Approximating the Nondominated Set of $\mathbb{R}_{\geqq}$-convex Bodies

MASTER THESIS

by

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Contents

Introduction

In real situations, we usually face problems that involve several optimization criteria and it is possible that these criteria can contradict each other. Cancer radiotherapy treatment, for example, tries to maximize the amount of cancer tissue exposed to radiation treatment while minimizing radiation exposure of healthy organs and, clearly, there is no easy compromise between both criteria.

Because several and often conflicting objectives have to be optimized, generally there exists no feasible solution that optimizes all criteria. However, it is possible to find solutions in which no criteria can be improved without worsening any other criteria.

In *Multicriteria Optimization* we seek for *efficient solutions*, i.e. solutions such that there is no other possible way of action which provide a better performance in all criteria. And due to the "efficiency property", when we compare two efficient solutions we will find that improvement in one criterion will always imply worsening on other criterion. This condition – improvement in some criteria resulting in worsening on some other criteria – is known as tradeoff between two efficient solutions.

We say that these efficient solutions – and, in fact, all feasible solutions from our problem – lie in the so called *Decision Space*. The evaluation of a solution in the decision space in each criterion produces a vector point in the so called *Outcome Space*.

Multicriteria programming identifies these efficient solutions and their corresponding outcomes. These relevant outcomes – the resulting image from efficient solutions – form the so called *Nondominated Set*.

In the context of multicriteria optimization there is an actor, the *Decision Maker*, who chooses a decision based on the information obtained concerning the efficient solutions and the nondominated set. The decision maker settles for a compromise related to the tradeoff of different efficient solutions.

Ultimately, the goal of a multicriteria program is to support the decision maker in his choice by letting him pick a compromise *a posteriori*, only after different possibilities and their respective compromises have already been explored.

However, it is often very difficult to characterize the whole nondominated set. Finding efficient solutions for some problems might be as well impossible due to the numerical complexity of the resulting optimization subproblems. Even if we are able to compute efficient solutions, an exact description will in most cases fail because the number of efficient solutions is either too large or even infinite. Furthermore, even in those cases where an exact description could be made, computing the whole nondominated set is generally computationally expensive.

Thus we need to approximate the nondominated set and the efficient solutions. Usually this is done by computing a *Representation Set*. The representation set is commonly a finite collection of points that approximate the nondominated set providing enough information to the decision maker.

Different ideas have been applied to generate representations of the nondominated set: nonnegative weighted scalarization, Tchebycheff weighted scalarization [Kaliszewski, 1987], block norms [Schandl et al., 2002], ϵ-constraints [Wiecek et al., 2001] and even evolutionary methods [Zitzler et al., 2001].

Following the work from Schandl et al. [2002], Klamroth et al. [2002] devised algorithms that compute inner and outer approximations of the nondominated set. In this paper we implement these algorithms and analyze their numerical complexity in higher dimensions. This work is structured as follows:

In Chapter 1 we formally define multicriteria optimization programming, efficient solutions and the nondominated set, we also introduce necessary notation and recall basic concepts from multicriteria optimization, these definitions and concepts will be needed in the rest of this paper.

We also define $\mathbb{R}_{\geq}$-convex bodies in Chapter 1. $\mathbb{R}_{\geq}$-convex bodies result as the outcome space of multiple convex objective functions evaluated over a convex decision space. $\mathbb{R}_{\geq}$-convex bodies have a nice characterization of their nondominated set which is exploited in order to generate nondominated points and their corresponding efficient solutions.

In the last section of Chapter 1, we consider *Quality Measures* for representation sets. These measures provide quantitative grounds to discern between different representations from the same nondominated set.

In Chapter 2, we define block norms and analyze their properties according to [Schandl et al., 2002]. Block norms' importance in multicriteria optimization is explained and their sufficiency to obtain all nondominated points is proved.

Block norms are the building blocks for the inner and outer approximation algorithms in [Klamroth et al., 2002]. In Chapter 3, we review these algorithms and propose three different variants. However, block norm based algorithms require to solve a sequence of subproblems, the number of subproblems becomes relatively high for the six criteria case and even intractable for real applications with nine criteria.

We try to overcome this disadvantage in Chapter 4, where we model our problem using bilevel linear programming to derive an algorithm that generates an approximation of the nondominated set.

We analyze and compare the approximation quality, running time and numerical convergence of such algorithms in Chapter 5 for three, four, five, six and nine multiple criteria.

Finally, we state our conclusions and suggest some directions worth of future investigation.

Chapter 1

Basic Notation and Concepts

In a multicriteria problem there is no "optimal" feasible solution but a set of "efficient" solutions which represent different compromises.

However, it is often unpractical to seek for all efficient solutions, and therefore representation sets are used to approximate the set of efficient solutions. In order to compare different representation sets, we need to define quality measures that quantify the desired properties of such representations.

This chapter will provide the notation and definitions necessary to properly understand concepts as efficient solution and representation set, in order to devise algorithms to compute them.

1.1 Multicriteria Optimization

A *Multicriteria Optimization Problem* is a problem where given a *feasible set* that depicts a decision space and a set of objective functions representing different criteria, we strive to find *efficient* decisions which present good compromises between the given criteria.

Definition 1.1 (Multicriteria Optimization Problem)**.** *The problem*

$$\begin{aligned} min\ \{z_1 &= f_1(\mathbf{x})\} \\ &\vdots \\ min\ \{z_n &= f_n(\mathbf{x})\} \\ s.t.\ \ \mathbf{x} &\in X \end{aligned} \tag{1.1}$$

where $X \subseteq \mathbb{R}^m$ is the ***feasible set*** *– also called* ***decision space*** *– and $f_i : X \mapsto \mathbb{R}$, $i = 1, 2, \dots, n$ are real-valued functions, is called a* ***Multicriteria Optimization Problem*** *or MOP.*

Since we are confronted with multiple criteria, there exists in general no feasible solution that minimizes all criteria. Thus there is no optimal decision. We could find, for example, a solution $\mathbf{x}$ that minimizes $f_1(\mathbf{x})$ but performs badly in all other criteria.

We are comparing feasible solutions with respect to their outcomes in each objective function. Thus it is convenient to define the *outcome space*.

Definition 1.2. *Consider the MOP in Definition 1.1. We define the* ***set of all image points*** *– or* ***outcome space*** *– Z as follows:*

$$Z = \{\mathbf{z} \in \mathbb{R}^n : \mathbf{z} = \mathbf{f}(\mathbf{x}),\ \mathbf{x} \in X\} \tag{1.2}$$

where $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))^T$.

Thus we try to solve the problem $\min\{\mathbf{z} : \mathbf{z} \in Z \subseteq \mathbb{R}^n\}$. However the meaning of min has to be yet specified since there is no standard total order over the image space $\mathbb{R}^n$.

Definition 1.3. *A binary relation $\preceq$ is called a* ***partial order*** *over $\mathbb{R}^n$ if for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ we have that:*

- $\mathbf{a} \preceq \mathbf{a}$ *(reflexivity);*
- *if* $\mathbf{a} \preceq \mathbf{b}$ *and* $\mathbf{b} \preceq \mathbf{a}$ *then* $\mathbf{a} = \mathbf{b}$ *(antisymmetry);*
- *if* $\mathbf{a} \preceq \mathbf{b}$ *and* $\mathbf{b} \preceq \mathbf{c}$ *then* $\mathbf{a} \preceq \mathbf{b}$ *(transitivity)*

If in addition $\preceq$ satisfies that for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ either $\mathbf{a} \preceq \mathbf{b}$ or $\mathbf{b} \preceq \mathbf{a}$ then $\preceq$ is called a ***total order*** *over $\mathbb{R}^n$.*

Given an arbitrary total order relation $\preceq$ over $\mathbb{R}^n$ and a set $M \subseteq \mathbb{R}^n$, the vector $\mathbf{a} \in \mathbb{R}^n$ is called *minimal* or a *minimizer* w.r.t. $\preceq$ in M if $\mathbf{a} \in M$ and $\mathbf{a} \preceq \mathbf{b}$ for all $\mathbf{b} \in M$.

However, a total order is not suitable for a MOP because, by minimizing this total order over Z, we will obtain a single point in the outcome space and thus we are forced to make a compromise *a priori* without exploring other possible minimizers for different total orders. Therefore in multicriteria optimization we need to use a partial order, thus we define the concept of nondomination which is more general than minimization.

Given an arbitrary order relation $\preceq$ over $\mathbb{R}^n$ and a set $M \subseteq \mathbb{R}^n$, we say that the vector $\mathbf{a} \in \mathbb{R}^n$ *dominates* $\mathbf{b} \in \mathbb{R}^n$ if $\mathbf{a} \preceq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$. Moreover, we say $\mathbf{a}$ is *nondominated* in M if there is no $\mathbf{c} \in M, \mathbf{c} \neq \mathbf{a}$ such that $\mathbf{c} \preceq \mathbf{a}$.

We want to find nondominated points in the outcome space with respect to a partial order. And we want to define this partial order in a way that all relevant points are considered. For example, let $\mathbf{x}^1, \mathbf{x}^2 \in X$ and $f_i(\mathbf{x}^1) \leq f_i(\mathbf{x}^2)$ for all $i = 1, 2, \ldots, n$ then obviously $\mathbf{x}^2$ is not an efficient decision. Thus we want to define a partial order such that $\mathbf{z}^1 = \mathbf{f}(\mathbf{x}^1) \preceq \mathbf{z}^2 = \mathbf{f}(\mathbf{x}^2)$ if $f_i(\mathbf{x}^1) \leq f_i(\mathbf{x}^2)$ for all $i = 1, 2, \ldots, n$.

Definition 1.4. *Let $\mathbf{z}^1, \mathbf{z}^2 \in \mathbb{R}^n$ be two vectors, whose components are denoted by subscripts, i.e. $\mathbf{z}^1 = (z^1_1, \ldots, z^1_n)^T$ Then we write:*

- $\mathbf{z}^1 \leqq \mathbf{z}^2$ *if* $z^1_i \leq z^2_i\ \forall i \in \{1, 2, \ldots, n\}$.
- $\mathbf{z}^1 \leq \mathbf{z}^2$ *if* $\mathbf{z}^1 \leqq \mathbf{z}^2$ *and* $\mathbf{z}^1 \neq \mathbf{z}^2$.
- $\mathbf{z}^1 < \mathbf{z}^2$ *if* $z^1_i < z^2_i\ \forall i \in \{1, 2, \ldots, n\}$.

The symbols $>$, $\geq$, $\geqq$ are defined analogously.

Remark 1.5. *The binary relation* $\mathbf{z}^1 \leqq \mathbf{z}^2$ *for* $\mathbf{z}^1, \mathbf{z}^2 \in \mathbb{R}^n$ *as defined in Definition 1.4 is a partial order over* $\mathbb{R}^n$*, i.e. it is reflexive, antisymmetric and transitive.*

The partial order $\mathbf{z}^1 \leqq \mathbf{z}^2$ is closely related with the set $\mathbb{R}^n_{\leqq} := \{\mathbf{z} \in \mathbb{R}^n : \mathbf{z} \leqq \mathbf{0}\}$. Since for any $\mathbf{z}^1, \mathbf{z}^2 \in \mathbb{R}^n$, $\mathbf{z}^1 \leqq \mathbf{z}^2$ iff $\mathbf{z}^1 \in \mathbf{z}^2 + \mathbb{R}^n_{\leqq}$. Thus it is possible to use the set $\mathbb{R}^n_{\leqq}$ to define the partial order $\mathbf{z}^1 \leqq \mathbf{z}^2$.

Definition 1.6. *A set* $C \subseteq \mathbb{R}^n$ *is called a* ***cone*** *if for all* $\mathbf{u} \in C, \alpha > 0$ *then* $\alpha\mathbf{u} \in C$.

If C *is a convex cone and* $\mathbf{0} \in C$ *then* $\mathbf{0}$ *is the only point that may not be expressed like a convex combination of other elements in* C*. If this happens then* $\mathbf{0}$ *is an extreme point of* C *and* C *is called an acute convex cone.*

An acute convex cone which is used to define a partial order like $\mathbb{R}^n_{\leqq}$ is called an “ordering cone”.

Definition 1.7. *We define* $\mathbb{R}^n_{\leqq} := \{\mathbf{z} \in \mathbb{R}^n : \mathbf{z} \leqq \mathbf{0}\}$ *and* $\mathbb{R}^n_{<}, \mathbb{R}^n_{\geqq}, \mathbb{R}^n_{>}$ *analogously.*

Using the partial order $\leqq$ we seek for points in the outcome space that are nondominated and their corresponding points in the decision space which we call “efficient” solutions. These efficient solutions are points in the decision space whose evaluation in the outcome space is such that there exists no other decision which performs better in all criteria.

1.2 The Nondominated Set

In this section we define the *Nondominated Set*. The nondominated set is the set of relevant outcomes, those outcomes that are worth to consider. These relevant outcomes result from evaluating the so called “efficient” solutions.

Definition 1.8 (Nondominated Set)**.** *Consider the MOP in Definition 1.1. We define the* ***set of all nondominated points*** $N(Z)$ *and the* ***set of all efficient solutions*** $E(X)$ *as follows:*

$$\begin{aligned} N(Z) &= \{\mathbf{z} \in Z : \nexists \bar{\mathbf{z}} \in Z \ s.t.\ \bar{\mathbf{z}} \leq \mathbf{z}\} \\ E(X) &= \{\mathbf{x} \in X : \mathbf{z} = \mathbf{f}(\mathbf{x}), \mathbf{z} \in N(Z)\} \end{aligned} \tag{1.3}$$

where $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))^T$.

Notice that $\mathbf{z} \in N(Z)$ *iff* $Z \cap (\mathbf{z} + \mathbb{R}^n_{\leqq}) = \{\mathbf{z}\}$*, because* $\bar{\mathbf{z}} \leq \mathbf{z}$ *iff* $\bar{\mathbf{z}} \in (\mathbf{z} + \mathbb{R}^n_{\leqq})$.

The nondominance property of a point $\mathbf{z} \in Z$ implies that there exist no other point $\bar{\mathbf{z}} \in Z$ such that $\bar{\mathbf{z}} \leqq \mathbf{z}$ and $\bar{\mathbf{z}} \neq \mathbf{z}$, i.e. having a nondominated point means that there exists no other point which is better in all criteria. On the other hand, we call a point $\mathbf{z} \in Z$ dominated if there exists another point $\bar{\mathbf{z}} \in Z$ which performs better in all objective functions, i.e. $\bar{\mathbf{z}} \leq \mathbf{z}$, in this case we say that $\bar{\mathbf{z}}$ dominates $\mathbf{z}$.

In this paper we continue to use $\mathbb{R}^n_{\leqq}$ as our ordering cone in the outcome space, i.e. $\bar{\mathbf{z}} \leqq \mathbf{z}$ iff $\bar{\mathbf{z}} \in (\mathbf{z} + \mathbb{R}^n_{\leqq})$, and thus the resulting nondominated set is also known in economic sciences as the *Pareto Set*.

In a MOP we strive to find nondominated points, therefore it is important to know under which conditions the existence of $N(Z)$ is guaranteed.

Since we have that $N(Z) \subseteq Z$, we assume $Z \neq \emptyset$. Otherwise X is empty and thus Problem (1.1) is not feasible. We also assume that $f_i(\mathbf{x}) \geq u_i$ for all $\mathbf{x} \in X$ and all $i \in \{1, 2, \ldots, n\}$, i.e. each objective function is bounded from below in X (the feasible set).

The assumption that each criterion is bounded from below is not restrictive concerning real models because in a real model we do not expect a criterion to get arbitrarily good. Feasibility and boundness are sufficient to guarantee the existence of an optimal solution in a linear problem, but they do not guarantee the existence of an optimal solution in a nonlinear program nor the existence of a nondominated point in a MOP.

Example 1. *Consider the following problem.*

$$\begin{aligned} \min \{z_1 = x_1 + 1\} \\ \min \{z_2 = x_1 x_2\} \\ s.t. \;\; \mathbf{x} \in X \end{aligned} \tag{1.4}$$

where $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1 x_2 \geq 1\}$.

In this problem $X \neq \emptyset$ *and thus* $Z \neq \emptyset$*, e.g.* $(1, 1) \in X$ *and* $(2, 1) \in Z$*, moreover* $z_1 = x_1 + 1 \geq 1$ *and* $z_2 = x_1 x_2 \geq 1$*. Furthermore, we have* X *closed.*

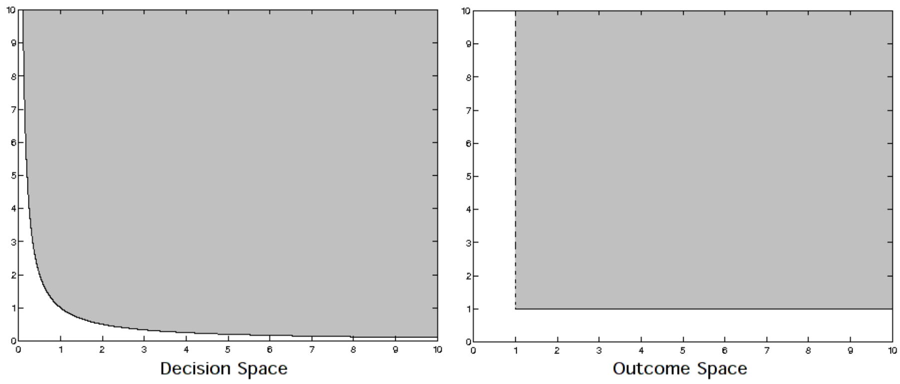

Figure 1.1: The decision space X and the outcome space Z of the MOP (1.4). The outcome space is not closed and all image points are dominated.

However, as seen in Figure 1.1, $N(Z) = \emptyset$ *and therefore feasibility, objective function boundness and decision space closeness do not guarantee the existence of the nondominated set.*

If Z is open then all points in Z are dominated, therefore in order to guarantee the existence of $N(Z)$ we could assume that Z is closed. However, it is sufficient for our concerns that the set $Z + \mathbb{R}^n_{\geqq}$ is closed, this follows from the result of the next lemma.

Lemma 1.9. *Let* Z *and* $N(Z)$ *be defined as in Definition 1.2 and Definition 1.8 respectively. We define* $\bar{Z} = Z + \mathbb{R}^n_{\geqq}$*. Then* $N(Z) = N(\bar{Z})$*.*

Proof. Result is trivial if $Z = \emptyset$.

To prove that $N(Z) \subseteq N(\bar{Z})$, let $\mathbf{z} \in N(Z) \subseteq Z$. Now we suppose for the sake of contradiction that $\mathbf{z} \notin N(\bar{Z})$. It follows that $\exists \bar{\mathbf{z}} \in Z, \mathbf{r} \in \mathbb{R}^n_{\geqq}$ s.t. $\bar{\mathbf{z}} + \mathbf{r} \leq \mathbf{z}$ and thus $\bar{\mathbf{z}} \leq \mathbf{z}$ and $\mathbf{z} \notin N(Z)$. Contradiction!

Conversely let $\bar{\mathbf{z}} \in N(\bar{Z})$, then $\bar{\mathbf{z}} = \mathbf{z} + \mathbf{r},\ \mathbf{z} \in Z,\ \mathbf{r} \in \mathbb{R}^n_{\geqq}$. It follows from $\bar{\mathbf{z}} \in N(\bar{Z})$ that $\mathbf{r} = 0$, since otherwise $\mathbf{z} \leq \bar{\mathbf{z}}$ and this would imply that $\bar{\mathbf{z}} \notin N(\bar{Z})$. Therefore $\bar{\mathbf{z}} \in Z$ as well.

Now suppose $\bar{\mathbf{z}} \notin N(Z)$ then $\exists \hat{\mathbf{z}} \in Z$ s.t. $\hat{\mathbf{z}} \leq \bar{\mathbf{z}}$ and $\hat{\mathbf{z}} + \mathbf{0} \leq \bar{\mathbf{z}}, (\hat{\mathbf{z}} + \mathbf{0}) \in \bar{Z}$ and thus $\bar{\mathbf{z}} \notin N(\bar{Z})$. Contradiction! □

Thus, the nondominated set of Z is identical as the nondominated set of $Z + \mathbb{R}^n_{\geqq}$. This fact will be very important in chapter 3, but for now it helps us to consider sets that are $\mathbb{R}_{\geqq}$-closed, i.e. sets $Z \subseteq \mathbb{R}^n$ such $Z + \mathbb{R}^n_{\geqq}$ is closed, instead of closed sets Z. Since all closed sets are $\mathbb{R}_{\geqq}$-closed then $\mathbb{R}_{\geqq}$-closeness of Z is a weaker condition than closeness of Z.

All these conditions on the structure of Z to guarantee the existence of $N(Z)$ can be summarized in the following assumption.

Assumption 1.10. *We assume that Z is non empty and that it is $\mathbb{R}_{\geqq}$-closed, i.e. $Z + \mathbb{R}^n_{\geqq}$ is closed. Furthermore we assume the existence of $\mathbf{u} \in \mathbb{R}^n$ such $Z - \mathbf{u} \subseteq \mathbb{R}^n_{\geqq}$, i.e. $\mathbf{z} \geq \mathbf{u}$ for all $\mathbf{z} \in Z$.*

Assumption 1.10 is sufficient to guarantee the existence of $N(Z)$.

Theorem 1.11 (Existence of the Nondominated Set)**.** *Let Z and $N(Z)$ be defined as in Definition 1.2 and Definition 1.8. If $Z \neq \emptyset$ and $\exists \hat{\mathbf{z}} \in Z$ such that $(\hat{\mathbf{z}} + \mathbb{R}^n_{\leqq}) \cap Z$ is compact then $N(Z) \neq \emptyset$.*

Proof. See [Borwein, 1983]. □

The basic idea of the previous theorem is to use the compactness of $(\hat{\mathbf{z}}+\mathbb{R}^n_{\leqq})\cap Z$ to guarantee the existence of $\mathbf{z}^1$, the solution for the problem $\min\{z_1 : \mathbf{z} \in (\hat{\mathbf{z}}+\mathbb{R}^n_{\leqq})\cap Z\}$. If $\mathbf{z}^1$ is the unique solution then we obtain a nondominated point. If this is not the case we consider the problem $\min\{z_2 : \mathbf{z} \in (\hat{\mathbf{z}}+\mathbb{R}^n_{\leqq})\cap Z, z_1 = z_1^1\}$, and so on until we obtain a unique solution.

The unique solution of the last procedure $\mathbf{z}$ is a nondominated point in $(\hat{\mathbf{z}} + \mathbb{R}^n_{\leqq}) \cap Z$ and thus is also nondominated in Z. If a point $\bar{\mathbf{z}} \in Z$ such that $\bar{\mathbf{z}} \leq \mathbf{z}$ existed, it would follow that $\bar{\mathbf{z}} \in (\hat{\mathbf{z}} + \mathbb{R}^n_{\leqq}) \cap Z$; which leads to a contradiction.

Note that $\mathbf{z}$ is the minimizer w.r.t. a total order called *lexicographical order* in Z. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, we say $\mathbf{a} \leq_{lex} \mathbf{b}$ if there exists $0 < m \leq n$ such that $a_i = b_i$ for all $i < m$ and $a_m < b_m$ or $\mathbf{a} = \mathbf{b}$. A minimizer w.r.t. a lexicographical order in Z is a nondominated point w.r.t. $\leqq$.

From Theorem 1.11 and Assumption 1.10 we get the following corollary.

Corollary 1.12. *Assumption 1.10, Lemma 1.9 and Theorem 1.11 guarantee the existence of $N(Z)$.*

Proof. Let Z be non empty and $\mathbb{R}_{\geqq}$-closed, and assume that there exists $\mathbf{u} \in \mathbb{R}^n$ such that $Z - \mathbf{u} \subseteq \mathbb{R}^n_{\geqq}$. Take $\hat{\mathbf{z}} \in Z$ and consider the set $(\hat{\mathbf{z}} + \mathbb{R}^n_{\leqq}) \cap (Z + \mathbb{R}^n_{\geqq})$. Both sets are closed and therefore their intersection is also closed. Moreover $(\hat{\mathbf{z}} + \mathbb{R}^n_{\leqq}) \cap (Z + \mathbb{R}^n_{\geqq})$ is bounded, i.e. $\mathbf{u} \leqq \mathbf{z} \leqq \hat{\mathbf{z}}$ for all $\mathbf{z} \in (\hat{\mathbf{z}} + \mathbb{R}^n_{\leqq}) \cap (Z + \mathbb{R}^n_{\geqq})$. By Theorem 1.11, $N(Z + \mathbb{R}^n_{\geqq})$ is non empty, and by Lemma 1.9 we finally get $N(Z) \neq \emptyset$. □

Now that we got sufficient conditions for the existence of $N(Z)$, our next concern is to define bounds for the nondominated set. We will define points $\mathbf{z}^*, \mathbf{z}^\times \in \mathbb{R}^n$ such that $\mathbf{z}^* \leqq \mathbf{z} \leqq \mathbf{z}^\times$ for all $\mathbf{z} \in N(Z)$. First we define the so called *ideal point*.

Definition 1.13. *The point* $\mathbf{z}^* \in \mathbb{R}^n$ *with:*

$$z_i^* = \min\{z_i : \mathbf{z} \in Z\} \quad i = 1, \ldots, n \tag{1.5}$$

is called the ***ideal point****. Notice that* $\mathbf{z}^*$ *is a tight lower bound for all* $\mathbf{z} \in Z$.

Note that if $\mathbf{z}^*$ were in the outcome space, then it would represent the ideal course of action. Since $\mathbf{z}^* \leqq \mathbf{z}\ \forall \mathbf{z} \in Z$ by definition, $(Z - \mathbf{z}^*) \subseteq \mathbb{R}^n_{\geqq}$. Moreover, since $N(Z) \subseteq Z$, we have that $\mathbf{z}^* \leqq \mathbf{z}\ \forall \mathbf{z} \in N(Z)$.

On the other hand, the *nadir point* – if it exists – is defined such that it represents an upper bound for all points $\mathbf{z} \in N(Z)$.

Definition 1.14. *The point* $\mathbf{z}^\times \in \mathbb{R}^n$ *with:*

$$z_i^\times = \max\{z_i : \mathbf{z} \in N(Z)\} \quad i = 1, \ldots, n \tag{1.6}$$

is called the ***nadir point****.*

Nondominated set bounds are needed for the proper implementation of the algorithms presented in Chapter 3. Since the ideal and nadir points are tight bounds for $N(Z)$, it is convenient to restrict our search for nondominated points to the cuboid defined by $(\mathbf{z}^* + \mathbb{R}^n_{\geqq}) \cap (\mathbf{z}^\times - \mathbb{R}^n_{\geqq})$.

Weighted sum scalarization, among other scalarization methods, is used to generate nondominated points. However, due to the geometry of the nondominated set, scalarization techniques are either not sufficient to compute all nondominated points or sufficient but not restricted to nondominated points.

1.3 Weakly and Properly Nondominated Sets

First we consider another set of points which are dominated, but that results in virtually all scalarization techniques that are sufficient to compute nondominated points.

Definition 1.15. *We define the* ***set of weakly nondominated solutions*** $N_w(Z)$ *as:*

$$N_w(Z) = \{\mathbf{z} \in Z : \nexists \tilde{\mathbf{z}} \in Z \ s.t.\ \tilde{\mathbf{z}} < \mathbf{z}\} \tag{1.7}$$

A point in the outcome space $\mathbf{z}$ is weakly nondominated iff $(\mathbf{z} + (\mathbb{R}^n_{<} \cup \{\mathbf{0}\})) \cap Z = \{\mathbf{z}\}$. Clearly $(\mathbf{z} + (\mathbb{R}^n_{<} \cup \{\mathbf{0}\})) \subseteq (\mathbf{z} + \mathbb{R}^n_{\leqq})$, thus if $\mathbf{z} \in N(Z)$ then $\mathbf{z} \in N_w(Z)$, i.e. $N(Z) \subseteq N_w(Z)$. However, a weakly nondominated point is not necessarily nondominated. Consider again Example 1, in this example $N(Z) = \emptyset$ while $N_w(Z) = \{\mathbf{z} \in \mathbb{R}^2 : z_1 > 1, z_2 = 1\}$.

Lemma 1.16. *Let Z and $N_w(Z)$ be defined as in Definition 1.2 and 1.15 respectively. We define $\bar{Z} = Z + (\mathbb{R}^n_> \cup \{\mathbf{0}\})$. Then $N_w(Z) = N_w(\bar{Z})$.*

Proof. Result is trivial if $Z = \emptyset$.

To prove that $N_w(Z) \subseteq N_w(\bar{Z})$ let $\mathbf{z} \in N_w(Z)$. Now suppose for the sake of contradiction that $\mathbf{z} \notin N_w(\bar{Z})$. It follows that $\exists \mathbf{z}' \in Z, \mathbf{r} \in (\mathbb{R}^n_> \cup \{\mathbf{0}\})$ such that $\mathbf{z}' + \mathbf{r} < \mathbf{z}$ thus $\mathbf{z}' < \mathbf{z}$ and $\mathbf{z} \notin N_w(Z)$. Contradiction!

Conversely, to prove $N_w(\bar{Z}) \subseteq N_w(Z)$ let $\bar{\mathbf{z}} \in N_w(\bar{Z})$, then $\bar{\mathbf{z}} = \mathbf{z} + \mathbf{r}, \mathbf{z} \in Z, \mathbf{r} \in (\mathbb{R}^n_> \cup \{\mathbf{0}\})$. It follows from $\bar{\mathbf{z}} \in N_w(\bar{Z})$ that $\mathbf{r} = \mathbf{0}$ and then $\bar{\mathbf{z}} \in Z$ as well.

Now suppose $\bar{\mathbf{z}} \notin N_w(Z)$ then $\exists \mathbf{z}' \in Z$ such that $\mathbf{z}' < \bar{\mathbf{z}}$, but then $\mathbf{z}' + \mathbf{0} < \bar{\mathbf{z}}, (\mathbf{z}' + \mathbf{0}) \in \bar{Z}$ and therefore $\bar{\mathbf{z}} \notin N_w(\bar{Z})$. Contradiction! □

We need to be careful if we sum the whole positive orthant to the set Z and then try to obtain the weakly nondominated set of Z, i.e. $N_w(Z)$, using $N_w(Z + \mathbb{R}^n_{\geqq})$.

Lemma 1.17. *Let Z and $N_w(Z)$ be defined as in Definition 1.2 and 1.15 respectively. Then $N_w(Z) \subseteq N_w(Z + \mathbb{R}^n_{\geqq})$.*

Proof. Result is trivial if $Z = \emptyset$.

Let $\mathbf{z} \in N_w(Z)$, suppose $\mathbf{z} \notin N_w(Z + \mathbb{R}^n_{\geqq})$ then $\exists \hat{\mathbf{z}} \in Z, \mathbf{r} \in \mathbb{R}^n_{\geqq}$ such that $\hat{\mathbf{z}} + \mathbf{r} < \mathbf{z}$, but then it follows $\hat{\mathbf{z}} < \mathbf{z}$ and $\mathbf{z} \notin N_w(Z)$. Contradiction! □

In general $N_w(Z + \mathbb{R}^n_{\geqq}) \nsubseteq N_w(Z)$. We can easily visualize this by taking the singleton $Z = \{\mathbf{0}\}$. It is obvious that $N_w(Z) = \{\mathbf{0}\}$. However $N_w(\mathbb{R}^n_{\geqq}) = \{\alpha\mathbf{z} : \mathbf{z} = \mathbf{e}^i, \alpha \geq 0, i = 1, 2, \ldots, n\}$, where:

$$e^i_j = \begin{cases} 0 & \text{if } j \neq i \\ 1 & \text{if } j = i \end{cases}$$

and therefore $\mathbf{e}^i, i = 1, 2, \ldots, n$ is the usual unit vector basis of $\mathbb{R}^n$.

We will see in the next section that when we are able to generate all nondominated points via weighted scalarization we may run into a point in $N_w(Z) \setminus N(Z)$. On the other hand, it is possible to exclude all points in $N_w(Z) \setminus N(Z)$ from our weighted scalarization; however, in this case, scalarization is not sufficient to generate the whole nondominated set.

If we exclude weakly nondominated solutions, weighted scalarization only considers points $\mathbf{z} \in N(Z)$ that have a bounded *partial tradeoff* w.r.t. all other image points.

When we compare two different nondominated points $\mathbf{z}^1, \mathbf{z}^2 \in N(Z)$ there are at least two components $i, j \in \{1, 2, \ldots, n\}, i \neq j$ such that $z^2_i < z^1_i$ and $z^2_j > z^1_j$. This condition – improvement in some criteria resulting in worsening on some other criteria – is known as *tradeoff* between two nondominated solutions. In this case, the fraction:

$$\frac{z^1_i - z^2_i}{z^2_j - z^1_j} \tag{1.8}$$

which is the ratio between the improvement and the worsening, is known as *partial tradeoff* from $\mathbf{z}^1$ w.r.t. $\mathbf{z}^2$.

Points $\mathbf{z} \in N(Z)$ whose partial tradeoff, compared to all other image points, is measurable are called properly nondominated points. The set of properly nondominated solutions is defined according to Geoffrion [1968].

Definition 1.18. *A point $\bar{\mathbf{z}} \in N(Z)$ is called* ***properly nondominated*** *if there exists $M > 0$ such that for each $i = 1, 2, \ldots, n$ and each $\mathbf{z} \in Z$ that satisfies $z_i < \bar{z}_i$ there is a j such that $z_j > \bar{z}_j$ and:*

$$\frac{\bar{z}_i - z_i}{z_j - \bar{z}_j} \leq M \tag{1.9}$$

Otherwise $\bar{\mathbf{z}} \in N(Z)$ is called ***improperly nondominated****. The set of all properly nondominated points is denoted by $N_p(Z)$.*

Properly nondominated points are nondominated points which have a measurable partial tradeoff, i.e. worsening a criterion from a properly nondominated point $\bar{\mathbf{z}}$ would improve another at most by a fixed ratio w.r.t. any other $\mathbf{z} \in Z$.

Example 2. *Consider the following MOP.*

$$\begin{aligned} \min\ & \{z_1 = x_1\} \\ \min\ & \{z_2 = x_2\} \\ s.t.\ & \mathbf{x} \in X \end{aligned} \tag{1.10}$$

where $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq -1, x_2 \leq 0, x_1 x_2 \leq 1\}$.

We have that $z_2 = x_2 \geq -1$ so the second objective is bounded from below; however the first criterion is not bounded.

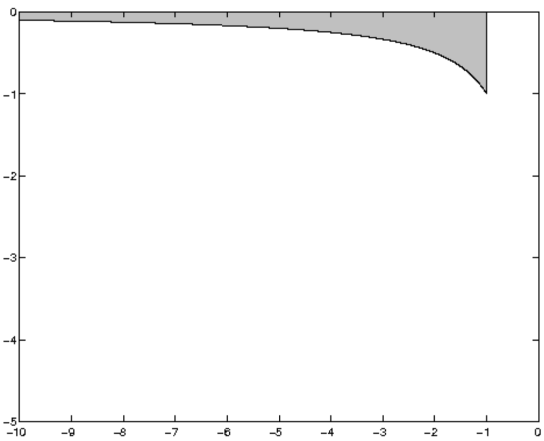

Figure 1.2: In the outcome space Z of the MOP (1.10), the first criterion is not bounded from below. In this MOP, all nondominated points are improperly nondominated, i.e. $N_p(Z) = \emptyset$.

In this example, $N(Z) = \{(z_1, z_2) \in \mathbb{R}^2 : z_1 \leq -1, z_2 = 1/z_1\}$. Let $\bar{\mathbf{z}}$ be a nondominated point, i.e. $\bar{\mathbf{z}} = (\bar{z}_1, 1/\bar{z}_1)$ for $\bar{z}_1 \leq -1$. Suppose that $\bar{\mathbf{z}} \in N_p(Z)$, then $\exists M > 0$ such that for any $\mathbf{z} \in Z$ such that $z_1 < \bar{z}_1$ we have:

$$\frac{\bar{z}_1 - z_1}{z_2 - \frac{1}{\bar{z}_1}} \leq M \tag{1.11}$$

Let $\mathbf{z} = (M/\bar{z}_1 + \bar{z}_1 + 1/\bar{z}_1, 0) \in Z$ *and substitute* $\mathbf{z}$ *in* (1.11)*. It follows:*

$$\frac{\bar{z}_1 - \frac{M}{\bar{z}_1} - \bar{z}_1 - \frac{1}{\bar{z}_1}}{0 - \frac{1}{\bar{z}_1}} = M + 1 \leq M \tag{1.12}$$

Contradiction! Therefore $\bar{\mathbf{z}} \notin N_p(Z)$ *for any* $\bar{\mathbf{z}} \in N(Z)$ *and thus* $N_p(Z) = \emptyset$*.*

In the previous example, it is possible to improve one objective by any arbitrarily big amount while worsening another criterion by only a fixed quantity, from any nondominated point. Thus the partial tradeoff from all $\mathbf{z} \in N(Z)$ is unbounded and therefore $N_p(Z) = \emptyset$.

The fraction (1.9) may be unbounded even in the case of bounded criteria. Consider the following example.

Example 3. *Consider the following MOP.*

$$\begin{aligned} &\min \{z_1 = x_1\} \\ &\min \{z_2 = x_2\} \\ &s.t. \quad \mathbf{x} \in X \end{aligned} \tag{1.13}$$

where $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$*.*

In this problem both objectives are bounded from below, i.e $z_1 \geq -1$ *and* $z_2 \geq -1$*. Furthermore,* $N(Z) = \{(z_1, z_2) \in \mathbb{R}^2 : z_2 = -\sqrt{1 - z_1^2}, -1 \leq z_1 \leq 0\}$*.*

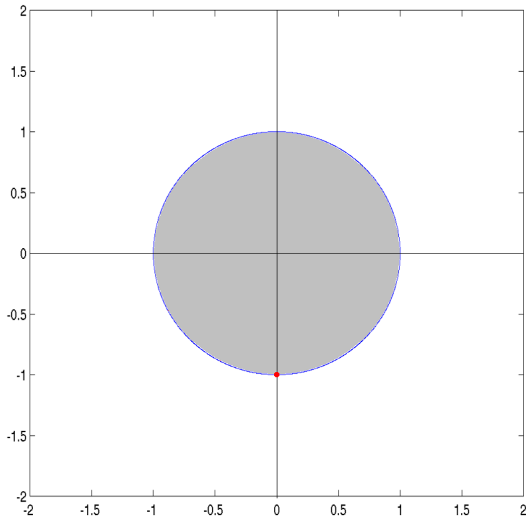

Figure 1.3: The outcome space Z of the MOP (1.13). The point $(0, -1)$ is improperly nondominated.

Let $\bar{\mathbf{z}}$ *be the nondominated point* $(0, -1)$*. Take a sequence of image points* $\{\mathbf{z}^k = (z_1^k, -\sqrt{1 - {z_1^k}^2})\}_{k \in \mathbb{N}}$ *such that:*

$$\lim_{k \to \infty} z_1^k = 0 \tag{1.14}$$

Take the limit of the partial tradeoff from $\bar{\mathbf{z}}$ *w.r.t. the sequence* $\{\mathbf{z}^k\}_{k\in\mathbb{N}}$.

$$\lim_{k\to\infty} \frac{-z_1^k}{-z_2^k+1} = \frac{1}{\left.\frac{d(-\sqrt{1-y^2})}{dy}\right|_{y=0}} = \infty \tag{1.15}$$

therefore $\bar{\mathbf{z}} \notin N_p(Z)$.

Benson presented an equivalent definition for properly nondominated solutions, but before we state Benson's theorem we need to introduce the following definition.

Definition 1.19. *The* conical hull $cone(U)$ *for* $U \subseteq \mathbb{R}^n$ *is defined as:*

$$cone(U) := \{\alpha\mathbf{u} : \alpha \geq 0, \mathbf{u} \in U\}$$

Thus, $cone(U)$ *is the smallest cone* C *such that* $U \subseteq C$.

Now we state Benson's definition of properly nondominance.

Theorem 1.20. *A solution* $\mathbf{z} \in Z$ *is properly nondominated if and only if*

$$\overline{cone(Z + \mathbb{R}^n_{\geqq} - \mathbf{z})} \cap \mathbb{R}^n_{\leqq} = \{\mathbf{0}\} \tag{1.16}$$

Proof. See [Benson, 1979]. □

Benson's definition for proper nondominance considers one point $\mathbf{z} \in Z$ and all other points $\bar{\mathbf{z}} \in Z + \mathbb{R}^n_{\geqq}$, then includes all $\alpha(\bar{\mathbf{z}} - \mathbf{z}), \alpha \geq 0$ in a cone C. Finally, it takes the closure of C and intersects it with the negative orthant. If the intersection consists of just the $\mathbf{0}$ vector, then it is possible to extend an acute convex closed cone which includes the negative orthant in its interior and whose intersection with $\overline{cone(Z + \mathbb{R}^n_{\geqq} - \mathbf{z})}$ is still the $\mathbf{0}$ vector; due to the fact that $\overline{cone(Z + \mathbb{R}^n_{\geqq} - \mathbf{z})}$ is closed by definition.

This implies that the improvement obtained in other points is at most directly proportional to the worsening in any objective, which corresponds to the Geoffrion's definition.

Henig [1982] provides yet another equivalent definition for a properly nondominated solution, which will be useful when we look at the properties of the so called *block norms* in Chapter 2.

Theorem 1.21. *A point vector* $\mathbf{z}$ *is properly nondominated iff there exists a convex cone* C *with* $\mathbb{R}^n_{\leq} \subseteq int(C)$ *so that:*

$$(Z - \mathbf{z}) \cap C = \{\mathbf{0}\} \tag{1.17}$$

Note that $\mathbf{0} \in C, C \neq \mathbb{R}^n$ *and the last equation can be rewritten as:*

$$(\mathbf{z} + C) \cap Z = \{\mathbf{z}\} \tag{1.18}$$

Proof. See [Henig, 1982]. □

We have all the definitions and results we need to continue into the next section where we discuss the weighted sum scalarization method to compute nondominated solutions.

1.4 Weighted sum scalarization

In this section we look at the *weighted sum scalarization* method as a way to obtain nondominated points and its sufficiency to generate the whole nondominated set of a $\mathbb{R}_{\geqq}$-convex body.

Convexity and closeness of $Z \subseteq \mathbb{R}^n$ allow us to find any point $\mathbf{z} \in \partial Z$ via a single objective optimization problem. This is a consequence of the *separation theorem.*

Theorem 1.22 (Separation Theorem). *Let A, B be convex non-empty subsets of $\mathbb{R}^n$ with empty intersection, i.e. $A \cap B = \emptyset$.*

1. *If A is open then there exist $\boldsymbol{\lambda} \in \mathbb{R}^n, \boldsymbol{\lambda} \neq \mathbf{0}$ and $\gamma \in \mathbb{R}$ such that*

$$\boldsymbol{\lambda}^T \mathbf{a} > \gamma \geq \boldsymbol{\lambda}^T \mathbf{b}$$

for all $\mathbf{a} \in A$ and $\mathbf{b} \in B$.

2. *If A is compact and B is closed then there exist $\boldsymbol{\lambda} \in \mathbb{R}^n, \boldsymbol{\lambda} \neq \mathbf{0}$ and $\gamma \in \mathbb{R}$ such that*

$$\boldsymbol{\lambda}^T \mathbf{a} > \gamma > \boldsymbol{\lambda}^T \mathbf{b}$$

for all $\mathbf{a} \in A$ and $\mathbf{b} \in B$.

Proof. See Rockafellar [1970], Section 11, Pages 97-101. □

Let $\bar{\mathbf{z}} \in \partial Z$ and Z closed and convex. It follows that $\{\bar{\mathbf{z}}\} \cap int(Z) = \emptyset$ and $int(Z)$ is open by definition. Therefore, by the Separation Theorem, there is $\boldsymbol{\lambda} \in \mathbb{R}^n, \boldsymbol{\lambda} \neq 0$ such that $\boldsymbol{\lambda}^T \mathbf{z} > \boldsymbol{\lambda}^T \bar{\mathbf{z}}$ for all $\mathbf{z} \in int(Z)$. Since $\boldsymbol{\lambda}^T \mathbf{z} : \mathbb{R}^n \mapsto \mathbb{R}$ is a linear continuous mapping, we have $\boldsymbol{\lambda}^T \mathbf{z} \geq \boldsymbol{\lambda}^T \bar{\mathbf{z}}$ for all $\mathbf{z} \in Z$.

Because $N(Z) \subseteq \partial Z$, if Z is closed and convex it is possible to find all nondominated points by solving a single objective minimization problem. However, the outcome space Z is not necessarily convex even for convex X and convex functions $f_i, i = 1, \ldots, n$.

Do not despair! Since $N(Z) = N(Z + \mathbb{R}^n_{\geqq})$, convexity of $Z + \mathbb{R}^n_{\geqq}$ it is sufficient for our concerns.

Definition 1.23. *A set $Z \subseteq \mathbb{R}^n$ is called $\mathbb{R}_{\geqq}$**-convex** if the set $Z + \mathbb{R}^n_{\geqq}$ is convex.*

Furthermore, the image set of convex functions defined in a convex domain is $\mathbb{R}^n_{\geqq}$-convex.

Lemma 1.24. *If X is a convex set and $f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})$ are convex functions then Z defined as in definition 1.2 is $\mathbb{R}_{\geqq}$-convex.*

Proof. Let $\mathbf{z}^1, \mathbf{z}^2 \in (Z + \mathbb{R}^n_{\geqq})$. i.e. $\mathbf{z}^1 = \hat{\mathbf{z}}^1 + \mathbf{r}^1, z^2 = \hat{\mathbf{z}}^2 + \mathbf{r}^2$ with $\hat{\mathbf{z}}^1, \hat{\mathbf{z}}^2 \in Z$ and $\mathbf{r}^1, \mathbf{r}^2 \in \mathbb{R}^n_{\geqq}$. Then there exist $\mathbf{x}^1, \mathbf{x}^2 \in X$ such that $\hat{\mathbf{z}}^1 = \mathbf{f}(\mathbf{x}^1), \hat{\mathbf{z}}^2 = \mathbf{f}(\mathbf{x}^2)$.

For any $\lambda \in [0, 1]$ we know that $\tilde{\mathbf{x}} = \lambda \mathbf{x}^1 + (1 - \lambda)\mathbf{x}^2 \in X$ by convexity of X. Furthermore $\tilde{\mathbf{z}} = \mathbf{f}(\tilde{\mathbf{x}}) \in Z$ and $\tilde{\mathbf{z}} \leqq \lambda \mathbf{f}(\mathbf{x}^1) + (1 - \lambda)\mathbf{f}(\mathbf{x}^2) = \lambda \hat{\mathbf{z}}^1 + (1 - \lambda)\hat{\mathbf{z}}^2$ by convexity of $f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})$.

It follows that there exists $\mathbf{r} \in \mathbb{R}^n_{\geqq}$ such that $\tilde{\mathbf{z}} + \mathbf{r} = \lambda \hat{\mathbf{z}}^1 + (1 - \lambda)\hat{\mathbf{z}}^2$. Therefore $\tilde{\mathbf{z}} + \mathbf{r} + \lambda \mathbf{r}^{\mathbf{1}} + (1 - \lambda)\mathbf{r}^2 = \lambda(\hat{\mathbf{z}}^1 + \mathbf{r}^1) + (1 - \lambda)(\hat{\mathbf{z}}^2 + \mathbf{r}^2) = \lambda \mathbf{z}^1 + (1 - \lambda)\mathbf{z}^2$. Finally $\lambda \mathbf{z}^1 + (1 - \lambda)\mathbf{z}^2 \in (Z + \mathbb{R}^n_{\geqq})$, because $\mathbf{r} + \lambda \mathbf{r}^1 + (1 - \lambda)\mathbf{r}^2 \in \mathbb{R}^n_{\geqq}$. □

$\mathbb{R}_{\geqq}$-convex bodies are especially interesting due to the nice characterization of their nondominated set. This nondominated set can be computed easily by means of weighted scalarization of the functions.

Definition 1.25. *Let $\boldsymbol{\lambda} \in \mathbb{R}^n_{\geq}$. Then:*

$$\begin{aligned} \min \sum_{k=1}^{n} \lambda_k f_k(\mathbf{x}) \\ s.t\ \mathbf{x} \in X \end{aligned} \tag{1.19}$$

is the ***weighted sum scalarization*** *of the Multicriteria Optimization Problem* (1.1).

The weighted scalarization method consists of assigning nonnegative weights to each objective function and then solving a single objective optimization problem. Since the trivial case $\boldsymbol{\lambda} = \mathbf{0}$ is not included in definition 1.25 it follows that w.l.o.g. we can set $\|\boldsymbol{\lambda}\|_1 = \sum \lambda_i = 1$.

Definition 1.26. *For a given $Z \subseteq \mathbb{R}^n$, $\boldsymbol{\lambda} \in \mathbb{R}^n_{\geq}$ we define:*

$$\begin{aligned} S(\boldsymbol{\lambda}, Z) = \{\hat{\mathbf{z}} \in Z : \boldsymbol{\lambda}^T \hat{\mathbf{z}} = \{\min\ \boldsymbol{\lambda}^T \mathbf{z}\ s.t.\ \mathbf{z} \in Z\}\} \\ S(Z) = \bigcup_{\boldsymbol{\lambda} \in \mathbb{R}^n_{>}} S(\boldsymbol{\lambda}, Z) \\ S_0(Z) = \bigcup_{\boldsymbol{\lambda} \in \mathbb{R}^n_{\geq}} S(\boldsymbol{\lambda}, Z) \end{aligned} \tag{1.20}$$

Without loss of generality $\boldsymbol{\lambda}$ satisfies $\|\boldsymbol{\lambda}\|_1 = 1$.

Because $\mathbb{R}^n_{>} \subseteq \mathbb{R}^n_{\geq}$, it is clear from Definition 1.26 that $S(Z) \subseteq S_0(Z)$.

The next lemma establishes the connection between the (weakly) nondominated set and the weighted sum scalarization of an MOP.

Lemma 1.27. *$S_0(Z) \subseteq N_w(Z)$ and $S(Z) \subseteq N(Z)$ for any $Z \subseteq \mathbb{R}^n$.*

Proof. $S_0(Z) \subseteq N_w(Z)$: Take $\hat{\mathbf{z}} \in S_0(Z)$ then $\exists \boldsymbol{\lambda} \geq \mathbf{0}$ such that $\boldsymbol{\lambda}^T \hat{\mathbf{z}} = \min\{\boldsymbol{\lambda}^T \mathbf{z} : \mathbf{z} \in Z\}$.

Suppose $\hat{\mathbf{z}} \notin N_w(Z)$ then $\exists \mathbf{z} \in Z$ s.t. $\mathbf{z} < \hat{\mathbf{z}}$ but this clearly implies $\boldsymbol{\lambda}^T \mathbf{z} < \boldsymbol{\lambda}^T \hat{\mathbf{z}}$. Contradiction!

$S(Z) \subseteq N(Z)$: Let be $\hat{\mathbf{z}} \in S(Z)$, this implies that $\exists \boldsymbol{\lambda} > 0$ such that $\boldsymbol{\lambda}^T \hat{\mathbf{z}} = \min\{\boldsymbol{\lambda}^T \mathbf{z} : \mathbf{z} \in Z\}$.

Suppose $\hat{\mathbf{z}} \notin N(Z)$ then $\exists \mathbf{z} \in Z$ with $\mathbf{z} \leq \hat{\mathbf{z}}$ but, since $\boldsymbol{\lambda} > 0$, then $\boldsymbol{\lambda}^T \mathbf{z} < \boldsymbol{\lambda}^T \hat{\mathbf{z}}$. Contradiction! □

Therefore we can obtain weakly nondominated points by means of nonnegative weights in the weighted scalarization. Analogously, we obtain nondominated points by using positive weights.

It is clear that we are mainly interested in nondominated points; however, as we will see in the next results, in general if we seek to obtain all nondominated points using weighted scalarization, we need to generate weakly nondominated points too.

Theorem 1.28. *$S_0(Z) = N_w(Z)$ if Z is $\mathbb{R}_{\geqq}$-convex.*

Proof. We only need to prove that if Z is $\mathbb{R}_{\geqq}$-convex then $N_w(Z) \subseteq S_0(Z)$. The other inclusion was proved in Lemma 1.27.

Let $\hat{\mathbf{z}} \in N_w(Z)$, we know by Lemma 1.17 that $\hat{\mathbf{z}} \in N_w(Z + \mathbb{R}^n_{\geqq})$. Then it follows that $(Z + \mathbb{R}^n_{\geqq} - \hat{\mathbf{z}}) \cap (\mathbb{R}^n_{<}) = \emptyset$ by Definition 1.15.

$(Z + \mathbb{R}^n_{\geqq} - \hat{\mathbf{z}})$ is convex – due to convexity of $Z + \mathbb{R}_{\geqq}$ –, $\mathbb{R}^n_{<}$ is open and convex. Then we apply the Separation Theorem 1.22 that ensures the existence of $\boldsymbol{\lambda} \in \mathbb{R}^n, \boldsymbol{\lambda} \neq \mathbf{0}$ and $c \in \mathbb{R}$ such that $\boldsymbol{\lambda}^T(\mathbf{z} + \mathbf{d} - \hat{\mathbf{z}}) \geq c \geq \boldsymbol{\lambda}^T \hat{\mathbf{d}}$ for all $\mathbf{z} \in Z, \mathbf{d} \in \mathbb{R}^n_{\geqq}, \hat{\mathbf{d}} \in \mathbb{R}^n_{<}$. By taking $\mathbf{z} = \hat{\mathbf{z}}, \mathbf{d} = \mathbf{0}$ we get that $c \leq 0$.

We have $\boldsymbol{\lambda}^T \hat{\mathbf{d}} \leq 0 \ \forall \hat{\mathbf{d}} < \mathbf{0}$. It is clear that this is only possible if $\boldsymbol{\lambda} \geq \mathbf{0}$. On the other side we have that $\boldsymbol{\lambda}^T \mathbf{z} + \boldsymbol{\lambda}^T \mathbf{d} \geq \boldsymbol{\lambda}^T \hat{\mathbf{z}}$ and from $\boldsymbol{\lambda} \geq \mathbf{0}$ we obtain that $\boldsymbol{\lambda}^T \mathbf{d} \geq 0 \ \forall \mathbf{d} \geqq \mathbf{0}$.

Finally we have $\boldsymbol{\lambda}^T \mathbf{z} \geq \boldsymbol{\lambda}^T \hat{\mathbf{z}} \ \forall \mathbf{z} \in Z$ which implies $\hat{\mathbf{z}} \in S(\boldsymbol{\lambda}, Z)$ and together with $\boldsymbol{\lambda} \geq \mathbf{0}$ we get $\hat{\mathbf{z}} \in S_0(Z)$. □

We know by Definition 1.15 that $N(Z) \subseteq N_w(Z)$. Furthermore, we know from Lemma 1.27 that $S(Z) \subseteq N(Z)$ and then, using Theorem 1.28, we get that $S(Z) \subseteq N(Z) \subseteq S_0(Z)$ when Z is $\mathbb{R}_{\geqq}$-convex.

Corollary 1.29. *If Z is $\mathbb{R}_{\geqq}$-convex then $S(Z) \subseteq N(Z) \subseteq S_0(Z)$.*

Proof. Follows directly from Lemma 1.27, Definition 1.15, and Theorem 1.28. □

Thus in an $\mathbb{R}_{\geqq}$-convex body we can, in principle, compute the whole non-dominated set by means of nonnegative weighted scalarization. However, we will thereby generate points that are weakly nondominated.

Lemma 1.30. *If $\hat{\mathbf{z}}$ is the unique element in $S(\boldsymbol{\lambda}, Z)$, i.e. $S(\boldsymbol{\lambda}, Z) = \{\hat{\mathbf{z}}\}$, for some $\boldsymbol{\lambda} \in \mathbb{R}^n_{\geq}$ then $\hat{\mathbf{z}} \in N(Z)$.*

Proof. Suppose $\hat{\mathbf{z}} \notin N(Z)$ then $\exists \mathbf{z} \in Z$ such that $\mathbf{z} \neq \hat{\mathbf{z}}, \mathbf{z} \leq \hat{\mathbf{z}}$. Then $\boldsymbol{\lambda}^T \mathbf{z} \leq \boldsymbol{\lambda}^T \hat{\mathbf{z}}$ but $\hat{\mathbf{z}} \in S(\boldsymbol{\lambda}, Z)$, thus $\mathbf{z} \in S(\boldsymbol{\lambda}, Z)$. Contradiction! □

If a weighted scalarization with nonnegative weights has a unique solution $\mathbf{z}$, then $\mathbf{z} \in N(Z)$. Furthermore, for all $\boldsymbol{\lambda} \geq \mathbf{0}$ there is at least one $\hat{\mathbf{z}} \in N(Z)$ such that $\hat{\mathbf{z}} \in S(\boldsymbol{\lambda}, Z)$. One $\hat{\mathbf{z}} \in N(Z), \hat{\mathbf{z}} \in S(\boldsymbol{\lambda}, Z)$ can be found by solving a second phase problem, for example $\min\{\sum z_i : \mathbf{z} \in S(\boldsymbol{\lambda}, Z)\}$. Obviously if $\hat{\mathbf{z}}$ is the minimizer of the last problem then $\hat{\mathbf{z}} \in N(Z)$. Otherwise $\exists \bar{\mathbf{z}} \leq \hat{\mathbf{z}}$ but then $\bar{\mathbf{z}} \in S(\boldsymbol{\lambda}, Z)$ and $\sum \bar{z}_i < \sum \hat{z}_i$, contradicting the minimality of $\hat{\mathbf{z}}$.

Because $S(Z) \subseteq N(Z)$, using strictly positive weights in the weighted scalarization avoids weak dominance. However, positive weights do not suffice for $N(Z)$, because $S(Z) \subseteq N_p(Z)$.

Theorem 1.31. *$S(Z) \subseteq N_p(Z)$ for all $Z \subseteq \mathbb{R}^n$.*

Proof. To prove that $S(Z) \subseteq N_p(Z)$ for all $Z \subseteq \mathbb{R}^n$ we use Geoffrion's definition (1.18) of properly nondominated points.

Suppose $\hat{\mathbf{z}} \in S(Z)$, then there exists $\boldsymbol{\lambda} > \mathbf{0}$ such that $\boldsymbol{\lambda}^T \hat{\mathbf{z}} \leq \boldsymbol{\lambda}^T \mathbf{z}$ for all $\mathbf{z} \in Z$. Now we suppose for the sake of contradiction that $\hat{\mathbf{z}} \notin N_p(Z)$.

This would imply that $\exists \mathbf{z}^* \in Z, i \in \{1, 2, \ldots, n\}$ such $z^*_i < \hat{z}_i$ and $\hat{z}_i - z^*_i > M(z^*_j - \hat{z}_j)$ for all $j \neq i$ for any arbitrary $M > 0$, this is true in the case $z^*_j - \hat{z}_j > 0$ because $\hat{\mathbf{z}} \notin N_p(Z)$ and also trivially true if $z^*_j - \hat{z}_j \leq 0$.

Let

$$M := \max_{i,j} \frac{\lambda_j}{\lambda_i}(n-1) > 0 \tag{1.21}$$

Since $\boldsymbol{\lambda} > \mathbf{0}$ then M is well defined. Substituting (1.21) in Definition 1.18 yields a contradiction.

$$\hat{z}_i - z_i^* > \frac{\lambda_j}{\lambda_i}(n-1)(z_j^* - \hat{z}_j) \quad \forall j \neq i \tag{1.22}$$

multiplying each of these equations by $\frac{\lambda_i}{n-1}$ and adding them all we have that

$$\lambda_i(\hat{z}_i - z_i^*) > \sum_{j \neq i} \lambda_j (z_j^* - \hat{z}_j) \tag{1.23}$$

This is equivalent to $\boldsymbol{\lambda}^T \hat{\mathbf{z}} > \boldsymbol{\lambda}^T \mathbf{z}^*$. Contradiction! □

So there are nondominated points that cannot be generated by means of positive weighted scalarization even in the $\mathbb{R}_{\geqq}$-convex case. E.g. take Z to be the circumference with radius 1 centered in the origin, as in Figure 1.3, although $(-1,0)$ and $(0,-1)$ are nondominated points they cannot be obtained via positive weighted scalarization.

As in the case of $N_w(Z)$, where we proved that it is equivalent to the set $S_0(Z)$ if Z is $\mathbb{R}_{\geqq}$-convex, a similar result for $N_p(Z)$ is derived. However, in order to prove the next theorem we need an intermediate result about convex cones.

Lemma 1.32. *Let $C \subseteq \mathbb{R}^n$ be a convex cone. We define $C^\circ := \{\boldsymbol{\mu} \in \mathbb{R}^n : \boldsymbol{\mu}^T \mathbf{y} \geq 0 \ \forall \mathbf{y} \in C\}$. Then C° is also a convex cone, furthermore $C^{\circ\circ} := \{\boldsymbol{\mu} \in \mathbb{R}^n : \boldsymbol{\mu}^T \mathbf{y} \geq 0 \ \forall \mathbf{y} \in C^\circ\} \subseteq \overline{C}$.*

Proof. Let $\boldsymbol{\mu}^1, \boldsymbol{\mu}^2 \in C^\circ$, then it is clear that $\alpha\boldsymbol{\mu}^1 \in C^\circ \ \forall \alpha \geq 0$ and that $\alpha\boldsymbol{\mu}^1 + (1-\alpha)\boldsymbol{\mu}^2 \in C^\circ \ \forall \alpha \in [0,1]$. Thus C° is a convex cone.

Now let $\hat{\mathbf{y}} \notin \overline{C}$ then by the Separation Theorem 1.22 there exists $\boldsymbol{\lambda} \neq \mathbf{0}$ and $c \in \mathbb{R}$ such $\boldsymbol{\lambda}^T \mathbf{y} > c > \boldsymbol{\lambda}^T \hat{\mathbf{y}} \ \forall \mathbf{y} \in \overline{C}$.

We have strict inequality because $\overline{C}$ is closed by definition, the singleton $\{\hat{\mathbf{y}}\}$ is compact and $\{\hat{\mathbf{y}}\} \cap \overline{C} = \emptyset$.

Since $\mathbf{0} \in \overline{C}$, then it follows that $c < 0$ and because $\overline{C}$ is a cone, then $\boldsymbol{\lambda}^T \mathbf{y} \geq 0 \ \forall \mathbf{y} \in \overline{C}$, otherwise we take $\mathbf{y}' \in \overline{C}$ such that $\boldsymbol{\lambda}^T \mathbf{y}' < 0$; but then for $\alpha \to \infty$ we have $\boldsymbol{\lambda}^T \alpha \mathbf{y}' < -M$ for any arbitrary $M > 0$.

Thus, it follows from $\boldsymbol{\lambda}^T \mathbf{y} \geq 0 \ \forall \mathbf{y} \in C$ that $\boldsymbol{\lambda} \in C^\circ$. Furthermore, from $\boldsymbol{\lambda}^T \hat{\mathbf{y}} < 0$ we get $\hat{\mathbf{y}} \notin C^{\circ\circ}$.

So $\hat{\mathbf{y}} \notin \overline{C}$ implies that $\hat{\mathbf{y}} \notin C^{\circ\circ}$ and therefore $C^{\circ\circ} \subseteq \overline{C}$. □

Now we are ready to prove the next theorem.

Theorem 1.33. *$S(Z) = N_p(Z)$ if Z is $\mathbb{R}^n_{\geqq}$-convex.*

Proof. It was proved that $S(Z) \subseteq N_p(Z)$ for any $Z \subseteq \mathbb{R}^n$, so we just need to check that $N_p(Z) \subseteq S(Z)$ if Z is $\mathbb{R}^n_{\geqq}$-convex. Here we use Benson's definition of proper nondominance, i.e. Theorem 1.20.

Let $\hat{\mathbf{z}} \in N_p(Z), C = \overline{cone(Z + \mathbb{R}^n_{\geqq} - \hat{\mathbf{z}})}$. If there exists a $\boldsymbol{\lambda} > \mathbf{0}$ such that $\boldsymbol{\lambda}^T \mathbf{z} \geq 0 \ \forall \mathbf{z} \in C$, then $\boldsymbol{\lambda}^T \mathbf{z} \geq \boldsymbol{\lambda}^T \hat{\mathbf{z}} \ \forall \mathbf{z} \in Z$, because $Z - \hat{\mathbf{z}} \subseteq C$; and thus $\hat{\mathbf{z}} \in S(Z)$.

For the sake of contradiction we suppose that there is no $\boldsymbol{\lambda} \in \mathbb{R}^n_>$ such $\boldsymbol{\lambda}^T \mathbf{z} \geq 0 \ \forall \mathbf{z} \in C$. Thus the intersection of C° as defined in Lemma 1.32 and $\mathbb{R}^n_>$ is empty.

Because C° and $\mathbb{R}^n_>$ are convex sets, the Separation Theorem 1.22 ensures the existence of $\mathbf{y} \neq \mathbf{0}$ and $c \in \mathbb{R}$ such that $\mathbf{y}^T\mathbf{z} \geq c \geq \mathbf{y}^T\mathbf{d} \ \forall \mathbf{z} \in C^\circ, \mathbf{d} \in \mathbb{R}^n_>$.

Because $\mathbf{0} \in C^\circ$, we have that $c \leq 0$ and by the fact that C° is a cone, we get that $\mathbf{y}^T\mathbf{z} \geq 0 \ \forall \mathbf{z} \in C^\circ$. Otherwise, if there was a $\mathbf{z'} \in C^\circ$ such that $\mathbf{y}^T\mathbf{z'} < 0$ then for $\alpha \to \infty$ we would have that $\mathbf{y}^T\alpha\mathbf{z'} < -M$ for any arbitrary $M > 0$.
So $\mathbf{y}^T\mathbf{z} \geq 0 \ \forall \mathbf{z} \in C^\circ$ and therefore $\mathbf{y} \in C^{\circ\circ}$.

By Lemma 1.32 and since C is closed and convex then $\mathbf{y} \in C$. However $\mathbf{y}^T\mathbf{d} \leq 0 \ \forall \mathbf{d} \in \mathbb{R}^n_>$ implies that $\mathbf{y} \in \mathbb{R}^n_{\leqq}$ and thus $\mathbf{y} \in C \cap \mathbb{R}^n_{\leqq}, \mathbf{y} \neq 0$ which contradicts the initial assumption $\hat{\mathbf{z}} \in N_p(Z)$ ($\hat{\mathbf{z}} \in N_p(Z)$ iff $C \cap \mathbb{R}^n_{\leqq} = \{\mathbf{0}\}$). Contradiction! □

Therefore in $\mathbb{R}_{\geqq}$-convex bodies we can obtain the whole weakly nondominated set by means of nonnegative weighted sum scalarization and the whole properly nondominated set by positive weighted sum scalarization of the objective function vector.

Corollary 1.34. *$S(Z) = N_p(Z) \subseteq N(Z) \subseteq N_w(Z) = S_0(Z)$ if Z is $\mathbb{R}^n_{\geqq}$-convex.*

However, in order to obtain the nondominated set in a $\mathbb{R}^n$-convex Z we need to consider both nonnegative and positive weighted sum scalarization. Positive weights are not sufficient to generate all nondominated points. On the other hand, nonnegative weights are sufficient but we cannot be sure that our solution is not just weakly nondominated unless it is the unique solution. To ensure nondominance of the solution we will need to consider a second phase problem.

1.5 Representation Sets

Although we have a way to generate $N(Z)$, it is impossible and inconvenient in practice to output the complete nondominated set. Finding nondominated points for some problems might be impossible due to their numerical complexity. Even if we are able to compute nondominated points and their corresponding efficient solutions, an exact description will fail in most cases since the cardinality of $N(Z)$ is either very large or even infinite. Furthermore, even in those cases where an exact description could be made, computing the whole nondominated set is generally computationally expensive.

We strive to generate nondominated solutions so the *Decision Maker* – an actor who chooses a decision based on the information obtained concerning the efficient solutions $E(X)$ and the nondominated set $N(Z)$ – has sufficient information to settle for a compromise related to the tradeoff between different efficient solutions.

Thus, another reason to not generate an exhaustive description of the nondominated set is "information overflow", i.e. working over such large sets of information slows down the decision making process. Therefore, our ultimate goal is to identify points in $N(Z)$ that are relevant to the decision maker.

Instead of $N(Z)$, we output a so called *Representation Set* $R(Z)$ which is usually a finite subset of $N(Z)$, i.e. $R(Z) \subseteq N(Z), |R(Z)| < \infty$. Typically $R(Z)$

fulfills a series of special requirements, i.e. maximum number of points, quality of approximation.

Definition 1.35. *A* ***Representation Set*** *$Rep(Z) \subseteq Z$ is a finite pointwise or point based approximation of $N(Z)$, in most cases $Rep(Z) \subseteq N(Z) + B_\epsilon(\mathbf{0})$ and $|Rep(Z)| < \infty$, where $B_\epsilon(\bar{\mathbf{z}}) := \{\mathbf{z} \in \mathbb{R}^n : \|\bar{\mathbf{z}} - \mathbf{z}\| \ < \epsilon, \epsilon > 0\}$.*

The decision maker should be able to distinguish relevant areas of $N(Z)$, make a choice in $Rep(Z)$ or specify an interesting area for further exploration where more nondominated points are desirable, using the information given by $Rep(Z)$.

Different properties of $Rep(Z)$ determine its usefulness for the decision making process and are consequentally used as *quality measures*.
We consider the following quality measures.

- `Representation point size`, i.e. $|Rep(Z)|$: The number of points considered in the representation set has to be sufficiently large to convey the structure of the nondominated set or to meet a desired approximation quality value. However, if the number of points is too big a proper information processing is no longer possible.

- `Approximation Quality in the outcome space`: This value can be obtained for example using the Hausdorff distance between $Rep(Z)$ – or a body related to $Rep(Z)$ – and $N(Z)$.

- `Algorithmic Quality Criteria`: We expect that an algorithm which generates nondominated points complies with a certain rule or a certain quality criterion, i.e. a nondominated point which maximizes a metric induced by a norm in the outcome space [Klamroth et al., 2002], a measure volume of non explored regions in the outcome space or uniform discretization in a parameter space [Kouvelis, October 2006]. Because the Haussdorf distance between $N(Z)$ and $Rep(Z)$ is hard to compute, these algorithmic quality criteria are often used to estimate the approximation quality in the outcome space.

- `Clustering Quality`: Clustering is the undesirable event when points in the representation set are too close to each other. Usually these points provide redundant information, and do not contribute to the approximation quality.

Furthermore, we are interested in the *complexity measures* to generate a representation set. These measures compare the time or space complexity of different algorithms.

- `Running Time`: Obviously we want to develop methods to obtain a sufficient number of points or to generate points that meet a desired quality approximation within a computing running time that is feasible for the decision process.

- `Solved Subproblems`: Usually, to generate a representation set we need to solve a sequence of optimization subproblems. The complexity and the number of solved subproblems influence the algorithm complexity.

In the next chapter we study the so called *block norms*. Klamroth et al. [2002] devised block norm based algorithms that generate inner and outer approximations of the nondominated set.

These algorithms are reviewed in Chapter 3. The algorithms and proposed variants are implemented and their computed representation set is compared. Because the approximations are point based, i.e. a full dimensional body is constructed using the points in the representation set, `clustering quality` is not relevant for our comparison.

Therefore, in Chapter 5, we limit the `size of the representation sets` and compare the `running time` and different `algorithmic quality criteria` between the implemented algorithms.

Chapter 2

Block Norms and Multicriteria Optimization

The results from the previous chapter justify the sufficiency of weighted sum scalarizations to compute the nondominated set of a $\mathbb{R}_{\geqq}$-convex body. However, there is no way to know *a priori* which weighted scalarizations are necessary to meet a required approximation quality of the representation set.

Block norms, introduced by Schandl, Klamroth, and Wiecek [2002], centered at a dominated point $\mathbf{u} \in Z + \mathbb{R}^n_{\geqq}$ are used to generate nondominated solutions and, at the same time, to approximate the structure of a $\mathbb{R}_{\geqq}$-convex outcome space. Thus allowing us to identify areas where more nondominated points are needed to enhance the approximation quality.

In this chapter we recall definitions and results presented in [Schandl et al., 2002] about block norms and their application to Multicriteria Optimization. We will find that, when used to approximate $N(Z)$, block norms centered at a dominated point are strongly related to weighted sum scalarization.

2.1 Polyhedral Gauges and Block Norms

We begin this chapter with the following theorem.

Theorem 2.1 (Minkowski-Weyl's Theorem). *For a $P \subseteq R^n$, the following statements are equivalent:*

1. *P is a **polyhedron**, i.e. $P = \{\mathbf{z} \in \mathbb{R}^n : \mathbf{A}\mathbf{z} \leqq \mathbf{b}\}$ for some $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$;*

2. *There are so called **extreme point** vectors $\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^p$ and so called **extreme ray** vectors $\mathbf{r}^1, \mathbf{r}^2, \ldots, \mathbf{r}^s$ in $\mathbb{R}^n$ such that*

$$P = \sum_{i=1}^{p} \lambda_i \mathbf{v}^i + \sum_{j=1}^{s} \mu_i \mathbf{r}^j$$

where $\lambda_i \geq 0, i = 1, \ldots, p$, $\sum_{i=1}^{p} \lambda_1 = 1$ and $\mu_j \geq 0, j = 1, \ldots, s$.

*Thus, every polyhedron has two representations of type (1) and (2), known as (halfspace) **H-representation** and (vertex) **V-representation**, respectively.*

Proof. See Padberg [1991] Section 7.3 Pages 155-156. □

We define polyhedral gauges according to Minkowski [1911].

Definition 2.2. *Let B be a **polytope**, i.e. a bounded polyhedron, in $\mathbb{R}^n$ containing the origin in its interior and let $\mathbf{z} \in \mathbb{R}^n$. The **polyhedral gauge** $\gamma : \mathbb{R}^n \mapsto \mathbb{R}$ of $\mathbf{z}$ is defined as:*

$$\gamma(\mathbf{z}) := \min\{\lambda \geq 0 : \mathbf{z} \in \lambda B\} \tag{2.1}$$

*If B is symmetric with respect to the origin, i.e. $\mathbf{z} \in B$ iff $-\mathbf{z} \in B$, then γ is called a **block norm**.*

*The vectors defined by the extreme points of B are called **fundamental vectors** and are denoted by $\mathbf{v}^i, i = 1, \ldots, p$. The fundamental vectors defined by the extreme points of a facet of B span a so called **fundamental cone** which is denoted by $C_j, j = 1, \ldots, m$.*

If a polytope, i.e. a bounded polyhedron, B satisfies $\mathbf{0} \in int(B)$ and $\mathbf{z} \in B$ iff $-\mathbf{z} \in B$ then the block norm γ, defined using B as in Definition 2.2, is indeed a norm in $\mathbb{R}^n$ [Minkowski, 1967].

Remark 2.3. *In Definition 2.2, for all $\mathbf{z} \in B$ we have $\gamma(\mathbf{z}) \leq 1$. Within this context, B is usually referred to as the unit ball of the block norm γ.*

The next theorem provides a way to calculate the polyhedral gauge γ of a point $\bar{\mathbf{z}}$ that lies in a fundamental cone.

Theorem 2.4. *Let γ be a polyhedral gauge with unit ball $B \subseteq \mathbb{R}^n$. Let $\bar{\mathbf{z}} \in C$ where C is the fundamental cone generated by k fundamental vectors $\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^k$ where $k \geq n$. Let*

$$\bar{\mathbf{z}} = \sum_{i=1}^{k} \lambda_i \mathbf{v}^i$$

be a representation of $\bar{\mathbf{z}}$ in terms of $\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^k$. Then

$$\gamma(\bar{\mathbf{z}}) = \sum_{i=1}^{k} \lambda_i$$

Furthermore, if $\mathbf{n}$ is the normal of the B polytope's facet such that $\mathbf{n}^T\mathbf{v}^i = 1\ \forall i = 1, \ldots, k$ then $\gamma(\bar{\mathbf{z}}) = \mathbf{n}^T\bar{\mathbf{z}}$.

Proof. Since $\mathbf{0}$ is in the interior of B, there exists $\mathbf{n} \in \mathbb{R}^n$ such $\mathbf{n}^T\mathbf{v} = 1$ for all $\mathbf{v} \in \{\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^k\}$. Because $\bar{\mathbf{z}}$ belongs to the fundamental cone C, $\exists \hat{\mathbf{z}}$ in the facet defined by $\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^k$ such that $\bar{\mathbf{z}} = \gamma(\bar{\mathbf{z}})\hat{\mathbf{z}}$.

It follows that:

$$\begin{aligned} \mathbf{n}^T\bar{\mathbf{z}} &= \mathbf{n}^T\left(\sum_{i=1}^{k} \lambda_i \mathbf{v}^i\right) \\ &= \sum_{i=1}^{k} \lambda_i \mathbf{n}^T\mathbf{v}^i \qquad (2.2) \\ &= \sum_{i=1}^{k} \lambda_i \end{aligned}$$

furthermore,

$$\begin{aligned} \mathbf{n}^T\bar{\mathbf{z}} &= \mathbf{n}^T(\gamma(\bar{\mathbf{z}})\hat{\mathbf{z}}) \\ &= \gamma(\bar{\mathbf{z}})\mathbf{n}^T\hat{\mathbf{z}} \\ &= \gamma(\bar{\mathbf{z}}) \end{aligned} \tag{2.3}$$

therefore $\gamma(\bar{\mathbf{z}}) = \mathbf{n}^T\bar{\mathbf{z}} = \sum_{i=1}^{k} \lambda_i$. □

For the further presentation, it is convenient to write $B = \{\mathbf{z} \in \mathbb{R}^n : {\mathbf{n}^j}^T\mathbf{z} \leq 1, j = 1, \ldots, m\}$, where $\mathbf{n}^j, j = 1, \ldots, m$ denote each of the B facets' normals.

Remark 2.5. *A polytope B is a bounded polyhedron, thus there exists a unique representation (up to scalar multiples) of B such $B = \{\mathbf{z} \in \mathbb{R}^n : \mathbf{A}\mathbf{z} \leqq \mathbf{b}\}$ where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Moreover, since each row of $\mathbf{A}$ uniquely represents each of the polytope facets' normal, none of the rows of $\mathbf{A}$ is $\mathbf{0}$.*

Furthermore, if $\mathbf{0} \in int(B)$ then $b_j \neq 0$ for all $j = 1, \ldots, m$. Therefore, there exists $\mathbf{N} \in \mathbb{R}^{m \times n}$ such that $B = \{\mathbf{z} \in \mathbb{R}^n : \mathbf{N}\mathbf{z} \leqq \mathbf{e}\}$ where $\mathbf{e} = (1, 1, \ldots, 1)^T \in \mathbb{R}^m$.

Block norms are not necessarily *absolute*. A norm $\rho : \mathbb{R}^n \mapsto \mathbb{R}$ is called *absolute* iff $\rho(\mathbf{a}) = \rho(\mathbf{b})$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ such that $|a_i| = |b_i| \; \forall i = 1, \ldots, n$.

The unit ball B_ρ of an absolute norm ρ, i.e. $B_\rho := \{\mathbf{z} \in \mathbb{R}^n : \rho(\mathbf{z}) \leq 1\}$, is a convex body that has the same structure in every orthant of $\mathbb{R}^n$. To see that B_ρ is convex, let $\mathbf{a}, \mathbf{b} \in B_\rho, \lambda \in [0, 1]$. Then $\rho(\lambda\mathbf{a}+(1-\lambda)\mathbf{b}) \leq \lambda\rho(\mathbf{a})+(1-\lambda)\rho(\mathbf{b}) \leq 1$ and therefore $\lambda\mathbf{a}+(1-\lambda)\mathbf{b} \in B_\rho$. The euclidian norm $\|\cdot\|_2$ in $\mathbb{R}^n$ is an example of an absolute norm and its unit ball is the unit sphere.

Block norms that have the same structure in every orthant are very useful in multicriteria optimization. In Sections 2.2 and 2.3, we will see that is possible to generate nondominated points either by minimizing a block norm in the positive orthant $\mathbb{R}^n_{\geqq}$, or by maximizing a block norm in the negative one $\mathbb{R}^n_{\leqq}$.

In order to construct absolute block norms based on a given structure in one orthant, we need to replicate that structure in every orthant of $\mathbb{R}^n$. To simplify the notation we define the *reflection set* and denote the *convex hull* of a finite set U by $convex(U)$.

Definition 2.6. *For a finite set $U = \{\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^k\} \subseteq \mathbb{R}^n$ we define:*

$$convex(U) := \{\mathbf{z} \in \mathbb{R}^n : \mathbf{z} = \sum_{i=1}^{k} \lambda_i\mathbf{u}^i, \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1\} \tag{2.4}$$

which is the smallest convex set that contains U.

Definition 2.7. *Let $\mathbf{u} \in \mathbb{R}^n$. The **reflection set** R **of a vector point** $\mathbf{u}$ is defined as:*

$$R(\mathbf{u}) := \{\mathbf{w} \in \mathbb{R}^n : |w_i| = |u_i| \;\; \forall \; i = 1, 2, \ldots, n\} \tag{2.5}$$

*and the **reflection set** R of **a set** $U \subseteq \mathbb{R}^n$ is defined as the following:*

$$R(U) := \bigcup_{\mathbf{u} \in U} R(\mathbf{u}) \tag{2.6}$$

Reflection sets will allow us to characterize and construct block norms whose unit ball structure is the same in every orthant.
We define an *absolute block norm* using the reflection set.

Definition 2.8. *A block norm γ is called **absolute** if for all $\mathbf{u} \in \mathbb{R}^n$:*

$$\gamma(\mathbf{w}) = \gamma(\mathbf{u}) \ \forall \mathbf{w} \in R(\mathbf{u}) \tag{2.7}$$

Observation 2.9. *As a direct consequence of Definition 2.8, a polytope B is the unit ball of an absolute block norm γ iff $R(\mathbf{z}) \subseteq B$ for all $\mathbf{z} \in B$.*

The unit ball of an absolute block norm has the same structure in every orthant. This property is *ad hoc* with our intended use of absolute block norms where all interesting points are either in the positive or in the negative orthant.

To visualize the geometry of a unit ball B that defines an absolute block norm, observe that for every point $\mathbf{z} \in B$ we have $R(\mathbf{z}) \subseteq B$ and, by convexity of B, the whole hypercube whose vertices are precisely $R(\mathbf{z})$ is contained in B, i.e. $convex(R(\mathbf{z})) \subseteq B$ for all $\mathbf{z} \in B$.

A consequence of the previous paragraph is that every hypercube whose center coincides with the origin defines an absolute block norm, in this case a weighted Tchebycheff norm [Sayin and Kouvelis, 2005]. As we maximize or minimize an absolute block norm whose unit ball is a hypercube we will find that we run into weakly nondominated points, in a similar way to nonnegative weighted scalarization.

Consider a polyhedral gauge γ with unit ball B, if we define $\tilde{\gamma}$ to be the polyhedral gauge whose unit ball is $\alpha B, \alpha > 0$ then $\gamma(\mathbf{z}) = \alpha\tilde{\gamma}(\mathbf{z})$ for all $\mathbf{z} \in \mathbb{R}^n$ – observe that $\mathbf{z} \in \lambda B$ iff $\mathbf{z} \in (\lambda/\alpha)\alpha B$, therefore $\gamma(\mathbf{z}) = \lambda$ iff $\tilde{\gamma}(\mathbf{z}) = \lambda/\alpha$ –. Furthermore, the symmetric and absoluteness properties of B are the same for $\alpha B, \alpha > 0$.

Corollary 2.10. *If γ is an absolute block norm with unit ball B then $\tilde{\gamma}$ with unit ball $\alpha B, \alpha > 0$ is also an absolute block norm.*

In the following, absolute block norm properties are derived and analyzed within the positive orthant. These properties are easily extended to all other orthants.

Lemma 2.11. *An absolute block norm γ with unit ball B has the following property:*

$$(\mathbf{z} - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \subseteq \gamma(\mathbf{z})B \cap \mathbb{R}^n_{\geqq} \quad \forall \mathbf{z} \in \mathbb{R}^n_{\geqq} \tag{2.8}$$

Proof. For $\mathbf{z} = \mathbf{0}$ the property is trivial.

First, let $\mathbf{z} \in \partial B \cap \mathbb{R}^n_{\geqq}$. Then $\gamma(\mathbf{z}) = 1$ and since γ is an absolute block norm we have that $R(\mathbf{z}) \subseteq B$.

By convexity of B it follows that $convex(R(\mathbf{z})) \subseteq B$ but:

$$(\mathbf{z} - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} = convex(R(\mathbf{z})) \cap \mathbb{R}^n_{\geqq} \tag{2.9}$$

and finally we get $(\mathbf{z} - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \subseteq B \cap \mathbb{R}^n_{\geqq}$.

The general case $\mathbf{z} \in \mathbb{R}^n_{\geqq}$ is a consequence of Corollary 2.10, by taking the absolute block norm $\tilde{\gamma}$ with unit ball $\gamma(\mathbf{z})B$. □

Corollary 2.12. *Lemma 2.11 implies that if γ is an absolute block norm, then all normals $\mathbf{n}^j$ to the facets of its unit ball B that are necessary to describe B in the positive orthant, i.e. $B \cap \mathbb{R}^n_{\geqq} = \{\mathbf{z} \geqq \mathbf{0} : {\mathbf{n}^j}^T \mathbf{z} \leq 1, j = 1, \ldots, m\}$, have nonnegative components.*

Proof. Let $B_{\geqq} := B \cap \mathbb{R}^n_{\geqq}$ and denote $B_{\geqq}$ by its facets such that $B_{\geqq} = \{\mathbf{z} \geqq \mathbf{0} : \mathbf{N}\mathbf{z} \leqq \mathbf{e}\}, \mathbf{N} \in \mathbb{R}^{m \times n}$. Let $\bar{\mathbf{z}} \in B_{\geqq}, \bar{\mathbf{z}} > \mathbf{0}$ be a inner point in one of the facets of $B_{\geqq}$, i.e. ${\mathbf{n}^j}^T \bar{\mathbf{z}} = 1$ for some $j \in \{1, \ldots, m\}$.

Assume for the sake of contradiction that $n^j_i < 0$. Then, for a sufficiently small $\epsilon > 0$, we have that $\hat{\mathbf{z}} = \bar{\mathbf{z}} - \epsilon \mathbf{e}^i \in \mathbb{R}^n_{\geqq}$ and ${\mathbf{n}^j}^T \hat{\mathbf{z}} > 1$, therefore $\hat{\mathbf{z}} \notin B_{\geqq}$, this is a contradiction to Lemma 2.11. □

Thus all normals to the facets necessary to describe the unit ball B of an absolute block norm in the positive orthant have nonnegative components. Therefore, in the positive orthant, B may have facets that are parallel to the axes of $\mathbb{R}^n$, i.e. B may have facets that do not intersect an axis.

We saw in Section 1.5 that by minimizing an hyperplane with nonnegative components in the outcome space Z, i.e. a nonnegative weighted sum scalarization of the objective functions, we are able to compute weakly nondominated points. We will find a similar result when using absolute block norms, whose facets' normals in the positive orthant have nonnegative components. Thus, we strengthen the requirements for an absolute block norm in the following way:

Definition 2.13. *An absolute block norm γ is called **oblique** if its unit ball B has the following property:*

$$(\mathbf{z} - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \cap \partial B = \{\mathbf{z}\} \quad \forall \mathbf{z} \in (\partial B \cap \mathbb{R}^n_{\geqq}) \tag{2.10}$$

Observation 2.14. *The unit ball B of an oblique block norm has the same structure in every orthant. Definition 2.13 implies that $convex(R(\mathbf{z})), \mathbf{z} \in \partial B$, i.e. the hypercube whose vertices are $R(\mathbf{z})$, is almost completely contained in the topological interior of B.*

The only exception are the vertices of the hypercube – they are in ∂B by definition –. In other words, $convex(R(\mathbf{z})) \setminus R(\mathbf{z}) \subseteq int(B)$.

As with the symmetric and absoluteness properties, the obliqueness property of B is maintained if we consider the scaled unit ball $\alpha B, \alpha > 0$.

Corollary 2.15. *If γ is an oblique block norm with unit ball B then $\tilde{\gamma}$ with unit ball $\alpha B, \alpha > 0$ is also an oblique block norm.*

We get a corresponding property for oblique block norms as in Lemma 2.11.

Lemma 2.16. *An oblique block norm γ with unit ball B has the following property:*

$$(\mathbf{z} - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \cap \partial(\gamma(\mathbf{z})B) = \{\mathbf{z}\} \quad \forall \mathbf{z} \in \mathbb{R}^n_{\geqq} \tag{2.11}$$

Proof. For $\mathbf{z} = \mathbf{0}$ the property is trivial. For $\mathbf{z} \neq \mathbf{0}$ it is obvious that $\mathbf{z} \in \partial(\gamma(\mathbf{z})B)$. Property (2.11) follows directly from Definition 2.13. □

Analogously to absolute block norms, all normals to the facets that are necessary to describe the unit ball B of an oblique block norm in the positive orthant have positive components.

Corollary 2.17. *Lemma 2.16 implies that if γ is an oblique block norm, then all normals $\mathbf{n}^j$ to the facets of its unit ball B that are necessary to describe B in the positive orthant, i.e. $B \cap \mathbb{R}^n_{\geqq} = \{\mathbf{z} \geqq \mathbf{0} : {\mathbf{n}^j}^T \mathbf{z} \leq 1, j = 1, \ldots, m\}$, have positive components.*

Proof. Let $B_{\geqq} := B \cap \mathbb{R}^n_{\geqq}$ and denote $B_{\geqq}$ by its facets such that $B_{\geqq} = \{\mathbf{z} \geqq \mathbf{0} : \mathbf{N}\mathbf{z} \leqq \mathbf{e}\}, \mathbf{N} \in \mathbb{R}^{m \times n}$. Let $\bar{\mathbf{z}} \in B_{\geqq}, \bar{\mathbf{z}} > \mathbf{0}$ be an inner point in one of the facets of $B_{\geqq}$, i.e. ${\mathbf{n}^j}^T \bar{\mathbf{z}} = 1$ and ${\mathbf{n}^k}^T \bar{\mathbf{z}} < 1$ for all $k \neq j$.

Assume for the sake of contradiction that $n^j_i = 0$ (it cannot be that $n^j_i < 0$ since an oblique norm is an absolute norm). Then, for a sufficiently small $\epsilon > 0$, we have that $\hat{\mathbf{z}} = \bar{\mathbf{z}} - \epsilon \mathbf{e}^i \in \mathbb{R}^n_{\geqq}$ and, because ${\mathbf{n}^j}^T \hat{\mathbf{z}} = 1$, $\hat{\mathbf{z}} \in \partial B$; this is a contradiction to Lemma 2.16. □

We found that a property of (oblique) absolute block norms is that the facets' normals of their unit ball restricted to the positive orthant have only (positive) nonnegative components.

Conversely, we want to know if it is possible to construct an (oblique) absolute block norm by means of a polyhedron $P = \{\mathbf{z} : \mathbf{A}\mathbf{z} \leqq \mathbf{e}\}$, $P \cap \mathbb{R}^n_{\geqq}$ bounded, where $\mathbf{A} \in \mathbb{R}^{m \times n}$ has only (positive) nonnegative components.

By taking such polyhedra and using the reflection set to replicate its positive orthant structure in all other orthants in $\mathbb{R}^n$, we construct a unit ball that fullfills the necessary properties to define an (oblique) absolute block norm.

Theorem 2.18. *Let P be a polyhedron with $\mathbf{0} \in int(P)$ and $P \cap \mathbb{R}^n_{\geqq}$ bounded. Furthermore, let $P = \{\mathbf{z} : \mathbf{A}\mathbf{z} \leqq \mathbf{e}\}$ be its unique representation, where $\mathbf{A} \in \mathbb{R}^{m \times n}$ has only (positive) nonnegative components and its rows correspond one to one to the facets of P. Then $B = R(P \cap \mathbb{R}^n_{\geqq})$ defines a polytope that can be used as a unit ball to generate an (oblique) absolute block norm.*

Proof. First consider the case when $\mathbf{A}$ has only nonnegative components.

1. $B \subseteq P$:
   Let $\mathbf{z} \in P$. Then $\mathbf{z} - \mathbb{R}^n_{\geqq} \subseteq P$, because all inequalities defining P consist of nonnegative components. We know that for $\mathbf{z} \in P \cap \mathbb{R}^n_{\geqq}$:

   $$R(\mathbf{z}) \subseteq \mathbf{z} - \mathbb{R}^n_{\geqq} \subseteq P \tag{2.12}$$

   it follows immediately that $B \subseteq P$.

2. B is convex:
   For the sake of contradiction we suppose there exist $\mathbf{z}^1, \mathbf{z}^2 \in B$ and $\mathbf{z} \notin B$ where $\mathbf{z} = \lambda \mathbf{z}^1 + (1 - \lambda)\mathbf{z}^2$ for some $\lambda \in (0, 1)$.

   Since B was defined using a reflection set, $\mathbf{z} \notin B$ implies $B \cap R(\mathbf{z}) = \emptyset$. Thus, we can assume w.l.o.g. that $\mathbf{z} \in \mathbb{R}^n_{\geqq}$.

   But $B \cap \mathbb{R}^n_{\geqq} = P \cap \mathbb{R}^n_{\geqq}$ so $\mathbf{z} \notin B$ means $\mathbf{z} \notin P$ but we have $\mathbf{z}^1, \mathbf{z}^2 \in B \subseteq P$ and P is convex. This leads to a contradiction since we get, using convexity of P, that $\mathbf{z} \in P$. Contradiction!

3. B is a polytope:
   B is closed and bounded because P is closed and bounded on the positive orthant. Since $P \cap \mathbb{R}^n_{\geqq}$ has a finite number of extreme points then B also has a finite number of extreme points and so B is a polytope.

4. $R(\mathbf{z}) \subseteq B$ for all $\mathbf{z} \in B$:
   Since B is defined using a reflection set then $R(\mathbf{z}) \subseteq B$ for all $\mathbf{z} \in B$.

Thus B is a polytope that satisfies Observation 2.9 and therefore the polyhedral gauge γ with unit ball B is an absolute block norm.

Now we consider the case where the components of $\mathbf{A}$ are strictly positive.

Since $B \cap \mathbb{R}^n_{\geqq} = P \cap \mathbb{R}^n_{\geqq}$, Let $\bar{\mathbf{z}} \in \partial P \cap \mathbb{R}^n_{\geqq}$ then there exists a set of inequalities I such:

$$\begin{aligned} \mathbf{A_I}\bar{\mathbf{z}} &= \mathbf{e} \\ \mathbf{A}\bar{\mathbf{z}} &\leqq \mathbf{e} \end{aligned} \tag{2.13}$$

Let $\hat{\mathbf{z}} \neq \bar{\mathbf{z}}, \hat{\mathbf{z}} \in (\bar{\mathbf{z}} - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq}$, we notice that due to all components being strictly positive we have $\mathbf{A}\hat{\mathbf{z}} < \mathbf{e}$. Thus $\hat{\mathbf{z}} \in int(P) \cap \mathbb{R}^n_{\geqq}$, which is equivalent to $\hat{\mathbf{z}} \in int(B) \cap \mathbb{R}^n_{\geqq}$. Therefore

$$(\mathbf{z} - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \cap \partial B = \{\mathbf{z}\} \ \forall \mathbf{z} \in (\partial B \cap \mathbb{R}^n_{\geqq})$$

and B is as in Definition 2.13. □

In the next two sections we will investigate how block norms can be used to generate nondominated points.

2.2 Block Norms centered at the ideal point

[Schandl et al., 2002] obtained several interesting results concerning the possibility of using absolute block norms centered in the ideal point to generate the nondominated set. In this section we summarize and present some of those results.

As we will see in this section, nondominated points are points in Z that minimize a block norm, "centered" in the ideal point. Intuitively, this means that nondominated points are, w.r.t. the distance defined by a block norm, "closer" to the ideal point than dominated points.
Consider the following problem.

$$\min\{\gamma(\mathbf{z}) \text{ s.t. } \mathbf{z} \in Z\} \tag{2.14}$$

where γ is an absolute block norm. Assume w.l.o.g. that the ideal point is the $\mathbf{0}$ vector, i.e. $\mathbf{z}^* = \mathbf{0}$.
If $\mathbf{z}^* = \mathbf{0}$, then $Z \subseteq \mathbb{R}^n_{\geqq}$ and problem (2.14) can be rewritten as:

$$\min\{\gamma(\mathbf{z}) \text{ s.t. } \mathbf{z} \in Z \cap \mathbb{R}^n_{\geqq}\} \tag{2.15}$$

Now we look at the relationship between the weakly nondominated set and the solution set of problem (2.15).

Theorem 2.19. *Let* $\hat{\mathbf{z}} \in N(Z)$*. Then there exists an absolute block norm* γ *such that* $\hat{\mathbf{z}}$ *uniquely minimizes problem* (2.15)*.*

Proof. Let $\hat{\mathbf{z}} \in N(Z)$. Define the block norm γ using the polytope $B = convex(R(\hat{\mathbf{z}}))$ as its unit ball. Obviously, γ is an absolute block norm.

Because $\hat{\mathbf{z}} \in \partial B$, we have that $\gamma(\hat{\mathbf{z}}) = 1$. If there exists $\mathbf{z} \in \mathbb{R}^n_{\geqq}, \mathbf{z} \neq \hat{\mathbf{z}}$ such that $\gamma(\mathbf{z}) \leq \gamma(\hat{\mathbf{z}}) = 1$, then $\mathbf{z} \in convex(R(\hat{\mathbf{z}})) \cap \mathbb{R}^n_{\geqq}$.

Thus $\mathbf{z} \in \hat{\mathbf{z}} - \mathbb{R}^n_{\geq}$, but then $\mathbf{z} \leq \hat{\mathbf{z}}$, this is a contradiction to the fact that $\hat{\mathbf{z}} \in N(Z)$. Contradiction! □

Theorem 2.19 guarantees that for each nondominated point $\mathbf{z} \in N(Z)$, there exists an absolute block norm γ centered at the ideal point such that $\mathbf{z}$ uniquely minimizes γ.

Conversely, if γ is an absolute block norm then by solving problem (2.15) we compute a weakly nondominated point.

Theorem 2.20. *If* $\hat{\mathbf{z}}$ *is a minimizer of problem* (2.15)*, where* γ *is an absolute block norm, then* $\hat{\mathbf{z}}$ *is weakly nondominated.*

Proof. We know that $\hat{\mathbf{z}} \in \partial(\gamma(\hat{\mathbf{z}})B)$, where B is the unit ball of γ, and that there exists no $\mathbf{z} \in Z$ such $\mathbf{z} \in int(\gamma(\hat{\mathbf{z}})B)$.

Because γ is an absolute block norm it follows that:

$$convex(R(\hat{\mathbf{z}})) \cap \mathbb{R}^n_{\geqq} \subseteq \gamma(\hat{\mathbf{z}})B \cap \mathbb{R}^n_{\geqq}$$

and thus,

$$\nexists \mathbf{z} \in Z : \mathbf{z} \in int(convex(R(\hat{\mathbf{z}})) \cap \mathbb{R}^n_{\geqq}$$

Therefore,

$$\nexists \mathbf{z} \in Z : \mathbf{z} \in (\hat{\mathbf{z}} - \mathbb{R}^n_{>}) \cap \mathbb{R}^n_{\geqq}$$

and $\hat{\mathbf{z}} \in N_w(Z)$. □

In theorem 2.19, we use the polyhedron $convex(R(\hat{\mathbf{z}}))$ to construct a block norm γ such that $\hat{\mathbf{z}} \in N(Z)$ minimizes γ. Because $\nexists \mathbf{z} \in Z$ such that $\mathbf{z} \in \hat{\mathbf{z}} + \mathbb{R}^n_{\leqq}$, thus we have that $\nexists \mathbf{z} \in Z$ such that $\mathbf{z} \in int(convex(R(\hat{\mathbf{z}})))$.

However, $convex(R(\hat{\mathbf{z}}))$ cannot be used to construct an oblique block norm. To construct an oblique norm, as seen in Theorem 2.18, we need a polyhedron $P = \{\mathbf{z} : \mathbf{A}\mathbf{z} \leqq \mathbf{b}\}$ such that $P \cap \mathbb{R}^n_{\geqq}$ is bounded and $\mathbf{A} \in \mathbb{R}^{m \times n}$ has only positive components. We can construct such P by means of a *polyhedral cone* C such that $\mathbb{R}^n_{\leq} \subseteq int(C)$.

Definition 2.21. *A* ***polyhedral cone*** $C \subseteq \mathbb{R}^n$ *is a cone such that* $C = \{\mathbf{z} : \mathbf{A}\mathbf{z} \leqq \mathbf{0}\}, \mathbf{A} \in \mathbb{R}^{m \times n}$*.*

Observation 2.22. *Although a polyhedron is defined by* $P = \{\mathbf{z} \in \mathbb{R}^n : \mathbf{A}\mathbf{z} \leqq \mathbf{b}\}$ *with* $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$*.* P *is a cone iff* $\mathbf{b} = \mathbf{0}$*.*

In the next theorem and throughout this paper we use the following notation.

Definition 2.23. *For a finite set $U = \{\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^k\} \subseteq \mathbb{R}^n$ we define:*

$$polycone(U) := \{\mathbf{z} \in \mathbb{R}^n : \mathbf{z} = \sum_{i=1}^{k} \lambda_i \mathbf{u}^i, \lambda_i \geq 0\} \tag{2.16}$$

which is the smallest convex cone that contains U.

Theorem 2.24. *Let $C \subseteq \mathbb{R}^n$ be a convex cone containing the origin with $\mathbb{R}^n_{\leq} \subseteq int(C)$. Then there exists a closed polyhedral cone $\bar{C} \subseteq C$ with $\mathbb{R}^n_{\leq} \subseteq int(\bar{C})$.*

Proof. Let $-\mathbf{e}^i \in \mathbb{R}^n$ and $\mathbf{c}^i \in \mathbb{R}^n$ for $i = 1, 2, \ldots, n$ be the vectors with the following components.

$$-e^i_j = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad c^i_j = \begin{cases} -1 & \text{if } i = j \\ \delta & \text{if } i \neq j \end{cases} \tag{2.17}$$

where $\delta > 0$.

Note that $\mathbb{R}^n_{\leqq} = polycone(\{-\mathbf{e}^1, \ldots, -\mathbf{e}^n\})$ and, according to [Tuy, 1998], the set $\bar{C} = polycone(\{\mathbf{c}^1, \ldots, \mathbf{c}^n\})$ is a closed polyhedral cone.

We use the fact that $int(C)$ is open to find $\delta > 0$ such that $\mathbf{c}^i \in C, i = 1, \ldots, n$. By convexity of C, it follows that $\bar{C} \subseteq C$. Observe that $-\mathbf{e}^i \in int(\bar{C})$ for all $i = 1, \ldots, n$.

Thus $polycone(\{-\mathbf{e}^1, \ldots, -\mathbf{e}^n\}) \setminus \{\mathbf{0}\} \subseteq int(\bar{C})$. Therefore $\mathbb{R}^n_{\leq} \subseteq int(\bar{C})$. □

The next corollary is a consequence from Theorems 2.24 and 1.21. It represents an equivalent definition for proper nondominated points.

Theorem 1.21 in Chapter 1 states that a vector $\mathbf{z}$ is properly nondominated iff there exists a convex cone C with $\mathbb{R}^n_{\leq} \subseteq int(C)$ so that:

$$(Z - \mathbf{z}) \cap C = \{\mathbf{0}\}$$

However, in Theorem 2.24, we proved that if C is a convex cone such that $\mathbb{R}^n_{\leq} \subseteq int(C)$ then there exists a polyhedral cone $\bar{C} \subseteq C$ with $\mathbb{R}^n_{\leq} \subseteq int(\bar{C})$. Thus we derive the following corollary.

Corollary 2.25. *A vector $\bar{\mathbf{z}}$ is properly nondominated iff there exists a closed polyhedral cone $\bar{C}$ with $\mathbb{R}^n_{\leq} \subseteq int(\tilde{C})$ such that:*

$$(Z - \bar{\mathbf{z}}) \cap \bar{C} = \{\mathbf{0}\} \tag{2.18}$$

or equivalently:

$$Z \cap (\bar{\mathbf{z}} + \bar{C}) = \{\bar{\mathbf{z}}\} \tag{2.19}$$

The connection between properly nondominated points and oblique block norms will be explained later in this section. However, we already identify a relationship between polyhedral cones which contain the negative orthant in its interior and oblique block norms.

Remark 2.26. *If C is a closed polyhedral cone such that $\mathbb{R}^n_{\leq} \subseteq int(C)$ then there exists $\mathbf{A} \in \mathbb{R}^{m \times n}$ such that $C = \{\mathbf{z} : \mathbf{A}\mathbf{z} \leqq \mathbf{0}\}$. Furthermore, all the components of $\mathbf{A}$ are strictly positive. Otherwise there would be $\mathbf{z} \in \mathbb{R}^n_{\leq}$ such that $\mathbf{z} \in \partial C$ or $\mathbf{z} \notin C$.*

Thus $\mathbf{z}+C$ for $\mathbf{z} \in \mathbb{R}^n_{\geq}$ is a polyhedron such that $\mathbf{0} \in int(\mathbf{z}+C)$, $(\mathbf{z}+C) \cap \mathbb{R}^n_{\geqq}$ bounded. Therefore, we apply theorem 2.18 to conclude that $R((\mathbf{z}+C) \cap \mathbb{R}^n_{\geqq})$ is a polytope that defines an oblique block norm.

If $\hat{\mathbf{z}} \in N_p(Z)$ in Theorem 2.19, then we know by Corollary 2.25 that there exists a polyhedral cone C such that $\mathbb{R}^n_{\geq} \subseteq int(C), (Z - \hat{\mathbf{z}}) \cap C = \{\mathbf{0}\}$. This polyhedral cone C can be used to create an oblique block norm as in Remark 2.26.

Theorem 2.27. *Let $\hat{\mathbf{z}} \in N_p(Z)$. Then there exists an oblique norm γ such that $\hat{\mathbf{z}}$ uniquely minimizes problem* (2.15)*.*

Proof. Let $\hat{\mathbf{z}} \in N_p(Z)$, then according to Corollary 2.25, there is a polyhedral cone C with $\mathbb{R}^n_{\leq} \subseteq int(C)$ such that $(\hat{\mathbf{z}} + C) \cap Z = \{\hat{\mathbf{z}}\}$. Using Remark 2.26, we construct a polytope $B := R((C + \hat{\mathbf{z}}) \cap \mathbb{R}^{\mathbf{n}}_{\geqq})$ that defines an oblique block norm γ such that $\hat{\mathbf{z}} \in \partial B$.

Since there is no $\mathbf{z} \in Z, \mathbf{z} \neq \hat{\mathbf{z}}, \mathbf{z} \in B$ therefore $\hat{\mathbf{z}}$ is the unique minimizer of problem (2.15). □

Conversely, if we use an oblique block norm γ in problem (2.15) then we obtain a properly nondominated solution.

Theorem 2.28. *If $\hat{\mathbf{z}} \in \mathbb{R}^n_{>}$ is a minimizer of problem* (2.15) *where γ is an oblique norm then $\hat{\mathbf{z}}$ is properly nondominated.*

Proof. Let $\hat{\mathbf{z}} \in \mathbb{R}^n_{>}$ be a minimizer of problem (2.15) where γ is an oblique block norm. It follows that there is no $\mathbf{z} \in Z, \mathbf{z} \in int(\gamma(\hat{\mathbf{z}})B)$, where B is the unit ball of γ.

We know, by Lemma 2.16 and $\mathbf{z}^i \in \hat{\mathbf{z}} - \mathbb{R}^n_{\geqq}$, that:

$$\mathbf{z}^i := (\hat{z}_1, \ldots, \hat{z}_{i-1}, 0, \hat{z}_{i+1}, \ldots, \hat{z}_n)^T \in int(\gamma(\hat{\mathbf{z}})B) \quad \forall i = 1, \ldots, n$$

thus, it follows that there exists $\delta > 0$ such that

$$\bar{\mathbf{z}}^i := (\hat{z}_1 + \delta, \ldots, \hat{z}_{i-1} + \delta, 0, \hat{z}_{i+1} + \delta, \ldots, \hat{z}_n + \delta)^T \in int(\gamma(\hat{\mathbf{z}})B)$$

for all $i = 1, \ldots, n$.

Define $C := polycone(\{\bar{\mathbf{z}}^1 - \hat{\mathbf{z}}, \ldots, \bar{\mathbf{z}}^n - \hat{\mathbf{z}}\})$. Observe that

$$\bar{\mathbf{z}}^i - \hat{\mathbf{z}} = (\delta, \ldots, \delta, -\hat{z}_i, \delta, \ldots, \delta)^T$$

and $\hat{z}_i > 0$. Thus, as in Theorem 2.24, $-\mathbf{e}^i \in int(C)$ for all $i = 1, \ldots, n$ and

$$polycone(\{-\mathbf{e}^1, \ldots, -\mathbf{e}^n\}) \setminus \{\mathbf{0}\} \subseteq int(C)$$

Therefore C is a closed polyhedral cone such that $\mathbb{R}^n_{\leq} \subseteq int(C)$.

But $(C + \hat{\mathbf{z}}) \cap \mathbb{R}^n_{\geqq} \subseteq int(\gamma(\hat{\mathbf{z}})B) \cup \{\hat{\mathbf{z}}\}$. So finally we have $(\hat{\mathbf{z}} + C) \cap Z = \{\hat{\mathbf{z}}\}$. It follows from Corollary 2.25 that $\hat{\mathbf{z}} \in N_p(Z)$. □

Thus we can generate all weakly nondominated points using absolute block norms centered in the ideal point. Also, we can generate the whole properly nondominated set by means of oblique block norms. However, in order to generate all nondominated solutions we need to consider absolute block norms; thereby we may compute points in $N_w(Z) \setminus N(Z)$. Similarly to the weighted sum scalarization, to ensure nondominance a second phase problem is needed.

2.3 Block Norms centered at a dominated point

We have seen in the previous section that nondominated points minimize the distance to the ideal point measured by a block norm. Our intuition tell us that nondominated points maximize a block norm's distance from an interior point of Z in the direction of the ordering cone.

By maximizing block norms centered at a point in $Z+\mathbb{R}^n_{\geqq}$, we get an insight of the outcome space structure. If Z is $\mathbb{R}_{\geqq}$-convex then the scaled block norm is also an inner approximation of the outcome space.

So assuming w.l.o.g. that $\mathbf{0} \in Z + \mathbb{R}^n_{\geqq}, \mathbf{u} = \mathbf{0}$, we consider the following problem:

$$\max\{\gamma(\mathbf{z}) \text{ s.t. } \mathbf{z} \in Z \cap \mathbb{R}^n_{\leqq}\} \tag{2.20}$$

where γ is an absolute block norm.

Theorem 2.29. *If $\hat{\mathbf{z}}$ is a solution of problem* (2.20)*, where γ is an absolute block norm, then $\hat{\mathbf{z}}$ is weakly nondominated.*

Conversely, if Z is $R_{\geqq}$-convex, then for any $\hat{\mathbf{z}} \in N_w(Z) \cap \mathbb{R}^n_{\leqq}$ there exists an absolute block norm γ such that $\hat{\mathbf{z}}$ is a solution of problem (2.20)*.*

Proof. Let $\hat{\mathbf{z}}$ be a solution of problem (2.20) where γ is an absolute block norm with unit ball B and suppose that $\hat{\mathbf{z}} \notin N_w(Z)$.

Thus there exists $\mathbf{z} \in Z, \mathbf{z} < \hat{\mathbf{z}}$. Since γ is an absolute block norm we have, by Lemma 2.11, that

$$(\mathbf{z} + \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\leqq} \subseteq \gamma(\mathbf{z})B \cap \mathbb{R}^n_{\leqq}$$

by taking $int(\mathbb{R}^n_{\geqq})$ and $int(\gamma(\mathbf{z})B)$, we get

$$(\mathbf{z} + \mathbb{R}^n_{>}) \cap \mathbb{R}^n_{\leqq} \subseteq int(\gamma(\mathbf{z})B) \cap \mathbb{R}^n_{\leqq}$$

since $\hat{\mathbf{z}} \in (\mathbf{z} + \mathbb{R}^n_{>}) \cap \mathbb{R}^n_{\leqq}$, we finally conclude that $\gamma(\mathbf{z}) > \gamma(\hat{\mathbf{z}})$. Contradiction!

Conversely, let Z be $\mathbb{R}_{\geqq}$-convex and $\hat{\mathbf{z}} \in N_w(Z) \cap \mathbb{R}^n_{\leqq}$. By Theorem 1.28, there exists $\boldsymbol{\lambda} \in \mathbb{R}^n_{\geq}$ such that $\boldsymbol{\lambda}^T\hat{\mathbf{z}} \leq \boldsymbol{\lambda}^T\mathbf{z}$ for all $\mathbf{z} \in Z$. Define $P = \{\mathbf{z} \in \mathbb{R}^n : \mathbf{z} \leqq -\mathbf{z}^*, \boldsymbol{\lambda}^T\mathbf{z} \leq \boldsymbol{\lambda}^T(-\hat{\mathbf{z}})\}$ where $\mathbf{z}^* \in \mathbb{R}^n_{<}$ is the ideal point. Due to the first set of inequalities $\mathbf{z} \leqq -\mathbf{z}^*$, P is a bounded polyhedron in $\mathbb{R}^n_{\geqq}$. Furthermore, since all coefficients in the inequalities are nonnegative, P generates an absolute block norm γ, as stated in Theorem 2.18.

Since $\boldsymbol{\lambda}^T(-\hat{\mathbf{z}}) = \boldsymbol{\lambda}^T(-\hat{\mathbf{z}})$ then $\hat{\mathbf{z}} = \partial B, B = R(P \cap \mathbb{R}^n_{\geqq})$ and hence $\gamma(\hat{\mathbf{z}}) = 1$. The existence of a point $\mathbf{z} \in Z \cap \mathbb{R}^n_{\leqq}$ such that $\gamma(\mathbf{z}) > \gamma(\hat{\mathbf{z}})$ implies that $-\mathbf{z} \leqq -\mathbf{z}^*$ and $\boldsymbol{\lambda}^T(-\mathbf{z}) > \boldsymbol{\lambda}^T(-\hat{\mathbf{z}})$. However, $\boldsymbol{\lambda}^T(-\mathbf{z}) > \boldsymbol{\lambda}^T(-\hat{\mathbf{z}})$ is equivalent to $\boldsymbol{\lambda}^T\mathbf{z} < \boldsymbol{\lambda}^T\hat{\mathbf{z}}$. Contradiction! □

Theorem 2.30. *If $\hat{\mathbf{z}}$ is a solution of problem* (2.20)*, where γ is an oblique block norm, then $\hat{\mathbf{z}}$ is nondominated.*

Conversely, if Z is $\mathbb{R}_{\geqq}$-convex then for any $\hat{\mathbf{z}} \in N_p(Z) \cap \mathbb{R}^n_{\leqq}$ there exists an oblique block norm γ such that $\hat{\mathbf{z}}$ is a solution of problem (2.20)*.*

Proof. Let $\hat{\mathbf{z}}$ be a solution of problem (2.20) where γ is an oblique block norm with unit ball B. For the sake of contradiction suppose that $\hat{\mathbf{z}} \notin N(Z)$.

Thus there exists $\mathbf{z} \in Z, \mathbf{z} \leq \hat{\mathbf{z}}$. Since γ is an oblique block norm it follows that

$$(\mathbf{z} + R^n_{\geqq}) \cap R^n_{\leqq} \subseteq \gamma(\mathbf{z})B \cap R^n_{\leqq}$$

and furthermore

$$(\mathbf{z} + R^n_{\geqq}) \cap R^n_{\leqq} \cap \partial(\gamma(\mathbf{z})B) = \{\mathbf{z}\}$$

by Lemma 2.16.

Then $\hat{\mathbf{z}} \in int(\gamma(\mathbf{z})B)$ because $\hat{\mathbf{z}} \in \mathbf{z} + R^n_{\geqq}$. Therefore $\gamma(\mathbf{z}) > \gamma(\hat{\mathbf{z}})$. Contradiction!

Conversely, let Z be $R_{\geqq}$-convex and $\hat{\mathbf{z}} \in N_p(Z)$. According to Theorem 1.33 there exists $\boldsymbol{\lambda} \in \mathbb{R}^n_{>}$ such that $\boldsymbol{\lambda}^T\hat{\mathbf{z}} \leq \boldsymbol{\lambda}^T\mathbf{z}$ for all $\mathbf{z} \in Z$. Define $P = \{\mathbf{z} \in \mathbb{R}^n : \boldsymbol{\lambda}^T\mathbf{z} \leq \boldsymbol{\lambda}^T(-\hat{\mathbf{z}})\}$. Because $\boldsymbol{\lambda} > \mathbf{0}$, it follows that P is a bounded polyhedron in $\mathbb{R}^n_{\geqq}$. Therefore, $B = R(P \cap \mathbb{R}^n_{\geqq})$ generates an oblique block norm γ, as stated in Theorem 2.18.

Because $\boldsymbol{\lambda}^T(-\hat{\mathbf{z}}) = \boldsymbol{\lambda}^T(-\hat{\mathbf{z}})$, it follows that $\hat{\mathbf{z}} \in \partial B$ and hence $\gamma(\hat{\mathbf{z}}) = 1$. Suppose there is a point $\mathbf{z} \in Z \cap \mathbb{R}^n_{\leqq}$ such that $\gamma(\mathbf{z}) > \gamma(\hat{\mathbf{z}})$, then $\boldsymbol{\lambda}^T(-\mathbf{z}) > \boldsymbol{\lambda}^T(-\hat{\mathbf{z}})$. However, $\boldsymbol{\lambda}^T(-\mathbf{z}) > \boldsymbol{\lambda}^T(-\hat{\mathbf{z}})$ is equivalent to $\boldsymbol{\lambda}^T\mathbf{z} < \boldsymbol{\lambda}^T\hat{\mathbf{z}}$. Contradiction! □

Notice that solving problem (2.20) with an oblique norm does not guarantee that the obtained point is properly nondominated. The reason is that because problem (2.20) does not consider all points $\mathbf{z} \in N(Z)$, unless our point $\mathbf{u} \in Z + \mathbb{R}^n_{\geqq}$ is such that $\mathbf{u} \geqq \mathbf{z}^\times$ where $\mathbf{z}^\times$ is the nadir point.

Example 4. *We present a situation where the maximizer w.r.t. a oblique block norm in $Z \cap \mathbb{R}^n_{\leqq}$ is improperly nondominated.*

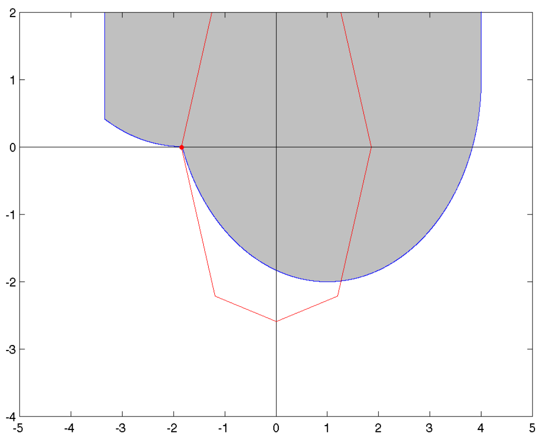

Figure 2.1: The maximizer of an oblique norm in the negative orthant is not necessarily a properly nondominated point.

Remark 2.31. *To consider all nondominated points in problem* (2.20) *we need to center an absolute block norm at a point $\mathbf{u} \geqq \mathbf{z}^\times$. However, computation of* *the nadir point is often difficult and therefore it is usually estimated [Ehrgott, 2003].*

As in the previous section and according to the results derived for weighted scalarization methods we need to consider both oblique and absolute block norms if we need to cover the whole the nondominated set.

The results of this section are the building blocks for the algorithms presented in Chapter 3. In these algorithms, a (weakly) nondominated point is obtained by solving problem (2.20) using a block norm γ. The block norm is updated by adding the newly founded point in its unit ball. By repeating these steps successively, we construct an polyhedral approximation of $N(Z)$.

Chapter 3

Inner and Outer Approximation

In this chapter we will review and propose some modifications to the Inner and Outer Approximations presented in Klamroth et al. [2002]. For the sake of simplicity we first describe the algorithms in the simple case of bicriteria problems. Then we will extend the idea of the algorithms idea to higher dimensions.
Throughout this chapter we will make the following assumption:

Assumption 3.1. *Let $Z \subseteq \mathbb{R}^n$, we assume that Z is $\mathbb{R}_{\geqq}$-convex. Furthermore we consider a so called* ***reference point*** $\mathbf{z}^0 \in Z + \mathbb{R}^n_{\geqq}$*, and w.l.o.g. assume that* $\mathbf{z}^0 = \mathbf{0}$.

3.1 Inner Approximation

The idea of an *inner approximation* is to generate a polyhedron P_I such that $P_I \subseteq (Z + \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\leqq}$, where all extreme points $\mathbf{v}^i$ of P_I are nondominated with respect to Z, i.e. $\mathbf{v}^i \in N(Z) \cap \mathbb{R}^n_{\leqq} = N(Z + \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\leqq}$.

Let $\mathbf{z}^1, \ldots, \mathbf{z}^m \in \mathbb{R}^n$ such that

$$\{\mathbf{z}^1, \ldots, \mathbf{z}^m\} \subseteq N(Z) \cap \mathbb{R}^n_{\leqq}$$

then $P_I = convex(\{\mathbf{z}^0, \mathbf{z}^1, \ldots, \mathbf{z}^m\})$, i.e the convex hull of $\mathbf{z}^1, \ldots, \mathbf{z}^m$ and $\mathbf{z}^0 = \mathbf{0}$, is a polyhedron such that $P_I \subseteq (Z + \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\leqq}$ by convexity of $Z + \mathbb{R}^n_{\geqq}$ and $\mathbb{R}^n_{\leqq}$.

P_I is called *inner polyhedron* of Z or *inner polyhedral approximation* of $N(Z)$ and the set $\{\mathbf{z}^1, \ldots, \mathbf{z}^m\} \subseteq N(Z)$ is called *finite point approximation* of $N(Z)$.

Remark 3.2. *If* $\{\mathbf{z}^1, \ldots, \mathbf{z}^m\} \subseteq N(Z) \cap \mathbb{R}^n_{\leqq}$ *and* $P_I = convex(\{\mathbf{z}^0, \mathbf{z}^1, \ldots, \mathbf{z}^m\})$ *then* $\{\mathbf{z}^1, \ldots, \mathbf{z}^m\} \subseteq \partial P_I$.

Let $\mathbf{z} \in N(Z) \cap int(P_I)$*. If* $\mathbf{z} \in int(P_I)$ *it follows that there exists* $\hat{\mathbf{z}} \in P_I \subseteq Z + \mathbb{R}^n_{\geqq}$ *such that* $\hat{\mathbf{z}} < \mathbf{z}$*. So* $\mathbf{z} \notin N(Z + \mathbb{R}^n_{\geqq}) = N(Z)$ *which yields a contradiction.*

3.1.1 Bicriteria Inner Approximation Algorithm

In this subsection we derive an algorithm to obtain an inner approximation of the nondominated set in the bicriteria case.

Let us construct an inner polyhedral approximation P_I of $N(Z)$ with $P_I = convex(\{\mathbf{z}^0, \mathbf{z}^1, \ldots, \mathbf{z}^m\})$ and $\mathbf{z}^1, \ldots, \mathbf{z}^m \in N(Z)$. A natural way to start our approximation is to let $\mathbf{z}^1$ be a solution of the problem $lexmin\{(z_1, z_2) : \mathbf{z} \in Z\}$, and let $\mathbf{z}^2$ be a solution of $lexmin\{(z_2, z_1) : \mathbf{z} \in Z\}$.

Here, $lexmin\{(z_1, z_2) : \mathbf{z} \in Z\}$ indicates that we are minimizing w.r.t. the *lexicographical order* of the vector $\hat{\mathbf{z}} = (z_1, z_2)^T, \mathbf{z} \in Z$ as defined in Chapter 1.

Let the reference point $\mathbf{z}^0 = (z_1^2, z_2^1)^T$. Then the nadir point $\mathbf{z}^\times \geqq \mathbf{z}^0$ by definition. Furthermore, any $\mathbf{z} \in Z, z_1 \geq z_1^0$ is dominated by $\mathbf{z}^2$ and, similarly, any $\mathbf{z} \in Z, z_2 \geq z_2^0$ is dominated by $\mathbf{z}^1$. Therefore $\mathbf{z}^0 = \mathbf{z}^\times$.

We assume w.l.o.g. that $\mathbf{z}_0 = \mathbf{0}$, then our initial approximation looks like $P_I = convex\{\mathbf{z}^0, \mathbf{z}^1, \mathbf{z}^2\}$. From $\mathbf{z}^0 = \mathbf{z}^\times$, it is clear that $N(Z) \subseteq \mathbb{R}^2_{\leqq}$.

Lemma 3.3. *In the bicriteria case, let* $\mathbf{n}$ *be the normal to a hyperplane supporting at least two nondominated points, w.l.o.g.* $\mathbf{z}^i$ *and* $\mathbf{z}^j$*, of the inner polyhedral approximation* P_I *of* $N(Z) \cap \mathbb{R}^n_{\leqq}$*. Equivalently,*

$$\begin{aligned} \mathbf{n}^T \mathbf{z}^i = \mathbf{b}, \mathbf{n}^T \mathbf{z}^j = \mathbf{b} \\ \mathbf{n}^T \mathbf{z} \leq \mathbf{b}, \forall \mathbf{z} \in P_I \\ \mathbf{z}^i, \mathbf{z}^j \in N(Z) \cap \mathbb{R}^2_{\leqq} \end{aligned} \tag{3.1}$$

Then $\mathbf{n} < \mathbf{0}$.

Proof. Since $\mathbf{0} \in P_I, \mathbf{z}^i \neq \mathbf{0}$ it follows that $b > 0$. Thus, from equation (3.1), we have that $\mathbf{n} \notin \mathbb{R}^2_{\geqq}$, otherwise $\mathbf{n}^T \mathbf{z}^i < 0$.

Suppose $n_1 > 0, n_2 < 0$. Then $\mathbf{n}^T \mathbf{z}^i = b, \mathbf{n}^T \mathbf{z}^j = b$ implies either $\mathbf{z}^i < \mathbf{z}^j$ or $\mathbf{z}^j < \mathbf{z}^i$, this is a contradiction to $\mathbf{z}^i, \mathbf{z}^j \in N(Z) \cap \mathbb{R}^2_{\leqq}$.

By a similar argument $n_1 < 0, n_2 > 0$ yields a contradiction. Therefore $\mathbf{n} < \mathbf{0}$ □

Thus, all facets of P_I, i.e. ${\mathbf{n}^j}^T \mathbf{z} \leq b_j, i = 1, \ldots, m$, that do not contain the $\mathbf{0}$ vector satisfy, by Lemma 3.3, $\mathbf{n} < \mathbf{0}$.

Then, according to Theorem 2.18, we can use $R(P_I)$ as the unit ball for an oblique block norm γ. Thus, by solving the problem:

$$\max\{\gamma(\mathbf{z}) \text{ s.t. } \mathbf{z} \in Z \cap \mathbb{R}^n_{\leqq}\} \tag{3.2}$$

we obtain a new nondominated point $\hat{\mathbf{z}} \in N(Z)$, following the result of Theorem 2.30.

This new point $\hat{\mathbf{z}} \in N(Z)$ maximizes the gauge γ and thus is the "furthest" nondominated point from P_I, since $\gamma(\mathbf{z}) \leq 1, \forall \mathbf{z} \in P_I$, according to the gauge γ.

By maximizing the block norm γ with unit ball $R(P_I)$ in $Z \cap \mathbb{R}^n_{\leqq}$, the structure of the inner polyhedron P_I is used to compute the furthest nondominated point from it. Furthermore, we include $\hat{\mathbf{z}}$ to the finite point representation and compute a new inner polyhedron $\hat{P}_I$ where $\hat{P}_I = convex\{P_I \cup \{\hat{\mathbf{z}}\}\}$.

This procedure is be applied iteratively to obtain a better inner approximation. The algorithm generates the furthest nondominated point from the current inner polyhedral approximation P_I and updates P_I iteratively with the newest point $\hat{\mathbf{z}}$.

Each new point added to the finite point representation of P_I improves the quality of the inner approximation, as we will see in the following lemma.

Lemma 3.4. *Let $Z \subseteq \mathbb{R}^n$ be $\mathbb{R}_{\geqq}$-convex and γ^k be an approximating gauge constructed from k nondominated points or points on the boundary of Z, where γ^k's unit ball is $B = R(convex(\{\mathbf{z}^0, \mathbf{z}^1, \ldots, \mathbf{z}^k\}) \cap \mathbb{R}^n_{\leqq})$. Let $\hat{\mathbf{z}}$ be the solution of the problem.*

$$\begin{aligned} \max \quad & \gamma^k(\mathbf{z}) \\ s.t \quad & \mathbf{z} \in Z \cap \mathbb{R}^n_{\leqq} \end{aligned} \tag{3.3}$$

Let γ^{k+1} be the updated gauge including the new point $\hat{\mathbf{z}}$. Then

$$\gamma^{k+1}(\mathbf{z}) \leq \gamma^k(\mathbf{z}) \quad \forall \mathbf{z} \in Z \cap \mathbb{R}^n_{\leqq} \tag{3.4}$$

Proof. Let B^k and B^{k+1} be the unit balls of γ^k and γ^{k+1}, respectively.

$B^k \subseteq B^{k+1}$ since B^{k+1} contains all extreme points of B^k, it follows that for all $\mathbf{z} \in Z \cap \mathbb{R}^n_{\leqq}$:

$$\gamma^{k+1}(\mathbf{z}) = \min\{\lambda \geq 0 : \mathbf{z} \in \lambda B^{k+1}\} \leq \min\{\lambda \geq 0 : \mathbf{z} \in \lambda B^k\} = \gamma^k(\mathbf{z}) \tag{3.5}$$

□

If we have an inner polyhedron P_I^k such that $B^k = R(P_I^k)$ defines an absolute block norm γ^k, then for $\lambda^k = \max\{\gamma^k(\mathbf{z}) : \mathbf{z} \in Z \cap \mathbb{R}^n_{\leqq}\}$ we know that $N(Z) \cap \mathbb{R}^n_{\leqq} \subseteq Z \cap \mathbb{R}^n_{\leqq} \subseteq \lambda^k P_I^k$.

Let $B^{k+1} = R(P_I^{k+1})$ define an absolute block norm such that $\gamma^{k+1}(\mathbf{z}) \leq \gamma^k(\mathbf{z}) \;\; \forall \mathbf{z} \in Z \cap \mathbb{R}^n_{\leqq}$ then $\lambda^{k+1} = \max\{\gamma^{k+1}(\mathbf{z}) : \mathbf{z} \in Z \cap \mathbb{R}^n_{\leqq}\} \leq \lambda^k$, therefore $N(Z) \cap \mathbb{R}^n_{\leqq} \subseteq Z \cap \mathbb{R}^n_{\leqq} \subseteq \lambda^{k+1} P_I^{k+1}$.

However, problem (3.2) cannot be solved the way is formulated, thus we need an equivalent problem.

Lemma 3.5. *For an absolute block norm γ consider problem (3.2), and let*

$$\mathbf{n}^1, \mathbf{n}^2, \ldots, \mathbf{n}^s \in \mathbb{R}^n_{\leqq}$$

be the normal vectors to the facets of the unit ball B that generates γ such that $\{\mathbf{z} \leqq 0 : {\mathbf{n}^j}^T \mathbf{z} \leq 1, j = 1, \ldots, s\} = B \cap \mathbb{R}^n_{\leqq}$.

*Solve the following so called **hyperplane maximization over** Z problem for each normal $\mathbf{n}^j, j = 1, \ldots, s$:*

$$\begin{aligned} \max \quad & {\mathbf{n}^j}^T \mathbf{z} \\ s.t. \quad & \mathbf{z} \in Z \cap \mathbb{R}^n_{\leqq} \end{aligned} \tag{3.6}$$

with corresponding optimal solutions $\hat{\mathbf{z}}^j$ and maximum $\hat{\lambda}_j = {\mathbf{n}^j}^T \hat{\mathbf{z}}^j$. And let $\lambda^* = \max\{\hat{\lambda}_j, j = 1, \ldots, s\}$ and $\mathbf{z}^* = \hat{\mathbf{z}}^j$ such that $\lambda^* = \hat{\lambda}_j$.

Then $\lambda^* = \gamma(\mathbf{z}^*)$ is a solution to problem (3.2).

Proof. Let $\bar{\mathbf{z}} \in Z \cap \mathbb{R}^n_{\leqq}$ such that $\bar{\lambda} = \gamma(\bar{\mathbf{z}}) = \max\{\gamma(\mathbf{z}) : \mathbf{z} \in Z \cap \mathbb{R}^n_{\leqq}\}$. Thus,

$$\bar{\mathbf{z}} \in \{\mathbf{z} \leqq 0 : \mathbf{n}^{j^T}\mathbf{z} \leq \bar{\lambda}, j = 1, \dots, s\} = \bar{\lambda}B \cap \mathbb{R}^n_{\leqq}$$

and there exists $\bar{j}$ such that $\mathbf{n}^{\bar{j}^T}\bar{\mathbf{z}} = \bar{\lambda}$. But then, $\lambda^* \geq \hat{\lambda}_{\bar{j}} \geq \bar{\lambda}$.

Conversely, let j^* be such that $\lambda^* = \hat{\lambda}_{j^*} = \mathbf{n}^{j^{*T}}\mathbf{z}^*$. Thus,

$$\mathbf{z}^* \in \{\mathbf{z} \leqq 0 : \mathbf{n}^{j^T}\mathbf{z} \leq \lambda^*, j = 1, \dots, s\} = \lambda^* B \cap \mathbb{R}^n_{\leqq}$$

and $\lambda^* = \mathbf{n}^{j^{*T}}\mathbf{z}^*$. Therefore,

$$\mathbf{z}^* \in \partial(\lambda^* B) \cap \mathbb{R}^n_{\leqq}$$

and $\gamma(\mathbf{z}^*) = \lambda^* \leq \bar{\lambda}$. □

Thus, if we have an inner polyhedral approximation P_I that generates an absolute block norm – always true in the bicriteria case by Lemma 3.3 and Theorem 2.18 – we can solve a *hyperplane maximization problem* (3.6) for each of the facets of P_I.

However, we have a V-representation of $P_I = \mathit{convex}(\{\mathbf{z}^0, \mathbf{z}^1, \dots, \mathbf{z}^m\})$. Then, in order to obtain a H-representation

$$P_I = \{\mathbf{z} \leqq 0 : \mathbf{n}^{j^T}\mathbf{z} \leq 1, j = 1, \dots, s\}$$

we need a convex hull algorithm.

Furthermore, since we iteratively update P_I with the newest point $\mathbf{z}^*$, i.e. $\mathit{convex}(P_I \cup \{\mathbf{z}^*\})$, we need all new and modified facets of P_I after each iteration.

Klamroth et al. [2002] proposes to use a convex hull algorithm based in the *beneath and beyond theorem* by Grünbaum. Here we state a simplified version of the beneath and beyond theorem. Which is used in the case of points in general position but can be easily extended by triangulating all facets.

Theorem 3.6 (Simplified Beneath and beyond)**.** *Let H be a convex hull in $\mathbb{R}^d$ and let $\mathbf{p}$ be a point in $\mathbb{R}^d \setminus H$. Then the following holds true for all faces f of $\mathit{convex}(\mathbf{p} \cup H)$:*

1. *F is also a facet of H iff $\mathbf{p}$ is below F.*

2. *F is not a facet of H iff its apex is $\mathbf{p}$ and its base is a ridge of H with one incident facet below $\mathbf{p}$ and the other incident facet above $\mathbf{p}$.*

Proof. See [Grünbaum, 1963]. □

However, the beneath and beyond theorem implies that for the bicriteria case, the new point $\mathbf{z}^*$ solution of problem (3.2), is only above one facet F of P_I.

Remark 3.7. *The new nondominated point $\mathbf{z}^*$, the solution of problem* (3.2)*, is only above one facet F of P, i.e. the one with normal $\mathbf{n}^*$ such that $\mathbf{n}^{*T}\mathbf{z}^* = \lambda^*$. Otherwise the new convex hull would contain a nondominated point in its interior and this is a contradiction.*

Therefore, Theorem 3.6 implies the generation of at most two facets with each of the two nondominated points that define F.

Thus, in bicriteria problems, each update of the inner polyhedral approximation P_I in the bicriteria case, substitutes the facet that yielded the biggest distance by two new facets.

We present the following algorithm that yields an finite point approximation for bicriteria problems. The algorithm's input is a bicriteria optimization problem and two stopping criteria, i.e. the maximum number of nondominated points to generate $MaxPoints$ and a desired approximation quality $\epsilon > 0$ such that the algorithm stops if:

$$\max\{\gamma(\mathbf{z}) : \mathbf{z} \in Z \cap \mathbb{R}^n_{\leqq}\} \leq 1 + \epsilon \tag{3.7}$$

The algorithm's output is the computed nadir point $\mathbf{z}^0$ and a set of nondominated points $N = \{\mathbf{z}^1, \dots, \mathbf{z}^m\}, m < MaxPoints$

Algorithm 1. *Inner Approximation Bicriteria Case.*

---

input:
- *A Bicriteria Optimization Problem, i.e.* $\min\{z_1, z_2 : \mathbf{z} \in Z\}$*, where* Z *is* $\mathbb{R}_{\geqq}$*-convex.*
- *Stopping parameters* $MaxPoints$ *and* ϵ*.*

output:
- *A sequence of nondominated points* N *and the nadir point* $\mathbf{z}^0 \in Z + \mathbb{R}^2_{\geqq}$*.*

begin

Compute $\mathbf{z}^1$ *the solution of the problem lexmin*$\{(z_1, z_2) : \mathbf{z} \in Z\}$.
Compute $\mathbf{z}^2$ *the solution of the problem lexmin*$\{(z_2, z_1) : \mathbf{z} \in Z\}$.
$N \leftarrow \{\mathbf{z}^1, \mathbf{z}^2\}$
Set the nadir point $\mathbf{z}^0$*, i.e.* $\mathbf{z}^0 = (z^2_1, z^1_2)^T$*, w.l.o.g.* $\mathbf{z}^0 = \mathbf{0}$.
Obtain $\mathbf{n}^1$ *such convex*$(\{\mathbf{z}^0, \mathbf{z}^1, \mathbf{z}^2\}) = \{\mathbf{z} \leqq \mathbf{0} : {\mathbf{n}^1}^T \mathbf{z} \leq 1\}$.
Solve problem $\hat{\lambda}_1 = \max\{{\mathbf{n}^1}^T \mathbf{z} : \mathbf{z} \in Z \cap \mathbb{R}^2_{\leqq}\}$ *with solution* ${\mathbf{n}^1}^T \hat{\mathbf{z}}^1 = \hat{\lambda}_1$.
push $(\mathbf{n}^1, \hat{\mathbf{z}}^1, \hat{\lambda}_1)$ *into a priority queue* Q.
pop *the maximum* λ^* *from* Q*, thereby obtaining* $(\mathbf{n}^*, \mathbf{z}^*, \lambda^*)$.
while $\lambda^* - 1 > \epsilon$ ***and*** $|N| < MaxPoints$
    $N \leftarrow N \cup \{\mathbf{z}^*\}$
    Obtain the two hyperplanes generated by $\mathbf{z}^*$ *and the two supporting points of* $\mathbf{n}^*$ *respectively.*
    for-each *new facet with normal* $\mathbf{n}^j$
        Solve problem $\hat{\lambda}_j = \max\{{\mathbf{n}^j}^T \mathbf{z} : \mathbf{z} \in Z \cap \mathbb{R}^2_{\leqq}\}$.
        push $(\mathbf{n}^j, \hat{\mathbf{z}}^j, \hat{\lambda}_j)$ *into* Q.
    end-for-each
    pop $(\mathbf{n}^*, \mathbf{z}^*, \lambda^*)$ *from* Q.
end-while
output $(N, \mathbf{z}^0)$

end

Definition 3.8. *In Algorithm 1, the value*

$$\max\{\gamma(\mathbf{z}) : \mathbf{z} \in Z \cap \mathbb{R}^n_{\leqq}\} - 1$$

where γ is a block norm with unit ball $R(convex(\{\mathbf{z}^0, \mathbf{z}^1, \ldots, \mathbf{z}^m\}))$, is called ***inner gauge approximation error*** *between the inner polyhedral approximation and $N(Z)$. Notice that the* ***inner gauge approximation error*** *is used as a* ***algorithmic quality criterion*** *for the bicriteria inner approximation algorithm.*

In the next subsection we generalize the bicriteria inner approximation algorithm so it works for further dimensions.

3.1.2 Inner Approximation Algorithm

Unfortunately some of the results that we obtained in the two dimensional case do not hold in the higher dimension scenario. In the following, we consider the difficulties that appear when implementing the inner approximation algorithm for higher dimensions.

1. There is no way to easily compute the facet representation of the inner polyhedron P_I in each iteration, as done in Remark 3.7 for the bicriteria case, since the ridges are faces which are not points and thus each ridge is not forced to be contained in ∂P. Therefore we need to use general convex hull algorithms in our implementations.

2. The H-representation of the inner polyhedron P_I in higher dimensions may have facets whose normal vector has positive components, as opposed to Lemma 3.3. To illustrate this, we consider the unit sphere centered a the origin and the nondominated points $(-1,0,0)^T$,$(0,-1,0)^T$,$(0,0,-1)^T$, and $(-\sqrt{1/3}, -\sqrt{1/3}, -\sqrt{1/3})^T$. The three facets from H-representation of P_I contain positive components, as it can be seen in Figure 3.1.

3. The nadir point $\mathbf{z}^\times$, i.e. $\mathbf{z}_i^\times = \max\{z_i : \mathbf{z} \in N(Z)\}$, is difficult to compute in higher dimensions without characterizing the whole nondominated set.

The lack of an inner polyhedron $P_I = convex(\{\mathbf{z}^0, \mathbf{z}^1, \ldots, \mathbf{z}^m\})$, where $\mathbf{z}^0$ is the reference point and $\{\mathbf{z}^1, \ldots, \mathbf{z}^m\}$ is a finite point approximation of the nondominated set, which can be used to generate an absolute block norm is tackled by Klamroth, Tind, and Wiecek [2002] by proposing, instead, the solution of the following problem for each fundamental cone C_j of P_I with fundamental vectors $\{\mathbf{v}^i : i \in I_j\}$.

$$\begin{aligned} \max \quad & \sum_{i \in I_j} \lambda_i \\ \text{s.t} \quad & \sum_{i \in I_j} \lambda_i \mathbf{v}^i \geqq \mathbf{z} \\ & \lambda_i \geq 0 \ \forall i \in I_j \\ & \mathbf{z} \in Z \end{aligned} \tag{3.8}$$

We refer to problem (3.8) as *gauge maximization within a cone* problem.

Each fundamental cone corresponds to a facet of the H-representation of the inner polyhedron. Thus, as previously, because we have the V-representation, we need to change representations using a convex hull algorithm. The extreme

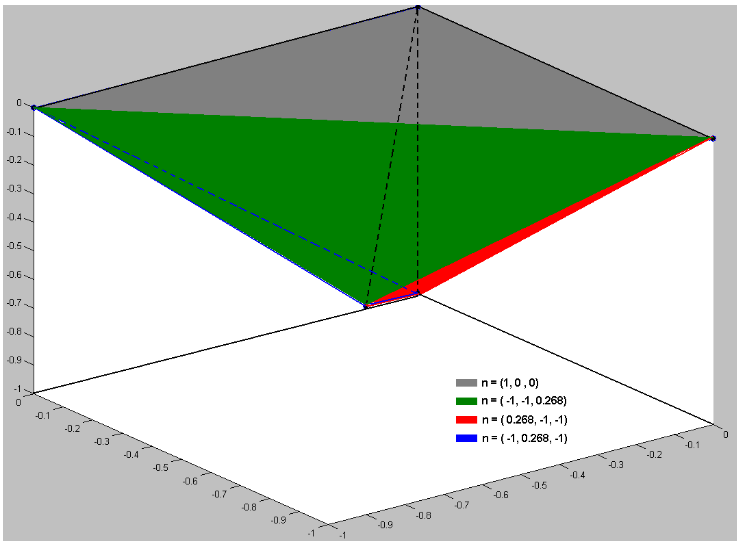

Figure 3.1: The resulting inner polyhedron in higher dimensions may have facets with positive components in their normals.

points $\mathbf{v}^i, i \in I_j$ which belong to the same cone C_j are in the same facet of the H-representation.

The *gauge maximization within a cone* problem finds the point $\sum_{i \in I_j} \hat{\lambda}_i \mathbf{v}^i$ inside the fundamental cone C_j which maximizes the polyhedral gauge and belongs to the set $Z + \mathbb{R}^n_{\geqq}$. It also obtains the corresponding point $\hat{\mathbf{z}}^j \in Z$ dominating $\sum_{i \in I_j} \hat{\lambda}_i \mathbf{v}^i$.

Note that $\sum_{i \in I_j} \lambda_i \mathbf{v}^i$ is a vector in $Z + \mathbb{R}^n_{\geqq}$ that by definition lies in the cone C_j. However, $\hat{\mathbf{z}}^j$ is not necessarily in C_j. Furthermore, If we substitute $\mathbf{z} \in Z$ with $\mathbf{z} \in Z + \mathbb{R}^n_{\geqq}$ we will get the same solution $\sum_{i \in I_j} \hat{\lambda}_i$, however the corresponding point $\hat{\mathbf{z}}^j$ will be in $Z + \mathbb{R}^n_{\geqq}$ thus it would in general be dominated by the solution in the original problem (3.8).

If problem (3.8) is solved for each fundamental cone C_j, and then taking the fundamental cone C_{j^*} and its solution to problem (3.8), $\mathbf{z}^*, \lambda^*_i$, such that $\sum_{i \in I_{j^*}} \lambda^*_i \geq \sum_{i \in I_j} \lambda_i$ for all fundamental cones C_j, i.e. taking the maximum of the solutions of problem (3.8) for all fundamental cones, we are indeed maximizing the polyhedral gauge with unit ball P_I over $(Z + \mathbb{R}^n_{\geqq}) \cap$ *polycone*$(\{\mathbf{z}^0, \mathbf{z}^1, \ldots, \mathbf{z}^m\})$.

To prove that the solution $\hat{\mathbf{z}}^j \in Z$ of the *gauge maximization within a cone* problem is in $N_w(Z)$, we will need the following theorem.

Theorem 3.9. *Let Z be $\mathbb{R}_{\geqq}$-convex and let C_j be a fundamental cone of the polyhedron:*

$$P = convex(\{\mathbf{z}^0, \mathbf{z}^1, \ldots, \mathbf{z}^m\}),$$

where $\mathbf{z}^0 = \mathbf{0}$ and $\mathbf{z}^1, \ldots, \mathbf{z}^m \in N_w(Z) \cap \mathbb{R}^n_{\leqq}$. Then the optimal solution of problem (3.8) is weakly nondominated.

Proof. Let $\hat{\mathbf{z}}^j, \hat{\lambda}_i \ \forall i \in I_j$ be the optimal solution of (3.8). Then there exist dual multipliers [Rockafeller 1970, Section 28, Pages 277-283], $\hat{\mathbf{u}} \geqq 0$ such that $\hat{\mathbf{z}}^j$ is the optimal solution of the problem:

$$\begin{aligned} \max \quad & \sum_{i \in I_j} \lambda_i - \hat{\mathbf{u}}^T(\mathbf{z} - \sum_{i \in I_j} \lambda_i \mathbf{v}^i) \\ \text{s.t} \quad & \mathbf{z} \in Z \\ & \lambda_i \geq 0 \ \forall i \in I_j \end{aligned} \tag{3.9}$$

Then the objective function can be rewritten as:

$$\sum_{i \in I_j} \lambda_i - \hat{\mathbf{u}}^T(\mathbf{z} - \sum_{i \in I_j} \lambda_i \mathbf{v}^i) = \sum_{i \in I_j} \lambda_i (1 + \hat{\mathbf{u}}^T \mathbf{v}^i) - \hat{\mathbf{u}}^T \mathbf{z} \tag{3.10}$$

The existence of the solution $\hat{\mathbf{z}}^j$ implies that $(1 + \hat{\mathbf{u}}^T \mathbf{v}^i) \leq 0$, otherwise the problem is unbounded. Moreover, an optimal solution of (3.10) must satisfy:

$$\sum_{i \in I_j} \lambda_i (1 + \hat{\mathbf{u}}^T \mathbf{v}^i) = 0$$

because $\sum_{i \in I_j} \lambda_i (1 + \hat{\mathbf{u}}^T \mathbf{v}^i) \leq 0$ and it does not depend on $\mathbf{z}$. Thus problem (3.8) can be replaced by:

$$\begin{aligned} \min \quad & \hat{\mathbf{u}}^T \mathbf{z} \\ \text{s.t} \quad & \mathbf{z} \in Z \end{aligned} \tag{3.11}$$

whose solution is weakly nondominated by Theorem 1.27. □

As in Lemma 1.30, we get the following remark.

Remark 3.10. *If $\hat{\mathbf{z}}^j$ is the unique solution of problem* (3.8) *then $\hat{\mathbf{z}}^j \in N(Z)$.*

Thus, with the above approach we can generate weakly nondominated points in the case where it is not possible to construct an absolute block norm from the inner polyhedron P_I. However, we still need to address the problem of choosing the reference point $\mathbf{z}^0$ and the initial finite point approximation of $N(Z)$ given by: $N = \{\mathbf{z}^1, \ldots, \mathbf{z}^m\}$.

Concerning the choice of $\mathbf{z}^0$, we cannot guarantee the "optimality" of $\mathbf{z}^0$. This means that, in general, we cannot compute $\mathbf{z}^\times$ and then let $\mathbf{z}^0 = \mathbf{z}^\times$ as we did in the bicriteria case.
Thus, there are two choices concerning the election of the reference point.

1. We try to estimate the nadir point, e.g. using a payoff table, and then let $\mathbf{z}^0 \simeq \mathbf{z}^\times$. The multicriteria problem may benefit because less feasible solutions are considered for problem (3.8). However, we cannot ensure that $N(Z) \subseteq (\mathbf{z}^0 + \mathbb{R}^n_{\leqq})$.

2. We could take an upper bound for the outcome space that is implicit in the original model, i.e. we let $\mathbf{z}^0$ be such that $Z \subseteq (\mathbf{z}^0 + \mathbb{R}^n_{\leqq})$.

In this paper we ensure that $Z \subseteq (\mathbf{z}^0 + \mathbb{R}^n_{\geqq})$. Unfortunately, even in this case there is no appropriate election of $\{\mathbf{z}^1, \ldots, \mathbf{z}^m\}$ such that:

$$N(Z) \subseteq polycone(\{\mathbf{z}^1, \ldots, \mathbf{z}^m\})$$

because, otherwise, it would imply that we know all nondominated point in the boundary of $N(Z)$. Thus, there is no way to ensure "full coverage" of $N(Z)$.

Now we present the corresponding generalized Inner Approximation Algorithm for more than two criteria. In this case we also include a stopping parameter $MaxCones$ to limit the maximum number of *gauge maximization within a cone* problems (3.8) which are solved by the algorithm.

Algorithm 2. *Inner Approximation Algorithm.*

---

input:
- *A Multicriteria Optimization Problem, i.e.* $\min\{\ \mathbf{z} :\ \mathbf{z} \in Z\}$*, where* Z *is* $\mathbb{R}_{\geqq}$*-convex.*
- *A point* $\mathbf{z}^0$*, assumed to be* $\mathbf{0}$ *and an initial approximation given by* $\{\mathbf{z}^1, \ldots, \mathbf{z}^m\} \subseteq N_w(Z) \cap \mathbb{R}^n_{\geqq}$.
- *Stopping parameters* $MaxPoints$, $MaxCones$ *and* ϵ.

output:
- *A sequence of weakly nondominated points* N.

begin

1: Use a convex hull algorithm to obtain the H-representation of $P_I = convex\{\mathbf{z}^0, \mathbf{z}^1, \ldots, \mathbf{z}^m\}$.
2: $c \leftarrow 0$
3: ***for-each*** *fundamental cone* C_j *of* P_I
4: *Solve problem* (3.8) *with optimal solution* $\hat{\mathbf{z}}^j, \hat{\lambda}_i, i \in I_j$ *and let* $\sigma_j = \sum_{i \in I_j} \hat{\lambda}_i$.
5: $c \leftarrow c + 1$
6: ***push*** $(C_j, \hat{\mathbf{z}}^j, \sigma_j)$ *into priority queue* Q.
7: ***end-for-each***
8: ***pop*** *the maximum* σ^* *from* Q*, therefore obtaining* $(C_{j^*}, \mathbf{z}^*, \lambda^*)$.
9: ***while*** $\sigma^* - 1 > \epsilon$ ***and*** $|N| < MaxPoints$ ***and*** $c < MaxCones$
10: $N \leftarrow N \cup \{\mathbf{z}^*\}$
11: *Using a convex hull algorithm, take the H-representation of* $P_I' = convex(P_I \cup \{\mathbf{z}^*\})$ *and identify new and modified fundamental cones.*
12: ***for-each*** *new and modified fundamental cone* C_j *in* P_I' *with respect to* P_I
13: *Solve problem* (3.8) *with optimal solution* $\hat{\mathbf{z}}^j, \hat{\lambda}_i, i \in I_j, \sigma_j = \sum_{i \in I_j}$.
14: $c \leftarrow c + 1$
15: ***push*** $(C_j, \hat{\mathbf{z}}^j, \sigma_j)$ *into priority queue* Q.
16: ***end-for-each***
17: $P \leftarrow P'$
18: ***pop*** $(C_{j^*}, \mathbf{z}^*, \sigma^*)$ *from* Q.
19: ***end-while***
20: ***output*** N

end

The two main differences between algorithm 1 and 2 are:

1. Since the inner polyhedron P is not necessarily *absolute*, we need to solve problem (3.8) instead of problem (3.6). Note that problem (3.8) has $n+|I_j|$ more constraints, i.e. the dimension of the outcome space plus the number of fundamental vectors in the fundamental cone.

2. We need to use of a general convex hull algorithm. In our numerical experiments we use *qhull* [Barber et al., 1996] which is based on the beneath and beyond theorem.

Definition 3.11. *In Algorithm 2, the value*

$$\sigma^* - 1 = \max_{C_j}\{\sigma^j = \sum_{i \in I_j} \hat{\lambda}_i\} - 1,$$

where $C_j, j = 1, \ldots, m$ are each of the fundamental cones of P_I and σ^j is the corresponding optimal value of problem (3.8) *for each cone C_j, is called* `pseudo inner gauge approximation error` *between the P_I and $N(Z)$. Notice that the* `pseudo inner gauge approximation error` *is an* `algorithmic quality criterion` *for the inner approximation algorithm.*

3.1.3 Positive Orthant Variant of the Inner Approximation Algorithm

The first proposed variant adds the positive orthant to the polyhedron $P_I = convex(\{\mathbf{z}^0\} \cup N)$. Thereby we get a polyhedron $P'_I = P_I + \mathbb{R}^n_{\geqq}$ that is a valid unit ball for an absolute block norm.

Remark 3.12. *The polyhedron $P'_I = (convex(\{\mathbf{z}^0\} \cup N) + \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\leqq}$, where N is our current finite point approximation of $N(Z)$, can be used to generate an absolute block norm. Then, by using P'_I as the inner polyhedron, it will be sufficient to solve a hyperplane maximization problem* (3.6)*, instead of a gauge maximization within a cone problem* (3.8)*.*

Since we use problem (3.6)*, we compute the* `inner gauge approximation error`*, as defined in Definition 3.8.*

Furthermore, $\mathbb{R}^n_{\leqq} \subseteq polycone(P'_I)$, thus providing "full coverage" of the negative orthant. This means that if $N(Z) \subseteq \mathbb{R}^n_{\leqq}$ then all points in $N(Z)$ can be, in principle, generated by the modified algorithm.

There are different ways to add the positive orthant to the inner polyhedron.

- To generate P'_I we may consider 2^n times the number of points in the convex hull algorithm, i.e. all the original points and their 2^n projections in each one of the faces that define the negative orthant. These points suffice to describe $(P_I + \mathbb{R}^n) \cap \mathbb{R}^n_{\leqq}$.

  In other words, instead of considering $P_I = convex\{\mathbf{z}^0, \mathbf{z}^1, \ldots, \mathbf{z}^m\}$ we take $P'_I = convex\{H(\mathbf{z}^0) \cup H(\mathbf{z}^1) \cup \ldots \cup H(\mathbf{z}^m)\}$, where

  $$H(\mathbf{z}) = \frac{1}{2}(R(\mathbf{z}) - \mathbf{z})$$

  is an hypercube such that two of its vertices are $\mathbf{z}^0 = \mathbf{0}$ and $\mathbf{z} \in \mathbb{R}^n_{\leqq}$.

- To obtain P'_I we may use a convex hull algorithm which computes the H-representation of a polyhedron given by their extreme points and their extreme rays, such as *cdd* [Fukuda], by Theorem 2.1

$$P_I + \mathbb{R}^n_{\geqq} = convex(\{\mathbf{z}^0, \mathbf{z}^1, \ldots, \mathbf{z}^m\}) + polycone(\{\mathbf{e}^1, \ldots, \mathbf{e}^n\})$$

.

- We may use points in the direction of the extreme rays $\mathbf{e}^1, \ldots, \mathbf{e}^n$ which are sufficiently far from the rest of the points; however we will need to correct some facets and probably discard some of them.

In our implementation, we consider the 2^n projections for each point that is added to the finite set representation. Although that not all these points need to be considered. In fact our implementation adds a smaller number of points that are sufficient to define an absolute block norm. In this context, it is interesting to know the total number of points which need to be considered. For our numerical examples, this number is computed and presented in Chapter 5.

3.1.4 Local Update Inner Approximation Algorithm

Another proposed variant is to use a local update instead of depending on the convex hull algorithm to identify all new and modified facets. Thus it will generate new points in less time; however, since point clustering may occur, a greater number of points will be needed to fullfill a desired approximation quality.

We skip the convex hull routine by performing a local update of facets; that is, we only consider the facets that are generated by taking the ridges of the optimal fundamental cone C_{j^*} as their base, and the new generated point, $\mathbf{z}^*$, as their apex.

This means that we run the convex hull algorithm with only the new point and the vertices of the optimal fundamental cone C_{j^*}.

The resulting inner approximation is not necessarily convex and may not be a polyhedron, but we can still use problem (3.8) in each new "pseudocone".

Remark 3.13. *Instead of computing $P_{Inew} = convex(P_I \cup \{\mathbf{z}^*\})$, in the local update we take $P'_{Inew} = P_I \cup convex(\{\mathbf{v}^i : i \in I_{j^*}\} \cup \{\mathbf{z}^*\})$. Obviously $P'_{Inew} \subseteq P_{Inew}$ and therefore our measured gauge γ' with unit ball P'_{Inew} is bigger than the gauge γ whose unit ball is P_{Inew}, i.e.*

$$\gamma'(\mathbf{z}) \geq \gamma(\mathbf{z}) \ \forall \mathbf{z} \in N(Z) \cap \mathbb{R}^n_{\leqq} \tag{3.12}$$

Hence, the real approximation quality is "better" than the one calculated by the local update algorithm.

The advantage of this variant is the local convex hull computation which speeds up the process of finding new points. However, a big disadvantage is the expected clustering and thus, the need of a greater number of points in order to meet a desired quality criterion.

The numeric comparisons with respect to the approximation quality and the required time of these variants of the inner approximation algorithm will be given in Chapter 5.

3.2 Outer Approximation

The *outer polyhedral approximation* is a polyhedron P_O such that $Z+\mathbb{R}^n_{\geqq} \subseteq P_O$. This polyhedron P_O can be obtained by the intersection of affine halfspaces. By convexity of $Z+\mathbb{R}^n_{\geqq}$, it follows that there exists $\mathbf{n} \leq \mathbf{0}$ for each $\bar{\mathbf{z}} \in N_w(Z)$ such that $\mathbf{n}^T\mathbf{z} \leq \mathbf{n}^T\bar{\mathbf{z}}\ \forall \mathbf{z} \in Z$ and thus $\mathbf{n}^T\mathbf{z} \leq \mathbf{n}^T\bar{\mathbf{z}}\ \forall \mathbf{z} \in Z+\mathbb{R}^n_{\geqq}$.

An outer approximation can be constructed from an initial set of weakly non-dominated points $N = \{\mathbf{z}^1, \dots, \mathbf{z}^m\}$ by taking the corresponding subgradient $\mathbf{n}^j \leq \mathbf{0}, j = 1, \dots, m$ and then constructing the polyhedron:

$$P_O = \{\mathbf{z} : {\mathbf{n}^j}^T\mathbf{z} \leq {\mathbf{n}^j}^T\mathbf{z}^j, j = 1, \dots m\}$$

Once we have P_O, we want to successively add a points to P_O to improve the quality of the approximation.

Thus, we want to find the maximum $0 < \lambda \leq 1$ such that $\lambda P_O \cap \mathbb{R}^n_{\leqq} \subseteq (Z+\mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\leqq}$.

Lemma 3.14. *If for all extreme points $\mathbf{v}^i, i = 1, \dots, p$ of the V-representation of the outer polyhedron $P_O \cap \mathbb{R}^n_{\leqq}$, we solve the following so called* ***direction search*** *problem:*

$$\begin{aligned} \max\ & \lambda \\ & \lambda\mathbf{v}^i \geqq \mathbf{z} \\ & \lambda \geq 0 \\ & \mathbf{z} \in Z \end{aligned} \tag{3.13}$$

and thus we obtain the solution of problem (3.13)*, i.e.* $(\lambda^i, \bar{\mathbf{z}}^i)$*, for each* $\mathbf{v}^i, i = 1, \dots, p$.

Then $\lambda^* = \min\{\lambda^i : i = 1, \dots, p\}$*.is the solution of problem* $\max\{\lambda : \lambda P_O \cap \mathbb{R}^n_{\leqq} \subseteq (Z+\mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\leqq}\}$

Proof. Let $\bar{\lambda} = \max\{\lambda : \lambda P_O \cap \mathbb{R}^n_{\leqq} \subseteq (Z+\mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\leqq}\}$, then for all extreme points $\mathbf{v}^i$ of $P_O \cap \mathbb{R}^n_{\leqq}$ we have $\bar{\lambda}\mathbf{v}^i = \mathbf{z}+\mathbf{r}, \mathbf{z} \in Z, \mathbf{r} \in \mathbb{R}^n_{\geqq}$.

Thus, $\lambda^i \geq \bar{\lambda}$ for all $\mathbf{v}^i, i = 1, \dots, p$ and therefore $\lambda^* \geq \bar{\lambda}$.

Conversely, we know that for all extreme points $\mathbf{v}^i, i = 1, \dots, p$ of $P_O \cap \mathbb{R}^n_{\leqq}$ we have that

$$\lambda^*\mathbf{v}^i \in (Z+\mathbb{R}^n_{\geqq})$$

Thus, it follows from $P_O \cap \mathbb{R}^n_{\leqq} = convex(\{\mathbf{v}^i : i = 1, \dots, p\})$ and convexity of $(Z+\mathbb{R}^n_{\geqq})$ that $\lambda^*(P_O \cap \mathbb{R}^n_{\leqq}) \subseteq (Z+\mathbb{R}^n_{\geqq})$. Therefore $\bar{\lambda} \geq \lambda^*$. □

Remark 3.15. *Note that in the direction search problem* (3.13)*,* $\lambda\mathbf{v}^i \in Z+\mathbb{R}^n_{\geqq}$*. Thus we have the same solution* λ *if we substitute* $\mathbf{z} \in Z$ *by* $\mathbf{z} \in Z+\mathbb{R}^n_{\geqq}$*.*

In the outer approximation, we take a generalized gauge $\hat{\gamma}$, whose unit ball is the reflection of a convex body, i.e $R((Z+\mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\leqq})$ and maximizing this gauge $\hat{\gamma}$ over the set $P_O \cap \mathbb{R}^n_{\leqq}$. This is the dual of the inner approximation algorithm, where we maximized the gauge whose unit ball is the reflection of the inner approximation, i.e. $R(P_I \cap \mathbb{R}^n_{\geqq})$ over the set $(Z + \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\leqq}$.

By maximizing the gauge of the norm with unit ball $R((Z + \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\leqq})$ over the outer approximation $P_O \cap \mathbb{R}^n_{\leqq}$, we obtain a point in the outer approximation which is furthest in the sense of the metric induced by the outcome set. The result provides a new (weakly) nondominated point which corresponds to the optimal solution of one of the *direction search* problem (3.13).

Theorem 3.16. *If problem* (3.13) *has a solution* $(\lambda, \bar{\mathbf{z}})$ *then* $\bar{\mathbf{z}} \in N_w(Z)$.

Proof. If $\bar{\mathbf{z}}$ is an optimal solution then there exist dual multipliers $\bar{\mathbf{u}} \geqq \mathbf{0}$ such that $\bar{\mathbf{z}}$ is the optimal solution of the problem:

$$\begin{aligned} \max \quad & \lambda - \bar{\mathbf{u}}^T(\mathbf{z} - \lambda \mathbf{v}^i) \\ \text{s.t} \quad & \mathbf{z} \in Z \\ & \lambda \geq 0 \end{aligned} \tag{3.14}$$

As in theorem 3.9, the objective function can be rewritten as:

$$\lambda - \bar{\mathbf{u}}^T(\mathbf{z} - \lambda \mathbf{v}^i) = \lambda(1 + \bar{\mathbf{u}}^T \mathbf{v}^i) - \bar{\mathbf{u}}^T z \tag{3.15}$$

The existence of a solution implies that $(1 + \bar{\mathbf{u}}^T \mathbf{v}^i) \leq 0$, otherwise the problem is unbounded; furthermore, at an optimal solution, we have that:

$$\lambda(1 + \bar{\mathbf{u}}^T \mathbf{v}^i) = 0$$

so problem (3.14) can be replaced by:

$$\begin{aligned} \min \quad & \bar{\mathbf{u}}^T z \\ \text{s.t} \quad & \mathbf{z} \in Z \end{aligned} \tag{3.16}$$

whose solution is weakly nondominated. □

The outer polyhedron approximation is given by it H-representation, in order to solve problem (3.13) we need to compute a V-representation via the dual of the convex hull algorithm. After obtaining the V-representation of the outer polyhedron, problem (3.13) is solved for each extreme point, hence computing an optimal direction $\mathbf{d}$ and an optimal point $\mathbf{z}^*$ in Z. Finally the outer polyhedral approximation P_O is again updated by intersecting it with the corresponding halfspace supporting $\mathbf{z}^* \in Z$.

However, *qhull* only outputs a H-representation given a V-representation of a polyhedron, thus we need to represent each facet of P_O as a point in a dual space. To do so, we choose an interior point of the outer polyhedron as the origin and represent each hyperplane $\mathbf{n}^T \mathbf{z} = d$ as a point $\mathbf{n}/-d$. Observe that if there exists a hyperplane such that $\mathbf{h}^T \mathbf{n}/-d = b$, it follows that $\mathbf{h}^T \mathbf{n}/-b = d$, which implies that $\mathbf{n}^T \mathbf{h}/b = d$. This means that by choosing an interior point, we can transform the inequality representation in a point representation, use the convex hull algorithm and then transform the resulting hyperplanes back to obtain the corresponding extreme points of P_O.

3.2.1 Bicriteria Outer Approximation Algorithm

Similar to the inner approximation, the geometry of the bicriteria case allows us to speed up the computation of V-representation of the outer polyhedron.

Observation 3.17. *Let $\mathbf{v}^i$ be an extreme point of the outer polyhedron resulting from the intersection of hyperplanes supporting the points $\{\mathbf{z}^1, \dots, \mathbf{z}^m\}$. Then due to $\{\mathbf{z}^1, \dots, \mathbf{z}^m\} \cup \{\mathbf{v}^i\} \subseteq P$, we have that $convex(\{\mathbf{z}^1, \dots, \mathbf{z}^m\} \cup \{\mathbf{v}^i\}) \subseteq P$.*

In the two dimensional case, there are two hyperplanes intersecting in $\mathbf{v}^i$ which support the corresponding points $\mathbf{z}^j$ and $\mathbf{z}^k$. Solving problem (3.13) for $\mathbf{v}^i$ provides a new weakly nondominated point $\hat{\mathbf{z}}$ on a supporting hyperplane $\boldsymbol{\lambda}^T\mathbf{z} + c = 0$ such that $\boldsymbol{\lambda}^T\mathbf{v}^i + c \geq 0$, because $\hat{\mathbf{z}} = \lambda\mathbf{v}^i, \lambda \leq 1$. Moreover, $\boldsymbol{\lambda}^T\mathbf{z}^j + c \leq 0$ and $\boldsymbol{\lambda}^T\mathbf{z}^k + c \leq 0$.

This implies the existence of a point $\mathbf{p} \in convex(\{\mathbf{z}^j, \mathbf{v}^i\})$ such that $\boldsymbol{\lambda}^T\mathbf{p} + c = 0$ and then $\mathbf{p}$ is an extreme point of the new outer polyhedron. This can also be applied to a point in $convex(\{\mathbf{z}^k, \mathbf{v}^i\})$. Moreover, there is no other way to generate further extreme points.

Thus, when we update P_O we just need to consider the intersection points of the two hyperplanes defining the best direction $\mathbf{v}^i = \mathbf{d}$ with the new hyperplane generated by taking the subgradient on the newly generated point $\mathbf{z}^*$.

Algorithm 3. *Outer Approximation Bicriteria Case.*

---

input:
- *A Bicriteria Optimization Problem, i.e.* $\min\{ z_1, z_2 : \mathbf{z} \in Z\}$, *where Z is $\mathbb{R}_{\geqq}$-convex.*
- *Stopping parameters $MaxPoints$ and ϵ.*

output:
- *A sequence of nondominated points N, and the nadir point $\mathbf{z}^0 \in Z + \mathbb{R}^2_{\geqq}$.*

begin
Compute $\mathbf{z}^1$, i.e. the solution of the problem $lexmin\{(z_1, z_2) : \mathbf{z} \in Z\}$.
Compute $\mathbf{z}^2$, i.e. the solution of the problem $lexmin\{(z_2, z_1) : \mathbf{z} \in Z\}$.
$N \leftarrow \{\mathbf{z}^1, \mathbf{z}^2\}$
Set the nadir point $\mathbf{z}^0$, i.e. $\mathbf{z}^0 = (z_1^2, z_2^1)^T$, w.l.o.g. $\mathbf{z}^0 = \mathbf{0}$.
$\mathbf{v}^1 \leftarrow (z_1^1, z_2^2)$
Solve problem (3.13) for $\mathbf{v}^1$ with solution $(\hat{\lambda}_1, \hat{\mathbf{z}}^1, \mathbf{v}^1)$.
push $(\mathbf{v}^1, \hat{\mathbf{z}}^1, \hat{\lambda}_1)$ *into a Priority Queue Q.*
pop *the minimum λ^* from Q, therefore obtaining $(\mathbf{v}^*, \mathbf{z}^*, \lambda^*)$.*
while $1 - \lambda^* > \epsilon$ ***and*** $|N| < MaxPoints$
  $N \leftarrow N \cup \{\mathbf{z}^*\}$
  Obtain the two new extreme points of the outer polyhedron which are defined by the intersection of the supporting hyperplane in $\mathbf{z}^*$ and the two hyperplanes that define $\mathbf{v}^*$.
  for-each *new extreme point $\mathbf{v}^i$*
    Solve problem (3.13) with solution $(\hat{\lambda}_i, \hat{\mathbf{z}}^i, \mathbf{v}^i)$.
    push $(\mathbf{v}^i, \hat{\mathbf{z}}^i, \hat{\lambda}_i)$ *into Q.*
  end-for-each
  pop $(\mathbf{v}^*, \mathbf{z}^*, \lambda^*)$ *from Q.*
end-while

output $(N, \mathbf{z}^0)$
end

3.2.2 Outer Approximation Algorithm

The outer approximation algorithm for higher dimensions follows directly from the scheme presented previously. The only difference with respect to the two dimensional case is the use of a general convex hull algorithm to compute the new extreme points.

The following is the pseudocode for the outer approximation in higher dimensions. As in the inner approximation we include a stopping parameter $MaxExtremePoints$ which limits the number of *direction search* problems (3.13) which are solved by the algorithm.

Algorithm 4. *Outer Approximation Algorithm.*

---

input:
- *A Multicriteria Optimization Problem, i.e.* $\min\{\ \mathbf{z} :\ \mathbf{z} \in Z\}$*, where* Z *is* $\mathbb{R}_{\geqq}$*-convex.*
- *A point* $\mathbf{z}^0$*, assumed to be* $\mathbf{0}$ *and an initial finite point approximation given by* $\{\mathbf{z}^1, \ldots, \mathbf{z}^m\} \subseteq N_w(Z) \cap \mathbb{R}^n_{\leqq}$*, and* $\{\mathbf{n}^1, \ldots, \mathbf{n}^m\}$ *for each* $i = 1, \ldots, m$ *such that* $\mathbf{n}^i \geq \mathbf{0}$ *and* ${\mathbf{n}^i}^T\mathbf{z} \geq {\mathbf{n}^i}^T\mathbf{z}^i\ \forall \mathbf{z} \in Z$ *for all* $i = 1, \ldots, m$
- *Stopping parameters* $MaxPoints$*,* $MaxExtremePoints$ *and* ϵ*.*

output:
- *A sequence of weakly nondominated points* N*.*

begin
Use dual convex hull algorithm to obtain the extreme points of:
$P_O \cap \mathbb{R}^n_{\leqq} = \{\mathbf{z} \leqq 0 : {\mathbf{n}^i}^T\mathbf{z} \geq {\mathbf{n}^i}^T\mathbf{z}^i, i = 1, \ldots, m\}$.
$c \leftarrow 0$
for-each *extreme point* $\mathbf{v}^i$ *of* $P_O \cap \mathbb{R}^n_{\leqq}$
    Solve problem (3.13) *with optimal solution* $(\hat{\mathbf{z}}^i, \hat{\lambda}^i)$.
    $c \leftarrow c + 1$
    push $(\mathbf{v}^i, \hat{\mathbf{z}}^i, \hat{\lambda}^i)$ *into priority queue* Q.
end-for-each
pop *the minimum* λ^* *from* Q*, therefore obtaining* $(\mathbf{v}^*, \mathbf{z}^*, \lambda^*)$.
while $1 - \lambda^* > \epsilon$ ***and*** $|N| < MaxPoints$ ***and*** $c < MaxExtremePoints$
    $N \leftarrow N \cup \{\mathbf{z}^*\}$
    Obtain the extreme points of $P'_O \cap \mathbb{R}^n_{\leqq} = P_0 \cap \mathbb{R}^n_{\leqq} \cap \{\mathbf{z} : {\mathbf{n}^*}^T\mathbf{z} \geq {\mathbf{n}^*}^T\mathbf{z}^*\}$.
    for-each *new and modified extreme point* $\mathbf{v}^i$ *in* $P'_O \cap \mathbb{R}^n_{\leqq}$ *w.r.t.* $P_O \cap \mathbb{R}^n_{\leqq}$
        Solve problem (3.13) *with optimal solution* $(\hat{\mathbf{z}}^i, \hat{\lambda}^i)$.
        $c \leftarrow c + 1$
        push $(\mathbf{v}^i, \hat{\mathbf{z}}^i, \hat{\lambda}^i)$ *into priority queue* Q.
    end-for-each
    $P_O \leftarrow P'_O$
    pop $(\mathbf{v}^*, \mathbf{z}^*, \lambda^*)$ *from* Q.
end-while

20: ***output*** *N*
end

In our implementation the initial approximation is provided by the individual minima in each component, hence the normals are $\mathbf{e}^i$ for $i = 1, \ldots, n$. Therefore, the initial set of extreme points that need to be solved is $2^n - 1$ (all vertices in an hypercube of dimension n except for the $\mathbf{0}$ vector).

Definition 3.18. *In Algorithms 3 and 4, the expression $1 - \lambda^* > \epsilon$ is the algorithmic quality criterion. The value*

$$\frac{1}{\lambda^*} - 1$$

is called `outer gauge approximation error` *between P_O and $N(Z)$. Note that the* `pseudo inner gauge approximation error` *is an* `algorithmic quality criterion` *for the inner approximation algorithm.*

3.2.3 Local Update Outer Approximation Algorithm

A variant is proposed in the same way as with the inner approximation algorithm. In this variant, the new extreme points are just computed by intersecting the newly founded affine halfspace with the affine halfspaces defining the extreme point $\mathbf{v}^*$, i.e. the furthest extreme point in the outer polyhedron from Z.

These new extreme points could be outside the updated outer approximation $P'_O \cap \mathbb{R}^n_{\leqq} = P_0 \cap \mathbb{R}^n_{\leqq} \cap \{\mathbf{z} : \mathbf{n}^{*T}\mathbf{z} \geq \mathbf{n}^{*T}\mathbf{z}^*\}$. For example, in the unit sphere centered at the origin, a local update after the second iteration will generate points that are no longer in the negative orthant. This means that these points lie outside the real outer approximation, and therefore the approximation quality measured for these points will be worse than the real outer polyhedron approximation quality. However, since we are not considering all extreme points, measured quality is expected to be overestimated.

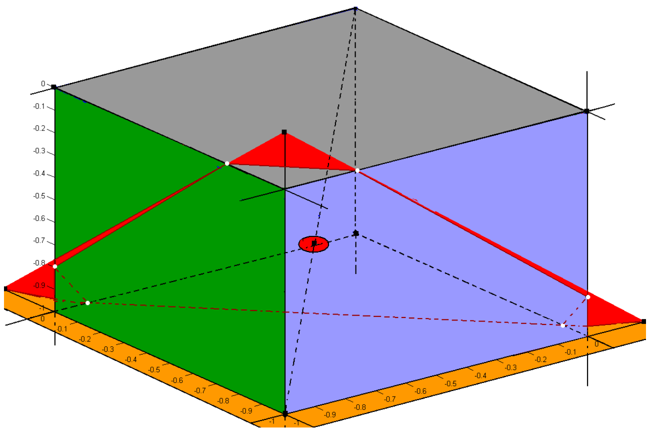

Figure 3.2: Local Update Outer Approximation Algorithm may generate points that are not in the outer polyhedron.

As in the local update inner approximation, the local convex hull computation speeds up the process of finding new points. However, we expect point clustering and thus, the need of a greater number of points are needed to meet a desired quality criterion.

3.3 Simultaneous Inner and Outer Approximation Algorithm

In Klamroth et al. [2002] a simultaneous application of the inner and outer approximation is proposed. Here the algorithm is started with an inner and an outer polyhedron, and we alternately obtain the best point according to the inner or the outer approximation. In each iteration we update both the inner and outer polyhedron.

Algorithm 5. *Simultaneous Inner and Outer Approximation Algorithm.*

---

input:
- *A Multicriteria Optimization Problem, i.e.* $\min\{\ \mathbf{z} :\ \mathbf{z} \in Z\}$*, where* Z *is* $\mathbb{R}_{\geqq}$*-convex.*
- *A point* $\mathbf{z}^0$*, assumed to be* $\mathbf{0}$ *and an initial approximation given by* $\{\mathbf{z}^1, \ldots, \mathbf{z}^m\} \subseteq N_w(Z) \cap \mathbb{R}^n_{\leqq}$*, and* $\{\mathbf{n}^1, \ldots, \mathbf{n}^m\}$ *for each* $i = 1, \ldots, m$ *such that* $\mathbf{n}^i \geq \mathbf{0}$ *and* ${\mathbf{n}^i}^T\mathbf{z} \geq {\mathbf{n}^i}^T\mathbf{z}^i\ \forall \mathbf{z} \in Z$ *for all* $i = 1, \ldots, m$
- *Stopping parameters* $MaxPoints$*,* $MaxCones$*,* $MaxExtPoints$*,* ϵ*.*

output:
- *A sequence of weakly nondominated points* N*.*

begin

Use convex hull algorithm to obtain the fundamental cones of $P_I = convex(\{\mathbf{z}^0, \mathbf{z}^1, \ldots, \mathbf{z}^m\})$*.*
Use dual convex hull algorithm to obtain the extreme points of $P_O = \{\mathbf{z} \leqq \mathbf{0} : {\mathbf{n}^i}^T\mathbf{z} \geq {\mathbf{n}^i}^T\mathbf{z}^i, i = 1, \ldots, m\}$*.*
$c_I \leftarrow 0$
for-each *fundamental cone* C_j *of* P_I
    Solve problem (3.8) *with optimal solution* $(\hat{\mathbf{z}}^j, \sigma^j = \sum_{i \in I_j} \hat{\lambda}_i)$*.*
    $c_I \leftarrow c_I + 1$
    push $(C_j, \hat{\mathbf{z}}^j, \sigma^j)$ *into priority queue* Q_I*.*
end-for-each
$c_O \leftarrow 0$
for-each *extreme point* $\mathbf{v}^i$ *of* $P_)$
    Solve problem (3.13) *with optimal solution* $(\hat{\mathbf{z}}^i, \hat{\lambda}^i)$*.*
    $c_O \leftarrow c_O + 1$
    push $(\mathbf{v}^i, \hat{\mathbf{z}}^i, \hat{\lambda}^i)$ *into priority queue* Q_O*.*
end-for-each
iteration $\leftarrow 0$
pop *the maximum* σ^* *from* Q_I*, therefore obtaining* $(C_{j^*}, \mathbf{z}^*, \sigma^*)$*.*
while $|N| < MaxPoints$ ***and*** $c_I < MaxCones$ ***and*** $c_O < MaxExtPoints$
    if *iteration is an even number* ***and*** $\sigma^* - 1 < epsilon$
        break-while

end-if
if *iteration is an odd number* ***and*** $1 - \lambda^* < epsilon$
  break-while
end-if
$N \leftarrow N \cup \{\mathbf{z}^*\}$
Obtain the fundamental cones of $P'_I = convex(P_I \cup \{\mathbf{z}^*\})$..
Obtain the extreme points of $P'_O = P_O \cap \{z : {\mathbf{n}^*}^T \mathbf{z} \geq {\mathbf{n}^*}^T \mathbf{z}^*\}$.
for-each *new and modified fundamental cone* C_j *in* P'_I *with respect to* P_1
  Solve problem (3.8) *with optimal solution* $(\hat{\mathbf{z}}^j, \sigma^j = \sum_{i \in I_j} \hat{\lambda}_i)$.
  $c_I \leftarrow c_I + 1$
  push $(C_j, \hat{\mathbf{z}}^j, \sigma^j)$ *into priority queue* Q_I.
end-for-each
for-each *new and modified extreme point* $\mathbf{v}^i$ *in* P'_O *with respect to* P_O
  Solve problem (3.13) *with optimal solution* $(\hat{\mathbf{z}}^i, \hat{\lambda}^i)$.
  $c_O \leftarrow c_O + 1$
  push $(\mathbf{v}^i, \hat{\mathbf{z}}^i, \hat{\lambda}^i)$ *into priority queue* Q_O.
end-for-each
$P_I \leftarrow P'_I$
$P_O \leftarrow P'_O$
$iteration \leftarrow iteration + 1$
if *iteration is even*
  pop *the maximum* σ^* *from* Q_I.
else
  pop *the minimum* λ^* *from* Q_O.
end-if
end-while
output N
end

A clear disadvantage of this algorithm is that it needs to run two different convex hull routines. We observe some difficulties implementing this with both convex libraries we are used. *Qhull* is designed to be run like an executable and not like a callable library. If we want to use more than one convex hull computation at a time, we need to save and restore all data through pointer manipulation. In doing this, it is reported that the computational cost increases by 8% [Barber et al., 1996]. *Cdd*, on the other hand, does not have a iterative procedure to add points to a previously calculated polyhedron.

3.4 Sandwiching between the inner and outer polyhedron

However, we do not need to run the two convex hull computations, since it is also possible to use the inner approximation as a substitute of Z in the *direction search* problem (3.13). This approach only needs the application of one convex hull computation and further solutions of linear programs. Moreover, according to Remark 3.15 the *direction search* problem (3.13) will add the positive orthant to the inner approximation.

Once we obtain the furthest extreme point in the outer polyhedron from the inner approximation, we need to solve problem (3.13) once more, using Z, to obtain a new (weakly) nondominated point of Z.

It is clear than we are solving as many *direction search* problems over Z as points generated by the algorithm. The rest is a dual convex hull algorithm implementation to obtain the extreme points of the outer approximation and computing the solution of a bunch of linear programs.

3.4.1 Sandwiching Algorithm

In this algorithm we update the outer approximation and use the inner approximation as a substitute of Z in the *direction search* problem (3.13). In doing this, we obtain the point in the outer approximation that maximizes the gauge whose unit ball is the inner polyhedron plus the positive orthant.

Once we obtain this point we solve a *direction search* problem (3.13) over Z, to obtain a new nondominated point. This point is use to update the inner and the outer polyhedron. The problem we need to solve for each extreme point $\mathbf{v}^i$ in the outer polyhedron is the so called *direction search over the inner polyhedron* problem:

$$\begin{array}{ll} \max & \alpha \\ & \alpha \mathbf{v}^i \geqq \mathbf{V}\boldsymbol{\lambda} \\ & \boldsymbol{\lambda} \geqq \mathbf{0} \\ & \mathbf{e}^T\boldsymbol{\lambda} \leq 1 \end{array}, \tag{3.17}$$

where

$$\mathbf{V} = (\mathbf{z}^1|\mathbf{z}^2|\dots|\mathbf{z}^m)$$

is the matrix whose columns are the current points of our finite point approximation.

The set:

$$\begin{array}{ll} & \mathbf{V}\boldsymbol{\lambda} \\ \text{s.t.} & \boldsymbol{\lambda} \geqq \mathbf{0} \\ & \mathbf{e}^T\boldsymbol{\lambda} \leq 1 \end{array} \tag{3.18}$$

corresponds to the inner polyhedral approximation, that is:

$$\{\mathbf{V}\boldsymbol{\lambda} : \boldsymbol{\lambda} \geqq \mathbf{0}, \mathbf{e}^T\boldsymbol{\lambda} \leq 1\} = convex(\{\mathbf{0}, \mathbf{z}^1, \dots, \mathbf{z}^m\})$$

and by Remark 3.15, because the *direction search over the inner polyhedron* problem has the solution if we take the set $\mathbf{V}\boldsymbol{\lambda} + \mathbb{R}^n_{\geqq}$, we are considering all points in $\mathbf{V}\boldsymbol{\lambda} + \mathbb{R}^n_{\geqq}$.

The Sandwiching Algorithm is very similar to Algorithm 4. In the Sandwiching Algorithm, instead of solving a *direction search problem over* Z (3.13), we solve a *direction search problem over* P_I (3.17) for each extreme point of the outer polyhedron P_O. In order to obtain a nondominated point, a *direction search problem over* Z (3.13) is solved only once at the end of each iteration.

Algorithm 6. *Sandwiching Algorithm.*

---

input:
- *A Multicriteria Optimization Problem, i.e.* $\min\{\ \mathbf{z} : \ \mathbf{z} \in Z\}$*, where* Z *is* $\mathbb{R}_{\geqq}$*-convex.*
- *A point* $\mathbf{z}^0$*, assumed to be* $\mathbf{0}$ *and an initial finite point approximation given by* $\{\mathbf{z}^1, \ldots, \mathbf{z}^m\} \subseteq N_w(Z) \cap \mathbb{R}^n_{\leqq}$*, and* $\{\mathbf{n}^1, \ldots, \mathbf{n}^m\}$ *for each* $i = 1, \ldots, m$ *such that* $\mathbf{n}^i \geq \mathbf{0}$ *and* ${\mathbf{n}^i}^T \mathbf{z} \geq {\mathbf{n}^i}^T \mathbf{z}^i \ \forall \mathbf{z} \in Z$ *for all* $i = 1, \ldots, m$
- *Stopping parameters* $MaxPoints$*,* $MaxExtremePoints$ *and* ϵ*.*

output:
- *A sequence of weakly nondominated points* N*.*

begin

Use dual convex hull algorithm to obtain the extreme points of:
$P_O \cap \mathbb{R}^n_{\leqq} = \{\mathbf{z} \leqq 0 : {\mathbf{n}^i}^T \mathbf{z} \geq {\mathbf{n}^i}^T \mathbf{z}^i, i = 1, \ldots, m\}$*.*
$c \leftarrow 0$
for-each *extreme point* $\mathbf{v}^i$ *of* $P_O \cap \mathbb{R}^n_{\leqq}$
    Solve the linear problem (3.17) *with optimal solution* $(\hat{\mathbf{z}}^i, \hat{\lambda}^i)$*.*
    $c \leftarrow c + 1$
    push $(\mathbf{v}^i, \hat{\mathbf{z}}^i, \hat{\lambda}^i)$ *into priority queue* Q*.*
end-for-each
pop *the minimum* λ' *from* Q*, therefore obtaining* $(\mathbf{v}', \mathbf{z}', \lambda')$*.*
Solve the problem (3.13) *for* $\mathbf{v}'$ *with optimal solution* $(\mathbf{z}^*, \lambda^*)$*.*
while $1 - \lambda^* > \epsilon$ ***and*** $|N| < MaxPoints$ ***and*** $c < MaxExtremePoints$
    $N \leftarrow N \cup \{\mathbf{z}^*\}$
    Obtain the extreme points of $P'_O \cap \mathbb{R}^n_{\leqq} = P_0 \cap \mathbb{R}^n_{\leqq} \cap \{\mathbf{z} : {\mathbf{n}^*}^T \mathbf{z} \geq {\mathbf{n}^*}^T \mathbf{z}^*\}$*.*
    for-each *new and modified extreme point* $\mathbf{v}^i$ *in* $P'_O \cap \mathbb{R}^n_{\leqq}$ *w.r.t.* $P_O \cap \mathbb{R}^n_{\leqq}$
        Solve the linear problem (3.17) *with optimal solution* $(\hat{\mathbf{z}}^i, \hat{\lambda}^i)$*.*
        $c \leftarrow c + 1$
        push $(\mathbf{v}^i, \hat{\mathbf{z}}^i, \hat{\lambda}^i)$ *into priority queue* Q*.*
    end-for-each
    $P_O \leftarrow P'_O$
    pop $(\mathbf{v}', \mathbf{z}', \lambda')$ *from* Q*.*
    Solve the problem (3.13) *for* $\mathbf{v}'$ *with optimal solution* $(\mathbf{z}^*, \lambda^*)$*.*
end-while
output N

end

The advantage of this algorithm is that it only solves as many *direction search problems* over Z as the number of points in the finite point representation. However, the `outer gauge approximation error` is overestimated.

3.4.2 Bicriteria Sandwiching Algorithm

For two dimensions, we have a nice result concerning the sandwiching algorithm. According to Observation 3.17, we do not need a convex hull algorithm to compute the extreme points of the outer polyhedron in the two dimensional case. However, in the context of the sandwiching algorithm, a stricter result is derived from the two dimension geometry.

Lemma 3.19. *An extreme point* $\mathbf{p}$ *of the outer polyhedron formed by the intersection of two hyperplanes supporting the nondominated points* $\mathbf{z}^i$ *and* $\mathbf{z}^j$ *is in the polyhedral cone formed by points* $\mathbf{z}^i$ *and* $\mathbf{z}^j$*, i.e.* $\mathbf{p} \in polycone(\{\mathbf{z}^i, \mathbf{z}^j\})$.

Proof. Consider the points $\mathbf{z}^i, \mathbf{z}^j \in N(Z) \cap \mathbb{R}^n_{\leqq}$ with the respective normals $\mathbf{n}^i, \mathbf{n}^j \geq \mathbf{0}$ such that

$$\mathbf{n}^{i^T}\mathbf{z} \geq \mathbf{n}^{i^T}\mathbf{z}^i$$

$$\mathbf{n}^{j^T}\mathbf{z} \geq \mathbf{n}^{j^T}\mathbf{z}^j$$

for all $\mathbf{z} \in Z$.

Since $\mathbf{n}^{i^T}\mathbf{z}^j \geq \mathbf{n}^{i^T}\mathbf{z}^i$, $\mathbf{n}^{i^T}\mathbf{z}^j < 0$ and $\mathbf{n}^{i^T}\mathbf{z}^i < 0$, it follows that there exists $\lambda \geq 1$ such that $n^{i^T}(\lambda z^j) = n^{i^T} z^i$. But we also know that

$$\mathbf{n}^{j^T}(\lambda\mathbf{z}^j) \leq \mathbf{n}^{j^T}\mathbf{z}^j$$

$$\mathbf{n}^{j^T}(\mathbf{z}^i) \geq \mathbf{n}^{j^T}\mathbf{z}^j$$

then there exists $\mathbf{y} \in convex(\{\mathbf{z}^i, \lambda\mathbf{z}^j\})$ such that

$$\mathbf{n}^{j^T}\mathbf{y} = \mathbf{n}^{j^T}\mathbf{z}^j$$

However, $\mathbf{n}^{i^T}\mathbf{y} = \mathbf{n}^{i^T}\mathbf{z}^i$ for all $\mathbf{y} \in convex(\{\mathbf{z}^i, \lambda\mathbf{z}^j\})$. Therefore $\mathbf{y} \in polycone(\{\mathbf{z}^i, \mathbf{z}^j\})$. □

Lemma 3.19 indicates that in the bicriteria case the matrix $\mathbf{V}$ in problem (3.17), only two columns need to be considered.

Chapter 4

Bilevel Models

In this chapter we consider a different formulation for the sandwiching algorithm. In the sandwiching approach we need to store the extreme points and the convex hull data structure; moreover, we need to solve a subproblem for each one of these extreme points.

Since the number of extreme points and the convex hull data structure grows considerably for every subsequent dimension, our approach is to solve the problem using a bilevel model.

Bilevel programming theory was motivated by the game theory of Von Stackelberg [Stackelberg, 1952] in the context of unbalance economics markets. In this type of problems the decision variables are partitioned amongst two problems.

First, the upper level problem – or leader problem – optimizes his objective; but then, after this, the lower level problem – called follower problem – reacts by optimizing his own objective function.

Because objectives and feasible sets for both problems can be interdependent, the leader problem decision affects the feasible set and optimal value on the follower and vice versa.

4.1 Bilevel Linear Problems

Most of the research and algorithms in literature are focused in Bilevel Linear Problems (BLLP).

Although all objectives and constraints of a Bilevel Linear Problem are linear, the upper level objective function is in general non-convex, since its values depend on the optimal solution of the lower level problem [Bialas and Karwan, 1984]. In fact Bilevel Linear Programming was proved to be NP-Hard Problem by Bard [1991] and Ben-Ayed and Blair [1990].

Definition 4.1. *[Bard, 1991]. For* $\mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m, F : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$ *and* $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$ *a* Bilevel Linear Problem *is defined as following:*

$$\begin{array}{lll} \min & F(\mathbf{x},\mathbf{y}) = {\mathbf{c}^1}^T\mathbf{x} + {\mathbf{d}^1}^T\mathbf{y} & \\ s.t. & \mathbf{A}^1\mathbf{x} + \mathbf{B}^1\mathbf{y} \leqq \mathbf{b}^1 & \\ & \quad \min \quad f(\mathbf{x},\mathbf{y}) = {\mathbf{c}^2}^T\mathbf{x} + {\mathbf{d}^2}^T\mathbf{y} & \\ & \quad s.t. \quad \mathbf{A}^2\mathbf{x} + \mathbf{B}^2\mathbf{y} \leqq \mathbf{b}^2 & \end{array} \tag{4.1}$$

where $\mathbf{c}^1, \mathbf{c}^2 \in \mathbb{R}^n$, $\mathbf{d}^1, \mathbf{d}^2 \in \mathbb{R}^m$, $\mathbf{b}^1 \in \mathbb{R}^p, \mathbf{b}^2 \in \mathbb{R}^q$, $\mathbf{A}^1 \in \mathbb{R}^{p\times n}, \mathbf{B}^1 \in \mathbb{R}^{p\times m}$ *and* $\mathbf{A}^2 \in \mathbb{R}^{q\times n}, \mathbf{B}^2 \in \mathbb{R}^{q\times m}$.

Definition 4.2. *We also define the following sets.*

1. *Constraint region of the BLLP problem:*

$$S = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m, \mathbf{A}^1\mathbf{x} + \mathbf{B}^1\mathbf{y} \leqq \mathbf{b}^1, \mathbf{A}^2\mathbf{x} + \mathbf{B}^2\mathbf{y} \leqq \mathbf{b}^2\}$$

   S considers all pairs $(\mathbf{x}, \mathbf{y})$ such they are in the feasible region of both problems.

2. *Projection of S onto the upper level problem's feasible region:*

$$S(X) = \{\mathbf{x} \in \mathbb{R}^n : \exists \mathbf{y} \in \mathbb{R}^m \text{ s.t. } (\mathbf{x}, \mathbf{y}) \in S\}$$

3. *Feasible set for the follower $\forall \mathbf{x} \in S(X)$:*

$$S(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{A}^2\mathbf{x} + \mathbf{B}^2\mathbf{y} \leqq \mathbf{b}^2\}$$

   Notice that $S(\mathbf{x}) \supseteq \{\mathbf{y} \in \mathbb{R}^m : (\mathbf{x}, \mathbf{y}) \in S\}$. Which means that for solving the lower level problem we are considering $\mathbf{y} \in \mathbb{R}^m$ that might be infeasible in the upper level problem.

4. *Lower problem's rational reaction set for $\mathbf{x} \in S(X)$:*

$$P(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = \arg\min\{f(\mathbf{x}, \hat{\mathbf{y}}) : \hat{\mathbf{y}} \in S(\mathbf{x})\}\}$$

   As before, it is important to notice that for $\mathbf{y} \in P(\mathbf{x})$, we may have $(\mathbf{x}, \mathbf{y}) \notin S$.

5. *Inducible region:*

$$\mathcal{R} = \{(\mathbf{x}, \mathbf{y}) \in S : \mathbf{y} \in P(\mathbf{x})\}$$

   Problem (4.1) can be rewritten as:

$$\min\{F(\mathbf{x}, \mathbf{y}) : (\mathbf{x}, \mathbf{y}) \in \mathcal{R}\} \tag{4.2}$$

A compact and non-empty S does not guarantee the existence of an optimal solution for the BLLP. To visualize that, we present the following example taken from [Shi et al., 2005].

Example 5. *Consider the following BLLP:*

$$\begin{array}{ll}
\min & x - 4y \\
s.t. & -x - y \leq -3 \\
& -3x + 2y \geq -4 \\
& x \geq 0 \\
& \quad \begin{array}{ll} \min & x + y \\ s.t. & -2x + y \leq 0 \\ & 2x + y \leq 12 \\ & y \geq 0 \end{array}
\end{array} \tag{4.3}$$

We have:

$$S = \{(x, y) \in \mathbb{R}^2 : -x - y \leq -3, -3x + 2y \geq -4, -2x + y \leq 0, 2x + y \leq 12\}$$

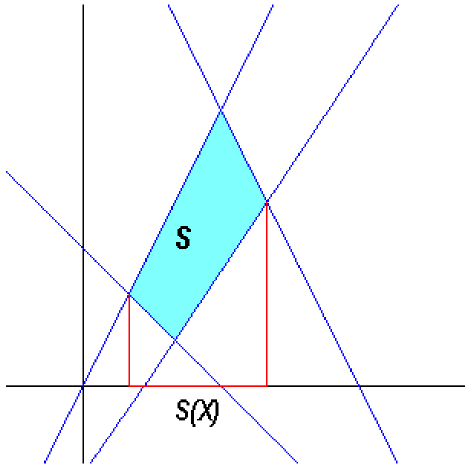

Figure 4.1: Constraint Region of an BLLP.

$$S(X) = \{x \in \mathbb{R} : 1 \leq x \leq 4\}$$

As we can see in the figure 4.1, S is non-empty and compact, and $0 \in S(x)$ for all $x \in S(X)$. We have that $P(x) = \{0\}$ for all $x \in S(X)$, but that means $\mathcal{R} = \emptyset$, so problem (4.3) does not have a solution.

In fact, some authors define the BLLP without constraints in the leader problem. By doing that we have $S(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^m : (\mathbf{x}, \mathbf{y}) \in S\}$ for all $\mathbf{x} \in S(X)$, and finally we have $(\mathbf{x}, \mathbf{y}) \in S$ for $\mathbf{y} \in P(\mathbf{x})$.

Another important issue in the BLLP definition is the fact that if the optimal solution for the lower level problem is not unique, then the objective function in the upper level problem is not well defined.

To deal with this situation we will assume that an optimal solution for the lower level problem is unique, or that we are able to take $\hat{\mathbf{y}} \in P(\mathbf{x})$ such that ${\mathbf{d}^1}^T \hat{\mathbf{y}} \leq {\mathbf{d}^1}^T \mathbf{y}$ for all $\mathbf{y} \in P(\mathbf{x})$.

There are many approaches used to solve Bilevel Linear Problems. They include:

1. Extreme-point search methods.
2. Kuhn-Tucker approach.
3. Complementary-pivot algorithm from Judice and Faustino.
4. Branch-and-bound algorithm from Bard and Falk.
5. Evolutionary methods such as Genetic Algorithms.

The Karush-Kuhn-Tucker approach is based in the following lemma:

Lemma 4.3. *A necessary condition that $(\mathbf{x}^*, \mathbf{y}^*)$ solves the bilevel linear program (4.1) locally is that there exists a vector $\boldsymbol{\lambda}^*$ such that $(\mathbf{x}^*, \mathbf{y}^*, \boldsymbol{\lambda}^*)$ locally* *solves:*

$$
\begin{aligned}
\min \quad & \mathbf{c}^{1^T}\mathbf{x} + \mathbf{d}^{1^T}\mathbf{y} \\
s.t. \quad & \mathbf{A}^1\mathbf{x} + \mathbf{B}^1\mathbf{y} \leqq \mathbf{b}^1 \\
& \mathbf{A}^2\mathbf{x} + \mathbf{B}^2\mathbf{y} \leqq \mathbf{b}^2 \\
& \boldsymbol{\lambda}^T\mathbf{B}^2 + \mathbf{d}^{2^T} = \mathbf{0} \\
& \boldsymbol{\lambda}^T(\mathbf{b}^2 - \mathbf{A}^2\mathbf{x} - \mathbf{B}^2\mathbf{y}) = 0 \\
& \boldsymbol{\lambda} \geqq \mathbf{0}
\end{aligned}
\tag{4.4}
$$

Proof. For the proof see [Bard, 1998]. □

This result is a simple consequence of the duality theory for linear programming. In linear programming the Karush-Kuhn-Tucker conditions are sufficient and necessary for optimality.

Remark 4.4. *The existence of $\boldsymbol{\lambda}^*$ as in Lemma 4.3 for a pair $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is not a sufficient condition for global optimality.*

4.2 Bilevel Linear Models

For each point $\mathbf{z}$ in the outer polyhedron P_O, we want to calculate the maximum $\alpha \geq 0$ such that $\alpha\mathbf{z} \in (P_I + \mathbb{R}^n_{\geqq})$. Equivalently, we want to solve the *direction problem over the inner polyhedron* (3.17)

$$
\begin{aligned}
\max \quad & \alpha \\
& \alpha\mathbf{z} \geqq \mathbf{V}\boldsymbol{\lambda} \\
& \boldsymbol{\lambda} \geqq \mathbf{0} \\
& \mathbf{e}^T\boldsymbol{\lambda} \leq 1
\end{aligned}
$$

where

$$\mathbf{V} = (\mathbf{z}^1|\mathbf{z}^2|\dots|\mathbf{z}^m) \in \mathbb{R}^{n\times m}$$

is the matrix whose columns are the points in the current finite point representation of $N(Z)$.

We want to find $\mathbf{z}^*$ the minimizer w.r.t. to the following problem:

$$\min\left\{\bar{\alpha} : z \in P_O \cap \mathbb{R}^n_{\leqq}, \bar{\alpha} = \max\left\{\alpha \geq 0 : \alpha z \in (P_I + \mathbb{R}^n_{\geqq})\right\}\right\}$$

In the framework of Bilevel Programming we can model this problem in the following way:

$$
\begin{aligned}
\min \quad & \alpha \\
\text{s.t.} \quad & \mathbf{Hz} \geqq \mathbf{b} \\
& \mathbf{z} \leqq \mathbf{0} \\
& \quad \min \quad -\alpha \\
& \quad \text{s.t.} \quad \mathbf{V}\boldsymbol{\lambda} \leqq \alpha\mathbf{z} \\
& \quad \qquad \boldsymbol{\lambda} \geqq \mathbf{0} \\
& \quad \qquad \mathbf{e}^T\boldsymbol{\lambda} \geqq \mathbf{0}
\end{aligned}
\tag{4.5}
$$

where:

$$\mathbf{H} = \begin{pmatrix} \mathbf{n}^{1T} \\ \hline \mathbf{n}^{2T} \\ \hline \vdots \\ \hline \mathbf{n}^{mT} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

$$\mathbf{b} = \left(\mathbf{n}^{1T}\mathbf{z}^1, \mathbf{n}^{2T}\mathbf{z}^2, \ldots, \mathbf{n}^{mT}\mathbf{z}^m\right)^T \in \mathbb{R}^m$$

In this problem $\mathbf{Hz} \geqq \mathbf{b}, \mathbf{z} \leqq \mathbf{0}$ represents the outer polyhedron, for each point of the outer polyhedron we solve a *direction search over the inner polyhedron* problem.

However, problem (4.5) is not a BLLP. Thus we need an equivalent formulation of the follower problem.

Lemma 4.5. *If $V \in \mathbb{R}^{n \times m}$ such that $v_{ij} \leq 0$ and $\mathbf{V} \neq \mathbf{0}$ and $\mathbf{p} \in \mathbb{R}^n_{\leq}$.*
Then

$$\begin{array}{rl} \min & \mathbf{e}^T \boldsymbol{\lambda} \\ s.t. & \mathbf{V}\boldsymbol{\lambda} \leqq \mathbf{p} \\ & \boldsymbol{\lambda} \geqq \mathbf{0} \end{array} \tag{4.6}$$

is an equivalent formulation of problem.

$$\begin{array}{rl} \max & \alpha \\ & \alpha \mathbf{p} \geqq \mathbf{V}\boldsymbol{\lambda} \\ & \boldsymbol{\lambda} \geqq \mathbf{0} \\ & \mathbf{e}^T \boldsymbol{\lambda} \leq 1 \end{array} \tag{4.7}$$

Proof. First, note that $\boldsymbol{\lambda} = \mathbf{0}$ is not feasible for problem (4.6) since $\mathbf{p} \in \mathbb{R}^n_{\leq}$. Furthermore, since $\mathbf{e}^T \boldsymbol{\lambda} > 0$ then we can consider the following problem:

$$\begin{array}{rl} \max & \dfrac{1}{\mathbf{e}^T \boldsymbol{\lambda}} \\ \text{s.t.} & \mathbf{V}\boldsymbol{\lambda} \leqq \mathbf{p} \\ & \boldsymbol{\lambda} \geqq \mathbf{0} \end{array} \tag{4.8}$$

Take the optimal solution to the problem (4.7), $(\alpha^*, \boldsymbol{\lambda}^*), \alpha^* > 0$. Notice that by defining:

$$\hat{\boldsymbol{\lambda}} := \frac{\boldsymbol{\lambda}^*}{\alpha^*}$$

It follows that $\hat{\boldsymbol{\lambda}}$ is a feasible solution for problem (4.8). Moreover,

$$\mathbf{e}^T \hat{\boldsymbol{\lambda}} \leq \frac{1}{\alpha^*}$$

so we have

$$\alpha^* \leq \frac{1}{\mathbf{e}^T \hat{\boldsymbol{\lambda}}}$$

Which means that the optimal solution $\hat{\boldsymbol{\lambda}}^*$ of problem (4.8) fullfills:

$$\frac{1}{\mathbf{e}^T \hat{\boldsymbol{\lambda}}^*} \geq \alpha^*$$

Conversely, let $\hat{\boldsymbol{\lambda}}^*$ be the optimal solution of problem (4.8). We have that:

$$\left(\frac{1}{\mathbf{e}^T\hat{\boldsymbol{\lambda}}^*}, \frac{\hat{\boldsymbol{\lambda}}^*}{\mathbf{e}^T\hat{\boldsymbol{\lambda}}^*}\right)$$

is a feasible solution for problem (4.7), which means that:

$$\alpha^* \geq \frac{1}{\mathbf{e}^T\hat{\boldsymbol{\lambda}}^*}$$

□

Thus, we formulate the following BLLP:

$$\begin{array}{lll}
\min & -\mathbf{e}^T\boldsymbol{\lambda} & \\
\text{s.t.} & \mathbf{Hz} \geqq \mathbf{b} & \\
& \mathbf{z} \leqq \mathbf{0} & \\
& \quad \min & \mathbf{e}^T\boldsymbol{\lambda} \\
& \quad \text{s.t.} & \mathbf{V}\boldsymbol{\lambda} \leqq \mathbf{z} \\
& & \boldsymbol{\lambda} \geqq \mathbf{0}
\end{array} \tag{4.9}$$

Problem (4.9) does not contain "connecting" constraints in the upper level problem, that is, there are no constraints in the upper level problem for the lower level problem variables. Therefore $(\mathbf{z}, \boldsymbol{\lambda}) \in S$ for all $\boldsymbol{\lambda} \in P(\mathbf{z}), \mathbf{z} \in S(Z)$, and thus the compactness and the non-emptiness of S are sufficient to guarantee existence of the optimal solution for problem (4.9).

Furthermore, since for any arbitrary $\hat{\boldsymbol{\lambda}} \in P(\mathbf{z})$, by definition of $P(\mathbf{z})$, we have $\mathbf{e}^T\hat{\boldsymbol{\lambda}} = \mathbf{e}^T\boldsymbol{\lambda}\ \forall \boldsymbol{\lambda} \in P(\mathbf{z})$. Thus the upper level objective of (4.9) is well defined.

In order to solve problem (4.9), we decided to use the KKT approach and include the KKT conditions of the follower problem in the leader one. In practice, the implementation of problem (4.9) yielded points which fullfill first optimality conditions but are not extreme points of the outer polyhedron. Thus, a different formulation was considered.

In the following problem, we restrict ourselves to the points in the cone of the inner approximation. In this sense, the resulting implementation is only comparable to the inner approximation algorithm presented in Section 3.1, because both of them are restricted to solutions within *polycone*(N), where N is the current finite point representation of $N(Z)$.

It is also possible to include more points in the inner approximation, to add the positive orthant, i.e. the projection of the representative points on the faces that define the negative orthant; exactly as in Section 3.1.3.

The reformulated problem looks as follows:

$$\begin{array}{lll}
\min & -\mathbf{e}^T(\boldsymbol{\lambda} - \mathbf{K}\boldsymbol{\eta}) & \\
\text{s.t.} & \mathbf{HV}\boldsymbol{\lambda} \geqq \mathbf{b} & \\
& \boldsymbol{\lambda} \geqq \mathbf{0} & \\
& \quad \min & -\mathbf{e}^T\mathbf{K}\boldsymbol{\eta} \\
& \quad \text{s.t.} & \mathbf{K}\boldsymbol{\eta} \leqq \boldsymbol{\lambda}
\end{array} \tag{4.10}$$

where $\mathbf{H}, \mathbf{V}, \mathbf{b}$ are defined as in problem (4.9) and $\mathbf{K}$ is a matrix whose column vectors span the nullspace of $\mathbf{V}$.

As previously, problem (4.10) has no "connecting constraints" and the follower problem is bounded for all feasible $\boldsymbol{\lambda}$ in the leader one. In addition, every $\boldsymbol{\eta} \in P(\boldsymbol{\lambda})$ has the same influence in the upper level objective, therefore the upper level objective is well defined.

The leader problem feasible set are all points in the intersection of the polyhedral cone of the inner polyhedron and the outer polyhedron. Each point $\mathbf{p}$ is represented by multipliers $\boldsymbol{\lambda} \geqq \mathbf{0}$ such that $\mathbf{p} = \mathbf{V}\boldsymbol{\lambda}$. Then we minimize the sum of the multipliers, i.e. $\mathbf{e}^T\boldsymbol{\lambda}$, by solving the follower problem. The follower computes $\mathbf{K}\boldsymbol{\eta} \leqq \boldsymbol{\lambda}$, where $\mathbf{K}$ spans the nullspace of $\mathbf{V}$. This means $\mathbf{p} = \mathbf{V}(\boldsymbol{\lambda} - \mathbf{K}\boldsymbol{\eta})\ \forall \boldsymbol{\eta}$; therefore, by maximizing $\mathbf{e}^T\mathbf{K}\boldsymbol{\eta}$ subject to $\mathbf{K}\boldsymbol{\eta} \leqq \boldsymbol{\lambda}$, we indeed compute a representation of $\mathbf{p} = \mathbf{V}(\boldsymbol{\lambda} - \mathbf{K}\boldsymbol{\eta}^*), (\boldsymbol{\lambda} - \mathbf{K}\boldsymbol{\eta}^*) \geqq \mathbf{0}$ that minimizes the sum of the multipliers.

Therefore the follower problem corresponds to problem (4.6), and thus we compute $\frac{1}{\alpha}$ for each point $\mathbf{p} \in \{\mathbf{z} \leqq \mathbf{0} : \mathbf{z} = \mathbf{V}\boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}, \mathbf{e}^T\boldsymbol{\lambda} \leq 1, \mathbf{H}\mathbf{z} \geq \mathbf{b}\}$, where α is the outer gauge of $\mathbf{p}$.

Using the KKT approach we obtain the following so called *BLLP nonlinear KKT resulting problem*:

$$
\begin{array}{ll}
\min & -\mathbf{e}^T(\boldsymbol{\lambda} - \mathbf{K}\boldsymbol{\eta}) \\
\text{s.t.} & \mathbf{H}\mathbf{V}\boldsymbol{\lambda} \geqq \mathbf{b} \\
& \boldsymbol{\lambda} \geqq \mathbf{0} \\
& \mathbf{K}\boldsymbol{\eta} \leqq \boldsymbol{\lambda} \\
& \boldsymbol{\mu}^T\mathbf{K} - \mathbf{e}^T\mathbf{K} = \mathbf{0} \\
& \boldsymbol{\mu}^T(\boldsymbol{\lambda} - \mathbf{K}\boldsymbol{\eta}) = 0 \\
& \boldsymbol{\mu} \geqq \mathbf{0}
\end{array}
\tag{4.11}
$$

4.3 Bilevel Algorithm

The *BLLP nonlinear KKT resulting problem* (4.11) is solved via a *Sequential Programming Quadratic* algorithm (SQP) with a LBFGS update, more details about the SQP implementation will be given in chapter 5.

However, every extreme point of the outer polyhedron is a local minimum for the *BLLP nonlinear KKT resulting problem* (4.11). Thus a solution of our resulting SQP algorithm does not guarantee global optimality.

The computed local minima highly depend in the starting point. Therefore, in the following algorithm, we solve problem (4.11) k times for a given finite point approximation of $N(Z)$, each time with a different randomly generated starting point. Then we consider the furthest computed point from the inner approximation and solve a *direction search problem* in each iteration. Numerical results are presented in Chapter 5.

Algorithm 7. *Bilevel Algorithm.*

---

input:

- *A Multicriteria Optimization Problem, i.e.* $\min\{\ \mathbf{z} :\ \mathbf{z} \in Z\}$*, where* Z *is* $\mathbb{R}_{\geqq}$*-convex.*
- *A point* $\mathbf{z}^0$*, w.l.o.g* $\mathbf{z}^0 = \mathbf{0}$ *and an initial finite point approximation given by* $\{\mathbf{z}^1, \ldots, \mathbf{z}^m\} \subseteq N_w(Z) \cap \mathbb{R}^n_{\leqq}$*, and* $\{\mathbf{n}^1, \ldots, \mathbf{n}^m\}$ *for each* $i = 1, \ldots, m$ *such that* $\mathbf{n}^i \geq \mathbf{0}$ *and* ${\mathbf{n}^i}^T\mathbf{z} \geq {\mathbf{n}^i}^T\mathbf{z}^i\ \forall \mathbf{z} \in Z$ *for all* $i = 1, \ldots, m$

- *Parameters* $MaxPoints$ *and* k.

output:

- *A sequence of weakly nondominated points* N.

begin

while $|N| < MaxPoints$
    $\mathbf{p} \leftarrow \mathbf{0}$
    $\gamma^* \leftarrow 0$
    for-each $i \in \{1, \ldots, k\}$
        Solve problem (4.11) *with optimal solution* $\mathbf{z} = \mathbf{V}\boldsymbol{\lambda}^*$ *and optimal value* $\gamma = \min\{\mathbf{e}^T\boldsymbol{\xi} : \boldsymbol{\xi} \geqq \mathbf{0}, \mathbf{V}\boldsymbol{\xi} = \mathbf{V}\boldsymbol{\lambda}^*\}$.
        if $\gamma > \gamma^*$ ***then***
            $\gamma^* \leftarrow \gamma$
            $\mathbf{p} \leftarrow \mathbf{z}$
        end-if
    end-for-each
    Solve a direction search problem (3.17) *to obtain a new (weakly) nondominated point* $\mathbf{z}^*$ *in the direction of* $\mathbf{p}$.
    $N \leftarrow N \cup \{\mathbf{z}^*\}$
end-while
output N

end

The *BLLP nonlinear KKT resulting problem* (4.11) has roughly $3 * |N| - dim$ variables and $5 * |N| - dim + 1$ constraints. Thus, the complexity depends on the number of computed nondominated points. Moreover, problem (4.11) has a complementary slackness constraint that has to be regularized.

Chapter 5

Implementation and Computational Results

Throughout this paper we considered eight different algorithms:

1. **Inner approximation algorithm**: It does not have "full coverage" of the negative orthant. The subproblems are *gauge maximizations within a cone* over Z. Its algorithmic quality criterion is the `pseudo inner gauge approximation error`.

2. **Positive orthant variant of the inner approximation algorithm**: Adds points to the inner polyhedron to generate an absolute block norm. Its subproblems are *hyperplane maximizations* over Z. The algorithm quality criterion is the `inner gauge approximation error`.

3. **Local update inner approximation algorithm**: An heuristic of the inner approximation. The inner polyhedron cones are updated locally. It solves fewer *gauge maximization within a cone* subproblems.

   It underestimates the `pseudo inner gauge approximation error`.

4. **Outer approximation algorithm**: The *direction search subproblems* find a nondominated point Z in a direction given by an extreme point of the outer polyhedron. Its algorithmic quality criterion is the `outer gauge approximation error`.

5. **Local update outer approximation algorithm**: The outer polyhedron extreme points are update locally, thus it solves fewer *direction search subproblems*.

   It mainly underestimates the `outer gauge approximation error`.

6. **Simultaneous inner and outer approximation algorithm**: The approximations update, by alternating the `pseudo inner gauge` and the `outer gauge approximation errors`.

7. **Sandwiching algorithm**: An heuristic of the outer approximation algorithm, it solves the linear *direction subproblems over the inner polyhedron.* Moreover, it solves only one *direction problem over Z* per iteration. It overestimates the `outer gauge approximation error`.

8. **Bilevel algorithm**: It does not solve a large number of subproblems, but a *BLLP nonlinear KKT resulting problem* used as an "educated" guess to obtain a point in the outer polyhedron. We perform a *direction search* over Z in each iteration.

We compare these algorithms with respect to the following quality and complexity measures:

1. `Pseudo inner gauge approximation error`.
2. `Inner gauge approximation error`.
3. `Outer gauge approximation error`.
4. `Running time`.
5. `Number of solved subproblems`.

after computing a fixed size finite point representation of the nondominated set for the following multicriteria problem:

$$\begin{aligned} &\min \{z_1 = x_1\} \\ &\quad\vdots \\ &\min \{z_n = x_n\} \\ \text{s.t.}\ &\frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + c \leq 0 \\ &\mathbf{x} \leqq \mathbf{0} \end{aligned} \tag{5.1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a positive definite matrix, $\mathbf{b}, \mathbf{u} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

Thus, the algorithms generate a representation set of the nondominated set of an ellipsoid in $\mathbb{R}^n$ intersected with the negative orthant, i.e. problem (5.1). In order to implement the mentioned algorithms, we need to solve five different types of subproblems:

1. *Hyperplane maximization over* Z, i.e. in problem (3.6). A nonlinear convex problem used in the **positive orthant variant**.
2. *Gauge maximization within a cone over* Z, i.e. problem (3.8). A nonlinear convex problem used in the **inner approximation**, **local update inner approximation** and **simultaneous algorithm**.
3. *Direction search over* Z, i.e. problem (3.13). A nonlinear convex subproblem used in the **outer** and **local update outer approximation**. It is also solved once per iteration in the **sandwiching** and **bilevel algorithm**.
4. *Direction search over the inner polyhedral approximation*, i.e. problem (3.17). A linear program used in the **sandwiching algorithm**.
5. *BLLP nonlinear KKT resulting problem*, i.e. problem (4.11). This a nonconvex problem, it needs to be solved in the **bilevel algorithm**.

To solve the nonlinear problems we use a Sequential Quadratic Programming Method [Boggs and Tolle, 1995].

5.1 Sequential Quadratic Programming

Consider the following optimization problem for $\mathbf{x} \in \mathbb{R}^n$:

$$\begin{array}{rll} \min & f(\mathbf{x}) & \\ \text{s.t.} & g_i(\mathbf{x}) \leq 0 & i = 1, \ldots, p \\ & h_i(\mathbf{x}) = 0 & i = p+1, \ldots, m \end{array} \tag{5.2}$$

Let $\boldsymbol{\phi}(\mathbf{x}, \boldsymbol{\lambda})$ be defined as:

$$\boldsymbol{\phi}(\mathbf{x}, \boldsymbol{\lambda}) = \begin{pmatrix} \nabla f(\mathbf{x}) + \sum_{i=1}^{p} \lambda_i \nabla g_i(\mathbf{x}) + \sum_{i=p+1}^{m} \lambda_i \nabla h_i(\mathbf{x}) \\ \lambda_1 g_1(\mathbf{x}) \\ \vdots \\ \lambda_p g_p(\mathbf{x}) \\ h_{p+1}(\mathbf{x}) \\ \vdots \\ h_m(\mathbf{x}) \end{pmatrix} \tag{5.3}$$

thus $\boldsymbol{\phi}(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$ is nonlinear system of equations, whose solution, together with the constraints $\lambda_i \geq 0, g_i(\mathbf{x}) \leq 0$ for $i = 1, \ldots, p$, represent a pair solution for $\mathbf{x}$ and the corresponding dual variables $\boldsymbol{\lambda}$ that fullfill the KKT conditions of problem (5.2).

To solve $\boldsymbol{\phi}(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$ using the Newton Method, let $\boldsymbol{\psi}(x, \lambda, B)$ be the jacobian of (5.3):

$$\boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{B}) = \begin{pmatrix} \mathbf{B} & \nabla g_1(\mathbf{x}) & \cdots & \nabla g_p(\mathbf{x}) & \nabla h_{p+1}(\mathbf{x}) & \cdots & \nabla h_m(\mathbf{x}) \\ \lambda_1 \nabla g_1(\mathbf{x})^T & g_1(\mathbf{x}) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda_p \nabla g_p(\mathbf{x})^T & 0 & \cdots & g_p(\mathbf{x}) & 0 & \cdots & 0 \\ \nabla h_{p+1}(\mathbf{x})^T & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \nabla h_m(\mathbf{x})^T & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $\mathbf{B}$ is a $n \times n$ positive definite matrix computed by the derivatives with respect to $\mathbf{x}$ of:

$$\nabla f(\mathbf{x}) + \sum_{i=1}^{p} \lambda_i \nabla g_i(\mathbf{x}) + \sum_{i=p+1}^{m} \lambda_i \nabla h_i(\mathbf{x})$$

or obtained via a BFGS update as done in the Quasinewton methods [Shanno, 1970].

Linearizing (5.3) we obtain:

$$\boldsymbol{\phi}(\mathbf{x} + \Delta\mathbf{x}, \boldsymbol{\lambda} + \Delta\boldsymbol{\lambda}) = \boldsymbol{\phi}(\mathbf{x}, \boldsymbol{\lambda}) + \boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{B})(\Delta\mathbf{x}, \Delta\boldsymbol{\lambda})^T$$

thus, if we set $\boldsymbol{\phi}(\mathbf{x} + \Delta\mathbf{x}, \boldsymbol{\lambda} + \Delta\boldsymbol{\lambda}) = \mathbf{0}$ we get the *newton step*:

$$\boldsymbol{\psi}(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{B})(\Delta\mathbf{x}, \Delta\boldsymbol{\lambda})^T = -\boldsymbol{\phi}(\mathbf{x}, \boldsymbol{\lambda})$$

In the Newton Method, we compute a newton step $(\mathbf{s}, \mathbf{u})^T$ and update iteratively the variables, i.e. from iteration k to next iteration $k+1$, in the following way:

$$\begin{aligned}\boldsymbol{\psi}(\mathbf{x}^k, \boldsymbol{\lambda}^k, \mathbf{B}^k)(\mathbf{s}, \mathbf{u})^T &= -\boldsymbol{\phi}(\mathbf{x}^k, \boldsymbol{\lambda}^k)\\ \mathbf{x}^{k+1} &= \mathbf{x}^k + \mathbf{s}\\ \boldsymbol{\lambda}^{k+1} &= \boldsymbol{\lambda}^k + \mathbf{u}\end{aligned}$$

However, instead of solving for the newton step, in the SQP method, we solve:

$$\boldsymbol{\psi}(\mathbf{x}^k, \boldsymbol{\lambda}^{k+1}, \mathbf{B}^k)(\mathbf{s}, \mathbf{u})^T = -\boldsymbol{\phi}(\mathbf{x}^k, \boldsymbol{\lambda}^k)$$

expanding the previous equation we get:

$$\begin{aligned}\mathbf{B}^k\mathbf{s} + \sum_{i=1}^{p} u_i \nabla g_i(\mathbf{x}^k) + \sum_{i=p+1}^{m} u_i \nabla h_i(\mathbf{x}^k) =&\\ - \nabla f(\mathbf{x}^k) - \sum_{i=1}^{p} \lambda_i^k \nabla g_i(\mathbf{x}^k) - \sum_{i=p+1}^{m} \lambda_i^k \nabla h_i(\mathbf{x}^k)&\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}\lambda_i^{k+1} \nabla g_i(\mathbf{x}^k)^T \mathbf{s} + u_i g_i(\mathbf{x}^k) &= -\lambda_i^k g_i(\mathbf{x}) \quad i = 1, \ldots, p\\ \nabla h_i(\mathbf{x}^k)^T \mathbf{s} &= -h_i(\mathbf{x}^k) \qquad i = p+1, \ldots, m\end{aligned} \tag{5.5}$$

furthermore, we add the constraints $\lambda_i^{k+1} \geq 0, g_i(\mathbf{x}^k) + \nabla g_i(\mathbf{x}^k)^T \mathbf{s} \leq 0, i = 1, \ldots, p$, i.e. we enforce positive dual multipliers for $g_i(\mathbf{x})$ and we let the linearization of $g_i(\mathbf{x})$ to be less or equal than 0.

$$\begin{aligned}\lambda_i^{k+1} \geq 0 \qquad i = 1, \ldots, p\\ g_i(\mathbf{x}^k) + \nabla g_i(\mathbf{x}^k)^T \mathbf{s} \leq 0 \qquad i = 1, \ldots, p\end{aligned} \tag{5.6}$$

Rearranging equations (5.4),(5.5) and (5.6) we get the following system

$$\begin{aligned}\mathbf{B}^k\mathbf{s} + \nabla f(\mathbf{x}^k) + \sum_{i=1}^{p} \lambda_i^{k+1} \nabla g_i(\mathbf{x}^k) + \sum_{i=p+1}^{m} \lambda_i^{k+1} \nabla h_i(\mathbf{x}^k) = 0\\ \nabla g_i(\mathbf{x}^k)^T \mathbf{s} + g_i(\mathbf{x}^k) \leq 0 \quad i = 1, \ldots, p\\ \nabla h_i(\mathbf{x}^k)^T \mathbf{s} + h_i(\mathbf{x}^k) = 0 \quad i = p+1, \ldots, m\\ \lambda_i^{k+1}(\nabla g_i(\mathbf{x}^k)^T \mathbf{s} + g_i(\mathbf{x}^k)) = 0 \quad i = 1, \ldots, p\\ \lambda_i^{k+1} \geq 0 \quad i = 1, \ldots, p\end{aligned} \tag{5.7}$$

Notice that $(\mathbf{s}, \boldsymbol{\lambda}^{k+1})$ in (5.7) is a KKT pair for the following convex problem:

$$\begin{aligned}\min \quad & \frac{1}{2}\mathbf{s}^T \mathbf{B}^k \mathbf{s} + \nabla f(\mathbf{x}^k)^T \mathbf{s}\\ \text{s.t.} \quad & \nabla g_i(\mathbf{x}^k)^T \mathbf{s} + g_i(\mathbf{x}^k) \leq 0 \quad i = 1, \ldots, p\\ & \nabla h_i(\mathbf{x}^k)^T s + h_i(\mathbf{x}^k) = 0 \quad i = p+1, \ldots, m\end{aligned} \tag{5.8}$$

Problem (5.8) is a Quadratic convex program if $\mathbf{B}^k$ is a symmetric positive definite matrix. Positive definiteness can be ensured using a BFGS update.

Let $\mathbf{s} = \mathbf{x}^{k+1} - \mathbf{x}^k$ and

$$\mathbf{v} = \nabla f(\mathbf{x}^{k+1}) + \sum_{i=1}^{p} \lambda_i^{k+1} \nabla g_i(\mathbf{x}^{k+1}) + \sum_{i=p+1}^{m} \lambda_i^{k+1} \nabla h_i(\mathbf{x}^{k+1})$$
$$-\nabla f(\mathbf{x}^{k}) + \sum_{i=1}^{p} \lambda_i^{k+1} \nabla g_i(\mathbf{x}^{k}) + \sum_{i=p+1}^{m} \lambda_i^{k+1} \nabla h_i(\mathbf{x}^{k})$$

thus, in a BFGS update,we update matrix $\mathbf{B}^k$ such that:

$$\mathbf{B}^{k+1} = \mathbf{B}^k + \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{s}^T\mathbf{v}} - \frac{\mathbf{B}^k\mathbf{s}\mathbf{s}^T\mathbf{B}^k}{\mathbf{s}^T\mathbf{B}^k\mathbf{s}},$$

if $\mathbf{B}^k$ is a positive definite and symmetric matrix and $\mathbf{s}^T\mathbf{v} > 0$ then the BFGS update guarantees positive definiteness and symmetry of $\mathbf{B}^{k+1}$ [Shanno, 1970].

So instead of solving a nonlinear system of equations or computing a newton step, SQP solves a Quadratic Convex Program to obtain a step which it is used to update $\mathbf{x}$ and the dual variables $\boldsymbol{\lambda}$.

As a consequence, we have an iterative procedure which, given a non-linear optimization problem (5.2), generates a pair $(\mathbf{x}, \boldsymbol{\lambda})$ which solves the system of equations $\boldsymbol{\phi}(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$ and satisfy $\lambda_i \geq 0, g_i(\mathbf{x}) \leq 0$ for $i = 1, \ldots, p$.

Algorithm 8. *Sequential Quadratic Programming Method.*

---

input:
- *A nonlinear optimization problem* (5.2) *and stopping parameters* $\epsilon > 0$, $MaxIterations > 0$.

output:
- *A point* $\mathbf{x}$ *and dual multipliers* $\boldsymbol{\lambda}$ *which satisfy* $\boldsymbol{\phi}(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$ *and* $\lambda_i \geq 0, g_i(\mathbf{x}) \leq 0$ *for* $i = 1, \ldots, p$.

begin

1: *Choose an initial vector* $\mathbf{x}^0 \in \mathbb{R}^n$ *and* $\boldsymbol{\lambda}^0 \in \mathbb{R}^m$ *with* $\lambda_i^0 > 0$ *for* $i = 1, \ldots, p$.
2: *Choose a symmetric pos. definite matrix* $\mathbf{B}^0$ *related to the derivatives w.r.t.* $\mathbf{x}$ *of each component of* $\nabla f(\mathbf{x}^0) + \sum_{i=1}^{p} \lambda_i^0 \nabla g_i(\mathbf{x}^0) + \sum_{i=p+1}^{m} \lambda_i^0 \nabla h_i(\mathbf{x}^0)$.
3: $k \leftarrow 0$
4: ***do***
5:   *Solve problem* (5.8) *to find a KKT pair* $(\mathbf{s}, \boldsymbol{\lambda})$.
6: $\mathbf{x}^{k+1} = \mathbf{x}^k + \mathbf{s}$
7: $\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}$.
8:   *Compute* $\mathbf{B}^{k+1}$.
9: $k \leftarrow k + 1$.
10: ***while*** $\|\mathbf{s}\| > \epsilon$ ***and*** $k < MaxIterations$
11: ***output*** $(\mathbf{x}^k, \boldsymbol{\lambda}^k)$

end

To solve the quadratic convex problem (5.8), we implement the Goldfarb's Algorithm [Goldfarb and Liu, 1993].

5.2 Implementation Details

All codes were implemented in C++, and compiled on gcc version 4.1.0 under Linux i386 with a 64-bit architecture. For convex hull computations, both libraries *cdd* and *qhull* are used. *Qhull* is used to update the inner and outer polyhedra in the algorithms presented in Chapter 3. On the other hand, *cdd* is used for the local update heuristics.
To solve the optimization subproblems, we consider three cases:

1. *Linear Problems.*

   We use *cdd* internal linear solver to solve the linear subproblems in the **sandwiching algorithm**.

2. *Convex problems over the ellipsoid.*

   They result from the **inner**, **outer** and **simultaneous approximation algorithms**, where we solve subproblems (3.6), (3.8) and (3.13).

   These problems are easily solved by the SQP method. Because Z is an ellipsoid, to compute $\mathbf{B}^k$ in Algorithm 8 we just need to multiply matrix the $\mathbf{A}$ by the dual variable of the only nonlinear constraint.

3. *Nonconvex subproblem* (4.11) *in the Bilevel algorithm.*

   Notice that the resulting optimization problem in the **bilevel algorithm** is nonconvex and has a constraint that represents a complementary slackness condition.

   Because problem (4.11) is not convex, we compute $\mathbf{B}^k$ using a BFGS update. Furthermore, the complementary slackness constraint, which is forced to be equal to 0 in the original problem, is "regularized" by allowing it to be less or equal than a parameter $\epsilon > 0$. Moreover, we let ϵ get smaller by a constant factor in every iteration of the SQP method. This is commonly known as a *cooling scheme*.

   We also limit the number of iterations for the SQP method for this problem.

The ellipsoids were constructed randomly. Symmetry and positive definiteness of matrix $\mathbf{A}$ of (5.1) is enforced by letting $\mathbf{A} = \mathbf{L}^T\mathbf{L}$, where $\mathbf{L} \in \mathbb{R}^{n \times n}$ is a full rank randomly generated lower triangular matrix. $\mathbf{b}$ is also generated randomly and c is chosen so $\mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c \leq 0$ is feasible.
In the following section we summarize our computational results.

5.3 Computational Results

In the next subsections, we constantly refer to algorithms, quality measures and subproblems. Thus, we no longer use the boldface, typewriter and italic font distinction. =P

We compare the algorithms w.r.t. the `inner gauge approximation` and `outer gauge approximation errors`, total `running time` and the number of `solved subproblems` that result after the computation of a fixed size finite point representation of the nondominated set of different randomly constructed ellipsoids.

In each dimension, we present a table summarizing the considered quality and complexity measures. We compute the average and the standard deviation for the all measures among all tests.

We also present graphs depicting the numerical convergence of the `inner and outer gauge approximation error` for the **positive orthant variant** and the **outer approximation** respectively. Furthermore, we include a graph to visualize the point ratio, i.e. number of considered points divided by the number of points in the representation set, needed to add the positive orthant to the inner polyhedron P_I in the **positive orthant variant**.

5.3.1 Three and four criteria results

For three criteria, we ran 30 test cases and we fixed the number of points in the representation set to be 100. However, the bilevel algorithm proved to be very slow, thus we ran the bilevel algorithm until it generated 50 points.

The bilevel algorithm computed 4 tries, i.e. $k = 4$ in Algorithm 7, and ran with a SQP iteration limit of 300. Comparisons are made for 50 and 100 iterations.

Tables 5.1 and 5.3 show the obtained average and standard deviation for the pseudo inner gauge, inner gauge, outer gauge approximation errors and for the running time and number of solved subproblems.

Table 5.1: Quality approximation measures for different algorithms at iteration 50 and 100 for three criteria.

	Qty. Msr.	pseudo inner gauge		inner gauge		outer gauge	
Algorithm	It.	50	100	50	100	50	100
Inner Approx.	μ	0.01157	0.00444	0.01490	0.01155	0.01853	0.01495
	σ	0.00214	0.00057	0.00605	0.00804	0.00674	0.00781
Positive Orthant	μ	0.10974	0.10741	0.01035	0.00454	0.02074	0.01174
	σ	0.11831	0.12127	0.00149	0.00065	0.00715	0.00399
Local Inner	μ	0.02181	0.01822	0.02575	0.02360	0.03502	0.03167
	σ	0.00275	0.00234	0.00652	0.00817	0.00651	0.00780
Outer Approx.	μ	–	–	0.01990	0.00899	0.01102	0.00474
	σ	–	–	0.00560	0.00190	0.00186	0.00066
Local Outer	μ	–	–	0.04007	0.02356	0.03265	0.01900
	σ	–	–	0.00731	0.00462	0.00616	0.00350
Simul. Approx.	μ	–	–	0.01291	0.00566	0.01251	0.00549
	σ	–	–	0.00159	0.00091	0.00138	0.00078
Sandiw.	μ	–	–	0.01363	0.00646	0.01325	0.00617
	σ	–	–	0.00203	0.00094	0.00271	0.00120
Bilevel	μ	–	–	0.02922	–	0.03069	–
	σ	–	–	0.00851	–	0.00881	–

The pseudo inner gauge was computed for the inner approximation and its variants. Though it is expected that the inner approximation performs better than the positive orthant variant in respect to the pseudo inner gauge, the difference is considerable.

This is explained by the presence of cones whose hyperplanes have relatively big positive components and because the inner approximation algorithm ignores relevant areas of the nondominated set.

Thus the inner gauge is particularly misleading. It does not provide "full coverage" and, furthermore, in the polyhedral cone of the representation set it underestimates the real inner approximation. Moreover, the existence of fundamental cones whose hyperplane contain positive components provides a wrong lecture related to the desired areas of the outcome space where more representation points are needed.

The inner and the local update inner algorithms have bad approximation qualities when compared to the others. The bilevel algorithm performs worse than any other algorithm at 50 iterations.

The positive orthant variant and the outer approximation balance each other. Providing no grounds to favor one over the other. However, as expected, the simultaneous algorithm generates a representation set which has competitive inner and outer gauge when compared to the positive variant and the outer approximation algorithms.

Because it measures distances between the inner and the outer polyhedron, the sandwiching algorithm also provides desirable inner and outer gauges.

Table 5.2: Complexity measures for different algorithms at iteration 50 and 100 for three criteria.

	Qty. Msr.	running time(ms)		number subproblems	
Algorithm	It.	50	100	50	100
Inner	μ	598.037	1665.44	156.333	397.967
Approx.	σ	276.435	473.438	5.60993	12.13
Positive	μ	12.534	26.5349	212.133	455.267
Orthant	σ	3.27753	5.98756	18.6727	22.5097
Local	μ	1051.27	3717.16	111.367	226.8
Inner	σ	365.083	888.022	1.12903	1.91905
Outer	μ	28.6682	59.87	227.1	475.667
Approx.	σ	5.04728	8.0337	9.00326	12.7721
Local	μ	20.2674	38.5352	108.533	222.233
Outer	σ	6.29727	8.50082	1.69651	2.17641
Simul.	μ	388.424	1456.49	414.633	902.467
Approx.	σ	157.074	328.657	9.0458	11.578
Sandiw.	μ	256.816	3234.87	227.467	474.8
	σ	11.2361	89.4248	7.78032	11.7015
Bilevel	μ	2356420	–	–	–
	σ	182763	–	–	–

The positive orthant variation runs faster than any other algorithm and solves less subproblems than the outer approximation. Also, the local update outer algorithm runs faster than the outer approximation since it solves less subproblems. However, we do not see any improvement concerning the simplified convex hull computation in three dimensions, e.g. the local inner algorithm runs slower than the inner approximation (probably due to numerical issues in the subproblems given tight cones). As expected, the number of subproblems for the local updates is very steady.

The sandwiching algorithm solves a similar number of subproblems as the outer approximation, since it basically is an heuristic for the outer approximation. However, it runs considerably slow, we argue that solving a linear program with 100 constraints is more complex than minimizing over an ellipsoid in three dimensions.

On the other hand, the simultaneous algorithm complexity is based on the inner and outer algorithms, the number of subproblems and the running time is equivalent to the addition of the subproblems and the time for both inner and outer approximation algorithms.

The bilevel algorithm runs extremely slow, and we notice that its running time increases significantly with every new point. It is definetively not suitable for three dimensions.

Next figures depict the numerical convergence of the inner and outer gauges.

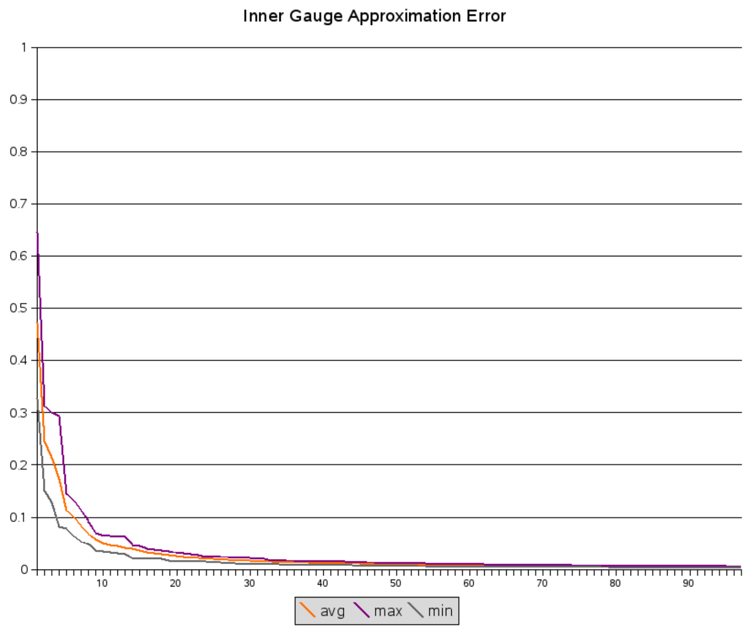

Figure 5.1: Inner gauge approximation error for three dimensions. At iteration 9 the inner gauge is already 0.1; at iteration 13, 0.05; and at iteration 34, 0.01.

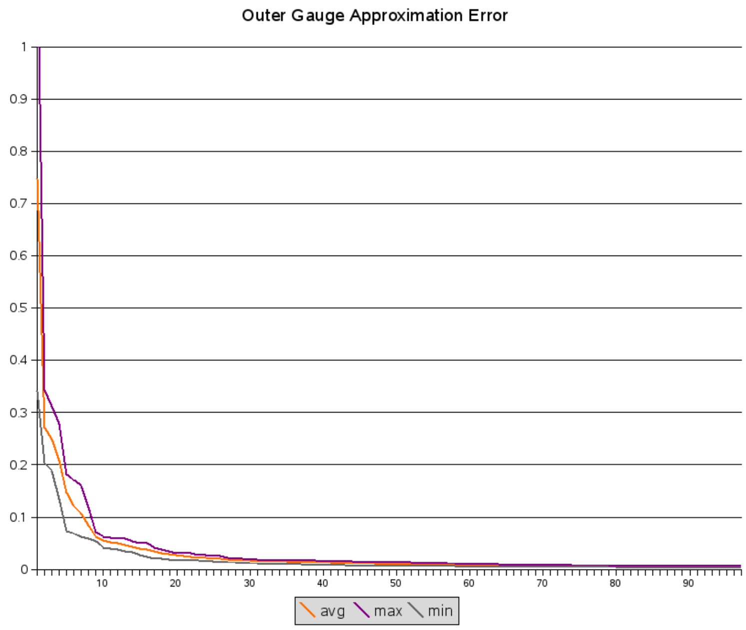

Figure 5.2: Outer gauge approximation error for three dimensions. At iteration 10 the outer gauge is already 0.1; at iteration 13, 0.05; and at iteration 36, 0.01.

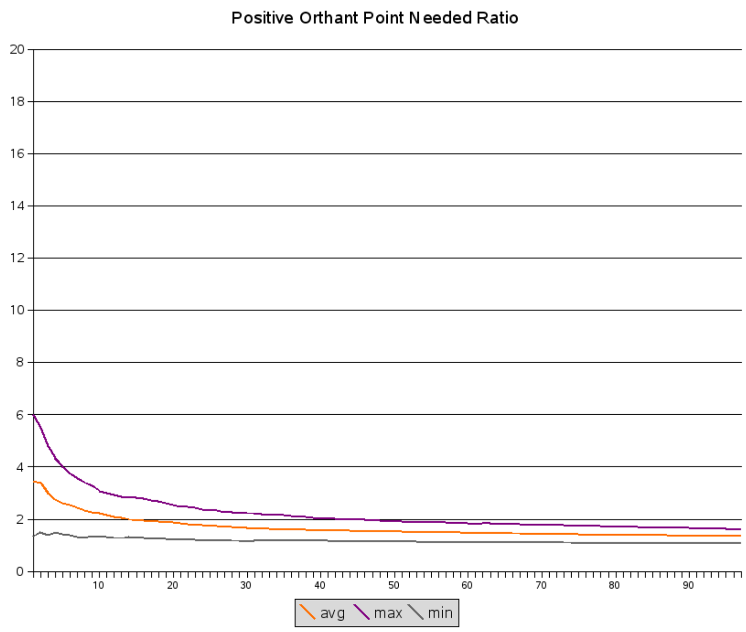

Figure 5.3: Point needed ratio to add the positive orthant. After 40 iterations we need less than the double of points in the representation.

In the four criteria case, we ran 30 test cases, and we fixed the number of points in the representation set to be 150. As previously, we ran the bilevel algorithm until it generated 50 points. Comparisons are made for 50 and 150 iterations.

Table 5.3: Quality approximation measures for different algorithms at iteration 50 and 150 for four criteria.

	Qty. Msr.	pseudo inner gauge		inner gauge		outer gauge	
Algorithm	It.	50	150	50	150	50	150
Inner Approx.	μ	0.04667	0.01735	0.05362	0.02938	0.06722	0.03546
	σ	0.00588	0.00216	0.00756	0.00932	0.01550	0.01215
Positive Orthant	μ	0.29266	0.26674	0.04031	0.01658	0.07501	0.03450
	σ	0.13016	0.15527	0.00479	0.00190	0.01403	0.00676
Local Inner	μ	0.11263	0.09557	0.11565	0.08029	0.15180	0.10459
	σ	0.01248	0.01025	0.01185	0.01021	0.01762	0.01445
Outer Approx.	μ	–	–	0.06874	0.03432	0.04246	0.01681
	σ	–	–	0.00788	0.00820	0.00449	0.00203
Local Outer	μ	–	–	0.14028	0.09335	0.14218	0.07834
	σ	–	–	0.01194	0.01347	0.01674	0.01441
Simul. Approx.	μ	–	–	0.07558	0.02647	0.05665	0.02129
	σ	–	–	0.01208	0.00415	0.00668	0.00247
Sandiw.	μ	–	–	0.05424	0.02328	0.05127	0.02229
	σ	–	–	0.00532	0.00159	0.00692	0.00391
Bilevel	μ	–	–	0.07708	–	0.08366	–
	σ	–	–	0.02065	–	0.02065	–

As in three criteria, the positive orthant variant balances w.r.t. the outer approximation. We also notice that the bilevel algorithm provide better approximation quality than the local heuristics at iteration 50.

In four dimensions, the sandwiching algorithm provides better quality approximations than the simultaneous algorithm. This is expectable, since the sandwiching algorithm adds a point in the outer polyhedron which is the furthest to the inner polyhedron with the addition of the positive orthant.

Table 5.4: Complexity measures for different algorithms at iteration 50 and 150 for four criteria.

	Qty. Msr.	running time(ms)		number subproblems	
Algorithm	It.	50	150	50	150
Inner Approx.	μ	1142.47	11092.7	348.7	1585.1
	σ	842.858	3431.34	15.6803	51.081
Positive Orthant	μ	44.1359	163.477	563.667	2090.83
	σ	7.96482	24.3153	44.7309	133.369
Local Inner	μ	1097.93	8837.22	128.5	391.333
	σ	834.471	2469.37	6.61112	8.8759
Outer Approx.	μ	119.607	448.295	635.533	2273.17
	σ	21.1243	76.6301	14.161	34.7901
Local Outer	μ	30.5351	91.8674	125.7	380.034
	σ	6.07866	16.6136	1.29055	3.06458
Simul. Approx.	μ	1167.27	7684.35	1151.87	4340.67
	σ	381.666	1641.32	42.6079	107.756
Sandiw.	μ	795.249	54593.9	637.133	2289.37
	σ	33.7378	1691.12	12.5058	42.8055
Bilevel	μ	2333460	–	–	–
	σ	165021	–	–	–

The positive orthant variant runs faster than outer, inner and local inner algorithms in four dimensions. The inner local runs faster that the inner approx. due to the number of solved subproblems. As previously, we see that the number of subproblems for the local heuristics is similar in all tests and it is very small compared to the others algorithms.

The outer approximation solves more subproblems than the positive orthant. The number of subproblems for the sandwiching algorithm is related to the number of subproblems in the outer approximation.

The sandwiching algorithm runs faster than the inner and local inner approximations at 50 iterations. That is, the linear direction search problem with 50 points is easier to solve that the gauge maximization within a cone problem.

The simultaneous algorithm solves a number of subproblems equivalent to the sum of the inner and outer approximations subproblems. On the other hand, the bilevel algorithm is considerably slow, but stays in the same order as in the three dimensional case.

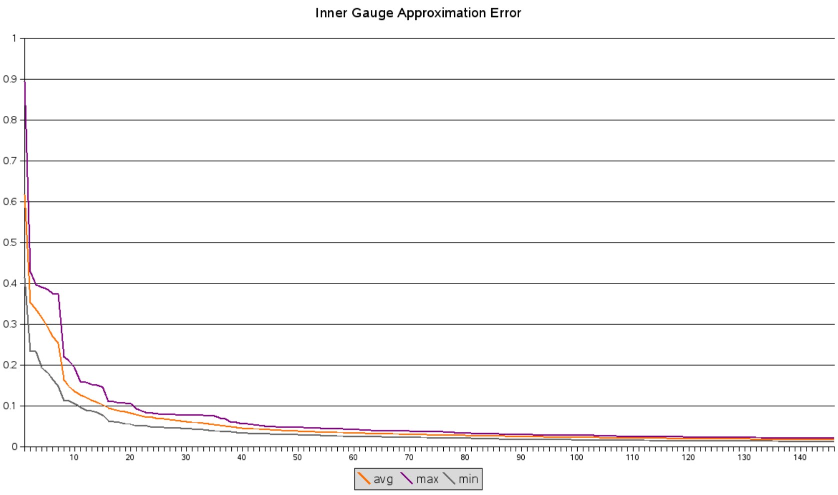

Figure 5.4: Inner gauge approximation error for four dimensions. At iteration 18 the inner gauge is already 0.1; at iteration 41, 0.05; at iteration 150, 0.02.

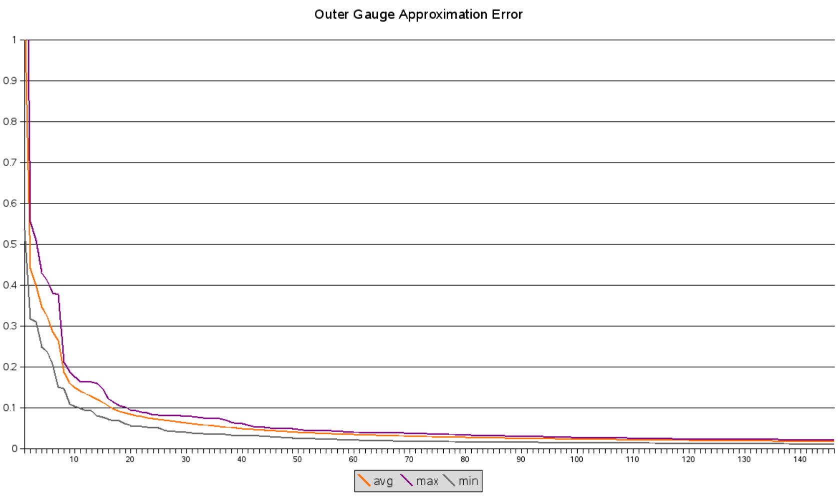

Figure 5.5: Outer gauge approximation error for four dimensions. At iteration 23 the outer gauge is already 0.1; at iteration 38, 0.05; at iteration 150, 0.02.

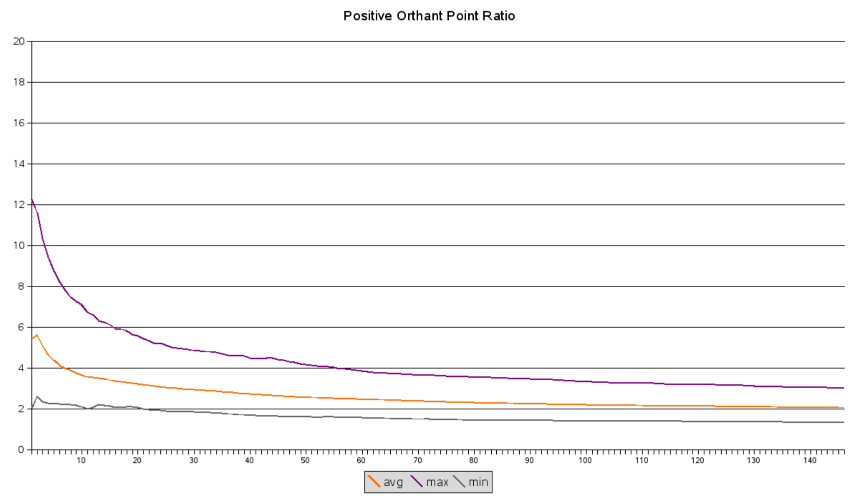

Figure 5.6: Point needed ratio to add the positive orthant. After 50 iterations we need less than four times the number of points in the representation.

From the results for three and four criteria, we conclude the following:

- The pseudo inner gauge is misleading as it only considers a partial area of the nondominated set. Thus, we no longer compute the pseudo inner gauge in the following dimensions.
- We no longer consider the simultaneous algorithm. Its complexity is related to the complexity of the inner and outer approximations. Moreover, its approximation quality is affected by the use of the pseudo inner gauge.
- We suggest to use the positive orthant variant in relatively easy problems and to use the sandwiching algorithm for more complex problems.
- The inner approximation and the local inner approximation are not suitable for representation set generation in three and four dimensions.
- In the case of ellipsoids, we do not get any further improvement of the inner and outer gauge after 100 and 200 points for 3 and 4 dimensions, respectively. A polyhedral representation of an ellipsoid based in 100 points in the three dimensional case should suffice for most applications.

5.3.2 Five and six criteria results

We ran 30 test cases in the five dimensional case, and fixed the number of points in the representation set to be 250. As in three and four dimensions, we ran the bilevel algorithm until it generated 50 points. Comparisons are made for 50 and 250 iterations.

Tables 5.5 and 5.6 summarize the results for five dimensions. In five dimensions, the bilevel algorithm provides competitive approximation quality w.r.t. the local heuristics.

Table 5.5: Quality approximation measures for different algorithms at iteration 50 and 250 for five criteria.

	Qty. Msr.	inner gauge		outer gauge	
Algorithm	It.	50	250	50	250
Inner Approx.	μ	0.114257	0.055182	0.145012	0.061194
	σ	0.017867	0.016061	0.031061	0.020931
Positive Orthant	μ	0.084042	0.028157	0.145934	0.057080
	σ	0.009445	0.002999	0.020152	0.008341
Local Inner	μ	0.251607	0.138755	0.290705	0.165630
	σ	0.042171	0.013307	0.045614	0.008658
Outer Approx.	μ	0.129224	0.056302	0.086070	0.028453
	σ	0.009561	0.006358	0.006857	0.002845
Local Outer	μ	0.343286	0.240319	0.285011	0.171292
	σ	0.040753	0.040689	0.039760	0.029948
Sandiw.	μ	0.111600	0.040926	0.105669	0.037490
	σ	0.008234	0.002915	0.009204	0.004220
Bilevel	μ	0.132494	–	0.149381	–
	σ	0.019880	–	0.031930	–

Table 5.6: Complexity measures for different algorithms at iteration 50 and 250 for five criteria.

	Qty. Msr.	running time(ms)		number subproblems	
Algorithm	It.	50	250	50	250
Inner Approx.	μ	6207.19	109429	1022.3	10350.6
	σ	4166.64	37042.6	52.4182	319.222
Positive Orthant	μ	330.154	2618.03	2859.23	19948.8
	σ	96.1021	599.035	831.959	4340.16
Local Inner	μ	287.885	5500.07	121.733	751.567
	σ	171.423	2853.67	0.691492	27.5151
Outer Approx.	μ	452.695	4101.46	1730.17	14329.7
	σ	104.162	538.408	104.162	376.995
Local Outer	μ	60.9369	330.175	182.567	764.038
	σ	12.8952	152.410	2.06253	20.9847
Sandiw.	μ	2428.55	1643860	1825.83	14532.9
	σ	149.178	57313.7	85.9631	381.452
Bilevel	μ	2967230	–	–	–
	σ	236805	–	–	–

There are no grounds to choose between positive orthant and outer approximation concerning the quality measures. Conveniently, the sandwiching algorithm presents good inner and outer gauge approximation quality.

However, concerning the time and number of solved subproblems, the positive orthant variant still runs faster than the outer approximation. Although it already solves a bigger number of subproblems.

The local update heuristics solved very few subproblems and thus their approximation quality is deficient. On the other hand, the sandwiching algorithm is faster than the inner approximation at 50 iterations.

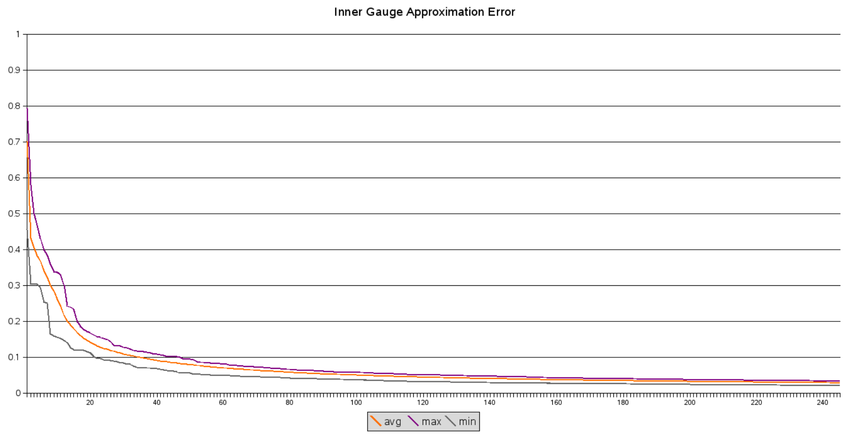

Figure 5.7: Inner gauge approximation error for five dimensions. At iteration 34 the inner gauge is already 0.1; at iteration 96, 0.05; at iteration 250, 0.03.

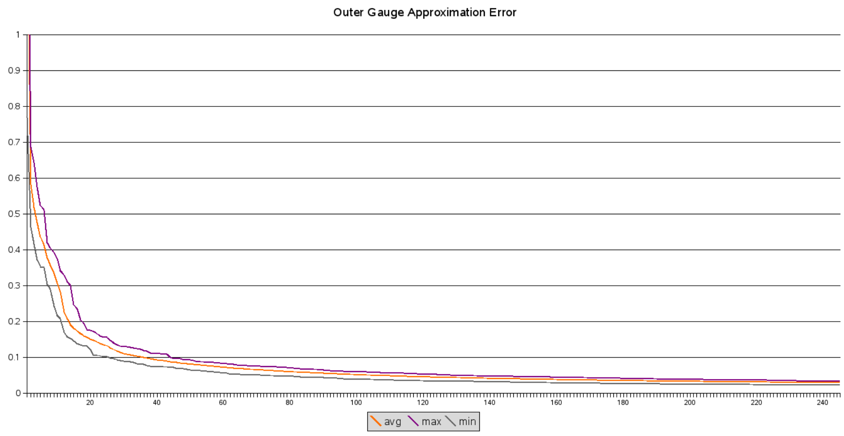

Figure 5.8: Outer gauge approximation error for five dimensions. At iteration 35 the outer gauge is already 0.1; at iteration 102, 0.05; at iteration 250, 0.03.

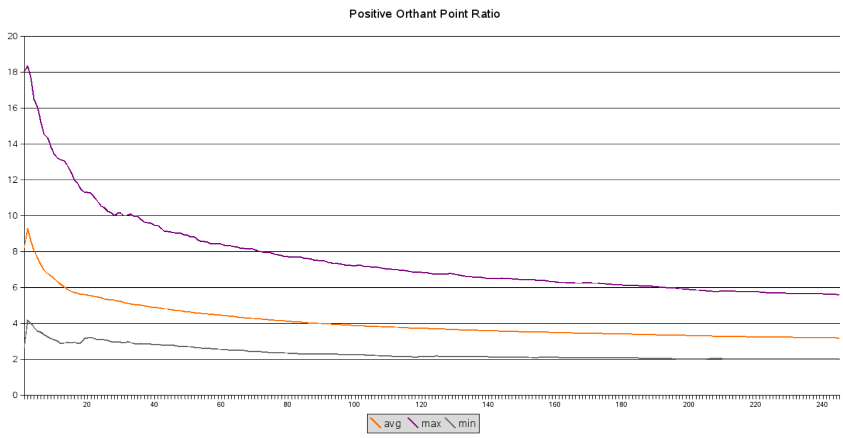

Figure 5.9: Point needed ratio to add the positive orthant. After 240 iterations we need less than six times the number of points in the representation.

Now we let the outcome space to be six dimensional, we ran 30 test cases, and fixed the number of points in the representation set to be 300. As previously, we ran the bilevel algorithm until it generated 50 points. Comparisons are made for 50 and 300 iterations.

Table 5.7: Quality approximation measures for different algorithms at iteration 50 and 300 for six criteria.

	Qty. Msr.	inner gauge		outer gauge	
Algorithm	It.	50	300	50	300
Inner Approx.	μ	0.243487	0.085137	0.351556	0.113183
	σ	0.039731	0.015237	0.124614	0.028684
Positive Orthant	μ	0.127208	0.044293	0.225750	0.090638
	σ	0.018784	0.007175	0.058277	0.016507
Local Inner	μ	0.336959	0.234717	0.538914	0.310528
	σ	0.037548	0.041115	0.075399	0.033220
Outer Approx.	μ	0.225048	0.089254	0.136255	0.046390
	σ	0.025967	0.009596	0.017499	0.006840
Local Outer	μ	0.446646	0.346363	0.493890	0.271242
	σ	0.061087	0.035415	0.057864	0.030619
Sandiw.	μ	0.181486	0.066571	0.171146	0.061850
	σ	0.013550	0.006185	0.017418	0.008443
Bilevel	μ	0.180906	–	0.247653	–
	σ	0.029319	–	0.129071	–

In six dimensions, the inner and local inner approximation quality is deficient due to their partial covering of the nondominated set. This issue worsens the quality of the representation for each further dimension. In fact, the approximation quality of the bilevel algorithm is better than the approximation quality of the inner approximation at iteration 50.

Table 5.8: Complexity measures for different algorithms at iteration 50 and 300 for six criteria.

	Qty. Msr.	running time(ms)		number subproblems	
Algorithm	It.	50	300	50	300
Inner Approx.	μ	34082	784365	3253.77	44516.4
	σ	40757	342798	325.337	2958.67
Positive Orthant	μ	3112.59	42395.2	16585.9	154933
	σ	1383.25	17616.9	6246.08	44811.9
Local Inner	μ	310.019	8716.28	165.6	806.733
	σ	101.731	20036.7	7.60036	27.3369
Outer Approx.	μ	1928.79	29436.4	4995.23	68177.3
	σ	306.041	3971.69	379.087	3038.77
Local Outer	μ	107.74	589.179	248.367	1145.33
	σ	22.2089	119.869	12.2825	47.6679
Sandiw.	μ	8440.66	14890700	5188.8	69900.2
	σ	743.276	934098	426.359	3817.88
Bilevel	μ	3030480	–	–	–
	σ	246948	–	–	–

The outer approximation runs faster than the positive orthant variant in six dimensions. This is because the number of subproblems increases faster in the positive orthant variant. In addition, the inclusion of more points (and the corresponding routine to verify its inclusion) requires more computing time in six dimensions.

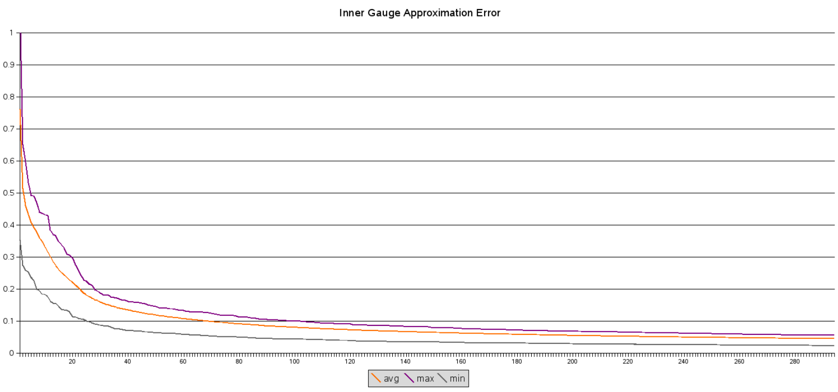

Figure 5.10: Inner gauge approximation error for six dimensions. At iteration 68 the inner gauge is 0.1; at iteration 232, 0.05. We obtain an approximation of 0.04 at iteration 300.

However, the positive orthant variant still runs faster than the inner approximation. On the other hand, the deficient approximation quality of the local heuristics and their faster running times are both justified by the small number of solved subproblems.

Although the sandwiching algorithm is still slower than the all others but the bilevel and the inner approximation algorithms, its running time at 50 iterations is in the same order as the positive orthant running time, i.e. the complexity of ellipsoid minimization at dimension 6 seems to be similar as a 50 constraint linear problem.

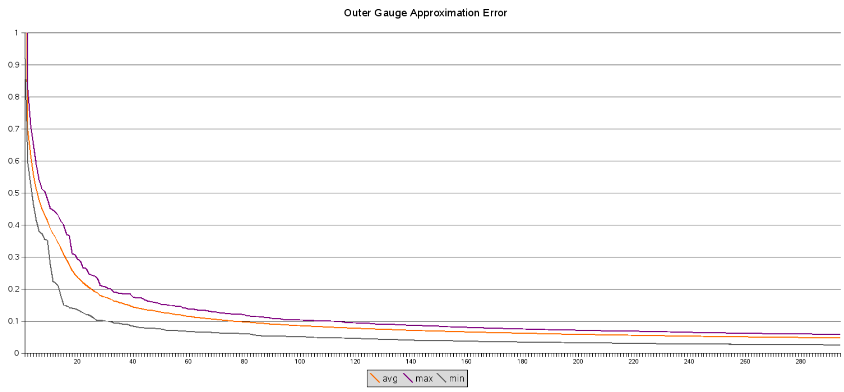

Figure 5.11: Outer gauge approximation error for six dimensions. At iteration 76 the outer gauge is smaller than 0.1; at iteration 252, 0.05. We obtain an approximation of 0.04 at iteration 300.

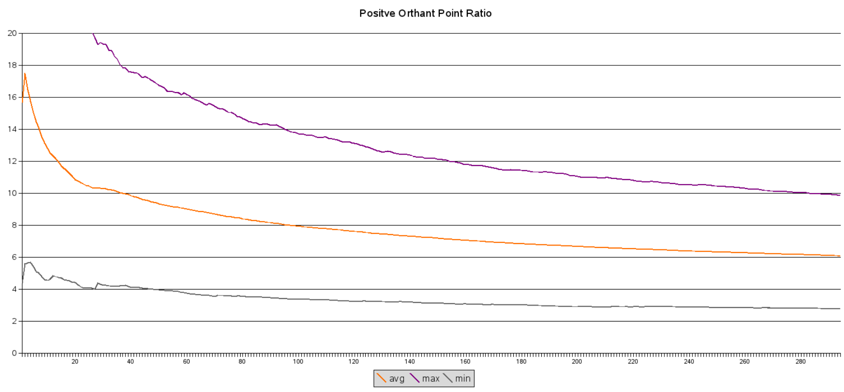

Figure 5.12: Point needed ratio for the inclusion of the positive orthant. Around 300 iterations we need ten times the number of points in the representation.

From the results in five and six dimensions we derive the following conclusions:

- Local update heuristics are fast but they do not provide a good quality approximation. This is explained by the small number of subproblems that are solved in these algorithms.
- The bilevel algorithm is slow compared to the other algorithms. However, its complexity is very similar for three, four, five and six dimensions. This suggests that the bilevel algorithm's complexity is mainly influenced by

the number of points in the represention instead of the outcome space dimension. This is related with the KKT derived constraints in the BLLP.

- The reasonable approximation quality of the bilevel algorithm opens the possibility to explore stochastic approaches even in six dimensions.
- It is convenient to use the outer approximation in six dimensions for relatively easy problems. Otherwise, the sandwiching algorithm is preferred. However, it requires more than 14890 secs in average to generate 300 points. Thus, it may not be convenient for applications which demand a faster response time.

5.3.3 Nine criteria results

Considering the previous results, it is interesting to observe the corresponding algorithmic behavior in nine dimensions. We ran 20 test cases to generate a finite representation set of 50 points in order to analyze the starting behavior of the algorithms.

Table 5.9: Quality measures for different algorithms at iteration 50 for nine criteria.

Algorithm	Qty. Msr.	inner gauge	outer gauge	runn. time (ms)	subpr. solved
Inner Approx.	μ	0.559861	0.823590	1503660	55878.3
	σ	0.077977	0.087633	1199020	14365.4
Positive Orthant	μ	0.352368	0.565271	31003300	1128290
	σ	0.0238239	0.0364359	29184400	538104
Local Inner	μ	0.597659	0.871494	1326.08	267.85
	σ	0.091712	0.119460	210.306	21.651
Outer Approx.	μ	0.537233	0.310210	59085.9	61625.4
	σ	0.052388	0.026833	22545.1	20068.5
Local Outer	μ	0.694655	0.731243	879.456	798.450
	σ	0.091551	0.096430	279.348	22.5306
Sandiw.	μ	0.434677	0.387655	161339	69695.2
	σ	0.023358	0.047076	48933.4	20276
Bilevel	μ	0.421104	0.520347	2358330	–
	σ	0.074055	0.080526	148355	–

The bilevel and the sandwiching algorithms provide a more balanced approximation quality than the positive orthant variant and the outer approximation for 50 points. In fact, the sandwiching algorithm has a very close approximation quality to the inner and outer approximations and in addition presents smaller deviation.

The positive orthant variant runs very slow due to the number of solved subproblems and the inclusion of points in order to add the positive orthant to the inner polyhedron. Actually, the bilevel algorithm even runs faster than the positive orthant variant. On the other hand, the sandwiching algorithm running time is in the same order as the inner and outer approximation. This suggests that the convex problems in 9 dimensions are more complex to solve than a 50 constraint linear program.

Figure 5.13: Gauge approximation errors for nine dimensions.

From these results we conclude the following:

- Convex hull computation does not seem to be an issue for 9 dimensions and 50 points; however, the number of subproblems is too big to handle it properly.
- The considered algorithms do not suffice for proper approximation in 9 dimensions.
- For small complex problems we may use the outer approximation, in order to use the positive orthant variant we need to add the positive orthant in a different way, e.g. projective geometry.
- For higher complex problems we need to consider randomized algorithms. The sandwiching algorithm may be worth to consider in 9 dimensions, if we use a heuristic version that separates the outcome space when the number of points surpasses a fixed quantity.

Conclusions and further considerations

In this paper we studied the application of block norms in Multicriteria Optimization. We began by introducing important definitions and notation in Chapter 1. We saw how nondominated points can be obtained using nonnegative weighted scalarization of the objective functions in a multicriteria problem.

However, this approach is not sufficient to generate a good representation of the nondominated set. Thus, following the work from Schandl et al. [2002], we looked at how block norms can be used to obtain nondominated points.

Although block norms centered at the ideal point are sufficient to generate all the nondominated set of the outcome space, we focused on block norms centered at a dominated point. In this way, we do not only generate nondominated points, but we are also able to approximate the structure of the nondominated set of $\mathbb{R}_{\geqq}$-convex bodies.

Klamroth et al. [2002] devised algorithms that approximate the nondominated set using an inner or an outer polyhedron derived from a finite point representation. In order to run, these algorithms need to alternate between the H-representation and the V-representation of the polyhedral approximation, thereby obtaining facets and extreme points which are used to solve a sequence of subproblems.

We reviewed these algorithms in Chapter 3 and proposed three variants for the inner approximation and outer approximation algorithms. We notice that the positive orthant variant is much faster than the inner approximation and provides better approximation quality. Furthermore, the addition of the positive orthant to the inner polyhedron does not represent a significant computational effort up to five dimensions.

The other two variants try to overcome the need to use a convex hull algorithm to switch between the H-representation and V-representation. They do this by performing a localized update, i.e. we compute new facets or extreme points taking a small subset of the current finite point representation. Although they run fast enough even in nine dimensions, the obtained representation set presents a deficient approximation quality.

We also considered a sandwiching approach, where we use the inner an outer polyhedron structure to estimate areas from the outcome space where more points are needed in order to improve the approximation quality. The sandwiching algorithm solves a sequence of linear programs and it only solves one direction search problem over the outcome space to obtain a new nondominated point at each iteration.

Although it estimates the distance between the outer polyhedron and the nondominated set, the sandwich algorithm generates representation sets whose approximation quality is superior to all presented algorithms in 5,6 and 9 dimensions. However, it required more than 14890 secs in average to generate 300 points in six dimensions. Thus, it may not be convenient for applications which demand a faster response time.

During the tests, we notice that the number of subproblems increases significantly because of the big number of facets and extreme points that are generated in higher dimensions (e.g. we needed to solve 2000000 subproblems to obtain 150 points in nine dimensions). Thus, we strove to devise an approach that would serve as a middle ground between deterministic algorithms (which still work efficiently for six dimensions) and stochastic methods (which are probably the *de facto* option for more than twelve criteria).

In Chapter 4, we used bilevel linear programming to estimate the distance between the inner and outer polyhedron without using the convex hull algorithm. Unfortunately, the resulting optimization problem from the Karush-Kuhn-Tucker conditions substitution turned out to be very slow compared to the other alternatives.

However, the bilevel algorithm's complexity is very similar for three, four, five and six dimensions. This suggests that it is mainly influenced by the number of points in the represention set and not by the outcome space dimension. The reasonable approximation quality of the bilevel algorithm opens the possibility to explore stochastic approaches even in six dimensions.

In Chapter 5, we presented the results which we got from the numerical testing of the algorithms and the numerical convergence of the positive orthant variant and the outer approximation algorithm.

We found that the considered algorithms do not suffice for proper approximation in 9 dimensions. The obtained insight from the numerical experience suggests the following ideas as prospects for future investigation.

- *Bilevel models solved by pivoting or extreme point enumeration method.*

  The bilevel algorithm presented a competitive approximation quality in six and nine dimensions. However, the KKT approach proved to be not convenient due to the complexity of the resulting optimization problem.

  Nevertheless, some other methods may be used. Particularly, those which traverse vertices of the outer polyhedron.

- *Clever localization of the convex hull computation.*

  The local update may be allowed to include a bigger subset of points to compute new extreme points from the updating polyhedron, thereby generating less but still relevant subproblems.

- *Space decomposition to reduce the number of subproblems, and their complexity.*

  The sandwiching algorithm runs efficiently until the number of points is big enough to affect the complexity of the subproblems and to significantly increase their number. In such case, we can decompose the outcome space in sections to be solved independently, thus generating in overall less subproblems and decreasing their complexity.

- *Random direction search driven by the inner/outer polyhedra structure.*

  Stochastic approaches may use the inner/outer polyhedra structure information to drive their search for extreme points of the outer polyhedron.

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DECLARATION

Kaiserslautern, January 31st, 2008

I hereby declare that I am the only author of this work and that no other sources than those that are listed have been used.

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