z_0) \otimes \bar{T}^n s\mathcal{A}(Z_0, Z_n) \otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_\Bbbk)(f_0, f_l) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \to s\underline{\mathsf{C}}_\Bbbk(X_k f_0, Z_n f_l),
\end{aligned}
$$

which is easily seen to be the restriction of

$$
(\operatorname{ev}^{\mathsf{A}_\infty} b^{\underline{\mathsf{C}}_\Bbbk} - (b^{\mathcal{A}} \boxtimes 1 + 1 \boxtimes B) \operatorname{ev}^{\mathsf{A}_\infty}) \operatorname{pr}_1 : Ts\mathcal{A} \boxtimes Ts\underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_\Bbbk) \to s\underline{\mathsf{C}}_\Bbbk
$$

to the summand $\bar{T}^k s\mathcal{A}(X_k, X_0) \otimes s\mathcal{A}(X_0, Z_0) \otimes \bar{T}^n s\mathcal{A}(Z_0, Z_n) \otimes \bar{T}^l s\underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_\Bbbk)(f_0, f_l)$ of the source. Since $\operatorname{ev}^{\mathsf{A}_\infty}$ is an A_∞-functor, it follows that $R = 0$, and the equation is proven. That finishes the proof of Lemma A.2. $\square$

Let $\mathcal{A}$ be an A_∞-category, and let $f : \mathcal{A} \to \underline{\mathsf{C}}_\Bbbk$ be an A_∞-functor. Denote by $\mathcal{M}$ the $\mathcal{A}$-module determined by f in Proposition 4.5.2. Denote

$$
\Upsilon = \mho_{00} : s\mathcal{E}(X, f) = Xf[1] \to s\underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_\Bbbk)(H^X, f) \tag{A.2.21}
$$

for the sake of brevity. The composition of Υ with the projection pr_n from (A.1.3) is given by the particular case $p = q = 0$ of (A.1.4):

$$
\begin{aligned}
\Upsilon_n = -\bigl[& s\mathcal{M}(X) \\
&\xrightarrow{\operatorname{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n), s\mathcal{A}(X, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n) \otimes s\mathcal{M}(X)) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1, \tau_c)} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n), \\
&\qquad\qquad\qquad\qquad\qquad s\mathcal{M}(X) \otimes s\mathcal{A}(X, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n)) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1, b^{\mathcal{M}}_{n+1})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n), s\mathcal{M}(Z_n)) \\
&\xrightarrow{(\underline{\varphi}^{\mathsf{C}_\Bbbk})^{-1}} \underline{\mathsf{C}}_\Bbbk(T^n s\mathcal{A}(Z_0, Z_n), \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X, Z_0), s\mathcal{M}(Z_n))) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1, [-1])} \underline{\mathsf{C}}_\Bbbk(T^n s\mathcal{A}(Z_0, Z_n), \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X, Z_0), \mathcal{M}(Z_n))) \\
&\qquad\qquad \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1, s)} \underline{\mathsf{C}}_\Bbbk(T^n s\mathcal{A}(Z_0, Z_n), s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X, Z_0), \mathcal{M}(Z_n)))\bigr],
\end{aligned}
$$

where $n \geqslant 0$, $\tau = \left(\begin{smallmatrix} 0 & 1 & \dots & n & n+1 \\ 1 & 2 & \dots & n+1 & 0 \end{smallmatrix}\right) \in \mathfrak{S}_{n+2}$. An element $r \in s\mathcal{M}(X)$ is mapped to an A_∞-transformation $(r)\Upsilon$ with components

$$
\begin{gathered}
(r)\Upsilon_n : T^n s\mathcal{A}(Z_0, Z_n) \to s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X, Z_0), \mathcal{M}(Z_n)), \quad n \geqslant 0, \\
z_1 \otimes \dots \otimes z_n \mapsto (z_1 \otimes \dots \otimes z_n \otimes r)\Upsilon'_n,
\end{gathered}
$$

where

$$\begin{aligned}\Upsilon'_n = -\bigl[&T^n s\mathcal{A}(Z_0,Z_n)\otimes s\mathcal{M}(X)\\ &\xrightarrow{\mathrm{coev}^{\underline{\mathsf{C}}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z_0), s\mathcal{A}(X,Z_0)\otimes T^n s\mathcal{A}(Z_0,Z_n)\otimes s\mathcal{M}(X))\\ &\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\tau_c)} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z_0), s\mathcal{M}(X)\otimes s\mathcal{A}(X,Z_0)\otimes T^n s\mathcal{A}(Z_0,Z_n))\\ &\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,b^{\mathcal{M}}_{n+1})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z_0), s\mathcal{M}(Z_n))\\ &\qquad\qquad\xrightarrow{[-1]} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z_0),\mathcal{M}(Z_n)) \xrightarrow{s} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z_0),\mathcal{M}(Z_n))\bigr].\end{aligned}$$

Since $\mho$ is a chain map by Lemma A.2, vanishing of (4.1.12) on $s\mathcal{M}(X)$ implies that the map

$$\mho_{00} = \Upsilon : (s\mathcal{M}(X), b^{\mathcal{E}}_{00} = s^{-1}d^{Xf}s = b^{\mathcal{M}}_0) \to (s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^X,f), b^{\mathcal{Q}}_{00} = B_1)$$

is a chain map as well.

A.3. Proposition. *Let $\mathcal{A}$ be a unital A_∞-category, let X be an object of $\mathcal{A}$, and let $f:\mathcal{A}\to\underline{\mathsf{C}}_\Bbbk$ be a unital A_∞-functor. Then the map Υ is homotopy invertible.*

When the author was almost done with writing up the dissertation, he learned that Proposition A.3 could also be found in Seidel's book [**48**, Lemma 2.12], where it is proven assuming that the ground ring $\Bbbk$ is a field. The proof is based on a spectral sequence argument. The proof presented here is considerably longer than that of Seidel, however it works in the case of an arbitrary commutative ground ring.

Proof. The A_∞-module $\mathcal{M}$ corresponding to f is unital by Proposition 4.5.4. Components of f are expressed via components of $b^{\mathcal{M}}$ as follows ($k\geqslant 1$):

$$\begin{aligned}f_k = \bigl[&T^k s\mathcal{A}(Z_0,Z_k) \xrightarrow{\mathrm{coev}^{\underline{\mathsf{C}}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{M}(Z_0), s\mathcal{M}(Z_0)\otimes T^k s\mathcal{A}(Z_0,Z_k)) \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,b^{\mathcal{M}}_k)}\\ &\underline{\mathsf{C}}_\Bbbk(s\mathcal{M}(Z_0), s\mathcal{M}(Z_k)) \xrightarrow{[-1]} \underline{\mathsf{C}}_\Bbbk(\mathcal{M}(Z_0),\mathcal{M}(Z_k)) \xrightarrow{s} s\underline{\mathsf{C}}_\Bbbk(\mathcal{M}(Z_0),\mathcal{M}(Z_k))\bigr]. \quad (\text{A.3.1})\end{aligned}$$

Define a map $\alpha : s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^X,f)\to s\mathcal{M}(X)$ as follows:

$$\begin{aligned}\alpha = \bigl[&s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^X,f) \xrightarrow{\mathrm{pr}_0} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,X),\mathcal{M}(X)) \xrightarrow{s^{-1}} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,X),\mathcal{M}(X))\\ &\xrightarrow{[1]} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,X), s\mathcal{M}(X)) \xrightarrow{\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1)} \underline{\mathsf{C}}_\Bbbk(\Bbbk, s\mathcal{M}(X)) = s\mathcal{M}(X)\bigr]. \quad (\text{A.3.2})\end{aligned}$$

The map α is a chain map. Indeed, pr_0 is a chain map, and

$$\begin{aligned}s^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1)b^{\mathcal{M}}_0 &= s^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1)\underline{\mathsf{C}}_\Bbbk(1,b^{\mathcal{M}}_0)\\ &= s^{-1}[1](-\underline{\mathsf{C}}_\Bbbk(1,b^{\mathcal{M}}_0)+\underline{\mathsf{C}}_\Bbbk(b_1,1))\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1)\\ &= s^{-1}[1]m^{\underline{\mathsf{C}}_\Bbbk}_1\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1)\\ &= s^{-1}m^{\underline{\mathsf{C}}_\Bbbk}_1[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1)\\ &= b^{\underline{\mathsf{C}}_\Bbbk}_1 s^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1),\end{aligned}$$

since ${}_X\mathbf{i}_0^{\mathcal{A}}$ is a chain map, and $[1]$ is a differential graded functor. Let us compute $\Upsilon\alpha$:

$$\begin{aligned}
\Upsilon\alpha &= \Upsilon_0 s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}}, 1) \\
&= -\bigl[s\mathcal{M}(X) \xrightarrow{\operatorname{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,X), s\mathcal{A}(X,X)\otimes s\mathcal{M}(X)) \\
&\qquad \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,c)} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,X), s\mathcal{M}(X)\otimes s\mathcal{A}(X,X)) \\
&\qquad\qquad \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,b_1^{\mathcal{M}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,X), s\mathcal{M}(X)) \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1)} \underline{\mathsf{C}}_{\Bbbk}(\Bbbk, s\mathcal{M}(X)) = s\mathcal{M}(X)\bigr] \\
&= \bigl[s\mathcal{M}(X) \xrightarrow{\operatorname{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,X), s\mathcal{A}(X,X)\otimes s\mathcal{M}(X)) \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1)} \\
&\qquad\qquad \underline{\mathsf{C}}_{\Bbbk}(\Bbbk, s\mathcal{A}(X,X)\otimes s\mathcal{M}(X)) \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,cb_1^{\mathcal{M}})} \underline{\mathsf{C}}_{\Bbbk}(\Bbbk, s\mathcal{M}(X)) = s\mathcal{M}(X)\bigr] \\
&= \bigl[s\mathcal{M}(X) \xrightarrow{\operatorname{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(\Bbbk, \Bbbk\otimes s\mathcal{M}(X)) \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,{}_X\mathbf{i}_0^{\mathcal{A}}\otimes 1)} \underline{\mathsf{C}}_{\Bbbk}(\Bbbk, s\mathcal{A}(X,X)\otimes s\mathcal{M}(X)) \\
&\qquad\qquad \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,cb_1^{\mathcal{M}})} \underline{\mathsf{C}}_{\Bbbk}(\Bbbk, s\mathcal{M}(X)) = s\mathcal{M}(X)\bigr] \\
&= \bigl[s\mathcal{M}(X) \xrightarrow{1\otimes {}_X\mathbf{i}_0^{\mathcal{A}}} s\mathcal{M}(X)\otimes s\mathcal{A}(X,X) \xrightarrow{b_1^{\mathcal{M}}} s\mathcal{M}(X)\bigr]. \qquad \text{(A.3.3)}
\end{aligned}$$

Since $\mathcal{M}$ is a unital A_∞-module by Proposition 4.5.4, it follows that $\Upsilon\alpha$ is homotopic to identity. Let us prove that $\alpha\Upsilon$ is homotopy invertible.

The graded $\Bbbk$-module $s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(H^X, f)$ is $V = \prod_{n=0}^{\infty} V_n$, where

$$V_n = \prod_{Z_0,\dots,Z_n\in\operatorname{Ob}\mathcal{A}} \underline{\mathsf{C}}_{\Bbbk}(\bar{T}^n s\mathcal{A}(Z_0,Z_n), s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,Z_0),\mathcal{M}(Z_n)))$$

and all products are taken in the category of graded $\Bbbk$-modules. In other terms, for $d\in\mathbb{Z}$, $V^d = \prod_{n=0}^{\infty} V_n^d$, where

$$V_n^d = \prod_{Z_0,\dots,Z_n\in\operatorname{Ob}\mathcal{A}} \underline{\mathsf{C}}_{\Bbbk}(\bar{T}^n s\mathcal{A}(Z_0,Z_n), s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,Z_0),\mathcal{M}(Z_n)))^d.$$

We consider V_n^d as Abelian groups with discrete topology. The Abelian group V^d is equipped with the topology of the product. Thus, its basis of neighborhoods of 0 is given by $\Bbbk$-submodules $\Phi_m^d = 0^{m-1}\times\prod_{n=m}^{\infty} V_n^d$. They form a filtration

$$V^d = \Phi_0^d \supset \Phi_1^d \supset \Phi_2^d \supset \dots.$$

We call a $\Bbbk$-linear map $a: V\to V$ of degree p continuous if the induced maps

$$a^{d,d+p} = a\big|_{V^d} : V^d \to V^{d+p}$$

are continuous for all $d\in\mathbb{Z}$. This holds if and only if for any $d\in\mathbb{Z}$ and $m\in\mathbb{N}$ there exists an integer $\kappa=\kappa(d,m)\in\mathbb{N}$ such that $(\Phi_\kappa^d)a\subset\Phi_m^{d+p}$. We may assume that

$$m' < m'' \quad \text{implies} \quad \kappa(d,m') \leqslant \kappa(d,m''). \qquad \text{(A.3.4)}$$

Indeed, a given function $m\mapsto\kappa(d,m)$ can be replaced with the function $m\mapsto\kappa'(d,m) = \min_{n\geqslant m}\kappa(d,n)$ and κ' satisfies condition (A.3.4). Continuous linear maps $a: V\to V$ of degree p are in bijection with families of $\mathbb{N}\times\mathbb{N}$-matrices $(A^{d,d+p})_{d\in\mathbb{Z}}$ of linear maps $A_{nm}^{d,d+p}: V_n^d\to V_m^{d+p}$ with finite number of non-vanishing elements in each column of $A^{d,d+p}$. Indeed, to each continuous map $a^{d,d+p}: V^d\to V^{d+p}$ corresponds the inductive limit over m of $\kappa(d,m)\times m$-matrices of maps $V^d/\Phi_{\kappa(d,m)}^d \to V^{d+p}/\Phi_m^{d+p}$. On the other hand, to each family $(A^{d,d+p})_{d\in\mathbb{Z}}$ of $\mathbb{N}\times\mathbb{N}$-matrices with finite number of non-vanishing elements in each column correspond obvious maps $a^{d,d+p}: V^d\to V^{d+p}$, and they are continuous.

Thus, $a = (a^{d,d+p})_{d\in\mathbb{Z}}$ is continuous. A continuous map $a : V \to V$ can be completely recovered from a $\mathbb{N}\times\mathbb{N}$-matrix $(a_{nm})_{n,m\in\mathbb{N}}$ of maps $a_{nm} = (A^{d,d+p}_{nm})_{d\in\mathbb{Z}} : V_n \to V_m$ of degree p. Naturally, not any such matrix determines a continuous map, however, if the number of non-vanishing elements in each column of (a_{nm}) is finite, then this matrix does determine a continuous map.

The differential $D \overset{\text{def}}{=} B_1 : V \to V$, $r \mapsto (r)B_1 = rb - (-)^r br$ is continuous and the function κ for it is simply $\kappa(d, m) = m$. Its matrix is given by

$$D = B_1 = \begin{bmatrix} D_{0,0} & D_{0,1} & D_{0,2} & \dots \\ 0 & D_{1,1} & D_{1,2} & \dots \\ 0 & 0 & D_{2,2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where

$$D_{k,k} = \underline{\mathsf{C}}_{\Bbbk}(1, b_1^{\mathsf{C}_{\Bbbk}}) - \underline{\mathsf{C}}_{\Bbbk}\big(\sum_{p+1+q=k} 1^{\otimes p} \otimes b_1 \otimes 1^{\otimes q}, 1\big) : V_k \to V_k,$$

$$r_k D_{k,k} = r_k b_1^{\mathsf{C}_{\Bbbk}} - (-)^r \sum_{p+1+q=k} (1^{\otimes p} \otimes b_1 \otimes 1^{\otimes q}) r_k,$$

(one easily recognizes the differential in the complex V_k),

$$r_k D_{k,k+1} = (r_k \otimes f_1) b_2^{\mathsf{C}_{\Bbbk}} + (H_1^X \otimes r_k) b_2^{\mathsf{C}_{\Bbbk}} - (-)^r \sum_{p+q=k-1} (1^{\otimes p} \otimes b_2 \otimes 1^{\otimes q}) r_k.$$

Further we will see that we do not need to compute other components.

Composition of $\alpha\Upsilon$ with pr_n equals

$$\begin{aligned} \alpha\Upsilon_n = -\big[s\underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_{\Bbbk})(H^X, f) &\xrightarrow{\mathrm{pr}_0} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,X), \mathcal{M}(X)) \\ &\xrightarrow{s^{-1}[1]} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,X), s\mathcal{M}(X)) \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1)} \underline{\mathsf{C}}_{\Bbbk}(\Bbbk, s\mathcal{M}(X)) = s\mathcal{M}(X) \\ &\xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n), \\ &\qquad\qquad s\mathcal{A}(X, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n) \otimes s\mathcal{M}(X)) \\ &\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\tau_c)} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n), \\ &\qquad\qquad s\mathcal{M}(X) \otimes s\mathcal{A}(X, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n)) \\ &\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,b_{n+1}^{\mathcal{M}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n), s\mathcal{M}(Z_n)) \\ &\xrightarrow{(\underline{\varphi}^{\mathsf{C}_{\Bbbk}})^{-1}} \underline{\mathsf{C}}_{\Bbbk}(T^n s\mathcal{A}(Z_0, Z_n), \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X, Z_0), s\mathcal{M}(Z_n))) \\ &\qquad \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,[-1]s)} \underline{\mathsf{C}}_{\Bbbk}(T^n s\mathcal{A}(Z_0, Z_n), s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X, Z_0), \mathcal{M}(Z_n)))\big]. \end{aligned}$$

Clearly, $\alpha\Upsilon$ is continuous (take $\kappa(d, m) = 1$). Its $\mathbb{N}\times\mathbb{N}$-matrix has the form

$$\alpha\Upsilon = \begin{bmatrix} * & * & * & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

A.4. Lemma. *The map $\alpha\Upsilon : V \to V$ is homotopic to a continuous map $V \to V$, whose $\mathbb{N}\times\mathbb{N}$-matrix is upper-triangular with the identity maps $\mathrm{id} : V_k \to V_k$ on the diagonal.*

Proof. Define a continuous $\Bbbk$-linear map $K : s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^X, f) \to s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^X, f)$ of degree -1 by its matrix

$$K = \begin{bmatrix} 0 & 0 & 0 & \dots \\ K_{1,0} & 0 & 0 & \dots \\ 0 & K_{2,1} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

so $\kappa(d, m) = m+1$, where $K_{k+1,k}$ maps the factor indexed by $(X, Z_0, \dots, Z_k)$ to the factor indexed by $(Z_0, \dots, Z_k)$ as follows:

$$\begin{aligned} K_{k+1,k} = \big[&\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X, Z_0) \otimes \bar{T}s\mathcal{A}(Z_0, Z_k), s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X, X), \mathcal{M}(Z_k))) \\ &\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,s^{-1}[1])} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X, Z_0) \otimes \bar{T}s\mathcal{A}(Z_0, Z_k), \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X, X), s\mathcal{M}(Z_k))) \\ &\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1))} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X, Z_0) \otimes \bar{T}s\mathcal{A}(Z_0, Z_k), \underline{\mathsf{C}}_\Bbbk(\Bbbk, s\mathcal{M}(Z_k))) \\ &\xrightarrow{(\underline{\varphi}^{\mathsf{C}_\Bbbk})^{-1}} \underline{\mathsf{C}}_\Bbbk(\bar{T}s\mathcal{A}(Z_0, Z_k), \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X, Z_0), s\mathcal{M}(Z_k))) \\ &\qquad\qquad \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,[-1]s)} \underline{\mathsf{C}}_\Bbbk(\bar{T}s\mathcal{A}(Z_0, Z_k), s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X, Z_0), \mathcal{M}(Z_k)))\big]. \end{aligned}$$

Other factors are ignored.

Composition of continuous maps $V \to V$ is continuous as well. In particular, one finds the matrices of B_1K and KB_1:

$$B_1K = \begin{bmatrix} D_{0,1}K_{1,0} & D_{0,2}K_{2,1} & D_{0,3}K_{3,2} & \dots \\ D_{1,1}K_{1,0} & D_{1,2}K_{2,1} & D_{1,3}K_{3,2} & \dots \\ 0 & D_{2,2}K_{2,1} & D_{2,3}K_{3,2} & \dots \\ 0 & 0 & D_{3,3}K_{3,2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$KB_1 = \begin{bmatrix} 0 & 0 & 0 & \dots \\ K_{1,0}D_{0,0} & K_{1,0}D_{0,1} & K_{1,0}D_{0,2} & \dots \\ 0 & K_{2,1}D_{1,1} & K_{2,1}D_{1,2} & \dots \\ 0 & 0 & K_{3,2}D_{2,2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We have $D_{k+1,k+1}K_{k+1,k} + K_{k+1,k}D_{k,k} = 0$ for all $k \geqslant 0$. Indeed, conjugating the expanded left hand side with $\underline{\mathsf{C}}_\Bbbk(1, [-1]s)$ we come to the following identity:

$$\begin{aligned} \big[&\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X, Z_0) \otimes \bar{T}s\mathcal{A}(Z_0, Z_k), \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X, X), s\mathcal{M}(Z_k))) \\ &\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,m_1^{\underline{\mathsf{C}}_\Bbbk})+\underline{\mathsf{C}}_\Bbbk(\sum_{p+q=k} 1^{\otimes p}\otimes b_1\otimes 1^{\otimes q},1)} \\ &\qquad\qquad \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X, Z_0) \otimes \bar{T}s\mathcal{A}(Z_0, Z_k), \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X, X), s\mathcal{M}(Z_k))) \\ &\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1))} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X, Z_0) \otimes \bar{T}s\mathcal{A}(Z_0, Z_k), \underline{\mathsf{C}}_\Bbbk(\Bbbk, s\mathcal{M}(Z_k))) \\ &\qquad\qquad \xrightarrow{(\underline{\varphi}^{\underline{\mathsf{C}}_\Bbbk})^{-1}} \underline{\mathsf{C}}_\Bbbk(\bar{T}s\mathcal{A}(Z_0, Z_k), \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X, Z_0), s\mathcal{M}(Z_k)))\big] \\ + \big[&\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X, Z_0) \otimes \bar{T}s\mathcal{A}(Z_0, Z_k), \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X, X), s\mathcal{M}(Z_k))) \\ &\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1))} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X, Z_0) \otimes \bar{T}s\mathcal{A}(Z_0, Z_k), \underline{\mathsf{C}}_\Bbbk(\Bbbk, s\mathcal{M}(Z_k))) \end{aligned}$$

$$\xrightarrow{(\underline{\varphi}^{\mathsf{C}_\Bbbk})^{-1}} \underline{\mathsf{C}}_\Bbbk(\bar{T}s\mathcal{A}(Z_0,Z_k),\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z_0),s\mathcal{M}(Z_k)))$$
$$\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,m_1^{\underline{\mathsf{C}}_\Bbbk})+\underline{\mathsf{C}}_\Bbbk(\sum_{p+q=k-1}1^{\otimes p}\otimes b_1\otimes 1^{\otimes q},1)} \underline{\mathsf{C}}_\Bbbk(\bar{T}s\mathcal{A}(Z_0,Z_k),\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z_0),s\mathcal{M}(Z_k)))\big] = 0.$$

After reducing all terms to the common form, beginning with $\underline{\mathsf{C}}_\Bbbk(1,\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1))$, all terms cancel each other, so the identity is proven.

Therefore, the chain map $a = \alpha\Upsilon + B_1K + KB_1$ is also represented by an upper-triangular matrix. Its diagonal elements are chain maps $a_{kk} : V_k \to V_k$. We are going to show that they are homotopic to identity maps.

Let us compute the matrix element $a_{00} : V_0 \to V_0 = \prod_{Z\in\operatorname{Ob}\mathcal{A}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z))$. We have

$$a_{00}\operatorname{pr}_Z = (\alpha\Upsilon_0 + B_1K_{1,0})\operatorname{pr}_Z : V_0 \to V_0 \xrightarrow{\operatorname{pr}_Z} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z)).$$

In the expanded form these terms are as follows:

$$\alpha\Upsilon_0\operatorname{pr}_Z = -\big[V_0 \xrightarrow{\operatorname{pr}_X} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,X),\mathcal{M}(X)) \xrightarrow{s^{-1}[1]} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,X),s\mathcal{M}(X))$$
$$\xrightarrow{\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1)} \underline{\mathsf{C}}_\Bbbk(\Bbbk,s\mathcal{M}(X)) = s\mathcal{M}(X)$$
$$\xrightarrow{\operatorname{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{A}(X,Z)\otimes s\mathcal{M}(X))$$
$$\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,cb_1^{\mathcal{M}})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z)) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z))\big],$$

$$B_1K_{1,0}\operatorname{pr}_Z = [(1\otimes f_1)b_2^{\underline{\mathsf{C}}_\Bbbk} + (H_1^X\otimes 1)b_2^{\underline{\mathsf{C}}_\Bbbk}]K_{1,0}\operatorname{pr}_Z,$$

$$(1\otimes f_1)b_2^{\underline{\mathsf{C}}_\Bbbk} = \big[V_0 \xrightarrow{\operatorname{pr}_X} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,X),\mathcal{M}(X))$$
$$\xrightarrow{\operatorname{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{A}(X,Z)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,X),\mathcal{M}(X)))$$
$$\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,c)} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,X),\mathcal{M}(X))\otimes s\mathcal{A}(X,Z))$$
$$\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes f_1)} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,X),\mathcal{M}(X))\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{M}(X),\mathcal{M}(Z)))$$
$$\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,b_2^{\underline{\mathsf{C}}_\Bbbk})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,X),\mathcal{M}(Z)))\big],$$

$$(H_1^X\otimes 1)b_2^{\underline{\mathsf{C}}_\Bbbk} = \big[V_0 \xrightarrow{\operatorname{pr}_Z} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z))$$
$$\xrightarrow{\operatorname{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{A}(X,Z)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z)))$$
$$\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,H_1^X\otimes 1)} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),$$
$$s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,X),\mathcal{A}(X,Z))\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z)))$$
$$\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,b_2^{\underline{\mathsf{C}}_\Bbbk})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,X),\mathcal{M}(Z)))\big],$$

$$\begin{aligned}
K_{1,0}\operatorname{pr}_Z = \big[V_1 &\xrightarrow{\operatorname{pr}_{X,Z}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z), s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,X),\mathcal{M}(Z))) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,s^{-1}[1])} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z), \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,X), s\mathcal{M}(Z))) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1))} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z), \underline{\mathsf{C}}_{\Bbbk}(\Bbbk, s\mathcal{M}(Z))) = \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z), s\mathcal{M}(Z)) \\
&\qquad\qquad\qquad\qquad\qquad\xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,Z),\mathcal{M}(Z))\big].
\end{aligned}$$

We claim that in the sum

$$a_{00}\operatorname{pr}_Z = \alpha\Upsilon_0\operatorname{pr}_Z + (1\otimes f_1)b_2^{\underline{\mathsf{C}}_{\Bbbk}}K_{1,0}\operatorname{pr}_Z + (H_1^X\otimes 1)b_2^{\underline{\mathsf{C}}_{\Bbbk}}K_{1,0}\operatorname{pr}_Z$$

the first two summands cancel each other, while the last, $(H_1^X\otimes 1)b_2^{\underline{\mathsf{C}}_{\Bbbk}}K_{1,0}$, is homotopic to identity. Indeed, $\alpha\Upsilon_0\operatorname{pr}_Z + (1\otimes f_1)b_2^{\underline{\mathsf{C}}_{\Bbbk}}K_{1,0}\operatorname{pr}_Z$ factors through

$$\begin{aligned}
-\big[&s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,X),\mathcal{M}(X)) \xrightarrow{s^{-1}[1]} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,X), s\mathcal{M}(X)) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1)} \underline{\mathsf{C}}_{\Bbbk}(\Bbbk, s\mathcal{M}(X)) = s\mathcal{M}(X) \\
&\quad\xrightarrow{\operatorname{coev}^{\underline{\mathsf{C}}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z), s\mathcal{A}(X,Z)\otimes s\mathcal{M}(X)) \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,cb_1^{\mathcal{M}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z), s\mathcal{M}(Z))\big] \\
+\big[&s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,X),\mathcal{M}(X)) \xrightarrow{\operatorname{coev}^{\underline{\mathsf{C}}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z), s\mathcal{A}(X,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,X),\mathcal{M}(X))) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,c)} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z), s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,X),\mathcal{M}(X))\otimes s\mathcal{A}(X,Z)) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,1\otimes f_1)} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z), s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,X),\mathcal{M}(X))\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{M}(X),\mathcal{M}(Z))) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,b_2^{\underline{\mathsf{C}}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z), s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,X),\mathcal{M}(Z))) \\
&\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,s^{-1}[1])} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z), \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,X), s\mathcal{M}(Z))) \\
&\quad\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1))} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z), \underline{\mathsf{C}}_{\Bbbk}(\Bbbk, s\mathcal{M}(Z))) = \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z), s\mathcal{M}(Z))\big].
\end{aligned}$$

It therefore suffices to prove that the above expression vanishes. By closedness, this is equivalent to the following equation:

$$\begin{aligned}
-\big[&s\mathcal{A}(X,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,X),\mathcal{M}(X)) \xrightarrow{1\otimes s^{-1}[1]} s\mathcal{A}(X,Z)\otimes\underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,X), s\mathcal{M}(X)) \\
&\qquad\qquad\qquad\xrightarrow{1\otimes\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1)} s\mathcal{A}(X,Z)\otimes s\mathcal{M}(X) \xrightarrow{cb_1^{\mathcal{M}}} s\mathcal{M}(Z)\big] \\
+\big[&s\mathcal{A}(X,Z)\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,X),\mathcal{M}(X)) \xrightarrow{c} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,X),\mathcal{M}(X))\otimes s\mathcal{A}(X,Z) \\
&\xrightarrow{1\otimes f_1} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,X),\mathcal{M}(X))\otimes s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{M}(X),\mathcal{M}(Z)) \xrightarrow{b_2^{\underline{\mathsf{C}}_{\Bbbk}}} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,X),\mathcal{M}(Z)) \\
&\qquad\xrightarrow{s^{-1}[1]} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,X), s\mathcal{M}(Z)) \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1)} \underline{\mathsf{C}}_{\Bbbk}(\Bbbk, s\mathcal{M}(Z)) = s\mathcal{M}(Z)\big] = 0.
\end{aligned}$$

Canceling c and transforming the left hand side using (4.5.2) we get:

$$\begin{aligned}
&-\big[s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,X),\mathcal{M}(X))\otimes s\mathcal{A}(X,Z) \xrightarrow{s^{-1}[1]\otimes 1} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,X),s\mathcal{M}(X))\otimes s\mathcal{A}(X,Z)\\
&\qquad \xrightarrow{1\otimes f_1 s^{-1}[1]} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,X),s\mathcal{M}(X))\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{M}(X),s\mathcal{M}(Z))\\
&\qquad\qquad \xrightarrow{m_2^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,X),s\mathcal{M}(Z)) \xrightarrow{\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1)} \underline{\mathsf{C}}_\Bbbk(\Bbbk,s\mathcal{M}(Z)) = s\mathcal{M}(Z)\big]\\
&+\big[s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,X),\mathcal{M}(X))\otimes s\mathcal{A}(X,Z) \xrightarrow{s^{-1}[1]\otimes 1} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,X),s\mathcal{M}(X))\otimes s\mathcal{A}(X,Z)\\
&\qquad \xrightarrow{1\otimes f_1 s^{-1}[1]} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,X),s\mathcal{M}(X))\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{M}(X),s\mathcal{M}(Z))\\
&\qquad\qquad \xrightarrow{\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes 1} \underline{\mathsf{C}}_\Bbbk(\Bbbk,s\mathcal{M}(X))\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{M}(X),s\mathcal{M}(Z)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}} s\mathcal{M}(Z)\big] = 0.
\end{aligned}$$

The above equation follows from the following identity which holds by properties of the closed monoidal category $\underline{\mathsf{C}}_\Bbbk$:

$$\begin{aligned}
&\big[\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,X),s\mathcal{M}(X))\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{M}(X),s\mathcal{M}(Z)) \xrightarrow{m_2^{\mathsf{C}_\Bbbk}}\\
&\qquad\qquad \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,X),s\mathcal{M}(Z)) \xrightarrow{\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1)} \underline{\mathsf{C}}_\Bbbk(\Bbbk,s\mathcal{M}(Z)) = s\mathcal{M}(Z)\big]\\
&= \big[\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,X),s\mathcal{M}(X))\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{M}(X),s\mathcal{M}(Z))\\
&\qquad \xrightarrow{\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes 1} \underline{\mathsf{C}}_\Bbbk(\Bbbk,s\mathcal{M}(X))\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{M}(X),s\mathcal{M}(Z))\\
&\qquad\qquad = s\mathcal{M}(X)\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{M}(X),s\mathcal{M}(Z)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}} s\mathcal{M}(Z)\big].
\end{aligned}$$

This is a particular case of identity (3.1.2) combined with (1.3.6).

Now we prove that $(H_1^X\otimes 1)b_2K_{1,0}$ is homotopic to identity. It maps each factor $s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z))$ into itself via the following map:

$$\begin{aligned}
&\big[s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z)) \xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{A}(X,Z)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z)))\\
&\qquad\qquad \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,P)} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z)) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z))\big],
\end{aligned}$$

where

$$\begin{aligned}
P = &-\big[s\mathcal{A}(X,Z)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z)) \xrightarrow{H_1^X s^{-1}[1]\otimes s^{-1}[1]}\\
&\qquad \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,X),s\mathcal{A}(X,Z))\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z)) \xrightarrow{m_2^{\mathsf{C}_\Bbbk}}\\
&\qquad\qquad \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,X),s\mathcal{M}(Z)) \xrightarrow{\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1)} \underline{\mathsf{C}}_\Bbbk(\Bbbk,s\mathcal{M}(Z)) = s\mathcal{M}(Z)\big]\\
= &\big[s\mathcal{A}(X,Z)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z)) \xrightarrow{1\otimes s^{-1}[1]} s\mathcal{A}(X,Z)\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z))\\
&\quad \xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}\otimes 1} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,X),s\mathcal{A}(X,X)\otimes s\mathcal{A}(X,Z))\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z))\\
&\qquad \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,b_2)\otimes 1} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,X),s\mathcal{A}(X,Z))\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z))\\
&\qquad\qquad \xrightarrow{\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes 1} \underline{\mathsf{C}}_\Bbbk(\Bbbk,s\mathcal{A}(X,Z))\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}} s\mathcal{M}(Z)\big]\\
= &-\big[s\mathcal{A}(X,Z)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z)) \xrightarrow{1\otimes s^{-1}[1]} s\mathcal{A}(X,Z)\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z))\\
&\qquad \xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}\otimes 1} \underline{\mathsf{C}}_\Bbbk(\Bbbk,\Bbbk\otimes s\mathcal{A}(X,Z))\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z))
\end{aligned}$$

$$\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,({}_X\mathbf{i}_0^{\mathcal{A}}\otimes 1)b_2)\otimes 1} s\mathcal{A}(X,Z)\otimes\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}} s\mathcal{M}(Z)\bigr]$$

$$\begin{aligned}
= -\bigl[s\mathcal{A}(X,Z)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z)) &\xrightarrow{1\otimes s^{-1}[1]} s\mathcal{A}(X,Z)\otimes\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z)) \\
&\xrightarrow{({}_X\mathbf{i}_0^{\mathcal{A}}\otimes 1)b_2\otimes 1} s\mathcal{A}(X,Z)\otimes\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}} s\mathcal{M}(Z)\bigr].
\end{aligned} \tag{A.4.1}$$

It follows that

$$\begin{aligned}
(H_1^X\otimes 1)b_2^{\mathsf{C}_\Bbbk}K_{1,0} = -\bigl[s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z)) &\xrightarrow{s^{-1}[1]} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z)) \\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{A}(X,Z)\otimes\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z))) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,({}_X\mathbf{i}_0^{\mathcal{A}}\otimes 1)b_2\otimes 1)} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{A}(X,Z)\otimes\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z))) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{ev}^{\mathsf{C}_\Bbbk})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z)) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z))\bigr].
\end{aligned}$$

Since $\mathcal{A}$ is a unital A_∞-category, there exists a homotopy $h'':s\mathcal{A}(X,Z)\to s\mathcal{A}(X,Z)$, a map of degree -1, such that $({}_X\mathbf{i}_0^{\mathcal{A}}\otimes 1)b_2 = -1 + h''b_1 + b_1h''$. Therefore, the map considered above is equal to

$$\begin{aligned}
\mathrm{id}_{s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z))} -\bigl[s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z)) &\xrightarrow{s^{-1}[1]} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z)) \\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{A}(X,Z)\otimes\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z))) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,(h''b_1+b_1h'')\otimes 1)} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{A}(X,Z)\otimes\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z))) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{ev}^{\mathsf{C}_\Bbbk})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z)) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z))\bigr] \\
&= \bigl(\mathrm{id}_{s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z))} + b_1^{\mathsf{C}_\Bbbk}K'_{00} + K'_{00}b_1^{\mathsf{C}_\Bbbk}\bigr),
\end{aligned}$$

where

$$\begin{aligned}
K'_{00} = \bigl[s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z)) &\xrightarrow{s^{-1}[1]} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z)) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(h'',1)} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z),s\mathcal{M}(Z)) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z),\mathcal{M}(Z))\bigr].
\end{aligned}$$

Indeed, $b_1^{\mathsf{C}_\Bbbk}K'_{00} + K'_{00}b_1^{\mathsf{C}_\Bbbk}$ is obtained by conjugating with $[-1]s$ the following expression:

$$\begin{aligned}
m_1^{\mathsf{C}_\Bbbk}\underline{\mathsf{C}}_\Bbbk(h'',1) + \underline{\mathsf{C}}_\Bbbk(h'',1)m_1^{\mathsf{C}_\Bbbk} &= (-\underline{\mathsf{C}}_\Bbbk(1,b_0^{\mathcal{M}}) + \underline{\mathsf{C}}_\Bbbk(b_1,1))\underline{\mathsf{C}}_\Bbbk(h'',1) \\
&+ \underline{\mathsf{C}}_\Bbbk(h'',1)(-\underline{\mathsf{C}}_\Bbbk(1,b_0^{\mathcal{M}}) + \underline{\mathsf{C}}_\Bbbk(b_1,1)) = -\underline{\mathsf{C}}_\Bbbk(b_1h''+h''b_1,1).
\end{aligned} \tag{A.4.2}$$

The rest is straightforward. Therefore, $(H_1^X\otimes 1)b_2^{\mathsf{C}_\Bbbk}K_{1,0}$ and a_{00} are homotopic to identity.

Now we are proving that diagonal elements $a_{kk}:V_k\to V_k$ are homotopic to identity maps for $k>0$. An element $r_{k+1}\in V_{k+1}$ is mapped to direct product over $Z_0,\dots,Z_k\in \operatorname{Ob}\mathcal{A}$ of

$$\begin{aligned}
r_{k+1}K_{k+1,k} = \mathrm{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{A}(X,Z_0),*}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z_0),r_{k+1}^{X,Z_0,\dots,Z_k}s^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1))[-1]s &: \\
\bar{T}s\mathcal{A}(Z_0,Z_k) \to s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z_0),\mathcal{M}(Z_k)).&
\end{aligned}$$

Here $r_{k+1}^{X,Z_0,\dots,Z_k}: s\mathcal{A}(X,Z_0)\otimes\bar{T}s\mathcal{A}(Z_0,Z_k)\to s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,X),\mathcal{M}(Z_k))$ is one of coordinates of r_{k+1}.

Thus, $r_k D_{k,k+1} K_{k+1,k}$ is the sum of three terms (A.4.3a)–(A.4.3c):

$$\begin{aligned}\operatorname{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{A}(X,Z_0),*}\,\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z_0),(r_k^{X,Z_0,\dots,Z_{k-1}}\otimes f_1)b_2^{\mathsf{C}_\Bbbk}s^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1))[-1]s:\\ \bar{T}s\mathcal{A}(Z_0,Z_k)\to s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z_0),\mathcal{M}(Z_k)),\quad \text{(A.4.3a)}\end{aligned}$$

where $r_k^{X,Z_0,\dots,Z_{k-1}}: s\mathcal{A}(X,Z_0)\otimes\bar{T}s\mathcal{A}(Z_0,Z_{k-1})\to s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,X),\mathcal{M}(Z_{k-1}))$. Since $b_2^{\mathsf{C}_\Bbbk}=(s\otimes s)^{-1}m_2s=-(s^{-1}\otimes s^{-1})m_2s$, and $[1]$ is a differential graded functor, we have

$$(r_k\otimes f_1)b_2^{\mathsf{C}_\Bbbk}s^{-1}[1]=-(r_ks^{-1}[1]\otimes f_1s^{-1}[1])m_2^{\mathsf{C}_\Bbbk}.$$

Identity (3.1.2) gives

$$\begin{aligned}(r_k\otimes f_1)b_2^{\mathsf{C}_\Bbbk}s^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1)&=-(r_ks^{-1}[1]\otimes f_1s^{-1}[1])m_2^{\mathsf{C}_\Bbbk}\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1)\\ =-(r_ks^{-1}[1]\otimes f_1s^{-1}[1])(\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes 1)m_2^{\mathsf{C}_\Bbbk}&=(r_ks^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes f_1s^{-1}[1])m_2^{\mathsf{C}_\Bbbk}\end{aligned}$$

(we have used the fact that $\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1)$ has degree -1 and f_1s^{-1} has degree 1).

$$\begin{aligned}\operatorname{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{A}(X,Z_0),*}\,\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z_0),(H_1^X\otimes r_k^{Z_0,\dots,Z_k})b_2^{\mathsf{C}_\Bbbk}s^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1))[-1]s:\\ \bar{T}s\mathcal{A}(Z_0,Z_k)\to s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z_0),\mathcal{M}(Z_k)),\quad \text{(A.4.3b)}\end{aligned}$$

where $r_k^{Z_0,\dots,Z_k}:\bar{T}s\mathcal{A}(Z_0,Z_k)\to s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z_0),\mathcal{M}(Z_k))$. Similarly to above

$$(H_1^X\otimes r_k)b_2^{\mathsf{C}_\Bbbk}s^{-1}[1]=(-)^{r+1}(H_1^Xs^{-1}[1]\otimes r_ks^{-1}[1])m_2^{\mathsf{C}_\Bbbk},$$

so that

$$\begin{aligned}(H_1^X\otimes r_k)b_2^{\mathsf{C}_\Bbbk}s^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1)&=(-)^{r+1}(H_1^Xs^{-1}[1]\otimes r_ks^{-1}[1])(\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes 1)m_2^{\mathsf{C}_\Bbbk}\\ &=(H_1^Xs^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes r_ks^{-1}[1])m_2^{\mathsf{C}_\Bbbk}\end{aligned}$$

($r_ks^{-1}[1]$ has degree $\deg r+1$ and $\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1)$ has degree -1).

For each p, q such that $p+q=k-1$,

$$\begin{aligned}\operatorname{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{A}(X,Z_0),*}\,\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z_0),(1^{\otimes p}\otimes b_2\otimes 1^{\otimes q})r_ks^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1))[-1]s:\\ \bar{T}s\mathcal{A}(Z_0,Z_k)\to s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z_0),\mathcal{M}(Z_k)),\quad \text{(A.4.3c)}\end{aligned}$$

where r_k means

$$r_k^{X,Z_0,\dots,Z_{p-1},Z_{p+1},\dots,Z_k}:\bar{T}s\mathcal{A}(X,Z_0,\dots,Z_{p-1},Z_{p+1},\dots,Z_k)\to s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,X),\mathcal{M}(Z_k)),$$

and $Z_{-1}=X$.

Thus, $r_kD_{k,k+1}K_{k+1,k}=\operatorname{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{A}(X,Z_0),*}\,\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z_0),\Sigma_1)[-1]s$, where

$$\begin{aligned}\Sigma_1&=(r_k^{X,Z_0,\dots,Z_{k-1}}s^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes f_1s^{-1}[1])m_2^{\mathsf{C}_\Bbbk}\\ &+(H_1^Xs^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes r_k^{Z_0,\dots,Z_k}s^{-1}[1])m_2^{\mathsf{C}_\Bbbk}\\ &-(-)^r\sum_{p+q=k-1}(1^{\otimes p}\otimes b_2\otimes 1^{\otimes q})r_k^{X,Z_0,\dots,Z_{p-1},Z_{p+1},\dots,Z_k}s^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}_0^{\mathcal{A}},1):\\ &\qquad\qquad s\mathcal{A}(X,Z_0)\otimes\bar{T}s\mathcal{A}(Z_0,Z_k)\to s\mathcal{M}(Z_k).\end{aligned}$$

Similarly, $r_kK_{k,k-1}D_{k-1,k}$ is the sum of three terms (A.4.4a)–(A.4.4c).

$$\begin{aligned}
&\bigl(\operatorname{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{A}(X,Z_0),*}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z_0),r_ks^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1))[-1]s\\
&\qquad\qquad\otimes\operatorname{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{M}(Z_{k-1}),s\mathcal{A}(Z_{k-1},Z_k)}\underline{\mathsf{C}}_\Bbbk(s\mathcal{M}(Z_{k-1}),b^{\mathcal{M}}_1)[-1]s\bigr)b^{\mathsf{C}_\Bbbk}_2\\
&=-\bigl(\operatorname{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{A}(X,Z_0),*}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z_0),r_ks^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1))\\
&\qquad\qquad\otimes\operatorname{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{M}(Z_{k-1}),s\mathcal{A}(Z_{k-1},Z_k)}\underline{\mathsf{C}}_\Bbbk(s\mathcal{M}(Z_{k-1}),b^{\mathcal{M}}_1)\bigr)m^{\mathsf{C}_\Bbbk}_2[-1]s\\
&=-\operatorname{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{A}(X,Z_0),*}\underline{\mathsf{C}}_\Bbbk\bigl(s\mathcal{A}(X,Z_0),(r_ks^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1)\otimes1)b^{\mathcal{M}}_1\bigr)[-1]s:\\
&\qquad\qquad\bar{T}s\mathcal{A}(Z_0,Z_k)\to s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z_0),\mathcal{M}(Z_k)),\quad\text{(A.4.4a)}
\end{aligned}$$

where r_k means $r^{X,Z_0,\dots,Z_{k-1}}_k:\bar{T}s\mathcal{A}(X,Z_0,\dots,Z_{k-1})\to s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,X),\mathcal{M}(Z_{k-1}))$. Here we use formulas (3.1.1) and (A.3.1).

$$\begin{aligned}
&\bigl(\operatorname{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{A}(X,Z_0),s\mathcal{A}(Z_0,Z_1)}\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z_0),b_2)[-1]s\\
&\qquad\qquad\otimes\operatorname{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{A}(X,Z_1),*}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z_1),r_ks^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1))[-1]s\bigr)b^{\mathsf{C}_\Bbbk}_2\\
&=(-)^r\bigl(\operatorname{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{A}(X,Z_0),s\mathcal{A}(Z_0,Z_1)}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z_0),b_2)\\
&\qquad\qquad\otimes\operatorname{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{A}(X,Z_1),*}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z_1),r_ks^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1))\bigr)m^{\mathsf{C}_\Bbbk}_2[-1]s\\
&=(-)^r\operatorname{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{A}(X,Z_0),*}\underline{\mathsf{C}}_\Bbbk\bigl(s\mathcal{A}(X,Z_0),(b_2\otimes1^{\otimes k-1})r_ks^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1)\bigr)[-1]s:\\
&\qquad\qquad\bar{T}s\mathcal{A}(Z_0,Z_k)\to s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z_0),\mathcal{M}(Z_k)),\quad\text{(A.4.4b)}
\end{aligned}$$

where r_k means $r^{X,Z_1,\dots,Z_k}_k:\bar{T}s\mathcal{A}(X,Z_1,\dots,Z_k)\to s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,X),\mathcal{M}(Z_k))$. We have used that $r_ks^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1)$ has degree $\deg r$ and formula (3.1.1).

$$\begin{aligned}
&(1^{\otimes p}\otimes b_2\otimes1^{\otimes q})\operatorname{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{A}(X,Z_0),*}\underline{\mathsf{C}}_\Bbbk\bigl(s\mathcal{A}(X,Z_0),r_ks^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1)\bigr)[-1]s\\
&=\operatorname{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{A}(X,Z_0),*}\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X,Z_0),1^{\otimes p+1}\otimes b_2\otimes1^{\otimes q})\\
&\qquad\qquad\underline{\mathsf{C}}_\Bbbk\bigl(s\mathcal{A}(X,Z_0),r_ks^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1)\bigr)[-1]s\\
&=\operatorname{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{A}(X,Z_0),*}\underline{\mathsf{C}}_\Bbbk\bigl(s\mathcal{A}(X,Z_0),(1^{\otimes p+1}\otimes b_2\otimes1^{\otimes q})r_ks^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1)\bigr)[-1]s:\\
&\qquad\qquad\bar{T}s\mathcal{A}(Z_0,Z_k)\to s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,Z_0),\mathcal{M}(Z_k)),\quad\text{(A.4.4c)}
\end{aligned}$$

where r_k is the map

$$r^{X,Z_0,\dots,Z_p,Z_{p+2},\dots,Z_k}_k:\bar{T}s\mathcal{A}(X,Z_0,\dots,Z_p,Z_{p+2},\dots,Z_k)\to s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X,X),\mathcal{M}(Z_k)).$$

Here we use the naturality of $\operatorname{coev}^{\mathsf{C}_\Bbbk}$.

Thus, $r_kK_{k,k-1}D_{k-1,k}=\operatorname{coev}^{\mathsf{C}_\Bbbk}_{s\mathcal{A}(X,Z_0),*}\underline{\mathsf{C}}_\Bbbk(h^XZ_0,\Sigma_2)[-1]s$, where

$$\begin{aligned}
\Sigma_2=&-(r^{X,Z_0,\dots,Z_{k-1}}_ks^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1)\otimes1)b^{\mathcal{M}}_1\\
&+(-)^r(b_2\otimes1^{\otimes k-1})r^{X,Z_1,\dots,Z_k}_ks^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1)\\
&-(-)^{r-1}\sum_{p+q=k-2}(1^{\otimes p+1}\otimes b_2\otimes1^{\otimes q})r^{X,Z_0,\dots,Z_p,Z_{p+2},\dots,Z_k}_ks^{-1}[1]\underline{\mathsf{C}}_\Bbbk({}_X\mathbf{i}^{\mathcal{A}}_0,1):\\
&\qquad\qquad s\mathcal{A}(X,Z_0)\otimes\bar{T}s\mathcal{A}(Z_0,Z_k)\to s\mathcal{M}(Z_k).
\end{aligned}$$

The element rK has degree $\deg r - 1$, so the sign $(-)^{r-1}$ arises. Combining this with the expression for $r_k D_{k,k+1} K_{k+1,k}$ we obtain

$$r_k D_{k,k+1} K_{k+1,k} + r_k K_{k,k-1} D_{k-1,k} = \mathrm{coev}^{\mathsf{C}_{\Bbbk}}_{s\mathcal{A}(X,Z_0),*}\, \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0),\Sigma)[-1]s,$$

where $\Sigma = \Sigma_1 + \Sigma_2$. We claim that $\Sigma = (H_1^X s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1) \otimes r_k^{Z_0,\dots,Z_k} s^{-1}[1]) m_2^{\mathsf{C}_{\Bbbk}}$. Indeed, first of all,

$$\begin{aligned}
&(b_2 \otimes 1^{\otimes k-1}) r_k^{X,Z_1,\dots,Z_k} s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1) \\
&\qquad + \sum_{p+q=k-2} (1^{\otimes p+1} \otimes b_2 \otimes 1^{\otimes q}) r_k^{X,Z_0,\dots,Z_p,Z_{p+2},\dots,Z_k} s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1) \\
&\qquad\qquad = \sum_{p+q=k-1} (1^{\otimes p} \otimes b_2 \otimes 1^{\otimes q}) r_k^{X,Z_0,\dots,Z_{p-1},Z_{p+1},\dots,Z_k} s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1),
\end{aligned}$$

so that

$$\begin{aligned}
\Sigma &= (r_k^{X,Z_0,\dots,Z_{k-1}} s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1) \otimes f_1 s^{-1}[1]) m_2^{\mathsf{C}_{\Bbbk}} \\
&\quad + (H_1^X s^{-1}\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1) \otimes r_k^{Z_0,\dots,Z_k} s^{-1}[1]) m_2^{\mathsf{C}_{\Bbbk}} - (r_k^{X,Z_0,\dots,Z_{k-1}} s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1) \otimes 1) b_1^{\mathcal{M}} : \\
&\qquad\qquad s\mathcal{A}(X,Z_0) \otimes \bar{T}s\mathcal{A}(Z_0,Z_k) \to s\mathcal{M}(Z_k).
\end{aligned}$$

Note that

$$\begin{aligned}
m_2^{\mathsf{C}_{\Bbbk}} = \mathrm{ev}^{\mathsf{C}_{\Bbbk}} &: s\mathcal{M}(Z_{k-1}) \otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{M}(Z_{k-1}), s\mathcal{M}(Z_k)) \\
&= \underline{\mathsf{C}}_{\Bbbk}(\Bbbk, s\mathcal{M}(Z_{k-1})) \otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{M}(Z_{k-1}), s\mathcal{M}(Z_k)) \to s\mathcal{M}(Z_k) = \underline{\mathsf{C}}_{\Bbbk}(\Bbbk, s\mathcal{M}(Z_k)),
\end{aligned}$$

therefore the first and the third summands cancel out, due to (4.5.2). Hence, only the second summand remains in $\Sigma = (H_1^X s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1) \otimes r_k^{Z_0,\dots,Z_k} s^{-1}[1]) m_2^{\mathsf{C}_{\Bbbk}}$. It follows that

$$\begin{aligned}
&r_k D_{k,k+1} K_{k+1,k} + r_k K_{k,k-1} D_{k-1,k} \\
&= \Bigl[\bar{T}s\mathcal{A}(Z_0,Z_k) \xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0), s\mathcal{A}(X,Z_0) \otimes \bar{T}s\mathcal{A}(Z_0,Z_k)) \\
&\quad \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,H_1^X s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes r_k s^{-1}[1])} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0), \\
&\qquad\qquad\qquad\qquad \underline{\mathsf{C}}_{\Bbbk}(\Bbbk, s\mathcal{A}(X,Z_0)) \otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0), s\mathcal{M}(Z_k))) \\
&\qquad\qquad \xrightarrow[\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})]{\underline{\mathsf{C}}_{\Bbbk}(1,m_2^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0), s\mathcal{M}(Z_k)) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,Z_0), \mathcal{M}(Z_k))\Bigr] \\
&= \Bigl[\bar{T}s\mathcal{A}(Z_0,Z_k) \xrightarrow{r_k s^{-1}[1]} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0), s\mathcal{M}(Z_k)) \\
&\quad \xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0), s\mathcal{A}(X,Z_0) \otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0), s\mathcal{M}(Z_k))) \\
&\quad \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,H_1^X s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1)\otimes 1)} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0), s\mathcal{A}(X,Z_0) \otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0), s\mathcal{M}(Z_k))) \\
&\qquad\qquad \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0), s\mathcal{M}(Z_k)) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,Z_0), \mathcal{M}(Z_k))\Bigr] \\
&= \Bigl[\bar{T}s\mathcal{A}(Z_0,Z_k) \xrightarrow{r_k s^{-1}[1]} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0), s\mathcal{M}(Z_k)) \\
&\quad \xrightarrow{\mathrm{coev}^{\mathsf{C}_{\Bbbk}}} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0), s\mathcal{A}(X,Z_0) \otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0), s\mathcal{M}(Z_k))) \\
&\quad \xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(H_1^X s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1),1)} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0), s\mathcal{A}(X,Z_0) \otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0), s\mathcal{M}(Z_k)))
\end{aligned}$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(1,\mathrm{ev}^{\mathsf{C}_{\Bbbk}})} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0),s\mathcal{M}(Z_k)) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,Z_0),\mathcal{M}(Z_k))\bigr]$$

$$= \bigl[\bar{T}s\mathcal{A}(Z_0,Z_k) \xrightarrow{r_k} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,Z_0),\mathcal{M}(Z_k)) \xrightarrow{s^{-1}[1]} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0),s\mathcal{M}(Z_k))$$

$$\xrightarrow{\underline{\mathsf{C}}_{\Bbbk}(H_1^X s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1),1)} \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{A}(X,Z_0),s\mathcal{M}(Z_k)) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,Z_0),\mathcal{M}(Z_k))\bigr].$$

Note that

$$H_1^X s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}({}_X\mathbf{i}_0^{\mathcal{A}},1) = -({}_X\mathbf{i}_0^{\mathcal{A}}\otimes 1)b_2 = 1 - h''b_1 - b_1h'' : s\mathcal{A}(X,Z_0) \to s\mathcal{A}(X,Z_0)$$

(compare with (A.4.1)). We see that for $k > 0$

$$a_{kk} = D_{k,k+1}K_{k+1,k} + K_{k,k-1}D_{k-1,k} = 1 + g :$$
$$\underline{\mathsf{C}}_{\Bbbk}(\bar{T}s\mathcal{A}(Z_0,Z_k), s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,Z_0),\mathcal{M}(Z_k))) \to \underline{\mathsf{C}}_{\Bbbk}(\bar{T}s\mathcal{A}(Z_0,Z_k), s\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,Z_0),\mathcal{M}(Z_k))),$$

where

$$g = -\underline{\mathsf{C}}_{\Bbbk}(\bar{T}s\mathcal{A}(Z_0,Z_k), s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}(h''b_1 + b_1h'',1)[-1]s).$$

More precisely, $D_{k,k+1}K_{k+1,k} + K_{k,k-1}D_{k-1,k}$ is a diagonal map, whose components are $1+g$. We claim that $g = m_1^{\mathsf{C}_{\Bbbk}}K'_{kk} + K'_{kk}m_1^{\mathsf{C}_{\Bbbk}}$, where

$$K'_{kk} = \underline{\mathsf{C}}_{\Bbbk}(\bar{T}s\mathcal{A}(Z_0,Z_k), s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}(h'',1)[-1]s).$$

Indeed, $m_1^{\mathsf{C}_{\Bbbk}} = \underline{\mathsf{C}}_{\Bbbk}(1,b_1^{\mathsf{C}_{\Bbbk}}) - \underline{\mathsf{C}}_{\Bbbk}(\sum_{p+q=k-1}(1^{\otimes p}\otimes b_1\otimes 1^{\otimes q}),1)$, so that

$$\begin{aligned}
m_1^{\mathsf{C}_{\Bbbk}}K'_{kk} &= \bigl(\underline{\mathsf{C}}_{\Bbbk}(1,b_1^{\mathsf{C}_{\Bbbk}}) - \underline{\mathsf{C}}_{\Bbbk}(\sum_{p+q=k-1} 1^{\otimes p}\otimes b_1\otimes 1^{\otimes q},1)\bigr)\underline{\mathsf{C}}_{\Bbbk}\bigl(1,s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}(h'',1)[-1]s\bigr)\\
&= \underline{\mathsf{C}}_{\Bbbk}\bigl(1,b_1^{\mathsf{C}_{\Bbbk}}s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}(h'',1)[-1]s\bigr)\\
&\quad+ \underline{\mathsf{C}}_{\Bbbk}\bigl(1,s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}(h'',1)[-1]s\bigr)\underline{\mathsf{C}}_{\Bbbk}\bigl(\sum_{p+q=k-1} 1^{\otimes p}\otimes b_1\otimes 1^{\otimes q},1\bigr),\\
K'_{kk}m_1^{\mathsf{C}_{\Bbbk}} &= \underline{\mathsf{C}}_{\Bbbk}\bigl(1,s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}(h'',1)[-1]s\bigr)\bigl(\underline{\mathsf{C}}_{\Bbbk}(1,b_1^{\mathsf{C}_{\Bbbk}}) - \underline{\mathsf{C}}_{\Bbbk}(\sum_{p+q=k-1} 1^{\otimes p}\otimes b_1\otimes 1^{\otimes q},1)\bigr)\\
&= \underline{\mathsf{C}}_{\Bbbk}\bigl(1,s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}(h'',1)[-1]sb_1^{\mathsf{C}_{\Bbbk}}\bigr)\\
&\quad- \underline{\mathsf{C}}_{\Bbbk}\bigl(1,s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}(h'',1)[-1]s\bigr)\underline{\mathsf{C}}_{\Bbbk}\bigl(\sum_{p+q=k-1} 1^{\otimes p}\otimes b_1\otimes 1^{\otimes q},1\bigr).
\end{aligned}$$

Therefore,

$$\begin{aligned}
K'_{kk}m_1^{\mathsf{C}_{\Bbbk}} + m_1^{\mathsf{C}_{\Bbbk}}K'_{kk} &= \underline{\mathsf{C}}_{\Bbbk}\bigl(1,b_1^{\mathsf{C}_{\Bbbk}}s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}(h'',1)[-1]s + s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}(h'',1)[-1]sb_1^{\mathsf{C}_{\Bbbk}}\bigr)\\
&= \underline{\mathsf{C}}_{\Bbbk}\bigl(1,s^{-1}[1](m_1^{\mathsf{C}_{\Bbbk}}\underline{\mathsf{C}}_{\Bbbk}(h'',1) + \underline{\mathsf{C}}_{\Bbbk}(h'',1)m_1^{\mathsf{C}_{\Bbbk}})[-1]s\bigr).
\end{aligned}$$

By (A.4.2) we have $m_1^{\mathsf{C}_{\Bbbk}}\underline{\mathsf{C}}_{\Bbbk}(h'',1) + \underline{\mathsf{C}}_{\Bbbk}(h'',1)m_1^{\mathsf{C}_{\Bbbk}} = \underline{\mathsf{C}}_{\Bbbk}(h''b_1 + b_1h'',1)$, so that

$$K'_{kk}m_1^{\mathsf{C}_{\Bbbk}} + m_1^{\mathsf{C}_{\Bbbk}}K'_{kk} = -\underline{\mathsf{C}}_{\Bbbk}(1,s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}(h''b_1 + b_1h'',1)[-1]s) = g.$$

Summing up, we have proved that

$$a = \alpha\Upsilon + B_1K + KB_1 = 1 + B_1K' + K'B_1 + N,$$

where $K' : s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(H^X,f) \to s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_{\Bbbk})(H^X,f)$ is a continuous $\Bbbk$-linear map of degree -1 determined by a diagonal matrix with the matrix elements

$$K'_{kk} = \underline{\mathsf{C}}_{\Bbbk}(\bar{T}s\mathcal{A}(Z_0,Z_k), s^{-1}[1]\underline{\mathsf{C}}_{\Bbbk}(h'',1)[-1]s) : V_k \to V_k,$$

$K'_{kl} = 0$ for $k \neq l$, and the matrix of the remainder N is strictly upper-triangular: $N_{kl} = 0$ for all $k \geqslant l$. Lemma A.4 is proven. $\square$

The continuous map of degree 0

$$\alpha\Upsilon + B_1(K - K') + (K - K')B_1 = 1 + N : V \to V,$$

obtained in Lemma A.4, is invertible (its inverse is determined by the upper-triangular matrix $\sum_{i=0}^{\infty}(-N)^i$, which is well-defined). Therefore, $\alpha\Upsilon$ is homotopy invertible. We have proved earlier that $\Upsilon\alpha$ is homotopic to identity. Viewing α, Υ as morphisms of the homotopy category $\mathcal{K}$, we see that both of them are homotopy invertible. Hence, they are homotopy inverse to each other. Proposition A.3 is proven. $\square$

The homotopy invertibility of $\Upsilon = \mho_{00}$ implies the invertibility of the cycle Ω_{00} up to boundaries. Hence the natural A_∞-transformation Ω is invertible and Theorem A.1 is proven. $\square$

A.5. Corollary. *There is a bijection between elements of $H^0(\mathcal{M}(X), d)$ and equivalence classes of natural A_∞-transformations $H^X \to f : \mathcal{A} \to \underline{\mathsf{C}}_{\Bbbk}$.*

The following representability criterion has been proven independently by Seidel [**48**, Lemma 3.1] in the case when the ground ring $\Bbbk$ is a field.

A.6. Corollary. *A unital A_∞-functor $f : \mathcal{A} \to \underline{\mathsf{C}}_{\Bbbk}$ is isomorphic to H^X for an object $X \in \operatorname{Ob}\mathcal{A}$ if and only if the $\mathcal{K}$-functor $\mathsf{k}f : \mathsf{k}\mathcal{A} \to \underline{\mathcal{K}} = \mathsf{k}\underline{\mathsf{C}}_{\Bbbk}$ is representable by X.*

Proof. The A_∞-functor f is isomorphic to H^X if and only if there is an invertible natural A_∞-transformation $H^X \to f : \mathcal{A} \to \underline{\mathsf{C}}_{\Bbbk}$. By Proposition A.3, this is the case if and only if there is a cycle $t \in \mathcal{M}(X)$ of degree 0 such that the natural A_∞-transformation $(ts)\Upsilon$ is invertible. By Lemma 3.4.11, the invertibility of $(ts)\Upsilon$ is equivalent to the invertibility modulo boundaries of the 0^{th} component $(ts)\Upsilon_0$ of $(ts)\Upsilon$. For each $Z \in \operatorname{Ob}\mathcal{A}$, the element ${}_Z(ts)\Upsilon_0$ of $\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,Z), \mathcal{M}(Z))$ is given by

$$\begin{aligned}
{}_Z(ts)\Upsilon_0 &= -\big[\mathcal{A}(X,Z) \xrightarrow{s} s\mathcal{A}(X,Z) \xrightarrow{(ts\otimes 1)b_1^{\mathcal{M}}} s\mathcal{M}(Z) \xrightarrow{s^{-1}} \mathcal{M}(Z)\big] \\
&= -\big[\mathcal{A}(X,Z) \xrightarrow{s} s\mathcal{A}(X,Z) \xrightarrow{ts\otimes f_1 s^{-1}} s\mathcal{M}(X) \otimes \underline{\mathsf{C}}_{\Bbbk}(\mathcal{M}(X), \mathcal{M}(Z)) \\
&\qquad \xrightarrow{1\otimes[1]} s\mathcal{M}(X) \otimes \underline{\mathsf{C}}_{\Bbbk}(s\mathcal{M}(X), s\mathcal{M}(Z)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_{\Bbbk}}} s\mathcal{M}(Z) \xrightarrow{s^{-1}} \mathcal{M}(Z)\big] \\
&= -\big[\mathcal{A}(X,Z) \xrightarrow{s} s\mathcal{A}(X,Z) \xrightarrow{ts\otimes f_1 s^{-1}} s\mathcal{M}(X) \otimes \underline{\mathsf{C}}_{\Bbbk}(\mathcal{M}(X), \mathcal{M}(Z)) \\
&\qquad \xrightarrow{s^{-1}\otimes 1} \mathcal{M}(X) \otimes \underline{\mathsf{C}}_{\Bbbk}(\mathcal{M}(X), \mathcal{M}(Z)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_{\Bbbk}}} \mathcal{M}(Z)\big] \\
&= \big[\mathcal{A}(X,Z) \xrightarrow{t\otimes sf_1 s^{-1}} \mathcal{M}(X) \otimes \underline{\mathsf{C}}_{\Bbbk}(\mathcal{M}(X), \mathcal{M}(Z)) \xrightarrow{\mathrm{ev}^{\mathsf{C}_{\Bbbk}}} \mathcal{M}(Z)\big].
\end{aligned}$$

By Proposition 2.1.6, the above composite is invertible in $\underline{\mathsf{C}}_{\Bbbk}(\mathcal{A}(X,Z), \mathcal{M}(Z))$ modulo boundaries, i.e., homotopy invertible, if and only if $\mathsf{k}f$ is representable by the object X. $\square$

A.7. Proposition. *The transformation Ω turns the pasting*

$$\begin{tikzcd}
\mathcal{A},\mathcal{A}^{\mathrm{op}} \arrow[r, "1{,}\mathscr{Y}"] \arrow[rrr, bend left=25, "\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}", "="'] \arrow[dr, "\mathscr{Y}^{\mathrm{op}}{,}\mathscr{Y}"'] \arrow[dr, phantom, bend left=15, "="] & \mathcal{A},\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk) \arrow[rr, "\mathrm{ev}^{\mathsf{A}_\infty}"] \arrow[d, "\mathscr{Y}^{\mathrm{op}}{,}1"'] & {} \arrow[dl, Rightarrow, "\Omega"'] & \underline{\mathsf{C}}_\Bbbk \\
& \underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)^{\mathrm{op}},\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk) \arrow[urr, "\mathrm{Hom}_{\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)}"'] & &
\end{tikzcd}$$

into the natural A_∞-transformation $r^{\mathscr{Y}}$ defined in Corollary 4.3.2. Equivalently, the homomorphism of $Ts\mathcal{A}^{\mathrm{op}}$-$Ts\mathcal{A}^{\mathrm{op}}$-bicomodules $t^{\mathscr{Y}} : \mathcal{R}_{\mathcal{A}^{\mathrm{op}}} \to {}_{\mathscr{Y}}\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)_{\mathscr{Y}}$ coincides with $(1\otimes 1\otimes \mathscr{Y})\cdot\mho : {}_1\mathcal{E}_{\mathscr{Y}} \to {}_{\mathscr{Y}}\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)_{\mathscr{Y}}$. In other terms,

$$\begin{aligned}
&\bigl[Ts\mathcal{A}^{\mathrm{op}}\otimes s\mathcal{A}^{\mathrm{op}}\otimes Ts\mathcal{A}^{\mathrm{op}} \xrightarrow{\mu_{Ts\mathcal{A}^{\mathrm{op}}}} Ts\mathcal{A}^{\mathrm{op}} \xrightarrow{\check{\mathscr{Y}}} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)\bigr]\\
&\quad= \bigl[Ts\mathcal{A}^{\mathrm{op}}\otimes s\mathcal{A}^{\mathrm{op}}\otimes Ts\mathcal{A}^{\mathrm{op}} \xrightarrow{1\otimes1\otimes\mathscr{Y}} Ts\mathcal{A}^{\mathrm{op}}\otimes s\mathcal{E}_{\mathscr{Y}}\otimes Ts\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk) \xrightarrow{\check{\mho}} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)\bigr].
\end{aligned}$$

Proof. Since $\mho_{kl}=0$ if $l>1$, the above equation reduces to two cases:

$$\begin{aligned}
&\bigl[T^ks\mathcal{A}^{\mathrm{op}}\otimes s\mathcal{A}^{\mathrm{op}}\otimes T^ms\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathscr{Y}_{k+1+m}} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)\bigr]\\
&= \bigl[T^ks\mathcal{A}^{\mathrm{op}}\otimes s\mathcal{A}^{\mathrm{op}}\otimes T^ms\mathcal{A}^{\mathrm{op}} \xrightarrow{1^{\otimes k}\otimes1\otimes\mathscr{Y}_m}\\
&\qquad\qquad T^ks\mathcal{A}^{\mathrm{op}}\otimes s\mathcal{E}_{\mathscr{Y}}\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk) \xrightarrow{\mho_{k1}} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)\bigr]
\end{aligned}$$

if $m>0$, and if $m=0$

$$\begin{aligned}
&\bigl[T^ks\mathcal{A}^{\mathrm{op}}\otimes s\mathcal{A}^{\mathrm{op}}\otimes T^0s\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathscr{Y}_{k+1}} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)\bigr]\\
&= \bigl[T^ks\mathcal{A}^{\mathrm{op}}\otimes s\mathcal{A}^{\mathrm{op}}\otimes T^0s\mathcal{A}^{\mathrm{op}} \xrightarrow{1\otimes1\otimes T^0\mathscr{Y}} T^ks\mathcal{A}^{\mathrm{op}}\otimes s\mathcal{E}_{\mathscr{Y}}\otimes T^0s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)\\
&\qquad\qquad \xrightarrow{\mho_{k0}} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)\bigr]. \qquad (A.7.1)
\end{aligned}$$

The first case expands to

$$\begin{aligned}
&\bigl[T^ks\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes s\mathcal{A}^{\mathrm{op}}(X_k,Y_0)\otimes T^ms\mathcal{A}^{\mathrm{op}}(Y_0,Y_m) \xrightarrow{\mathscr{Y}_{k+1+m}} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^{X_0},H^{Y_m})\bigr]\\
&= \bigl[T^ks\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes s\mathcal{A}^{\mathrm{op}}(X_k,Y_0)\otimes T^ms\mathcal{A}^{\mathrm{op}}(Y_0,Y_m)\\
&\qquad \xrightarrow{1^{\otimes k}\otimes1\otimes\mathscr{Y}_m} T^ks\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kH^{Y_0}[1]\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^{Y_0},H^{Y_m})\\
&\qquad\qquad \xrightarrow{\mho_{k1}} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^{X_0},H^{Y_m})\bigr].
\end{aligned}$$

The obtained equation is equivalent to the system of equations

$$\begin{aligned}
&\bigl[T^ks\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes s\mathcal{A}^{\mathrm{op}}(X_k,Y_0)\otimes T^ms\mathcal{A}^{\mathrm{op}}(Y_0,Y_m) \xrightarrow{\mathscr{Y}_{k+1+m}}\\
&\qquad s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^{X_0},H^{Y_m}) \xrightarrow{\mathrm{pr}_n} \underline{\mathsf{C}}_\Bbbk(T^ns\mathcal{A}(Z_0,Z_n),s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),\mathcal{A}(Y_m,Z_n)))\bigr]\\
&= \bigl[T^ks\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes s\mathcal{A}^{\mathrm{op}}(X_k,Y_0)\otimes T^ms\mathcal{A}^{\mathrm{op}}(Y_0,Y_m)\\
&\qquad \xrightarrow{1^{\otimes k}\otimes1\otimes\mathscr{Y}_m} T^ks\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kH^{Y_0}[1]\otimes s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^{Y_0},H^{Y_m})\\
&\qquad\qquad \xrightarrow{\mho_{k1;n}} \underline{\mathsf{C}}_\Bbbk(T^ns\mathcal{A}(Z_0,Z_n),s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),\mathcal{A}(Y_m,Z_n)))\bigr],
\end{aligned}$$

where $n \geqslant 0$, $Z_0, \dots, Z_n \in \operatorname{Ob} \mathcal{A}$. By closedness, each of these equations is equivalent to

$$\begin{aligned}
&\bigl[T^n s\mathcal{A}(Z_0, Z_n) \otimes T^k s\mathcal{A}^{\mathrm{op}}(X_0, X_k) \otimes s\mathcal{A}^{\mathrm{op}}(X_k, Y_0) \otimes T^m s\mathcal{A}^{\mathrm{op}}(Y_0, Y_m) \\
&\quad \xrightarrow{1^{\otimes n} \otimes \mathscr{Y}_{k+1+m}} T^n s\mathcal{A}(Z_0, Z_n) \otimes s\underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_\Bbbk)(H^{X_0}, H^{Y_m}) \\
&\quad \xrightarrow{1^{\otimes n} \otimes \mathrm{pr}_n} T^n s\mathcal{A}(Z_0, Z_n) \otimes \underline{\mathsf{C}}_\Bbbk(T^n s\mathcal{A}(Z_0, Z_n), s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0, Z_0), \mathcal{A}(Y_m, Z_n))) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \xrightarrow{\mathrm{ev}^{\mathsf{C}_\Bbbk}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0, Z_0), \mathcal{A}(Y_m, Z_n))\bigr] \\
&= \bigl[T^n s\mathcal{A}(Z_0, Z_n) \otimes T^k s\mathcal{A}^{\mathrm{op}}(X_0, X_k) \otimes s\mathcal{A}^{\mathrm{op}}(X_k, Y_0) \otimes T^m s\mathcal{A}^{\mathrm{op}}(Y_0, Y_m) \\
&\quad \xrightarrow{1^{\otimes n} \otimes 1^{\otimes k} \otimes 1 \otimes \mathscr{Y}_m} T^n s\mathcal{A}(Z_0, Z_n) \otimes T^k s\mathcal{A}^{\mathrm{op}}(X_0, X_k) \\
&\qquad\qquad\qquad\qquad\qquad \otimes X_k H^{Y_0}[1] \otimes s\underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_\Bbbk)(H^{Y_0}, H^{Y_m}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \xrightarrow{\Upsilon'_{k1;n}} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0, Z_0), \mathcal{A}(Y_m, Z_n))\bigr].
\end{aligned}$$

The left hand side equals $(1^{\otimes n} \otimes \mathscr{Y}_{k+1+m}) \mathrm{ev}^{\mathsf{A}_\infty}_{n1} = ((1, \mathscr{Y}) \mathrm{ev}^{\mathsf{A}_\infty})_{n,k+1+m} = (\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}})_{n,k+1+m}$ by the definition of the Yoneda A_∞-functor $\mathscr{Y} : \mathcal{A}^{\mathrm{op}} \to \underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_\Bbbk)$. The right hand side equals

$$\begin{aligned}
&(-)^{k+1}\bigl[T^n s\mathcal{A}(Z_0, Z_n) \otimes T^k s\mathcal{A}^{\mathrm{op}}(X_0, X_k) \otimes s\mathcal{A}^{\mathrm{op}}(X_k, Y_0) \otimes T^m s\mathcal{A}^{\mathrm{op}}(Y_0, Y_m) \\
&\quad \xrightarrow{1^{\otimes n} \otimes 1^{\otimes k} \otimes 1 \otimes \mathscr{Y}_m} \\
&\qquad T^n s\mathcal{A}(Z_0, Z_n) \otimes T^k s\mathcal{A}^{\mathrm{op}}(X_0, X_k) \otimes X_k H^{Y_0}[1] \otimes s\underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_\Bbbk)(H^{Y_0}, H^{Y_m}) \\
&\quad \xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), s\mathcal{A}(X_0, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n) \otimes T^k s\mathcal{A}^{\mathrm{op}}(X_0, X_k) \\
&\qquad\qquad\qquad\qquad\qquad \otimes X_k H^{Y_0}[1] \otimes s\underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_\Bbbk)(H^{Y_0}, H^{Y_m})) \\
&\quad \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{perm})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), X_k H^{Y_0}[1] \otimes T^k s\mathcal{A}(X_k, X_0) \otimes s\mathcal{A}(X_0, Z_0) \\
&\qquad\qquad\qquad\qquad\qquad \otimes T^n s\mathcal{A}(Z_0, Z_n) \otimes s\underline{\mathsf{A}_\infty}(\mathcal{A}; \underline{\mathsf{C}}_\Bbbk)(H^{Y_0}, H^{Y_m})) \\
&\quad \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes \mathrm{ev}^{\mathsf{A}_\infty}_{k+1+n,1})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), X_k H^{Y_0}[1] \otimes s\underline{\mathsf{C}}_\Bbbk(X_k H^{Y_0}, Z_n H^{Y_m})) \\
&\quad \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), X_k H^{Y_0}[1] \otimes \underline{\mathsf{C}}_\Bbbk(X_k H^{Y_0}[1], Z_n H^{Y_m}[1])) \\
&\qquad \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{ev}^{\mathsf{C}_\Bbbk})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), Z_n H^{Y_m}[1]) \xrightarrow{[-1]s} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0, Z_0), \mathcal{A}(Y_m, Z_n))\bigr] \\
&= (-)^{k+1}\bigl[T^n s\mathcal{A}(Z_0, Z_n) \otimes T^k s\mathcal{A}^{\mathrm{op}}(X_0, X_k) \otimes s\mathcal{A}^{\mathrm{op}}(X_k, Y_0) \otimes T^m s\mathcal{A}^{\mathrm{op}}(Y_0, Y_m) \\
&\quad \xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), s\mathcal{A}(X_0, Z_0) \otimes T^n s\mathcal{A}(Z_0, Z_n) \otimes T^k s\mathcal{A}^{\mathrm{op}}(X_0, X_k) \\
&\qquad\qquad\qquad\qquad\qquad \otimes s\mathcal{A}^{\mathrm{op}}(X_k, Y_0) \otimes T^m s\mathcal{A}^{\mathrm{op}}(Y_0, Y_m)) \\
&\quad \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{perm})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), X_k H^{Y_0}[1] \otimes T^k s\mathcal{A}(X_k, X_0) \otimes s\mathcal{A}(X_0, Z_0) \\
&\qquad\qquad\qquad\qquad\qquad \otimes T^n s\mathcal{A}(Z_0, Z_n) \otimes T^m s\mathcal{A}^{\mathrm{op}}(Y_0, Y_m)) \\
&\quad \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes(1^{\otimes k+1+n} \otimes \mathscr{Y}_m) \mathrm{ev}^{\mathsf{A}_\infty}_{k+1+n,1})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), X_k H^{Y_0}[1] \otimes s\underline{\mathsf{C}}_\Bbbk(X_k H^{Y_0}, Z_n H^{Y_m})) \\
&\quad \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), X_k H^{Y_0}[1] \otimes \underline{\mathsf{C}}_\Bbbk(X_k H^{Y_0}[1], Z_n H^{Y_m}[1])) \\
&\qquad \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{ev}^{\mathsf{C}_\Bbbk})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0, Z_0), Z_n H^{Y_m}[1]) \xrightarrow{[-1]s} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0, Z_0), \mathcal{A}(Y_m, Z_n))\bigr]
\end{aligned}$$

$$
\begin{aligned}
= (-)^{k+1}\big[&T^n s\mathcal{A}(Z_0,Z_n)\otimes T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes s\mathcal{A}^{\mathrm{op}}(X_k,Y_0)\otimes T^m s\mathcal{A}^{\mathrm{op}}(Y_0,Y_m) \\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), s\mathcal{A}(X_0,Z_0)\otimes T^n s\mathcal{A}(Z_0,Z_n)\otimes T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k) \\
&\qquad\qquad \otimes s\mathcal{A}^{\mathrm{op}}(X_k,Y_0)\otimes T^m s\mathcal{A}^{\mathrm{op}}(Y_0,Y_m)) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{perm})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), X_kH^{Y_0}[1]\otimes T^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0) \\
&\qquad\qquad \otimes T^n s\mathcal{A}(Z_0,Z_n)\otimes T^m s\mathcal{A}^{\mathrm{op}}(Y_0,Y_m)) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes(\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}})_{k+1+n,m})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), X_kH^{Y_0}[1]\otimes s\underline{\mathsf{C}}_\Bbbk(X_kH^{Y_0}, Z_nH^{Y_m})) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), X_kH^{Y_0}[1]\otimes \underline{\mathsf{C}}_\Bbbk(X_kH^{Y_0}[1], Z_nH^{Y_m}[1])) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{ev}^{\mathsf{C}_\Bbbk})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), Z_nH^{Y_m}[1]) \xrightarrow{[-1]s} \underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),\mathcal{A}(Y_m,Z_n))\big] \qquad (A.7.2)
\end{aligned}
$$

by the same argument. By (4.2.1), the composite $(1\otimes(\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}})_{k+1+n,m}s^{-1}[1])\,\mathrm{ev}^{\mathsf{C}_\Bbbk}$ from the above expression is given by

$$
\begin{aligned}
(-)^m\big[&s\mathcal{A}(Y_0,X_k)\otimes T^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0)\otimes T^n s\mathcal{A}(Z_0,Z_n)\otimes T^m s\mathcal{A}^{\mathrm{op}}(Y_0,Y_m) \\
&\xrightarrow{\mathrm{perm}} T^m s\mathcal{A}(Y_m,Y_0)\otimes s\mathcal{A}(Y_0,X_k)\otimes T^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0)\otimes T^n s\mathcal{A}(Z_0,Z_n) \\
&\qquad\qquad \xrightarrow{b^{\mathcal{A}}_{m+1+k+1+n}} s\mathcal{A}(Y_m,Z_n)\big],
\end{aligned}
$$

therefore (A.7.2) equals

$$
\begin{aligned}
(-)^{k+1+m}\big[&T^n s\mathcal{A}(Z_0,Z_n)\otimes T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes s\mathcal{A}^{\mathrm{op}}(X_k,Y_0)\otimes T^m s\mathcal{A}^{\mathrm{op}}(Y_0,Y_m) \\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), s\mathcal{A}(X_0,Z_0)\otimes T^n s\mathcal{A}(Z_0,Z_n)\otimes T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k) \\
&\qquad\qquad \otimes s\mathcal{A}^{\mathrm{op}}(X_k,Y_0)\otimes T^m s\mathcal{A}^{\mathrm{op}}(Y_0,Y_m)) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{perm})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), T^m s\mathcal{A}(Y_m,Y_0)\otimes s\mathcal{A}(Y_0,X_k)\otimes T^k s\mathcal{A}(X_k,X_0) \\
&\qquad\qquad \otimes s\mathcal{A}(X_0,Z_0)\otimes T^n s\mathcal{A}(Z_0,Z_n)) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,b^{\mathcal{A}}_{m+1+k+1+n})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), s\mathcal{A}(Y_m,Z_n)) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),\mathcal{A}(Y_m,Z_n))\big],
\end{aligned}
$$

which is $(\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}})_{n,k+1+m}$ by (4.2.1). The first case is proven.

Let us study the second case, which is equation (A.7.1). It expands to

$$
\begin{aligned}
&\big[T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes s\mathcal{A}^{\mathrm{op}}(X_k,Y)\otimes T^0 s\mathcal{A}^{\mathrm{op}}(Y,Y) \xrightarrow{\mathscr{Y}_{k+1}} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^{X_0},H^Y)\big] \\
&= \big[T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes X_kH^Y[1]\otimes T^0 s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^Y,H^Y) \\
&\qquad\qquad \xrightarrow{\mho_{k0}} s\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^{X_0},H^Y)\big]. \qquad (A.7.3)
\end{aligned}
$$

Composing this equation with pr_n and using closedness we turn it into another equation. By the previous case, the left hand side coincides with

$$
\begin{aligned}
(\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}})_{n,k+1} &= (-)^{k+1}\big[T^n s\mathcal{A}(Z_0,Z_n)\otimes T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes s\mathcal{A}^{\mathrm{op}}(X_k,Y) \\
&\xrightarrow{\mathrm{coev}^{\mathsf{C}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), s\mathcal{A}(X_0,Z_0)\otimes T^n s\mathcal{A}(Z_0,Z_n)\otimes T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes s\mathcal{A}^{\mathrm{op}}(X_k,Y)) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{perm})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), s\mathcal{A}(Y,X_k)\otimes T^k s\mathcal{A}(X_k,X_0)\otimes s\mathcal{A}(X_0,Z_0)\otimes T^n s\mathcal{A}(Z_0,Z_n)) \\
&\qquad \xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,b^{\mathcal{A}}_{k+n+2})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), s\mathcal{A}(Y,Z_n)) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),\mathcal{A}(Y,Z_n))\big]
\end{aligned}
$$

expressed via (4.2.1). This has to equal the right hand side which is

$$\begin{aligned}
\mho'_{k0;n} = (-)^{k+1}\big[&T^n s\mathcal{A}(Z_0,Z_n)\otimes T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes s\mathcal{A}^{\mathrm{op}}(X_k,Y) \\
&\xrightarrow{\mathrm{coev}^{\underline{\mathsf{C}}_\Bbbk}} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), s\mathcal{A}(X_0,Z_0)\otimes T^n s\mathcal{A}(Z_0,Z_n)\otimes T^k s\mathcal{A}^{\mathrm{op}}(X_0,X_k)\otimes s\mathcal{A}(Y,X_k)) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{perm})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), s\mathcal{A}(Y,X_k)\otimes T^k s\mathcal{A}(X_k,X_0)\otimes \bar{T}^{1+n} s\mathcal{A}(X_0,Z_0,\dots,Z_n)) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes H^Y_{k+1+n})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), s\mathcal{A}(Y,X_k)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(Y,X_k),\mathcal{A}(Y,Z_n))) \\
&\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,1\otimes s^{-1}[1])} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), s\mathcal{A}(Y,X_k)\otimes s\underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(Y,X_k),s\mathcal{A}(Y,Z_n))) \\
&\qquad\qquad\xrightarrow{\underline{\mathsf{C}}_\Bbbk(1,\mathrm{ev}^{\underline{\mathsf{C}}_\Bbbk})} \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(X_0,Z_0), s\mathcal{A}(Y,Z_n)) \xrightarrow{[-1]s} s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(X_0,Z_0),\mathcal{A}(Y,Z_n))\big]
\end{aligned}$$

obtained as (A.1.5) with $f = H^Y$. The required equation follows from the identity

$$\begin{aligned}
b^{\mathcal{A}}_{k+n+2} = \big[&s\mathcal{A}(Y,X_k)\otimes T^{k+1+n} s\mathcal{A}(X_k,Z_n) \\
&\xrightarrow{1\otimes H^Y_{k+1+n}} s\mathcal{A}(Y,X_k)\otimes s\underline{\mathsf{C}}_\Bbbk(\mathcal{A}(Y,X_k),\mathcal{A}(Y,Z_n)) \\
&\qquad\xrightarrow{1\otimes s^{-1}[1]} s\mathcal{A}(Y,X_k)\otimes \underline{\mathsf{C}}_\Bbbk(s\mathcal{A}(Y,X_k),s\mathcal{A}(Y,Z_n)) \xrightarrow{\mathrm{ev}^{\underline{\mathsf{C}}_\Bbbk}} s\mathcal{A}(Y,Z_n)\big],
\end{aligned}$$

which is an immediate consequence of (A.1.1) written for H^Y_{k+1+n}. We conclude that Proposition A.7 holds true. □

As a corollary, we obtain the following well-known result, cf. [**16**, Theorem 9.1], [**39**, Theorem A.11].

A.8. Corollary. *The A_∞-functor $\mathscr{Y} : \mathcal{A}^{\mathrm{op}} \to \underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)$ is homotopy fully faithful.*

Proof. By (A.7.3), we have

$$\mathscr{Y}_1 = \mho_{00} : s\mathcal{A}^{\mathrm{op}}(X,Y) \to s\underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)(H^X,H^Y),$$

for each pair $X, Y \in \operatorname{Ob}\mathcal{A}$. By Proposition A.3, the component $\mho_{00}$ is homotopy invertible, hence so is $\mathscr{Y}_1$. □

Let $\operatorname{Rep}(\mathcal{A},\underline{\mathsf{C}}_\Bbbk)$ denote the essential image of $\mathscr{Y} : \mathcal{A}^{\mathrm{op}} \to \underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)$, i.e., the full differential graded subcategory of $\underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)$ whose objects are representable A_∞-functors $(X)\mathscr{Y} = H^X : \mathcal{A} \to \underline{\mathsf{C}}_\Bbbk$, for $X \in \operatorname{Ob}\mathcal{A}$, which are unital by Remark 4.4.3. Thus, the Yoneda A_∞-functor $\mathscr{Y} : \mathcal{A}^{\mathrm{op}} \to \underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)$ takes values in the subcategory $\operatorname{Rep}(\mathcal{A},\underline{\mathsf{C}}_\Bbbk)$.

A.9. Corollary. *Let $\mathcal{A}$ be a unital A_∞-category. Then the restricted Yoneda A_∞-functor $\mathscr{Y} : \mathcal{A}^{\mathrm{op}} \to \operatorname{Rep}(\mathcal{A},\underline{\mathsf{C}}_\Bbbk)$ is an equivalence.*

In particular, each unital A_∞-category is A_∞-equivalent to a differential graded category.

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List of symbols

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