Selfish Bin Coloring

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Abstract. We introduce a new game, the so-called bin coloring game, in which selfish players control colored items and each player aims at packing its item into a bin with as few different colors as possible.
We establish the existence of Nash and strong as well as weakly and strictly Pareto optimal equilibria in these games in the cases of capacitated and uncapacitated bins. For both kinds of games we determine the prices of anarchy and stability concerning those four equilibrium concepts. Furthermore, we show that extreme Nash equilibria, those with minimal or maximal number of colors in a bin, can be found in time polynomial in the number of items for the uncapcitated case.

Keywords: algorithmic game theory, bin coloring, Nash equilibria, strong equilibria, weakly/ strictly Pareto optimal Nash equilibria, price of anarchy, price of stability, extreme Nash equilibria

1 Introduction

In 1999, Koutsoupias and Papadimitriou [18, 19] laid the basis for a new direction of research in theoretical computer science by introducing the coordination ratio, transferring the concept of approximation ratio, for NP-complete problems, and competitive ratio, for online problems, to games. This coordination ratio is defined as the quotient of the social cost of a worst Nash equilibrium and the optimal social cost and thus measures the loss in overall performance due to the lack of central coordination in non-cooperative games, where Nash equilibria are considered as the solution concept to these games. This resembles the lack of knowledge of future events in online scenarios and the lack of computation time and storage space for NP-complete problems.

With this seminal work Koutsoupias and Papadimitriou [18, 19] combined classical game theory with concepts of theoretical computer science resulting in the new field of algorithmic game theory. The motivation behind this new approach is to understand the development, structure and mode of operation of networks in which users with various interests interact. Especially, the internet with its tremendous impact on everyday life motivates to study systems with selfish users in order to get an idea how to positively influence the interactions of users in the world wide web.

The coordination ratio, as introduced in [18], is today mostly referred to as price of anarchy (PoA) due to Papadimitriou [24]. It is extensively studied for various problems starting from selfish job scheduling in the KP-Model, named after Koutsoupias and Papadimitriou [18, 19], for Nash equilibria in pure as well as mixed strategies (see among others [18, 19, 23, 10, 15, 14]).

Anshelevich et al. [2] add a more positive measure by taking the best Nash equilibrium into consideration. The resulting price of stability (PoS) indicates how much the performance of a system suffers in order to get a solution acceptable to all users.

Andelman et al. [1] use strong equilibira, as introduced by Aumann [3], as a solution concept and establish the strong price of anarchy (SPoA) as a measure of performance when coordination of the users is possible. Strong equilibria are stable against the deviation of coalitions (where each player in a coalition must improve its cost) and not only deviations of single users as in Nash equilibria. The authors state that in games where the SPoA is significantly smaller than the PoA, coordination of users can improve the overall performance of the system.

Chien and Sinclair [6] as well as Holzman and Law-Yone [17] use the concept of Pareto optimality, which is well studied in economics [22], to characterize certain Nash equilibria and the resulting price of anarchy and stability. In a (weakly) Pareto optimal Nash equilibrium there is no alternative strategy choice beneficial for all players. A strictly Pareto optimal Nash equilibrium is also stable against deviations in which some players do not benefit but are also not worse off and at least one player improves his personal cost.

A classical problem in combinatorial optimization is the bin packing problem in which items of different sizes are to be packed into as few bins, with a given capacity, as possible. This problem is known to be NP-hard (see Coffman and Csirik [7] for a general survey).

The analysis of bin packing in the algorithmic game theory context is started by Bilò [4]. In non-cooperative bin packing games, each item is controlled by a player and the cost of the player is proportional to the item's share in the bin space used. Hence, each player wants to minimize its cost by packing its item into the most filled bin it fits in. The social cost is the number of bins used and the goal is to keep this number as small as possible.

Bilò [4] proves the existence of a Nash equilibrium in pure strategies by showing that the greedy best response dynamics converge in finite time. He also establishes that there is always a Nash equilibrium with minimal number of bins, i.e., the price of stability is 1, but that finding such a good equilibrium is NP-hard. First upper and lower bounds on the price of anarchy are also given by Bilò [4].

Epstein and Kleiman [12] analyze the price of anarchy and stability as well as the strong price of anarchy and stability. They improve on the bounds of Bilò [4] and give nearly tight bounds for the price of anarchy. Additionally, the authors prove the existence of a strong Nash equilibrium in every bin packing game and establish that the strong price of anarchy and the strong price of stability are identical and given by the approximation ratio of the Subset Sum algorithm, a greedy algorithm for the classical bin packing problem. For years the exact approximation ratio of this algorithm was unknown although almost matching upper and lower bounds were given by Caprara and Pferschy [5]. Finally, Epstein, Kleiman and Mestre [13] succeeded in closing this gap.

Yu and Zhang [28] add that one pure equilibrium can be computed in polynomial time by a recursive variant of the First-Fit Decreasing algorithm for bin packing.

The classical bin packing problem is also studied in several variants: Csirik and Woeginger [9] give a survey on online bin packing and Csirik and Leung [8] give an overview of (other) variants of the bin packing problem.

One of those variants are so-called class-constrained packing problems, in which the items are grouped into different classes identified by colors. Each bin is equipped not only with a limited capacity concerning the size of items to be packed into it, but also with an upper bound on the number of different colors present in the bin. The goal in the optimization problem is to minimize the number of used bins. These problems were studied among others by Shachnai and Tamir [25, 26] as well as Xavier and Miyazawa [27].

Class-constrained packing problems are closely related to the bin coloring problem introduced by Krumke et al. [20] motivated by an application in a distribution center for office supply. In the offline model, unit-sized items of different colors are to be packed in a fixed order into bins, and at any time during the packing process, only up to a fixed number of bins may be partially filled. The aim is to pack as few different colors in a bin as possible. In the online variant the items arrive one after the other and have to be packed immediately and irrevocably, i.e., any algorithm has to pack the items without knowledge of further items. The authors establish that finding an optimal packing for the offline case is already NP-hard in a special case, where it is additionally known that the items exactly fit into a certain number of bins. Additionally, Krumke et al. [20] analyze the performance of different online algorithms and establish the surprising result that packing exactly one bin at a time yields a better competitive ratio than a natural greedy strategy.

Hiller and Vredeveld [16] use stochastic comparison for a probabilistic analysis of online bin coloring algorithms and this yields more intuitive results in the comparision of the two algorithms of Krumke et al. [20]. See also [11] for a comparison of these algorithms which yields more intuitive results as well.

Lin et al. [21] address other variants of bin coloring from the point of view of competitive analysis. The authors motivate their further study of this problem by an application in networks where the items of one color represent the packets of a single task and the bins correspond to channels. In the minimum bin coloring setting the number of bins is fixed, the items fit exactly into this number of bins and the aim is to minimize the maximal number of different colors in the bin. The online maximum bin coloring aims at packing the items online in such a way that the minimal number of different colors in the bins is maximized. Both problem variants aim at maximizing two different measures of fairness among the users.

In this paper we analyze this minimum bin coloring problem (of [21]) in the context of algorithmic game theory, i.e., when the items of the bin coloring problem are controlled by selfish players.

Contribution

We address a new class of non-cooperative games, so-called bin coloring games. In these games, selfish colored unit-sized items are packed into a fixed number of (uncapacitated or capacitated) bins. The items aim at being packed in a bin with items of as few other colors as possible. The social objective is to minimize the maximal number of different colors in a bin.

We establish the existence of optimal Nash and strong as well as weakly and stricly Pareto optimal Nash equilibria in capacitated and uncapacitated bin coloring games.

The price of anarchy in uncapacitated bin coloring games varies from unbounded for Nash equilibria in general over asymptotically 1 for strong and weakly Pareto optimal Nash equilibria to 1 for strictly Pareto optimal Nash equilibria. For the capacitated case we add that the price of anarchy is unbounded but given by the number of bins, if only games with a constant number of bins are considered.

For uncapacitated bin coloring games, we also give a simple polynomial algorithm to find a strong equilibrium and discuss best and worst Nash equilibria of single instances.

Roadmap

In Section 2 we formally define bin coloring games and introduce necessary notation. Nash equilibria in these games are addressed in Section 3. First, we establish their existence, then analyze the price of anarchy and stability, and add results on extreme Nash equilibria. Section 4 is concerned with the existence of strong equilibria as well as the strong price of anarchy and stability. Before we conclude the work in Section 6, we derive implications of the previously established results on weakly and strictly Pareto optimal Nash equilibria in Section 5.

2 Preliminaries and Notation

In bin coloring games selfish players control items of equal sizes but different colors. Each player controls a single item. The aim of the players is to put their item in a bin with as few colors as possible. The number of bins is limited and the load of the bins is either not bounded in the uncapacitated case or bounded in the capacitated one.

More formally an instance $\Gamma = [m, k, (n_i)_{i\in\{1,\dots,k\}}, U]$ of the bin coloring game is given by the number $m \in \mathbb{Z}^+$ of bins, the number $k \in \mathbb{Z}^+$ of colors implying the set $C := \{1, \dots, k\}$ of colors, the numbers $n_i \in \mathbb{Z}^+$, $i = 1, \dots, k$ of items of color i and the bin capacity $U \in \mathbb{Z}^+ \cup \{\infty\}$. Thus the number of items n in the game is given by $n = \sum_{i=1}^{k} n_i$. We refer to the set of all items by N and identify every player with its item. The strategy set of each player equals the set of bins.

Additionally, the sets of uncapacitated ($U = \infty$) and capacitated ($U < \infty$) bin coloring games are denoted by BC and BC_{cap}, respectively. In this paper, we always assume that every instance to the capacitated bin coloring has a feasible solution. That is, $mU \leq n$.

An outcome of a bin coloring game is given by a strategy profile $\sigma = (\sigma_1, \dots, \sigma_n)$, i.e., an assignment of items to bins with $\sigma_i \in \{1, \dots, m\}$ for all $i \in N$. We denote by σ_{-i} the strategy profile in which all players keep their choice as in σ but the item of player i is removed. Analogously, for a subset $I \subseteq N$ we refer to a strategy profile in which all players in I are ignored by σ_{-I}.

The load of bin $j \in \{1, \dots, m\}$ is given by

$$\ell_j(\sigma) = |\{i \in \{1, \dots, n\} | \sigma_i = j\}|$$

and equals the number of items in the bin. In contrast to that the color load, i.e., the number of colors in the bin, is referred to by $\text{cl}_j(\sigma)$. More formally, let z_i denote the color of item i, and let $\text{cl}_j(\sigma) = |\{z_i | \sigma_i = j\}|$. The personal cost $c_i(\sigma)$ of a player i with strategy σ_i is determined by $c_i(\sigma) = \text{cl}_{\sigma_i}(\sigma)$.

The social cost of a strategy profile is given by the maximal number of different colors in a bin:

$$\text{sc}(\sigma) = \max_{j=1,\dots,m} \text{cl}_j(\sigma) = \max_{i=1,\dots,n} c_i(\sigma).$$

The minimal social cost of a strategy profile in game Γ is referred to as $\mathsf{opt}(\Gamma)$.

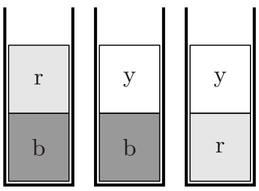
One concept of stability is to consider a strategy profile as stable if no player wants to deviate from its chosen bin. This idea is formalized by the concept of Nash equilibria.

Definition 1 (Nash equilibrium). *A strategy profile σ is a (pure) Nash equilibrium of the bin coloring game Γ =[m, k,$(n_i)_{i\in\{1,\ldots,k\}}$,$U$] if for all $i \in N$ and strategy choice $\bar{\sigma}_i \in \{1, \ldots, m\}$ s.t. $\ell_{\bar{\sigma}_i}(\sigma_{-i}, \bar{\sigma}_i) \leq U$*

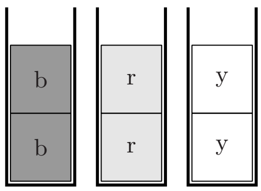
$$c_i(\sigma) = cl_{\sigma_i}(\sigma) \leq cl_{\bar{\sigma}_i}(\sigma_{-i}, \bar{\sigma}_i) = c_i(\sigma_{-i}, \bar{\sigma}_i)$$

In general, bin coloring games do not have a unique Nash equilibrium and we denote the set of Nash equilibria of an instance Γ of the bin coloring game by $\mathsf{NE}(\Gamma)$.

Example 1. Consider the following instance of the uncapacitated bin coloring game ($U = \infty$), 3 bins and 6 items, where there are two items of each one of the colors: red, blue and yellow. Two packings of these items are given in Figure 1. Both packings shown in Figures 1a



(a) A non-optimal coloring, which is a Nash equilibrium



(b) An optimal coloring (which is a Nash equilibrium)

Fig. 1: Comparison of different colorings

and 1b are Nash equilibria. However, the packing in Figure 1a has social cost 2, while the one in Figure 1b has optimal social cost 1.

Observe that if the players can cooperate, then the packing in Figure 1a is not stable, as all players are better off in the packing in Figure 1b and thus would have an incentive to jointly deviate.

The latter observation leads to the notion of strong equilibria as introduced by Aumann [3].

Definition 2 (Strong equilibrium). *A strategy profile σ is a strong equilibrium if for every coalition $I \subseteq N$ and any strategy choice $\bar{\sigma}_I$ of players in I with $\ell_j(\sigma_{-I}, \bar{\sigma}_I) \leq U$, $j \in \{1, \ldots, m\}$, there is a member $i \in I$ with*

$$c_i(\sigma_{-I}, \bar{\sigma}_I) \geq c_i(\sigma).$$

The definition states that for every coalition and every feasible strategy choice, at least one member of the coalition is not better off. An equivalent statement is that for no coalition there is a feasible strategy choice beneficial to all its members.

Note also that the set $\mathsf{SE}(\Gamma)$ of strong equilibria of the bin coloring game Γ is a subset of $\mathsf{NE}(\Gamma)$ as coalitions of size 1 can not improve by changing their strategy.

There are also intermediate concepts between Nash and strong equilibria, so-called weakly Pareto optimal Nash equilibria:

Definition 3 (Weakly/Strictly Pareto optimal Nash equilibrium). *A Nash equilibrium σ is weakly Pareto optimal if there is no strategy choice $\bar{\sigma}$ s.t. for all $i \in N$*

$$c_i(\bar{\sigma}) < c_i(\sigma).$$

A Nash equilibrium σ is strictly Pareto optimal if there is no strategy profile $\bar{\sigma}$ and $i^* \in N$ s.t. for all $i \in N \setminus \{i^*\}$

$$c_i(\bar{\sigma}) \leq c_i(\sigma) \quad \text{and} \quad c_{i^*}(\bar{\sigma}) < c_{i^*}(\sigma).$$

We denote by $\mathsf{wPNE}(\Gamma)$, $\mathsf{sPNE}(\Gamma)$ the set of weakly and strictly Pareto optimal Nash equilibria of the bin coloring game Γ, respectively.

In weakly Pareto optimal Nash equilibria the grand coalition of all players cannot deviate in such a way that all players are better off. In comparison to strong equilibria coalitions of more than one player but less than all players are not considered. So every strong equilibrium is also weakly Pareto optimal.

Strictly Pareto optimal Nash equilibria take into account that players might participate in a coalition without improving their personal cost, if they are not worse off and can help other players, such as i^*, to improve their cost. In general neither Nash nor strong equilibria are strictly Pareto optimal.

Another approach for classifying specific Nash equilibria and especially for addressing Nash equilibria with different social cost in one instance, as given in Example 1, is to ask for extreme representatives of the set of Nash equilibria and this gives rise to the following two problems:

Definition 4 (Extreme Nash Equilibrium Problems). *Finding Nash equilibria with best or worst social cost corresponds to solving two optimization problems:*

Best Nash Equilibrium Problem (BNE)

Given: *Instance $\Gamma = [m, k, (n_i)_{i\in\{1,\ldots,k\}}, \infty]$ of the uncapacitated Bin Coloring Game.*

Output: *A Nash equilibrium σ, with a minimum social cost.*

Worst Nash Equilibrium Problem (WNE)

Given: *Instance $\Gamma = [m, k, (n_i)_{i\in\{1,\ldots,k\}}, \infty]$ of the uncapacitated Bin Coloring Game.*

Output: *A Nash equilibrium σ, with a maximum social cost.*

Instead of focusing on a single instance, also the whole set of (un)capacitated bin coloring games can be characterized by determining the worst-case ratio of the optimal social cost and that of a best or worst Nash equilibrium. Thus we consider the following metrics.

Definition 5 (Price of Anarchy and Stability). *For $\mathcal{G} \in \{BC, BC_{cap}\}$ the price of anarchy $PoA(\mathcal{G})$ is given by*

$$PoA(\mathcal{G}) = \sup_{\Gamma\in\mathcal{G}} \sup_{\sigma\in NE(\Gamma)} \frac{sc(\sigma)}{opt(\Gamma)}.$$

If instead the best Nash equilibrium of every instance is considered this leads to the price of stability

$$PoS(\mathcal{G}) = \sup_{\Gamma\in\mathcal{G}} \inf_{\sigma\in NE(\Gamma)} \frac{sc(\sigma)}{opt(\Gamma)}.$$

This concept is applied analogously to strong equilibria as well as weakly/strictly Pareto optimal Nash equilibria yielding the strong *price of anarchy* $\mathsf{SPoA}(\mathcal{G})$ *and the strong price of stability* $\mathsf{SPoS}(\mathcal{G})$ *as well as the Pareto prices of anarchy* $\mathsf{wPPoA}(\mathcal{G})$, $\mathsf{sPPoA}(\mathcal{G})$ *and stability* $\mathsf{wPPoS}(\mathcal{G})$, $\mathsf{sPPoS}(\mathcal{G})$.

So far we have identified the players decisions directly using the strategy profile. Due to the structure of bin coloring games it suffices to represent the strategy choices of the players by an assignment $A = (a_{ij})$ of colors $i = 1, \ldots, k$ to bins $j = 1, \ldots, m$, where a_{ij} is the number of items of color i packed into bin j.

Fortunately, this compact representation suffices to describe Nash equilibria as well as strong ones due to the fact, that knowing the assignment allows to calculate the load $\ell_j(A) = \sum_{i=1}^{k} a_{ij}$ as well as the number of colors $\mathrm{cl}_j(A) = |\{i \in \{1, \ldots, k\} | a_{ij} > 0\}|$ of any bin $j \in \{1, \ldots, m\}$ and thus to evaluate the social cost.

As long as uncapacitated bin coloring games are considered, the number of items of one color can be neglected as well, as the load of the bin does not matter in that scenario. If an item with color i has an incentive to change from bin j_1 to bin j_2 so have all items of color i in bin j_1 and due to the unboundedness of the bins such a change is also possible.

We note that even if a Nash equilibrium is represented as an assignment of colors to bins (not items to bins), this information is not always polynomial in the input size of the game as we need a size which is $\Theta(mk)$ information to store this assignment (which is not polynomial in $\log(m)$, k, and $\log n$). If the number k of colors is greater than m, then $\mathcal{O}(mk) \subseteq \mathcal{O}(k^2)$ and the information is input-polynomial.

3 Nash Equilibria in Bin Coloring Games

In this section we address existence and computability of Nash equilibria as well as the performance of Nash equilibria in comparison to the optimal assignment.

First, in the uncapacitated case the optimal assignment of items to bins can be easily calculated:

Lemma 1. *An optimal assignment A of an uncapacitated bin coloring game $\Gamma = [m,\ k,\ (n_i)_{i\in\{1,\dots,k\}},\ \infty]$ has $sc(A) = \lceil k/m \rceil$.*

Proof. Packing all items of one color at once to the bin with least colors so far yields an assignment with the above maximum number of colors in a bin. As each color has to be assigned to at least one bin, this solution is optimal. □

Additionally, the assignment in the proof of Lemma 1 is also a Nash equilibrium, which establishes the following corollary:

Corollary 1. *Every uncapacitated bin coloring game $\Gamma = [m,\ k,\ (n_i)_{i\in\{1,\dots,k\}},\ \infty]$ admits at least one optimal Nash equilibrium.*

Price of Anarchy and Stability

Corollary 1 implies not only existence of a Nash equilibrium in every instance but also that $\mathsf{PoS}(\mathsf{BC}) = 1$. In addition, the price of anarchy is determined as follows:

Theorem 1. *The price of anarchy of uncapacitated bin coloring games with m bins equals the number of bins: $\mathsf{PoA}(\mathsf{BC}_m) = m$.*

Proof. Recalling Lemma 1, the upper bound is easy, as any assignment can not have more than all k different colors in one bin. For the lower bound consider the following example: We construct an instance with $k = m$ colors and $n_i = m$ items of each color $i \in \{1, \dots, k\}$. Assigning all items of one color to one bin obviously yields an optimal solution with only one color in every bin. But packing one item of every color into every bin yields a Nash equilibrium with m colors in every bin. □

For the whole class of uncapacitated bin coloring games, we conclude:

Corollary 2. *The price of anarchy $\mathsf{PoA}(\mathsf{BC})$ of uncapacitated bin coloring games is unbounded.*

Note that this lower bound of m on the price of anarchy holds also for the capacitated case as we can always consider the capacitated case with $U = n$, that is equivalent to the uncapacitated case. In addition, for any capacitated bin coloring game Γ the relation $\mathsf{opt}(\Gamma) \geq \lceil \frac{k}{m} \rceil$ holds as well. Hence, we conclude the following:

Corollary 3. *The price of anarchy* $\mathsf{PoA}(\mathsf{BC}_{cap})$ *of capacitated bin coloring games is unbounded but equals to the number of bins for games* $\mathsf{BC}_{cap,m}$ *with a fixed number* m *of bins.*

Computing Extreme Nash Equilibria

We have already seen that a best Nash equilibrium in the uncapacitated case can be calculated in time $\mathcal{O}(k)$ using the construction in the proof of Lemma 1 and thus BNE can be solved in linear time.

The situation is more involved for the worst Nash equilibrium: Our algorithm to solve WNE is based on the following structural lemma which characterizes how the number of colors of the bins can vary in a Nash equilibrium.

Lemma 2. *Let* A *be a Nash equilibrium of* $\Gamma = [m, k, (n_i)_{i\in\{1,\dots,k\}}, \infty]$ *with the social cost* $sc(A) = r + 1$ *for some value of* r*. Then, there is a* $p \in \{1, 2, \dots, m\}$ *such that the set of bins is partitioned into two sets* M_1, M_2 *with* $|M_1| = p$, $cl_j(A) = r + 1$ *for all* $j \in M_1$*, and* $cl_j(A) = r$ *for all* $j \in M_2$*.*

Moreover, in A *there is no color with items in bins in* M_1 *as well as in* M_2*, i.e., the set of colors is partitioned as well.*

Proof. Let M_1 denote the set of bins which contain items of $r + 1$ colors (by definition, $M_1 \neq \emptyset$), and let M_2 be the (possibly empty) set of bins which contain r colors.

If there is a bin j with $\mathrm{cl}_j(A) \leq r - 1$, then any item in a bin $j_1 \in M_1$ with $\mathrm{cl}_{j_1}(A) = r + 1$ has an incentive to deviate from bin j_1 to j. This deviation can increase the number of colors in j to at most r colors. Hence, $M_1 \cup M_2$ is a partition of $\{1, \dots, m\}$.

It remains to show that there are no two items with color i which are packed into bins $j_1 \in M_1$ and $j_2 \in M_2$, respectively. Assume for the sake of a contradiction that such a color i and bins $j_1 \in M_1$, $j_2 \in M_2$ exist. Then the item with color i in j_1 has an incentive to deviate and move to bin j_2 as $r + 1 = \mathrm{cl}_{j_1}(A) > \mathrm{cl}_{j_2}(A) = r$ and this deviation does not increase the number of colors in j_2. Thus, the claim follows. □

Motivated by Lemma 2, consider packings with the following properties: κ is the total number of colors, such that there are η_i

items of color i which are packed into $\tau_i \leq \min\{\pi, \eta_i\}$ bins. Moreover, all bins are packed with items of either α or $\alpha+1$ colors, where $0 \leq \beta \leq \pi - 1$ is the number of bins with $\alpha+1$ colors. In order to allow the existence of such a packing, the condition $\sum_{i=1}^{\kappa} \tau_i = \alpha\pi + \beta$ must be satisfied. We let $\overline{\tau} = (\tau_1, \tau_2, \ldots, \tau_\kappa)$ and $\overline{\eta} = (\eta_1, \eta_2, \ldots, \eta_\kappa)$. Such a packing is called a $(\pi, \kappa, \alpha, \beta, \overline{\tau}, \overline{\eta})$ packing.

To introduce a solution algorithm for WNE we start by defining a *round-robin packing* for π bins: Let $j = \left(\sum_{l=1}^{i-1} \tau_l \mod \pi\right) + 1$. The items of color i are assigned to bins starting from bin j, where bin j receives $\eta_i - \tau_i + 1$ items, and each successive bin receives one item. If at some stage bin π is reached, the next items, if existing, are packed into bins $1, 2, \ldots$.

Lemma 3. *For any $(\pi, \kappa, \alpha, \beta, \overline{\tau}, \overline{\eta})$ packing, this round-robin packing is an alternative $(\pi, \kappa, \alpha, \beta, \overline{\tau}, \overline{\eta})$ packing.*

Proof. To show the claim, it suffices to consider the number of colors in each bin. Since $\tau_i \leq \pi$ for all $1 \leq i \leq \kappa$, the process above defines a packing where the total number of colors in all bins is $\sum_{i=1}^{\kappa} \tau_i$. Since a round-robin packing method is used, and $\sum_{i=1}^{\kappa} \tau_i = \alpha\pi + \beta$, the β first bins have a load of $\alpha + 1$, while the other bins have a load of α. □

Therefore, we restrict our further analysis to round-robin packings.

Our Algorithm SolveWNE is based on the above Lemmata 2 and 3. It uses dynamic programming to create the partition of the colors into two sets C_1 and C_2, where items with colors in C_1 are packed into bins from $M_1 := \{1, \ldots, p\}$ and those with colors from C_2 into bins in $M_2 := \{p+1, \ldots, m\}$. Additionally, each bin $j_1 \in M_1$ will have $r+1$ colors and each bin $j_2 \in M_2$ only r ones, respectively. Note that a packing with these properties is always a Nash equilibrium, as no item has an incentive to change to another bin. Since we do not know p in advance, we will apply procedure $\mathsf{SolveWNE}_p$ for all values of $p \in \{1, \ldots, m\}$.

Taking Lemma 2 into account, we get a simple upper bound for the maximum number of colors in a bin in a Nash equilibrium

$r_{\max} := \lceil \frac{\sum_{i=1}^{k} \min\{n_i, m\}}{m} \rceil$. In general, this is not better than just k, but if there are only few items of the same color, then this bound is stronger.

The general idea of the dynamic programming $\mathsf{SolveWNE}_p$ is to test for a fixed value $p \in \{1, \ldots, m\}$ and for all $a, c \in \{0, 1, \ldots, r_{\max}\}$, $b \in \{0, 1, \ldots, p-1\}$, $d \in \{0, 1, \ldots, m-p-1\}$ and $i \in \{1, \ldots, k\}$, whether there is a partition of the first i colors in such a way, that the first p bins have a or $a+1$ colors and exactly b of them have $a+1$ colors and the bins of the second group have c or $c+1$ colors and exactly d bins have $c+1$ colors, i.e., determine the values

$$f_p(a, b, c, d, i) := \begin{cases} 1 & \text{if such a partition exists} \\ 0 & \text{else} \end{cases}$$

as follows:

1. For $i = 0$ and for all $a, c \in \{0, 1, \ldots, r_{\max}\}$, $b \in \{0, 1, \ldots, p-1\}$ and $d \in \{0, 1, \ldots, m-p-1\}$ set:

$$f_p(a, b, c, d, 0) = \begin{cases} 1 & \text{if } a = 0, b = 0, c = 0 \text{ and } d = 0 \\ 0 & \text{otherwise .} \end{cases}$$

2. The $(i+1)$-th color is assigned in a round-robin manner, i.e.,

$$f_p(a, b, c, d, i+1) =$$

$$\begin{cases} 1 & \text{if } f_p(a, b - t_{i+1}, c, d, i) = 1 \qquad (1) \\ & \quad \text{for some } 1 \leq t_{i+1} \leq \min\{n_{i+1}, b-1\} \\ & \text{or } f_p(a-1, p+b-t_{i+1}, c, d, i) = 1 \qquad (2) \\ & \quad \text{for some } b \leq t_{i+1} \leq \min\{n_{i+1}, p\} \\ & \text{or } f_p(a, b, c, d - t_{i+1}, i) = 1 \qquad (3) \\ & \quad \text{for some } 1 \leq t_{i+1} \leq \min\{n_{i+1}, d-1\} \\ & \text{or } f_p(a, b, c-1, m-p+d-t_{i+1}, i) = 1 \qquad (4) \\ & \quad \text{for some } d \leq t_{i+1} \leq \min\{n_{i+1}, m-p\} \\ 0 & \text{else} \end{cases}$$

In this recursion, the value of t_{i+1} ($1 \leq t_{i+1} \leq n_{i+1}$) is the number of bins in which items with color $i+1$ are packed in the underlying

round robin assignment. If the items of color $i+1$ are added to the first set of bins, then $t_{i+1} \leq p$ as well, and $t_{i+1} \leq m-p$ otherwise.
In the first case, there are two subcases: Either these items are used to fill bins that had only a colors such that the number b of bins with $a+1$ colors increases, this case occurs if $t_{i+1} \leq b-1$, and is described in (1), or the items packed increase a, hence all bins with only $a-1$ colors are filled to a colors and additional b bins were filled to $a+1$, in this case $t_{i+1} \geq b$, and this is described in (2). (3) and (4) describe the analogous cases where color $i+1$ is assigned to the second set of bins.
If $p = m$ and thus there is no real partition of the bins, we simply have to analyze $f_m(a, b, i)$ and the recursion only contains the first two cases.

Algorithm SolveWNE applies $\mathsf{SolveWNE}_p$ for all $p \in \{1, \ldots, m\}$ and afterwards finds the maximum value of $r+1$ such that there is a value of $p \in \{1, \ldots, m-1\}$, for which $f_p(r+1, 0, r, 0, k) = 1$ or $f_m(r+1, 0, k) = 1$.

Note that in order to construct a strategy profile, it is necessary to backtrack the values t_i, and the assignment of colors into the two sets.

We now analyze Algorithm SolveWNE and show, that if it does not yield a Nash equilibrium with social cost $r+1$, then there is none.

Lemma 4. *There is a value of $p \in \{1, \ldots, m-1\}$ for which $f_p(r+1, 0, r, 0, k) = 1$ or $f_m(r+1, 0, k) = 1$, if and only if there is a Nash equilibrium A with $sc(A) = r+1$.*

Proof. We establish by induction on i that $f_p(a, b, c, d, i) = 1$ if and only if there is a way to allocate the items of the first i colors into the two sets of bins such that the first p bins have a or $a+1$ colors and exactly b of them have $a+1$ colors and the second group of bins has c or $c+1$ colors and exactly d bins have $c+1$ colors. This claim is sufficient by Lemma 2.

The claim for $i = 0$ is trivial, since there are no such items and thus the only feasible values for a, b, c, d are all zeros.

Assume that the claim holds for i, and analyze the cases leading to $f_p(a, b, c, d, i+1) = 1$. Assume that this value is set to 1 due to a value of t_{i+1} using (1) or (2). Then, our solution packs the items of color $i+1$ into exactly t_{i+1} bins of the first set. By Lemma 3, we first use the b bins with $a+1$ colors, and if necessary, use also the other bins of the first set of bins. Otherwise, we set this value to 1 due to some value of t_{i+1} using (3) or (4). Then we pack the items of color $i+1$ into exactly t_{i+1} bins of the second set of bins. Again, we use first the d bins which have $c+1$ colors, and if necessary, we use also some of the other bins of the second set of bins. Note that by our conditions on t_{i+1}, there are sufficient bins of the corresponding set of bins, and enough items of color $i+1$ so that we can actually spread its items into t_{i+1} distinct bins. The claim now follows by the induction assumption.

To prove the other direction, assume that there is a way to allocate the items of the first $i+1$ colors into the two sets of bins such that the first p bins have a or $a+1$ colors and exactly b of them have $a+1$ colors and the bins of the second group have c or $c+1$ colors and exactly d bins have $c+1$ colors. Denote by t_{i+1} the number of bins used to pack the items of color $i+1$, and use this value in the recursion formula for setting the value of $f_p(a, b, c, d, i+1)$. Then, this value is 1 due to the line corresponding to the set of bins for which this partition of the items allocates the items of color $i+1$, using the value of t_{i+1}. □

Therefore, we have established the following theorem.

Theorem 2. *Algorithm SolveWNE computes a worst Nash equilibrium in time polynomial in the number of items.*

4 Strong Equilibria in Bin Coloring Games

Recall the proof of Lemma 1. The optimal assignment constructed in this proof is also strong. This establishes that every uncapacitated bin coloring game admits a strong equilibrium and that one strong equilibrium can be calculated in time $\mathcal{O}(k)$. Additionally, the strong price of stability $\mathsf{SPoS}(\mathsf{BC})$ is 1.

The analysis of the strong price of anarchy follows:

Theorem 3. *The strong price of anarchy* SPoA(BC) *is asymptotically* 1.

Proof. Given a strong Nash equilibrium define an undirected graph $G = (V, E)$ where $V = \{1, 2, \ldots, m\}$ corresponds to the set of bins and there is an edge $e = (u, v)$ between two vertices u and v (assume that $v > u$) if there is a color such that at least one item with this color is packed into u and one such item is packed into v and if none of the bins $u + 1, u + 2, \ldots, v - 1$ contains an item of this color. We associate the edge with the inducing color. Note that the edges associated with a color, are the edges of a simple path in G.

If there is an edge $e = (u, v)$ in G then the number of colors in u and v has to be equal, as otherwise the items of the color present in both bins want to change to the bin with fewer colors. The two bins u and v can also have only one color in common, as otherwise the items of the common colors can form a coalition and improve their situation by sorting the colors to the two bins.

Additionally, the graph G does not contain a cycle. The cycles with edges of a single color were removed in the construction and cycles with edges of at least two colors contradict the strong equilibrium property: Such a cycle allows the coalition of the items inducing the edges of the cycle to improve their situation by directing the cycle and shifting all items of a common color along the corresponding directed edge and thus decreasing the number of colors in each bin of the cycle by one.

Thus the graph G is a forest, in which every connected component has a common number of colors. These numbers of colors may differ by at most one, as otherwise the assignment of items to bins would not even be a Nash equilibrium.

Hence, keeping in mind that there are k colors and every edge in the graph corresponds to a color being used not in one but in two bins, we can conclude, that in the worst case G is a tree and the maximal number of different colors in a bin is bounded by

$$\left\lceil \frac{k + m - 1}{m} \right\rceil \leq \left\lceil \frac{k}{m} \right\rceil + 1$$

Using Lemma 1 this yields the desired result. □

The bound established in the proof of Theorem 3 is tight, as the following example shows:

Consider m bins, three colors {red, blue, green} and one red, one blue and $m-1$ green items. Obviously packing only three bins, one for each color is optimal. But assigning the red and the blue item to the first bin and one green item to any other bin is also a strong equilibrium, as the green items cannot improve their situation and thus are not willing to participate in a coalition.

Theorem 4. *Every capacitated bin coloring game $\Gamma=[m, k, (n_i)_{i\in\{1,\dots,k\}}, U<\infty]$ admits an optimal strong equilibrium, i.e., $\mathsf{PoS}(\mathsf{BC}_{cap}) = \mathsf{SPoS}(\mathsf{BC}_{cap}) = 1$.*

Proof. Let $\Gamma=[m, k, (n_i)_{i\in\{1,\dots,k\}}, U]$ be an instance of the capacitated bin coloring game and σ^*[4] an optimal strategy profile which is additionally lexicographically optimal in the following sense: The i-th component of this lexicographic (auxiliary) objective function is to minimize the number of items which are packed in bins with exactly $\mathrm{sc}(\sigma^*) - i + 1$ colors. We next argue that σ^* is a strong equilibrium.

Assume for the sake of a contradiction that there is a coalition $I \subseteq N$ which has an incentive to deviate from σ^* to $\bar{\sigma}_I$. Let p be the maximal cost of an item in I in strategy profile σ^*:

$$p = \max_{i\in I} c_i(\sigma^*) = \max_{i\in I} \mathrm{cl}_{\sigma_i^*}(\sigma^*).$$

For every $q \geq p$, no member of coalition I is packed in a bin with q colors in $(\sigma^*_{-I}, \bar{\sigma}_I)$, as otherwise it has no benefit from participating in the coalition. Hence, for every bin j with $\mathrm{cl}_j(\sigma^*) = q \geq p+1$, the color load stays the same after the deviation, i.e., $\mathrm{cl}_j(\sigma^*_{-I}, \bar{\sigma}_I) = q$. Thus, for every such bin j, the items assigned to it in σ^* and $(\sigma^*_{-I}, \bar{\sigma}_I)$ are identical.

However, there is an item $i \in I$ with

$$\mathrm{cl}_{\sigma_i}(\sigma^*) = p = c_i(\sigma^*) > c_i(\sigma^*_{-I}, \bar{\sigma}_I).$$

Hence, the number of items which are packed in bins with color load at least p strictly decreased by the deviation of coalition I. Hence,

[4] We use the notation of a strategy profile instead of an assignment because this eases notation concerning possible coalitions.

the new solution is lexicographic smaller than σ^* and this contradicts the lexicographic optimality of σ^*. □

The proof of Theorem 4 does not imply a polynomial time algorithm to find a strong equilibrium. Having such a non-constructive proof is justified by the the fact that finding an optimal assignment to the capacitated case is NP-hard by results of Krumke et al. [20].

The proof of Theorem 4 is a generic proof implying also that there is an optimal strong equilibrium if the bins of the game have different capacities or the game is asymmetric either in the way that the strategy sets of each item is restricted to several bins, or if some colors imply conflicts and cannot be packed together in one bin.

Theorem 5. *The strong price of anarchy $\mathsf{SPoA}(\mathsf{BC}_{cap})$ of capacitated bin coloring games is unbounded but equals to the number of bins if only games with a fixed number of bins are considered.*

Proof. Consider bin coloring games $\mathsf{BC}_{\mathrm{cap,m}}$ with a fixed number m of bins. The upper bound follows from Lemma 1. Hence, it suffices to prove the lower bound of m on the $\mathsf{SPoA}(\mathsf{BC}_{\mathrm{cap,m}})$. Consider the following instance where t is a large integer number to be specified later. We let $U = mt$, there is one color, say color 1, with $m(m-1)t$ items, and additional set of mt items each of which has a distinct color. The social optimal solution packs in each bin t items of the additional set of items together with $U - t = (m-1)t$ items of the first color. We next describe a strong equilibrium. In this solution the items of the first color are packed into the first $m-1$ bins. Each such bin is allocated mt such items and hence no further items can be packed into these bins. The additional mt items of the other colors are packed into the remaining bin. This solution has a cost of mt. It is clearly a strong equilibrium because the items of the first color have no incentive to deviate, and the other items cannot deviate due to the capacity constraint. Hence, we conclude that the strong price of anarchy of the capacitated bin coloring games is at least $\frac{mt}{t+1}$ and this ratio tends to m as t grows to infinity. □

5 Weakly/Strictly Pareto optimal Nash Equilibria in Bin Coloring Games

In the uncapacitated case, the optimal assignment give in the proof of Lemma 1 is also weakly and strictly Pareto optimal. For the capacitated case the lexicographically optimal assignment used in the proof of Theorem 4 is also strictly and weakly Pareto optimal.

This established not only existence of strictly and weakly Pareto optimal Nash equilibria in capacitated as well as uncapacitated bin coloring games but additionally the following prices of stability:

$$\mathsf{wPPoS}(\mathsf{BC}) = \mathsf{sPPoS}(\mathsf{BC}) = \mathsf{wPPoS}(\mathsf{BC}_{\text{cap}}) = \mathsf{sPPoS}(\mathsf{BC}_{\text{cap}}) = 1$$

For weakly Pareto optimal Nash equilibria, the lower bound on the price of anarchy established in Theorem 3 applies, as $\mathsf{wPNE}(\Gamma) \subseteq \mathsf{SE}(\Gamma)$ for every game Γ.

But for strictly Pareto optimal Nash equilibria the situation is different:

Theorem 6. *The strictly Pareto price of anarchy $\mathsf{sPPoA}(\mathsf{BC})$ of uncapacitated bin coloring games is* 1*. Moreover, social cost of every weakly Pareto optimal Nash equilibrium exceeds the optimal social cost by at most 1.*

Proof. Let Γ be an instance of the uncapacitated bin coloring game. Assume there is a strictly (respectively, weakly) Pareto optimal Nash equilibrium σ of Γ with $\mathrm{sc}(\sigma) \geq \mathsf{opt}(\Gamma) + 1$ ($\mathrm{sc}(\sigma) \geq \mathsf{opt}(\Gamma) + 2$).

We know that in a Nash equilibrium the color load of two bins can differ by at most 1. Hence, the color load of the bins in σ is at least $\mathsf{opt}(\Gamma)$ ($\mathsf{opt}(\Gamma) + 1$). Thus, jointly deviating to an optimal solution does not worsen (strictly improve) the situation for any item in the game. But those items assigned to the bin which determines the social cost of σ, are better off after the deviation. This contradicts the assumption of σ being strictly (weakly) Pareto optimal. □

To address the capacitated case, first look at games with a fixed number of bins m. The tight bound of m for the price of anarchy of capacitated bin coloring games with m bins can be transfered to weakly as well as strictly Pareto optimal Nash equilibria by using the upper bound for Nash equilibria given in Theorem 1 and the lower bound established for strong equilibria in Theorem 5. Thus, $\mathsf{wPPoA}(\mathsf{BC}_{\mathrm{cap,m}})$ and $\mathsf{sPPoA}(\mathsf{BC}_{\mathrm{cap,m}})$ equal m, and $\mathsf{wPPoA}(\mathsf{BC}_{\mathrm{cap}})$ and $\mathsf{sPPoA}(\mathsf{BC}_{\mathrm{cap}})$ are unbounded.

6 Conclusion and Further Research

The newly introduced bin coloring games admit optimal solutions that are Nash, strong as well as strictly and weakly Pareto optimal equilibria in the capacitated and the uncapacitated setting. Besides establishing existence, this determines the prices of stability for all four equilibrium concepts to be 1.

In the uncapacitated case an optimal strong equilibrium can be found in polynomial time. Additionally, a worst Nash equilibrium can be found in time polynomial in the number of items. The price of anarchy is unbounded, the strong price of anarchy as well as the weakly Pareto price of anarchy are asymptotically 1 and the strictly Pareto price of anarchy is exactly 1.

In contrast to that, in the capacitated case the strong price of anarchy as well as the Pareto prices of anarchy are unbounded and equal the number of bins if only games with a fixed number of bins are considered.

An overview of the results concerning the prices of anarchy and stability is given in Table 1.

Finding an optimal solution in the capacitated case is NP-hard due to results of Krumke et al. [20] and our proof of existence of an optimal strong equilibrium resembles this fact. The latter proof of existence is generic in the way that it also applies if the bins have different capacities or other additional restrictions are given. Moreover, the proofs of the upper bounds on the price of anarchy and price of stability does not change for all notions of equilibria studied in this paper, if we consider the model in which the capacity of a bin may be non-identical among all bins. Thus directions of further research are to analyze these variants of the model.

	BC		BC_{cap}	
	PoS	PoA	PoS	PoA
Nash equilibria	1	∞	1	∞
strong equilibria	1	asymp. 1	1	∞
weakly Pareto optimal Nash equilibria	1	asymp. 1	1	∞
strictly Pareto optimal Nash equilibria	1	1	1	∞

Table 1: Overview of results

References

1. Andelman, N., Feldman, M., Mansour, Y.: Strong price of anarchy. Games and Economic Behaviour **65**(2), 289–317 (2009)
2. Anshelevich, E., Dasgupta, A., Tardós, E., Wexler, T.: Near-optimal network design with selfish agents. In: Proceedings of the 35th Annual ACM Symposium on the Theory of Computing (STOC), pp. 511–520. ACM (2003)
3. Aumann, R.J.: Acceptable points in general cooperative n-person games. In: R.D. Luce, A.W. Tucker (eds.) Contributions to the Theory of Games IV, pp. 287–324. Princeton University Press (1959)
4. Bilò, V.: On the packing of selfish items. In: Proceedings of the 20th International Parallel Symposium on Distributed Processing (IPDPS), pp. 25–29. IEEE (2006)
5. Caprara, A., Pferschy, U.: Worst-case analyses of the subset sum algorithm for bin packing. Operations Research Letters **32**(2), 159–166 (2004)
6. Chien, S., Sinclair, A.: Strong and pareto price of anarchy in congestion games. In: S. Albers, A. Marchetti-Spaccamela, Y. Matias, S. Nikoletseas, W. Thomas (eds.) Proceedings of the 36th International Colloquium on Automata, Languages and Programming (ICALP), *LNCS*, vol. 5555, pp. 279–291. Springer (2009)
7. Coffman Jr., E.G., Csirik, J.: Performance guarantees for one-dimensional bin packing. In: T.F. Gonzales (ed.) Handbook of Approximation Algorithms and Metaheuristics, chap. 32. Chapman & Hall/Crc (2007)
8. Csirik, J., Leung, J.Y.T.: Variants of classical one-dimensional bin packing. In: T.F. Gonzalez (ed.) Handbook of Approximation Algorithms and Metaheuristics, chap. 33. Chapman & Hall/Crc (2007)
9. Csirik, J., Woeginger, G.J.: On-line packing and covering problems. In: A. Fiat, G.J. Woeginger (eds.) Online Algorithms: The state of the Art, *LNCS*, vol. 1442, chap. 7, pp. 147–177. Springer (1998)
10. Czumaj, A., Vöcking, B.: Tight bounds for worst-case equilibria. ACM Transactions on Algorithms **3**(1), Article 4 (2007)
11. Epstein, L., Favrholdt, L.M., Kohrt, J.S.: Comparing online algorithms for bin packing problems. Journal of Scheduling **online first** (2009). DOI 10.1007/s10951-009-0129-5
12. Epstein, L., Kleiman, E.: Selfish bin packing. Algorithmica **online first** (2009). DOI 10.1007/s00453-009-9348-6
13. Epstein, L., Kleiman, E., Mestre, J.: Parametric packing of selfish items and the subset sum algorithm. In: Proceedings of the 5th International Workshop on Internet and Network Economics (WINE) (2009). To appear

14. Feldmann, R., Gairing, M., Lücking, T., Monien, B., Rode, M.: Nashification and the coordination ratio for a selfish routing game. In: J.C.M. Baeten, J.K. Lenstra, J. Parrow, G.J. Woeginger (eds.) Proceedings of the 30th International Colloquium on Automata, Languages and Programming (ICALP), *LNCS*, vol. 2719, pp. 514–526. Springer (2003)
15. Fischer, S., Vöcking, B.: On the structure and complexity of worst-case equilibria. Theoretical Computer Science **378**(2), 165–174 (2007)
16. Hiller, B., Vredefeld, T.: Probabilistic analysis of online bin coloring algorithms via stochastic comparison. In: D. Halperin, K. Mehlhorn (eds.) Proceedings of the 16th Annual European Symposium on Algorithms (ESA), *LNCS*, vol. 5193, pp. 528–539. Springer (2008)
17. Holzman, R., Law-Yone, N.: Strong equilibrium in congestion games. Games and Economic Behaviour **21**(1-2), 85–101 (1997)
18. Koutsoupias, E., Papadimitriou, C.: Worst-case equilibria. In: C. Meinel, S. Tison (eds.) Proceedings of the 16th International Symposium on Theoretical Aspects of Computer Science (STACS), *LNCS*, vol. 1563, pp. 404–413. Springer (1999)
19. Koutsoupias, E., Papadimitriou, C.: Worst-case equilibria. Computer Science Review **3**, 65–69 (2009)
20. Krumke, S.O., de Paepe, W.E., Stougie, L., Rambau, J.: Bincoloring. Theoretical Computer Science **407**(1–3), 231–241 (2008)
21. Lin, M., Lin, Z., Xu, J.: Almost optimal solutions for bin coloring problems. Journal of Combinatorial Optimization **16**(1), 16–27 (2008)
22. Mas-Colell, A., Whinston, M.D., Green, J.G.: Microeconomic Theory. Oxford University Press (1995)
23. Mavronicolas, M., Spirakis, P.: The price of selfish routing. Algorithmica **48**(1), 91–126 (2007)
24. Papadimitriou, C.: Algorithms, games, and the internet. In: Proceedings of the 33rd Annual ACM Symposium on the Theory of Computing (STOC), pp. 749–753 (2001)
25. Shachnai, H., Tamir, T.: Polynomial time approximation schemes for class-constrained packing problems. Journal of Scheduling **4**(6), 313–338 (2001)
26. Shachnai, H., Tamir, T.: Tight bounds for online class-constrained packing. Theoretical Computer Science **321**(1), 103–123 (2004)
27. Xavier, E.C., Miyazawa, F.K.: The class constrained bin packing problem with applications to video-on-demand. Theoretical Computer Science **393**(1-3), 240–259 (2008)
28. Yu, G., Zhang, G.: Bin packing of selfish items. In: C. Papadimitriou, S. Zhang (eds.) Proceedings of the 4th International Workshop on Internet and Network Economics (WINE), *LNCS*, vol. 5385, pp. 446–453. Springer (2008)