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Vorwort

Das Tätigkeitsfeld des Fraunhofer-Instituts für Techno- und Wirtschaftsmathematik ITWM umfasst anwendungsnahe Grundlagenforschung, angewandte Forschung sowie Beratung und kundenspezifische Lösungen auf allen Gebieten, die für Techno- und Wirtschaftsmathematik bedeutsam sind.

In der Reihe »Berichte des Fraunhofer ITWM« soll die Arbeit des Instituts kontinuierlich einer interessierten Öffentlichkeit in Industrie, Wirtschaft und Wissenschaft vorgestellt werden. Durch die enge Verzahnung mit dem Fachbereich Mathematik der Universität Kaiserslautern sowie durch zahlreiche Kooperationen mit internationalen Institutionen und Hochschulen in den Bereichen Ausbildung und Forschung ist ein großes Potenzial für Forschungsberichte vorhanden. In die Berichtreihe sollen sowohl hervorragende Diplom- und Projektarbeiten und Dissertationen als auch Forschungsberichte der Institutsmitarbeiter und Institutsgäste zu aktuellen Fragen der Techno- und Wirtschaftsmathematik aufgenommen werden.

Darüber hinaus bietet die Reihe ein Forum für die Berichterstattung über die zahlreichen Kooperationsprojekte des Instituts mit Partnern aus Industrie und Wirtschaft.

Berichterstattung heißt hier Dokumentation des Transfers aktueller Ergebnisse aus mathematischer Forschungs- und Entwicklungsarbeit in industrielle Anwendungen und Softwareprodukte – und umgekehrt, denn Probleme der Praxis generieren neue interessante mathematische Fragestellungen.

Prof. Dr. Dieter Prätzel-Wolters
Institutsleiter

Kaiserslautern, im Juni 2001

Pareto navigation – systematic multi-criteria-based IMRT treatment plan determination

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Abstract

Background and purpose

Inherently, IMRT treatment planning involves compromising between different planning goals. Multi-criteria IMRT planning directly addresses this compromising and thus makes it more systematic. Usually, several plans are computed from which the planner selects the most promising following a certain procedure. Applying Pareto navigation for this selection step simultaneously increases the variety of planning options and eases the identification of the most promising plan.

Material and methods

Pareto navigation is an interactive multi-criteria optimization method that consists of the two navigation mechanisms “selection” and “restriction”. The former allows the formulation of wishes whereas the latter allows the exclusion of unwanted plans. They are realized as optimization problems on the so-called plan bundle – a set constructed from precomputed plans. They can be approximately reformulated so that their solution time is a small fraction of a second. Thus, the user can be provided with immediate feedback regarding his or her decisions.

Results and discussion

Pareto navigation was implemented in the MIRA navigator software and allows real-time manipulation of the current plan and the set of considered plans. The changes are triggered by simple mouse operations on the so-called navigation star and lead to real-time updates of the navigation star and the dose visualizations. Since any Pareto-optimal plan in the plan bundle can be found with just a few navigation operations the MIRA navigator allows a fast and directed plan determination. Besides, the concept allows for a refinement of the plan bundle, thus offering a middle course between single plan computation and multi-criteria optimization.

Conclusion

Pareto navigation offers so far unmatched real-time interactions, ease of use and plan variety, setting it apart from the multi-criteria IMRT planning methods proposed so far.

Keywords: convex, interactive multi-objective optimization, intensity modulated radiotherapy planning

1 Introduction

Treatment planning for IMRT is usually done using so-called inverse planning. Here, the planner makes certain prescriptions for the desired dose distribution, which internally are used to set up an optimization problem. The solution to this problem is presented to the planner as a plan proposal. If the plan does not meet the planner's expectations, in current systems he has to re-run the process of optimization and evaluation. This "human iteration loop" usually ends, when the treatment plan cannot be improved anymore or a time limit has been reached.

In current research, some efforts are being made to shorten the "human iteration loop", thus reducing the time requirements and fully exploiting the possibilities of IMRT. One promising approach is the introduction of multi-criteria optimization.

Multi-criteria optimization treats the different objectives – in our case achieving high and homogeneous doses in the tumour volumes and low doses in the organs at risk – as separate quality measures of a solution. Consequentially, the function modelling quality of a dose distribution is vector-valued. There is no natural way to judge for two arbitrary vectors, which of the two is smaller, i.e. better. Thus, a definition of optimality for vectors is needed. We will work with the most common definition known as Pareto-optimality:

A solution is Pareto-optimal, if no criteria can be improved without worsening at least one of the remaining criteria.

Looking at the criterion values for the feasible plans the values for the set of Pareto-optima is a part of the boundary. In particular, the set of Pareto-optimal plans is infinite for all but degenerate cases. By the definition of Pareto-optimality each such plan is a – mathematically – best-possible compromise between the different criteria. So, multi-criteria optimization offers a huge number of promising choices.

Therefore, multi-criteria IMRT planning has been an active area of research for the past ten years. First, Yu ([Yu, 1997]) proposed a scheme for a systematic parameter choice based on multi-criteria decision theory to substitute the trial and error process. Haas et al. ([Haas *et al.*, 1997], [Haas *et al.*, 1998]) applied a multi-criteria genetic algorithms to simultaneously find the beam geometry and the intensities.

More recent activities applied deterministic approaches to approximate the Pareto-optimal set. Linear models ([Küfer *et al.*, 2000], [Holder, 2001], [Bortfeld *et al.*, 2002], [Hamacher and Küfer, 2002], [Thieke *et al.*, 2002], [Küfer *et al.*, 2003]) as well as non-linear models ([Cotrutz *et al.*, 2001], [Küfer *et al.*, 2003], [Lahanas *et al.*, 2003]) were under consideration.

The huge number of choices raises the need for systematic support for decision-making. Here, we will discuss a method to support the decision-making that is called Pareto navigation. It was specifically developed for IMRT planning and is part of a research software called MIRA – Multicriteria Interactive Radiotherapy Assistant ([Thieke *et al.*, 2007]). We will describe the user interactions of the MIRA navigator and how they are realized.

2 Material and methods

We are assuming the irradiation directions to be fixedly given. The intensities of the different beamlets over all beams is then denoted by the vector $\mathbf{x} \in \mathbb{R}^n_+$ taken from the non-negative orthant. One such $\mathbf{x}$ is often called solution or plan. The corresponding dose distribution $\mathbf{d}(\mathbf{x}) = \mathbf{Px}$ is given through the linear mapping $\mathbf{P}$ containing in its columns the dose deposited by the different beamlets for unit intensity. The quality of a dose distribution is judged by a vector-valued function $\mathbf{f}$, where the k components are assumed to be convex.

This restriction is necessary to be able to apply efficient deterministic optimization methods to the problems under consideration. Convex minimization techniques are able to check, whether they have reached the global optimum. In contrast to this, finding the global optimum for non-convex objective functions usually is not possible with a reasonable computational effort. Furthermore, the set of Pareto-optimal plans might not be connected any longer, a fact that will inevitably lead to volatile reactions with regard to small changes.

Examples of convex functions used in IMRT planning and functions that can be substituted by convex functions are given in [Romeijn *et al.*, 2004]. The framework they setup contains TCP, NTCP and Niemierko's EUD ([Niemierko, 1999]) among others. In this work, we will exclusively work with convex functions.

For our model we assume that some – possibly trivial – upper bounds $\mathbf{b} \in (\mathbb{R} \cup \{\infty\})^k$ for the criteria are given. Moreover, we assume that the values of the criteria are scaled and shifted such that same levels of quality are mapped to roughly the same values. The latter can e.g. be achieved by the use of aspired criterion values derived from the physicians prescription.

The multi-criteria problem to be solved then reads as

$$\operatorname{v-min} \big\{ \mathbf{f}(\mathbf{Px}) \; \big| \; \mathbf{f}(\mathbf{Px}) \leq \mathbf{b},\, \mathbf{x} \geq \mathbf{0} \big\}, \tag{1}$$

where v- min refers to vector-valued minimization. This problem is purely abstract. To solve it a method to identify or approximate the solution set must be chosen.

The most common method for solving multi-criteria optimization problems is to turn them into a family of single-criteria optimization problems – a technique called scalarization. So, let

$$\sigma \colon \mathbb{R}^k \times \mathbb{R}^l \longrightarrow \mathbb{R} \tag{2}$$

denote our scalarization function. Now, instead of solving (1), we will solve

$$\min \big\{ \sigma\big(\mathbf{f}(\mathbf{Px}), \mathbf{u}\big) \; \big| \; \mathbf{f}(\mathbf{Px}) \leq \mathbf{b},\, \mathbf{x} \geq \mathbf{0} \big\} \tag{3}$$

for yet to be determined choices of the parameters $\mathbf{u}$. Now, we can use the well-known methods for single-criteria optimization problems for solving (3). The choice of $\mathbf{u}$ now determines the resulting solution. Hence, the solution set is implicitly parameterized. For well-chosen scalarization functions this parameterization covers the whole solution set and is continuous ([Pascoletti and Serafini, 1984], [Wierzbicki, 1986], [Monz, 2006]).

2.1 Pareto navigation

"Interactive multi-criteria optimization" ([Hwang and Masud, 1979], [Miettinen, 1999]) is the class of multi-criteria methods, where preferences of the decision-maker – in our case the planner – are incorporated directly into the search for a clinically sound compromise. Here, the speed of interaction does not play a role in the classification. Interactive multi-criteria optimization methods are the methods of choice, when the final compromise cannot be anticipated prior to the calculation. A wide variety of methods exist (see [Miettinen, 1999] and the references therein) and is used for a wide range of applications ([White, 1990]).

Usually, the chosen interactive multi-criteria method is directly applied to problem (1). Here, preference information expressed by the planner is translated into a scalarization and some bounds. Then the corresponding scalar problem (3) is solved and the resulting plan is shown to the planner. The planner changes the preference information e.g. weights to cause a change in the current plan and issues another scalar optimization problem and the cycle is repeated.

This "human iteration loop" is tedious, since the time for executing one scalar optimization usually takes one to several minutes and only after the result is known, it will be clear whether the expectations are met. Thus, the process might consist of too many iterations or is even stopped prematurely.

In contrast to this we opt for a two stage approach that consists of a lengthy offline computation of a set of Pareto-optimal plans ([Küfer *et al.*, 2003], [Craft, 2006], [Monz, 2006]) that is done offline and a real-time decision-making phase – Pareto navigation – that we will describe here.

Pareto navigation only assumes the pre-computed plans to be Pareto optimal, but does not make any further assumptions on how they are created. Therefore, it is indifferent about the source of the pre-computed plans. They could be gained by manually setup optimization problems or even be calculated by a different inverse planning system, as long as the underlying optimization model is convex. So far, plans originating from KonRad and a linear and nonlinear MIRA solver were used in the MIRA navigator.

Note that, even though the source of the plans does not play a role, the completeness and accuracy with which the pre-computed plans approximate the set of Pareto optimal plans implies the variety and quality of the plans being accessible during Pareto navigation.

Henceforth, we will assume that a set $\mathbf{x}^{(i)}, i = 1, ..., m$ of pre-computed Pareto-optimal plans is given. For convenience we bundle them into the matrix

$$\mathbf{X} := \left(\mathbf{x}^{(1)} \,|\, ... \,|\, \mathbf{x}^{(m)}\right).$$

2.2 The plan bundle

If we were only considering the pre-computed plans themselves during Pareto navigation, not much would be gained. Comparing some dozen plans still is tedious work. Besides, we would still rely on the assumption that the best clinical compromise is contained in the set of pre-computed plans.

Hence, we do not only consider the pre-computed plans in Pareto navigation, but all convex combinations, i.e. mixes, of plans as well. Given a reasonable approximation of the set of Pareto-optimal plans, this set offers a variety of plans, whose quality is close to that of the Pareto-optimal plans. Nevertheless, working on the convex combination of pre-computed plans leads to a noticeable simplification of the optimization problems solved during the decision-making phase. It is thus a first important step towards real-time decision-making mechanisms.

So, the set of plans – called plan bundle – that we consider in Pareto navigation

$$\mathcal{X} := \left\{ \sum_{i=1}^{m} \mathbf{v}_i \mathbf{x}^{(i)} \,\Big|\, \sum_{i=1}^{m} \mathbf{v}_i = 1, \mathbf{v} \geq \mathbf{0} \right\}$$

is a good tradeoff between interaction speed and solution quality. Here, the vector $\mathbf{v}$ contains the percentages with which the plans are mixed.

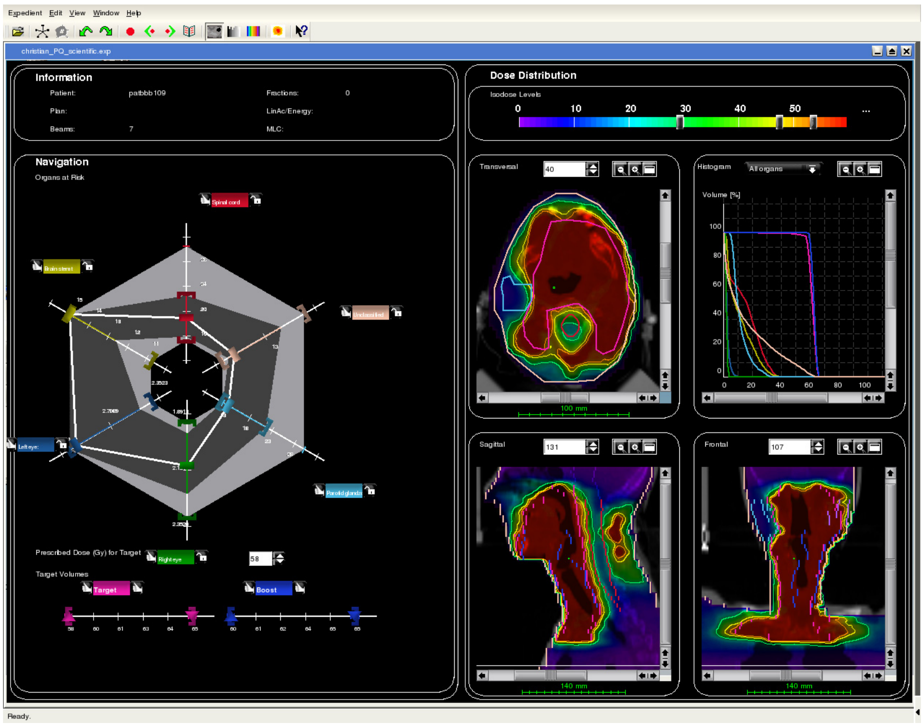
By the definition of convexity

$$\mathbf{f}\left(\mathbf{P}\sum_{i=1}^{m}\mathbf{v}_i\mathbf{x}^{(i)}\right) \;\le\; \sum_{i=1}^{m}\mathbf{v}_i\mathbf{f}\big(\mathbf{P}\mathbf{x}^{(i)}\big) \;\le\; \sum_{i=1}^{m}\mathbf{v}_i\mathbf{b} \;=\; \mathbf{b}.$$

mixed plans fulfil the upper bounds. Furthermore, the convex combination of intensities is non-negative, since the sum runs over non-negative vectors being multiplied with non-negative coefficients. Hence, every convex combination of feasible plans is feasible. This allows us to drop the feasibility constraints for problems with $\mathcal{X}$ as their feasible domain.

2.3 The user interaction

The user interaction has two directions. The information provided by Pareto navigation and its mechanisms for receiving user preference information.



Patient: patbbb109

Plan:

Beams: 7

Fractions: 0

LinAc/Energy:

MLC:

Prescribed Dose (Gy) for Target: 58

Transversal: 40

Sagittal: 131

Frontal: 107

Figure 1: A screen shot of the MIRA navigator.

The information provided comprises the visualization of the current plan with its DVH and dose distribution on transversal, sagittal and frontal slices and a visualization of the plan bundle in the so-called navigation star (see figure 1).

The navigation star features one axis per function. The range of function values covered by the plan bundle on the different axes is connected forming a donut-like gray area. This area visualizes the options contained in the plan bundle. The part of the area that is currently not accessible due to tightened constraints is displayed in a lighter gray. This area is called inactive planning horizon, the currently accessible part is called active planning horizon and the full set is called planning horizon. The tumour axes are displayed below the star, but work the same way as those contained in the star.

The display of the inactive planning horizon makes the implicit effects of constraints on some of the axis with regard to the other axes explicit. The current plan is embedded into the plan bundle visualization by connecting its function values for the different axes yielding a white polygon. This gives the user a visual clue on how much the current plan can still be changed with regard to the different criteria.

Towards the higher function values each axis contains a so-called restrictor. Moving the restrictor excludes the plans that have a function value higher than the one given by the restrictor's current position. A change in a restrictor position causes an update of the planning horizon and thus provides the decision maker with feedback on the effect of his or her decision. If sharpening a certain upper bound hardly has an effect on the remaining criteria, one could make it even tighter than originally intended. If on the other hand the effect is more dramatic than expected, one could directly revert the change partially.

Thus, the restriction mechanism allows the decision maker to enforce or loosen bounds *after* (s)he has examined feasible solutions and gained some insight on the effect of choosing a certain bound. Besides, it can be used to temporarily set a bound that has no specific meaning in terms of the modelling, but serves as a means of steering the selection mechanism.

One could argue that if good bounds exist, they should be included into the original problem formulation. But this only accounts for the situation where good bounds are known. There is often a certain *uncertainty* about the value the planner should enforce with the bound (s)he sets. If ineptly chosen it could even render the problem infeasible. The restriction mechanism is not allowed to exclude the current solution, so the upper bounds set by it can never render the problem infeasible. At most, the current solution is the only feasible solution left. In total, delaying the definition of upper bounds enables the planner to make a much more informed decision.

The points where the navigation polygon meets the axes features sliders called selectors. To change the current plan the planner can grab one of the sliders with the mouse and change the function value for this axis. The system then finds a plan with the specified function value on that axis and minimally worsened values for the other axes. In certain cases "minimally worsened" might mean "improved".

When a selector is moved, the changed function values for the other axes and the visualization of the current plan are updated, so that the planner will get immediate feedback concerning his or her changes. The provided information is gained by solving auxiliary optimization problems, which will be described in detail in the following sections.

2.4 Updating the planning horizon

The restriction mechanism allows the decision maker to change the upper bounds $\mathbf{b}$ for one of the axes. This has an effect on the ranges of potentially all other axes (see figure 2). Thus, best and worst values for each axis have to be recalculated after such a change.

In multi-criteria optimization the vector combining the individual minima for the different objective functions among the Pareto-optimal plans is called ideal point and the combination of the occurring maxima among the Pareto-optimal plans is called

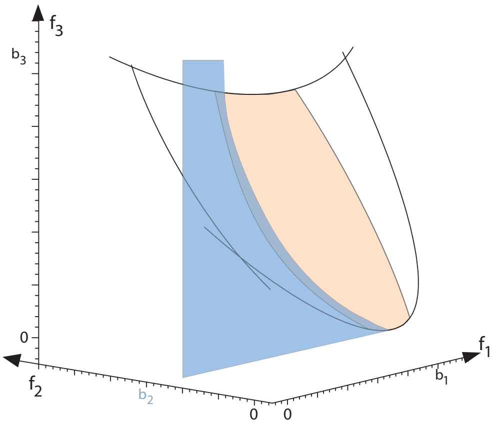


Figure 2: The figure illustrates the restriction mechanism for a problem with three criteria. The surface visualizes the function values for the set of Pareto-optimal solutions and the coloured part is the subset that conforms with the original upper bounds. The shaded plane indicates the position of the changed upper bound, that cuts away the best possible solutions for the first criterion in this case.

nadir point. We will therefore look at how the ideal point and nadir point can be computed or approximated.

The ideal point can be computed by solving the k optimization problems

$$\mathbf{y}_i^I(\mathbf{b}) \;:=\; \min\left\{\mathbf{f}_i(\mathbf{Px}) \mid \mathbf{x} \in \mathcal{X},\, \mathbf{f}(\mathbf{Px}) \leq \mathbf{b}\right\} \quad \text{for} \;\; i = 1, ..., k, \tag{4}$$

each yielding one component of the ideal point. Here, we use the fact, that the minima over the Pareto-optimal plans are the same as the corresponding minima over the full set.

The minimization over $\mathcal{X}$ is formal way of writing the problem. If we substitute the the definition of $\mathcal{X}$ into the problem,

$$\mathbf{y}_i^I(\mathbf{b}) \;:=\; \min\left\{\mathbf{f}_i(\mathbf{PXw}) \;\middle|\; \mathbf{f}(\mathbf{PXw}) \leq \mathbf{b},\, \sum_{l=1}^{m} \mathbf{w}_l = 1,\, \mathbf{w} \geq \mathbf{0}\right\} \tag{5}$$
$$\text{for} \;\; i = 1, ..., k,$$

we see that the variables to be found are the convex combination coefficients $\mathbf{w}$.

The problem of finding the components of the nadir point

$$\max\left\{\mathbf{f}_i(\mathbf{Px}) \mid \mathbf{x} \in \mathcal{X},\, \mathbf{f}(\mathbf{Px}) \leq \mathbf{b}, \mathbf{x} \text{ Pareto-optimal}\right\} \tag{6}$$

is much more involved. First, it is a convex *maximization* problem – a problem class that is much more complex than the class of convex minimization problems. Second, its domain is nonconvex, since here we can not use the full set of plans instead of the Pareto-optimal ones. Therefore, the problem is difficult to solve in three or more dimensions (see e.g. the abstract of [Benson and Sayin, 1994]). For more on how to compute it see the survey article [Yamamoto, 2002] and the articles [Dauer, 1991], [Horst *et al.*, 1997], [Horst and Thoai, 1997], [Luc and Muu, 1997], [Korhonen *et al.*, 1997] and [Ehrgott and Tenfelde-Podehl, 2003].

A practical approach to approximate it is the use of the so-called payoff table. In the payoff table the function values for the results of the ideal point computation (5) are collected and the nadir point is estimated as the combination of the maxima occurring for each component. The rationale is that due to the definition of Pareto-optimality one has to pay for improvements and that achieving the best possible value for one component will cost a high price in the others.

If the vectors in the payoff table correspond to Pareto-optimal points, the maximum is found over a subset of the Pareto-optimal points instead of the full set and the values are thus smaller or equal to the true values. If the vectors are not Pareto-optimal the resulting estimate can be too small or too big ([Ehrgott and Tenfelde-Podehl, 2003]). We are therefore using a payoff table estimate with Pareto-optimal points, which can be computed along with the ideal point with just a small extra effort.

2.5 The selection mechanism

The selection mechanism is the tool to communicate wishes and thereby change the current solution. The decision maker chooses one criterion and changes it to any feasible value that (s)he finds desirable. The solution given back in response to this action attains the specified value in the chosen criterion and has the most favourable distance to the remaining criteria. In particular, it tries to improve beyond the previous level when this is possible. Furthermore, it observes the upper bounds on the criteria introduced by the restriction mechanism.

Internally, the selection mechanism executes the following optimization problem:

$$\min \Big\{ \max_{i \in \mathcal{K} \setminus \{j\}} \Big\{ \mathbf{f}_i(\mathbf{P}\mathbf{x}) - \mathbf{y}_i^R + \mathbf{s}_i \Big\} \;\Big|\; \mathbf{f}_j(\mathbf{P}\mathbf{x}) = \tau,\ \mathbf{f}(\mathbf{P}\mathbf{x}) \leq \mathbf{b},\ \mathbf{x} \in \mathcal{X},\ \mathbf{s} \geq \mathbf{0} \Big\}, \tag{7}$$

where j is the index of the chosen criterion, τ the selected value and $\mathcal{K} := \{1, ..., k\}$ the set of possible indices. The problem is a variation of a reference point based scalarization introduced in the article [Pascoletti and Serafini, 1984] and is a convex minimization problem.

It searches for the smallest point in

$$\mathcal{Y} := \big\{ \mathbf{f}(\mathbf{P}\mathbf{x}) + \mathbf{s} \,\big|\, \mathbf{x} \in \mathcal{X},\ \mathbf{s} \geq \mathbf{0} \big\}$$

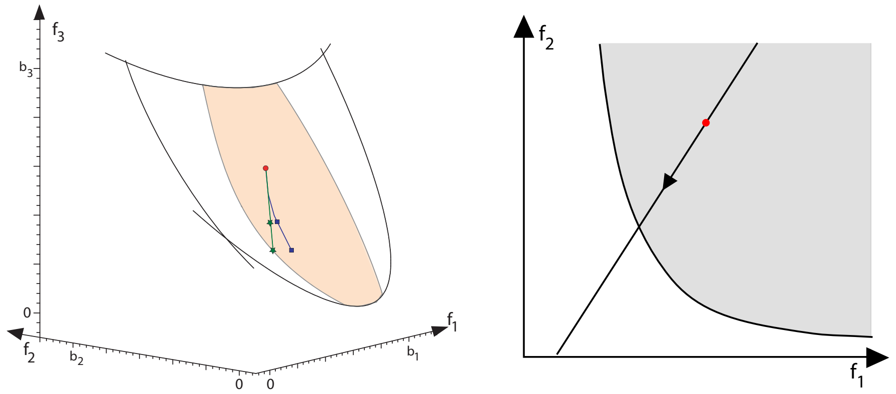
along the line $\mathbf{y}^R + t\mathbf{1}$ anchored at the reference point $\mathbf{y}^R$ (see figure 3(b)).

The problem (7) can be equivalently reformulated as

$$\min \Big\{ \max_{i \in \mathcal{K} \setminus \{j\}} \Big\{ \mathbf{y}_i - \mathbf{y}_i^R \Big\} \;\Big|\; \mathbf{y} = \mathbf{f}(\mathbf{P}\mathbf{X}\mathbf{w}) + \mathbf{s},\ \mathbf{y} \leq \mathbf{b},\ \mathbf{y}_j = \tau,\ \sum_{i=1}^{m} \mathbf{w}_i = 1,\ \mathbf{w}, \mathbf{s} \geq \mathbf{0} \Big\}, \tag{8}$$

using the definition of $\mathcal{X}$. So, the main variables to be found are the convex combination coefficients $\mathbf{w}$.

As the reference point $\mathbf{y}^R$ in (8) we use the vector of criterion values of the plan that the planner started to change. So, we are enforcing the τ value for criterion j and look for best distance to the previous plan in the remaining criteria. For an analysis of the mathematical properties of the selection mechanism see [Monz, 2006].



(a) (b)

Figure 3: Starting from the reference point – the dot – the planner decreases the value for the third criterion. The points marked by asterisks are equal to the starting point except the changed component and are projected to the squared points (figure 3(a)). For the projections the changing axes is restricted to the given value, thus in effect intersecting the feasible area with a hyperplane. On the resulting lower dimensional domain we search along the line passing through the projection of the starting point for the best point still contained in the feasible area (figure 3(b)).

2.6 Making the mechanisms real-time feasible

Interactive multi-criteria optimization methods are especially useful when the response times of the system to user input are short. If the planner has to wait noticeable fractions of a minute for the system to answer the interactivity is somewhat lost and the decision process becomes awkward.

Working on the restricted set $\mathcal{X}$ already leads to a noticeable speed-up for the problems to be solved during Pareto navigation. But for true interactivity the solution time of some fraction of a minute is still too long. Hence, we aim at an slightly coarser approximation that yields the desired speed-up.

We substitute

$$\mathbf{f}(\mathbf{PXw}) \;=\; \mathbf{f}\Big(\mathbf{P}\sum_{i=1}^{m}\mathbf{w}_i\mathbf{x}^{(i)}\Big) \qquad \text{by} \qquad \sum_{i=1}^{m}\mathbf{w}_i\mathbf{f}\big(\mathbf{Px}^{(i)}\big).$$

The definition of convexity ensures that the second expression is always greater than or equal to the first.

Thus, the upper bound constraint after the substitution

$$\sum_{i=1}^{m}\mathbf{w}_i\mathbf{f}\big(\mathbf{Px}^{(i)}\big) \;\le\; \mathbf{b} \quad\Longrightarrow\quad \mathbf{f}(\mathbf{PXw}) \;\le\; \mathbf{b}$$

implies the previous upper bound constraint. Due to the resulting smaller feasible domain and higher values in the objective functions, we get a slightly pessimistic approximation of the original problem.

The deviation of the feasible set

$$\mathcal{Y}^L \;:=\; \Big\{\sum_{i=1}^{m}\mathbf{w}_i\mathbf{f}\big(\mathbf{Px}^{(i)} + \mathbf{s}_i \;\Big|\; \sum_{i=1}^{m}\mathbf{w}_i = 1,\; \mathbf{w},\mathbf{s} \ge \mathbf{0}\Big\}$$

from the feasible set $\mathcal{Y}$ can be controlled with the use of special approximation algorithms ([Solanki *et al.*, 1993] [Klamroth *et al.*, 2002] [Craft, 2006]). The worst-case-error introduced by the above approximation is known a-priori in this case.

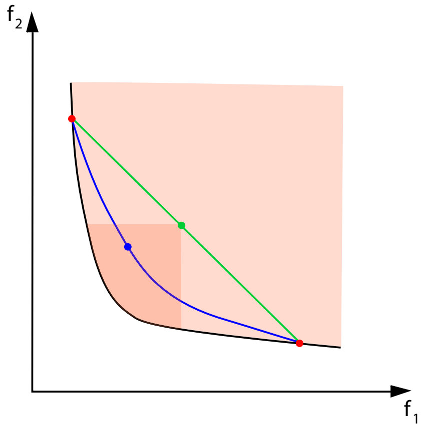


Figure 4: The upper line is the set of Pareto-optimal points of $\mathcal{Y}^L$ and the curve the one for $\mathcal{Y}$. For the same convex combination coefficients $\mathbf{w}$, the point on the curve $\mathbf{f}(\mathbf{PXw})$ is by convexity contained in the darker area $\{\mathbf{y} \in \mathcal{Y} \,|\, \mathbf{y} \le \mathbf{Yw}\}$.

We will now have a look at the effect of the approximation on the planning horizon update problems and the selection problem. To make the notion more convenient we will bundle the criteria values for the precomputed plans into the matrix

$$\mathbf{Y} \;:=\; \left(\mathbf{f}\big(\mathbf{Px}^{(1)}\big) \,\middle|\, \dots \,\middle|\, \mathbf{f}\big(\mathbf{Px}^{(m)}\big) \right).$$

Applying the substitution to the ideal point problems

$$\min \left\{ (\mathbf{Yw})_i \;\middle|\; \mathbf{Yw} \le \mathbf{b},\; \sum_{l=1}^{m} \mathbf{w}_l = 1,\; \mathbf{w} \ge \mathbf{0} \right\} \tag{9}$$

yields a linear program that computes an approximation of the minimum value for the i^{th} criterion value. Besides the potentially smaller feasible region that might increase the result the linearization overestimates the value of the objective function, so that the overall estimate is an upper bound for the original value. To improve the estimate, $\mathbf{f}_i\big(\mathbf{PXw}^{(i)}\big)$ can be evaluated for the optimal $\mathbf{w}^{(i)}$ giving a sharper estimate of the minimum.

The linear program implementing (9) has m variables – the number of precomputed solutions – and k restrictions, where k as above is the number of criteria. This is very small for a linear program and consequentially solving problem (9) takes a very small fraction of a second. With hardly any additional effort we can ensure the solutions to be Pareto-optimal for $\mathcal{Y}^L$ and then carry out the payoff table heuristic to get a nadir point estimate.

Simplifying the selection mechanism with the same substitution results in the following optimization problem:

$$\min \left\{ \max_{i \in \mathcal{K} \setminus \{j\}} \left\{ \mathbf{y}_i - \mathbf{y}_i^R \right\} \;\middle|\; \begin{array}{l} \mathbf{y} = \mathbf{Yw} + \mathbf{s},\; \mathbf{y} \le \mathbf{b}, \\ \mathbf{y}_j = \tau,\; \sum_{i=1}^{m} \mathbf{w}_i = 1,\; \mathbf{w}, \mathbf{s} \ge \mathbf{0} \end{array} \right\}, \tag{10}$$

The equality constraint $\mathbf{y}_j = \tau$ is certainly much easier to implement in the linearized case, since in effect it restricts the *variables* $\mathbf{w}$ to an affine linear subspace. So, one can enforce it by only allowing changes, for which the variables stay in this subspace.

But the equality constraint is only an approximate. If we reevaluate the optimal convex combination $\hat{\mathbf{w}}$ with the original convex functions $\mathbf{f}(\mathbf{PX}\hat{\mathbf{w}})$, the convexity of $\mathbf{f}_j$ ensures that τ is an upper bound for $\mathbf{f}_j(\mathbf{PX}\hat{\mathbf{w}})$. In general it will be smaller than τ. Nevertheless, the deviation is bounded by the distance of the approximation to the original problem, so that for a good approximation the deviation is small.

To get rid of the above maximum we replace it

$$\max_{i \in \mathcal{K} \setminus \{j\}} \left\{ \mathbf{y}_i - \mathbf{y}_i^R \right\} = \min \left\{ z \in \mathbb{R} \;\middle|\; \mathbf{y}_i - \mathbf{y}_i^R + \bar{\mathbf{s}}_i = z \text{ for } i \in \mathcal{K} \setminus \{j\},\ \bar{\mathbf{s}} \geq \mathbf{0} \right\}$$

by a small optimization problem. This problem can be seamlessly integrated into problem (10) adding up to the following linear program:

$$\min \Big\{ z \in \mathbb{R} \;\Big|\; (\mathbf{Yw})_i - \mathbf{y}_i^R + \mathbf{s}_i = z,\ i \in \mathcal{K} \setminus \{j\}, \quad (\mathbf{Yw})_j = \tau,\ \mathbf{Yw} \leq \mathbf{b},\ \sum_{i=1}^{m} \mathbf{w}_i = 1,\ \mathbf{s} \geq \mathbf{0} \Big\}, \tag{11}$$

where we fused the $\mathbf{s}$ and the $\bar{\mathbf{s}}$ vectors.

The above linear program (11) has $2k + 1$ constraints and $k + m + 1$ variables, where again m is the number of precomputed solutions and k is the number of criteria. The above dimensions are very small for linear programming. Consequentially, the problems are solved within a very small fraction of a second. Even with the effort of evaluating $\mathbf{f}(\mathbf{PX}\hat{\mathbf{w}})$ for the optimal $\hat{\mathbf{w}}$ and the time spent for updating the visualization of the current plan the whole process can be executed several times a second.

3 Results and discussion

In the MIRA navigator (see figure 1) the two navigation mechanisms are constructed to follow a grab-move-release user interaction pattern. So, the planner clicks on either a restrictor or selector, moves it while keeping the mouse pressed and releases it at its final position. When one of the controls is grabbed the corresponding linear programs are setup and in case of the selector the starting point is fixed.

While the control is moved the corresponding linear programs are solved several times a second for the current position of the control and the corresponding visualizations are concurrently updated. This process stops, when the mouse is released.

To further improve its usability, the MIRA navigator features a so-called plan pool, where promising plans can be put aside. This way, several promising alternatives can be stored for later in-depth inspection. Besides, they can be used to re-start the navigation at that specific point.

The full strength of Pareto navigation becomes apparent, when the restriction and selection mechanism are combined. It is shown in [Monz, 2006] that every Pareto-optimal plan in the plan bundle that we head for can be found with at most k selection and k restriction operations, where k is the number of criteria.

The different possible operations can be combined in arbitrary order, allowing every planner to use it in his or her preferred way. So, it does not aim at supervising the planner, but tries to support the planner as good as possible.

3.1 Extensions

The linearization of the navigation mechanisms makes them real-time feasible and thus enables immediate feedback. Yet their results could visibly differ from the original results, if the pre-computed plans do not form a good approximation of the set of Pareto-optimal plans. This could happen, if the number of criteria k exceeds five or six, in which case computing a good approximation needs too many plans, i.e. will cost too much time.

So, if the distance between the approximation and the set of Pareto-optimal plans is potentially large in an area that is of interest to the planner, (s)he could decide to invest some time into improving the approximation. If s(he) does so, the selection problem (7) would be executed for the full set of feasible intensities yielding another Pareto-optimal plan. This plan can then be added to the linearized problems and greatly improves the local approximation (see figure 5). After the plan is added, the planning horizon is updated and the newly computed point is set to be the current plan.

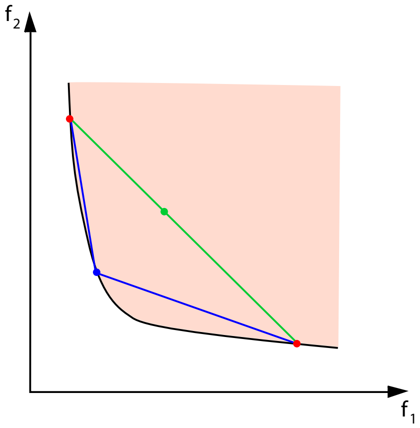


Figure 5: The upper line is the set of Pareto-optimal points of $\mathcal{Y}^L$ prior to adding the new point and the piece-wise linear curve the one after adding the new point.

The decision making process could thus start with a relatively coarse approximation and be refined in regions of interest to the planner. This establishes a trade-off between the effort spent in pre-computation and the effort spent during the decision-making phase.

Adding new plans to the approximation also works for plans lying outside the area approximated so far. So, a new plan could be added to increase the variety of plans considered in the navigation. This way, the iterative approach can be combined with Pareto navigation.

So, one would start with a few plans resulting from manually setup optimization problems. Then, navigation would be used to identify the preferred plan among the convex combination of pre-computed plans. If no clinically acceptable compromise is found, another plan could be added, thus adding more variety to the navigation.

The above approach should only be taken, if the planner is sure to come close to his or her preferred compromise. For all other cases a systematic approximation method will most likely yield a plan bundle of superior quality. In particular, for more complex cases the plan bundle might contain planning options that the planner did not forsee.

4 Conclusion

In this article we introduced a interactive multi-criteria optimization method called Pareto navigation. Distinctive features of Pareto navigation are its real-time interactions, the infinite number of plans its mechanisms work on, its indifference about the source of the plans and the possibility of freely combining iterative planning and Pareto navigation. Altogether, it allows for a more directed and informed determination of treatment plans that will result in shorter planning times and more satisfactory results.

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