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Computation of the central elements and centralizers of sets of elements in non-commutative polynomial algebras

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Contents

Preface

Abstract. *In this thesis we present the implementation of libraries* `center.lib` *and* `perron.lib` *for the non-commutative extension* SINGULAR:PLURAL *of the Computer Algebra System* SINGULAR.

The library `center.lib` *was designed for the computation of elements of the centralizer of a set of elements and the center of a non-commutative polynomial algebra. It also provides solutions to related problems.*

The library `perron.lib` *contains a procedure for the computation of relations between a set of pairwise commuting polynomials.*

The thesis comprises the theory behind the libraries, aspects of the implementation and some applications of the developed algorithms.

Moreover, we provide extensive benchmarks for the computation of elements of the center. Some of our examples were never computed before.

Motivation for studying and computing centers

The center is a kind of a heart of the algebra, especially in the case when the center is nontrivial. In this section we briefly list some applications of the center and centralizers.

Invariant operators in physics

There is a natural connection, first discovered by Eugene Wigner, between the properties of particles, the representation theory of Lie groups and Lie algebras, and the symmetries of the universe. This postulate states that *each particle "is" an irreducible representation of the symmetry group of the universe* (cf. `http://en.wikipedia.org/wiki/Particle_physics_and_representation_theory`).

Many (but not all) symmetries, form Lie groups. Representations of a Lie group is closely related to representation of its Lie algebra; since the latter is usually simpler to compute, that is the way it is usually done.

Lie groups and Lie algebras appear in physics in many different guises. Specific Lie groups may appear as consequences of specific dynamics. Consider any physical system with dynamics described by a system of ordinary or partial differential equations. This system of equations will be invariant under some local Lie group of local point transformations, taking solutions into solutions. This group is a Lie group and its Lie algebra can be determined in an algorithmic manner.

An important problem arising in the representation theory of a Lie group or Lie algebra, and especially in physical applications, is the determination of functions of the generators commuting with all generators, i.e., the **invariant functions**. From the mathematical point of view their importance is due to the following circumstances. They can be used to label irreducible representations of a given Lie group or Lie algebra and to split reducible representations into irreducible ones. This topic will be discussed below in more details. Further, basis functions for irreducible representations of a Lie group can be constructed so as to correspond to the reduction of the group to a given chain of subgroups. The basis functions in such a case will be the common eigenfunctions of the invariant operators of all the groups in the chain. Invariant operators also play a crucial role in special function theory. Indeed, the entire theory of special functions can be based on group representation theory and different functions occur as the eigenfunctions of different sets of invariant operators.

In physics, invariant operators of the symmetry group of a physical system and of its subgroups provide quantum numbers. Indeed, the eigenvalues of the invariant operators of the entire symmetry group will be the quantum numbers, characterizing the system as such (e.g., the particle mass and spin in the case of the Poincaré group). The invariant operators of subgroups will then characterize states of the system (its energy, linear or angular momentum, etc). In other applications, invariant operators of dynamical groups provide mass formulas, energy spectra, and in general characterize specific properties of physical systems.

Another application is related to possible symmetry breakings in nature. Thus, in an idealized situation a physical quantity may be characterized by the invariants of some group. When further interactions, breaking the the idealized symmetry are considered, the same quantity may also depend on the invariants of a subgroup or subgroups.

Important examples of invariant operators are Casimir operators.

For example, it is well known that in the case of semisimple Lie algebras all invariants can be written as functions of l polynomial invariants, where l is the rank of the algebra. These l basic invariants (Casimir operators) form an integrity basis, i.e., any polynomial invariant can be written as a polynomial in the basic invariants.

In the case of nilpotent algebras all invariants can again be written as functions of $r - R$ polynomial invariants (r is the dimension of the algebra, R the rank of the matrix of the commutation table). However, in this case the "basic invariants" do not necessarily form an integrity basis. Thus, higher-order polynomial invariants may exist which are functions of the lower ones, but not polynomial in them.

A **Casimir operator** is a polynomial in the generators of a Lie group and thus an element of the enveloping algebra of the corresponding Lie algebra, commuting with all the generators of the group. These operators are in one–to–one correspondence with polynomial invariants characterizing orbits of the coadjoint representation of a Lie algebra. In fact, the Casimir operators of a finite dimensional semisimple complex Lie algebra $\mathfrak{g}$ are a distinguished basis of the center $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ of the enveloping algebra of $\mathfrak{g}$ made of homogeneous polynomials.

In physical applications the Casimir operators are usually associated with quantities (such as the momentum-square or the Pauli-Lubanski vector) characterizing a certain physical system, rather than a specific state of this system. Moreover they may represent such important quantities as angular momentum, elementary particle's mass and spin, Hamiltonians of various physical systems etc.

Let us illustrate the physical use of Casimir operators by the following examples:

Example. Let us consider the group of all rotations about the origin of 3-dimensional Euclidean space $\mathbb{R}^3$. This group is called the **rotation group** and denoted by $SO(3)$.

One important property of $SO(3)$ is the existence of a Casimir operator J^2 which correspond, in this case, to the *total angular momentum.*

The Lie group $SO(3)$ corresponds to the simple complex Lie algebra $\mathfrak{so}_3$ with the enveloping algebra $\mathcal{U}(\mathfrak{so}_3)$ whose center (the Casimir operator of $SO(3)$) can be easily computed using our methods.

Example. The Fairlie–Odesskii algebra $\mathcal{U}'_q(\mathfrak{so}_3)$ is a non-standard q-deformation of the enveloping algebra $\mathcal{U}(\mathfrak{so}_3)$ of the Lie algebra $\mathfrak{so}_3$. As a matter of interest, this algebra arose naturally as the algebra of observables in quantum gravity in (2+1)-dimensional de Sitter space with space being torus. The parameter q is related to the Plank constant and the curvature of the de Sitter space. Thus it is important, from the point of view of physics, to study the structure (in particular, the center) of this algebra.

For $n > 3$, the algebras $\mathcal{U}'_q(\mathfrak{so}_n)$ are no less important. They serve as intermediate algebras in deriving the algebra of observables in 2+1 quantum gravity with 2D space of genus $g > 1$, so that $n = 2g + 2$. In order to obtain the algebra of observables, the q-deformed algebra $\mathcal{U}'_q(\mathfrak{so}_{2g+2})$ should be factorized by some ideal generated by (combinations of) Casimir elements of $\mathcal{U}'_q(\mathfrak{so}_{2g+2})$. This fact, along with others, motivates the study of Casimir elements of $\mathcal{U}'_q(\mathfrak{so}_n)$.

Example. Representations of $\mathcal{U}_q(\mathfrak{sl}_2)$ are also used in nuclear physics. For example, the rotational spectra of deformed nuclei can be described by a q-deformed rotator which corresponds to the Casimir element of $\mathcal{U}_q(\mathfrak{sl}_2)$. This rotator is defined by the Hamiltonian $H = (2I)^{-1}C'_q + E_0$, where I is the moment of inertia, E_0 is the bandhead energy and C'_q is the suitably chosen quadratic Casimir element.

Classification of $\Bbbk$-algebras

The center is an important invariant of a $\Bbbk$-algebra. Of course, centers of isomorphic $\Bbbk$-algebras are isomorphic.

Provided in certain algebras the notion of a minimal set of the generators of the center makes sense, it might be used for distinguishing non-isomorphic algebras.

Example. Let us consider some 5-dimensional real Lie algebras generated by $e_1, \dots e_5$ subject to the following relations:

- $A_{5,1} : [e_3, e_5] = e_1, [e_4, e_5] = e_2,$
- $A_{5,4} : [e_3, e_5] = e_1, [e_2, e_4] = e_1,$
- $A_{5,5} : [e_3, e_5] = e_2, [e_2, e_5] = e_1, [e_3, e_4] = e_1.$

The centers of these algebras are known:

- $\mathcal{Z}(A_{5,1}) = \mathbb{R}\langle e_1, e_2, e_2e_3 - e_1e_4\rangle,$
- $\mathcal{Z}(A_{5,4}) = \mathbb{R}\langle e_1\rangle,$
- $\mathcal{Z}(A_{5,5}) = \mathbb{R}\langle e_1\rangle.$

Therefore we can use the center to conclude that $A_{5,1} \not\cong A_{5,4}$ and $A_{5,1} \not\cong A_{5,5}$, but we can not distinguish $A_{5,4}$ and $A_{5,5}$ by these means.

Construction of algebras, associated to linear operators

Differential, shift, difference operators and their quantum analogues must not commute with anything different to the operator of scalar multiplication. Therefore, in algebras arising from classical operators like differential, shift, difference etc. and their quantum analogues the following important principle is implicitly used: the center of such an algebra must be trivial, that is, it must contain only the ground field.

Thus, computation of the center is important to recover identities between generators of an algebra which have to be taken into account, so that the resulting factor algebra has a trivial center.

Example. Let us consider the first Heisenberg Lie algebra $\mathfrak{h}_1$ generated by operators P, Q, C subject to $[P, Q] = C$. The center of its enveloping algebra $\mathcal{H}_1$ is generated by C. Therefore in order to obtain the Weyl algebra of differential operators we factorize $\mathcal{H}_1$ by two-sided ideal generated by $C - 1$:

$$\mathcal{W}_1 \cong \mathcal{H}_1 / \langle C - 1\rangle .$$

Representation theory

Let $\mathfrak{g}$ be a Lie algebra over a field $\Bbbk$. **Representation** of $\mathfrak{g}$ in a vector space V is a homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.

A representation ρ in V is called **finite-dimensional** if $\dim V < \infty$, and **irreducible** if there are no proper subspaces in V, that are invariant under all operators $\rho(g), g \in \mathfrak{g}$.

Due to Weyl's theorem, for any semisimple Lie algebra any finite dimensional representation is a direct sum of irreducible ones.

Every representation of $\mathfrak{g}$ can be uniquely extended to a representation of the enveloping algebra $\mathcal{U}(\mathfrak{g})$. So, one can say that representations of $\mathfrak{g}$ (i.e., $\mathfrak{g}$-modules) are the same things as $\mathcal{U}(\mathfrak{g})$-modules.

The importance of the center follows from a simple observation that is often used in linear algebra: if two operators commute, then an eigenspace for one of them is invariant under the other. This means that we can reduce representations by taking a joint eigenspace for $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$.

Assume now X is an irreducible Harish-Chandra $(\mathfrak{g}, K)$-module. Then every element of $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ acts on X by a scalar. This defines a homomorphism

$$\chi_X : \mathcal{Z}(\mathcal{U}(\mathfrak{g})) \to \mathbb{C}$$

of algebras, which is called the **infinitesimal character** of X. Due to a theorem of Harish-Chandra (cf. [23]), infinitesimal characters are important parameters for classifying irreducible $(\mathfrak{g}, K)$-modules. It turns out that for every fixed infinitesimal character, there are only finitely many irreducible $(\mathfrak{g}, K)$-modules with this infinitesimal character. Thus, elements of $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ can be used to label irreducible representations of the Lie algebra.

Zassenhaus variety

There are some simple facts which distinguish Lie algebras over fields of prime characteristic (we will say Lie algebras of prime characteristic) from Lie algebras over fields of characteristic 0. These are:

1. The degrees of the absolutely irreducible representations of a Lie algebra of prime characteristic are bounded whereas, according to a theorem of Weyl, the degrees of absolutely irreducible representations of a semisimple Lie algebra over a field of characteristic 0 can be arbitrary high.

2. For each Lie algebra of prime characteristic there are indecomposable representations which are not irreducible, whereas every indecomposable representation of a semisimple Lie algebra over a field of characteristic 0 is irreducible.

3. The quotient ring of the enveloping algebra of a Lie algebra of prime characteristic is a division algebra of finite dimension over its center, whereas this is not the case for characteristic 0.

4. There are faithful fully reducible representations of every Lie algebra of prime characteristic, whereas for characteristic 0 only ring sums of semisimple Lie algebras and abelian abelian Lie algebras admit faithful fully reducible representations.

These facts have been established for special cases for many years, and some of them have been considered in the general case by N. Jacobson (cf. [20]). They are at the basis of every investigation aiming at a theory of Lie algebras of prime characteristic embedded into their enveloping algebras.

In 1954 Zassenhaus (cf. [48]) showed that with each finite dimensional Lie algebra $\mathfrak{g}$ over an algebraically closed field $\mathbb{k}$ of characteristic $p > 0$ one can associate an algebraic variety $M_{\mathfrak{g}}$, which we will call the **Zassenhaus variety** of $\mathfrak{g}$. The center $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ of the enveloping algebra of $\mathfrak{g}$ is the coordinate ring of $M_{\mathfrak{g}}$. In [48] it was proved that $M_{\mathfrak{g}}$ is a normal irreducible affine variety whose dimension coincides with the dimension of $\mathfrak{g}$. This variety can be also defined to be the m-spectrum of $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$.

The Zassenhaus variety of the algebra $\mathfrak{g}$ is closely related to the irreducible representations of $\mathfrak{g}$. Namely, if V is a simple $\mathfrak{g}$-module and $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ is the center of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$, then by Shur's lemma we have $zv = \chi_V(z)v, z \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})), v \in V$, where χ_V is a homomorphism of $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ into $\mathbb{k}$, i.e., a point of the variety $M_{\mathfrak{g}}$. Thus, the problem of describing singular points of $M_{\mathfrak{g}}$ and the determination of equations determining $M_{\mathfrak{g}}$ is closely connected with the problem of the classification of the irreducible representations of $\mathfrak{g}$. Therefore, to classify the irreducible representations of a Lie algebra it is natural to study its Zassenhaus variety and the distribution of representations over its points. There is a surjective mapping which assigns a point on the Zassenhaus variety to each irreducible representation such that the preimage of any point of $M_{\mathfrak{g}}$ is finite and for the points of an open dense subset this preimage consists of one element (cf. [48]).

The Zassenhaus variety was studied in some special classes under certain conditions on p. In the case when $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{k})$ and $p > 2$ this approach was applied by Rudakov and Shafarevich who used the early work of Zassenhaus to organize all of the irreducible representations into a single geometric picture: the Zassenhaus variety, whose simple points correspond to representations of dimension p (cf. [41]). It was also shown that the Zassenhaus variety in this case has singularities of type A_1 for all prime $p > 2$.

Using `center.lib` one can explicitly compute generators of the center of any enveloping algebra and then, using `perron.lib`, algebraical dependence between them. In [26] our libraries (`center.lib` and `perron.lib`) were used to reproduce some of results of Rudakov and Shafarevich via direct computation and it was shown that the computed dependencies indeed have singularities of type A_1.

It would be very interesting to work out a similar description for higher ranks, but so far only a bare outline exists (cf. [25]).

Our methods, in particular, can be used in order to check the following conjecture which is due to Y. Drozd:

Conjecture. Let $\mathfrak{g}$ be a classical simple Lie algebra over an algebraically closed field of characteristic $p > 0$. Here "classical" means that it corresponds to a simple complex Lie algebra. Then the singularities of the Zassenhaus variety of $\mathfrak{g}$ are always *simple* and their types are just deformations of the type of $\mathfrak{g}$ (e.g., A-D-E).

Though this conjecture appeared first, in a different form, nearly 40 years ago it still remains open.

Importance

One studies algebras via their modules on the one side and via their subalgebras on the other side. As for centralizers, they are very useful to produce nontrivial subalgebras.

Commutative subalgebras play an important role in representation theory, namely one builds families of modules from them, parameterizing the action of subalgebra by constants. In particular, Cartan subalgebras are used to introduce Verma modules; centers are used in Whittaker modules; Gel'fand-Zetlin subalgebras give rise to Gel'fand-Zetlin modules and so on.

The structure of the center is studied theoretically but it is known deeply only for some special classes of algebras. Recipes for the computations of generators of the center are rare. That is why it is very important to be able to compute the center of any $\Bbbk$-algebra.

Overview

We develop algorithms for computing elements of the center and the centralizer of a finite set of elements. They can be applied to a wide classes of unital associative non-commutative algebras.

Our primary goal was to provide users of the Computer Algebra System SINGULAR with a library for the computation of elements of the center and of the centralizer of a finite set of elements.

This paper has the following structure:

Firstly, we list some preliminaries in chapter 1. Then in chapter 2 we formally describe our algorithms and prove their correctness in theorem 2.6. Further in chapter 3 we compute "by hands" several examples and check them with our library. Moreover, we describe several optimizations used in our implementation. Afterwards the subalgebra reduction is considered in chapter 4.

We list known theoretical facts about various $\Bbbk$-algebras in chapter 5. Further we consider some applications of the developed algorithms in chapters 6 and 7.

Finally, we provide the user's manual for developed libraries and give some benchmarks in appendix 8 .

The thesis comes with a CD with Singular (version 3-0-1), the libraries `center.lib`, `perron.lib` and `algebras.lib`, the modified SymbolicData and the electronic version of this thesis.

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Chapter 1

Preliminaries

Throughout this paper, $\mathbb{k}$ will stand for a commutative field.

1.1 Basic notions

Definition 1.1. An **algebra over** $\mathbb{k}$ (or simply **algebra**) is a $\mathbb{k}$-vector space $\mathcal{A}$ endowed with a bilinear multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ (denoted $(x, y) \mapsto x * y$).

The algebra $\mathcal{A}$ is called **associative** if the multiplication is associative, that is, $x*(y*z) = (x * y) * z$ for all $x, y, z \in \mathcal{A}$.

The algebra $\mathcal{A}$ is called **unital** if there is a unity element $1_{\mathcal{A}}$ in $\mathcal{A}$ that satisfies $1_{\mathcal{A}} * x = x * 1_{\mathcal{A}} = x$ for all $x \in \mathcal{A}$.

In what follows unital associative algebras over $\mathbb{k}$ are called **$\mathbb{k}$-algebras**.

The assumption that multiplication is bilinear is equivalent to the right and left distributive laws, together with the following condition:

$$(x * y) * a = x * (y * a) = (x * a) * y, \text{ for all } x, y \in \mathcal{A} \text{ and } a \in \mathbb{k}. \tag{1.1}$$

Any $\mathbb{k}$-algebra is a ring with unity. Conversely, if $\mathcal{A}$ is a $\mathbb{k}$-vector space and a ring, with unity, that satisfies (1.1), then $\mathcal{A}$ is a $\mathbb{k}$-algebra.

Definition 1.2. An algebra over $\mathbb{k}$ is called **finitely generated** if it is finitely generated as a ring over $\mathbb{k}$.

Definition 1.3. Let $\mathcal{A}$ be an algebra over $\mathbb{k}$. A **subalgebra** of $\mathcal{A}$ is a $\mathbb{k}$-vector subspace S of $\mathcal{A}$ which is closed under the multiplication of $\mathcal{A}$.

In the case of unital associative algebras we require additionally that $1_{\mathcal{A}} \in S$.

Definition 1.4. Let $\mathcal{A}$ be an algebra over $\Bbbk$. A $\Bbbk$-subspace $I \subset \mathcal{A}$ is called a **left ideal**(resp., a **right ideal**) if for any $a \in \mathcal{A}, x \in I$ one has $a * x \in I$ (resp., $x * a \in I$).

If I is a left ideal and a right ideal, it is called a **two-sided ideal**.

Definition 1.5. If e_i is a basis of an algebra $\mathcal{A}$ as a $\Bbbk$-vector space, then the product "$*$" is completely determined by the **structure constants** $f^i_{jk} \in \Bbbk$ defined by $e_j * e_k = \sum_i f^i_{jk} e_i$.

Definition 1.6. An **algebra homomorphism** from an algebra $\mathcal{A}$ to an algebra $\mathcal{B}$ is a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ such that $\Phi(x * y) = \Phi(x) * \Phi(y)$ for all $x, y \in \mathcal{A}$.

Definition 1.7. A $\Bbbk$-algebra $\mathcal{A}$ is called **filtered** if it admits an increasing sequence $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots$ of finite dimensional vector subspaces of $\mathcal{A}$ satisfying the following properties:

1. $1 \in \mathcal{A}_0$,
2. $\mathcal{A}_i * \mathcal{A}_j \subseteq \mathcal{A}_{i+j}, \quad i, j \geqslant 0$,
3. $\mathcal{A} = \cup \mathcal{A}_i$.

This sequence of vector spaces is called a **filtration** of $\mathcal{A}$.

Definition 1.8. A linear **representation** of an associative algebra $\mathcal{A}$ in a vector space V is an algebra homomorphism from $\mathcal{A}$ to the associative algebra $\mathrm{End}(V)$ of endomorphisms of V

$$\rho_V : \mathcal{A} \to \mathrm{End}(V).$$

The vector space V **caries the representation** and is called a **representation space** of $\mathcal{A}$ or a left $\mathcal{A}$-module.

The representation ρ is called **faithful** if ρ is injective.

An **invariant subspace** of the representation ρ_V is a subspace W of V such that $\rho_V(a)(W) \subset W$ for all $a \in \mathcal{A}$.

A representation is called **irreducible** or **simple** if it has no proper invariant subspaces.

Definition 1.9. Let $\mathcal{A}$ be an algebra. A **derivation** D of $\mathcal{A}$ is a linear endomorphism of $\mathcal{A}$ satisfying the Leibniz rule:

$$\mathrm{D}(x * y) = (\mathrm{D}\, x) * y + x * (\mathrm{D}\, y). \tag{1.2}$$

The set of all derivations of $\mathcal{A}$ is denoted $\mathrm{Der}(\mathcal{A})$. Clearly, $\mathrm{Der}(\mathcal{A})$ is a $\Bbbk$-vector subspace of $\mathrm{End}(\mathcal{A})$.

Remark 1.10. Let $\mathrm{D}_1, \mathrm{D}_2 \in \mathrm{Der}(\mathcal{A})$. Then $[\mathrm{D}_1, \mathrm{D}_2] := \mathrm{D}_1 \circ \mathrm{D}_2 - \mathrm{D}_2 \circ \mathrm{D}_1 \in \mathrm{Der}(\mathcal{A})$. The proof is by direct calculation.

Remark 1.11. As a consequence of (1.2) one has the **Leibniz formula**:

$$\mathrm{D}^k(x * y) = \sum_{i=0}^{k} \binom{k}{i} (\mathrm{D}^i\, x) * (\mathrm{D}^{k-i}\, y), \tag{1.3}$$

where $\mathrm{D}^k = \mathrm{D} \circ \ldots \circ \mathrm{D}$ (k factors).

Hence if $\operatorname{char} \mathbb{k} = p \neq 0$ then

$$\mathrm{D}^p(x * y) = (\mathrm{D}^p\, x) * y + x * (\mathrm{D}^p\, y), \tag{1.4}$$

i.e., D^p is a derivation.

Thus $\mathrm{Der}(\mathcal{A})$ is closed under the mapping $\mathrm{D} \mapsto D^p$ as well as the bracket composition (cf. remark 1.10).

1.2 Tensor algebra

Let V be a vector space over $\mathbb{k}$. The tensor algebra of V, denoted $\mathcal{T}(V)$, is the algebra of tensors on V (of any rank) with multiplication being the tensor product. The tensor algebra is, in a sense, the "most general" algebra containing V. This notion of generality is formally expressed by a certain universal property (see below).

Definition 1.12. The **tensor algebra**

$$\mathcal{T}(V) = \bigoplus_{n=0}^{\infty} \mathcal{T}_n(V)$$

is the graded $\mathbb{k}$-algebra with the n-th graded component given by n-th tensor power of V:

$$\mathcal{T}_n(V) = V^{\otimes n} = \overbrace{V \otimes \cdots \otimes V}^{n \text{ times}}, \quad n = 1, 2, \ldots,$$

and $\mathcal{T}_0(V) = \mathbb{k}$.

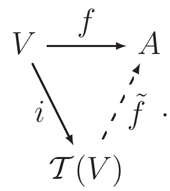
The multiplication $m : \mathcal{T}(V) \times \mathcal{T}(V) \to \mathcal{T}(V)$ is determined by the canonical isomorphism $\mathcal{T}_k(V) \otimes \mathcal{T}_l(V) \to \mathcal{T}_{k+l}(V)$ given by the tensor product:

$$m(a, b) = a \otimes b, \quad a \in V^{\otimes k},\ b \in V^{\otimes l}$$

which is then extended by linearity to all of $\mathcal{T}(V)$.

Remark 1.13. The construction generalizes in straightforward manner to the tensor algebra of any module M over a commutative ring R. If R is a non-commutative ring, one can still perform the construction for any R-R bimodule M. It does not work for ordinary R-modules because the iterated tensor products cannot be formed.

The fact that the tensor algebra is the most general algebra containing V is expressed by the following universal property: Any linear transformation $f : V \to A$ from V to a $\Bbbk$-algebra A can be uniquely extended to an algebra homomorphism from $\mathcal{T}(V)$ to A as indicated by the following commutative diagram:



Here i is the canonical inclusion of V into $\mathcal{T}(V)$. In fact, one can define the tensor algebra $\mathcal{T}(V)$ as the unique $\Bbbk$-algebra satisfying this universal property (moreover, $\mathcal{T}(V)$ is unique up to a unique isomorphism).

The tensor algebra $\mathcal{T}(V)$ is also called the **free associative algebra** on the vector space V.

If V has finite dimension n, its tensor algebra can be regarded as the "algebra of polynomials over $\Bbbk$ in n non-commuting variables". If we take basis vectors for V, those become non-commuting variables in $\mathcal{T}(V)$, subject to no constraints (beyond associativity, the distributive law and $\Bbbk$-linearity).

That is, we construct the free associative algebra of V in the following way: choose a basis $\{x_1, \ldots, x_n\}$ in V, and let $T = \mathcal{T}(V) = \Bbbk\langle x_1, \ldots, x_n\rangle$ be the algebra of non-commutative polynomials in variables $\{x_1, \ldots, x_n\}$ with coefficients in $\Bbbk$. As a vector space, it is generated by monomials in these variables which are finite sequences of x_i in arbitrary order (repetitions are allowed). The product is defined by concatenation of the monomials. The map $i : V \to T$ is the natural embedding ($x_i \mapsto x_i$).

Remark 1.14. The free associative algebra generated by $x_1, \ldots, x_n$ over $\Bbbk$:

$$\Bbbk\langle x_1, \ldots, x_n\rangle$$

is a finitely generated $\Bbbk$-algebra.

Finitely generated free associative $\Bbbk$-algebra is called **general non-commutative polynomial ring** over the field $\Bbbk$.

Obviously, the commutative polynomial ring $\Bbbk[x_1, \ldots, x_n]$ is a commutative finitely generated $\Bbbk$-algebra. In the case $n = 1$, $\Bbbk[x]$ and $\Bbbk\langle x\rangle$ coincide with the algebra of polynomials in one variable.

Remark 1.15. Because of the generality of the tensor algebra, many other algebras of interest are constructed by starting with the tensor algebra and then imposing certain relations on the generators, that is, by constructing certain factors of $\mathcal{T}(V)$. Examples of this construction are the **exterior algebra**, the **symmetric algebra**, **Clifford algebras** and **universal enveloping algebras**.

Obviously, any $\Bbbk$-algebra is isomorphic to a factor of a free associative $\Bbbk$-algebra by some two-sided ideal.

In what follows we identify the tensor algebra $\mathcal{T}(V)$ with the free associative algebra constructed above.

1.3 Lie theory

We will not discuss the Lie Theory in much details here, we refer the interested reader to [16], [6] and [20].

1.3.1 Lie groups and Lie algebras

Definition 1.16. A **Lie group** G is a group which is also an analytic manifold, such that the group operations are smooth, that is, the multiplication map from $G \times G$ into G and the inverse map from G into G are required to be analytic maps.

Example 1.17. While the Euclidean space $\mathbb{R}^n$ is a real Lie group (with ordinary vector addition as the group operation), more typical examples are given by **matrix Lie groups**, i.e. groups of invertible matrices (under the matrix multiplication). For instance, the group $SO(3)$ of all rotations in 3-dimensional space is a matrix Lie group.

Definition 1.18. A **Lie algebra** is an algebra $\mathfrak{g}$ over a field $\Bbbk$ with the product defined by:

$$\mathfrak{g} \times \mathfrak{g} \ni (x, y) \mapsto [x, y] \in \mathfrak{g} \tag{1.5}$$

satisfying

1. $[x, x] = 0$ (*antisymmetry*),
2. $[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$ (*Jacobi identity*),

for all $x, y, z \in \mathfrak{g}$. This operation in a Lie algebra is called a Lie bracket. By custom one sometimes refers to the Lie bracket as a commutator.

The Lie algebra $\mathfrak{g}$ is called **abelian** or **commutative** if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

Remark 1.19. To every Lie group, we can associate a Lie algebra which completely captures the local structure of the group, at least if the Lie group is connected. This is done as follows.

The Lie algebra $\mathfrak{g}$ of a Lie group G consists of left invariant vector fields on G. The left invariance condition means the following: let $l_g : G \to G$ be the left translation by $g \in G$, i.e., $l_g(h) = gh$ for all $h \in G$. A vector field X on G is left invariant if $(dl_g)_h X_h = X_{gh}$.

Clearly, $\mathfrak{g}$ is a vector space. It can be identified with the tangent space to G at the unit element e. Namely, to any left invariant vector field we can attach its value at e. Conversely, a tangent vector at e can be translated to all other points of G to obtain a left invariant vector field.

Note that we did not require our vector fields to be smooth; it is however a fact that a left invariant vector field is automatically smooth.

The operation making $\mathfrak{g}$ into a Lie algebra is the bracket of vector fields:

$$[X,Y]f = X(Yf) - Y(Xf),$$

for $X, Y \in G$ and f a smooth function on G. Here we identify vector fields with derivations of the algebra $C^\infty(G)$, i.e., think of them as first order differential operators.

The Lie algebra $\mathfrak{g}$ has the same dimension as the manifold G.

G acts on $\mathfrak{g}$ be conjugation: $Ad_g X = gXg^{-1}$. This is called the adjoint action. The differential of this map with respect to g is an action of $\mathfrak{g}$ on itself, $\operatorname{ad}_X Y = [X,Y]$, which is also called the adjoint action (we will consider later this action in more details).

Example 1.20. Every vector space becomes an abelian Lie algebra trivially if we define the Lie bracket to be identically zero.

Example 1.21. Euclidean space $\mathbb{R}^3$ becomes a Lie algebra with the Lie bracket given by the cross product of vectors.

Example 1.22. Let $\mathcal{A}$ be an arbitrary algebra over a field. Then $\mathrm{Der}(\mathcal{A})$ becomes a Lie algebra when $[\mathrm{D}_1, \mathrm{D}_2] := \mathrm{D}_1 \circ \mathrm{D}_2 - \mathrm{D}_2 \circ \mathrm{D}_1$ due to remark 1.10.

Notation. Let $\mathfrak{g}$ be a Lie algebra and $f \in \mathfrak{g}$. Let $\operatorname{ad}_f : \mathfrak{g} \to \mathfrak{g}$ be be a linear mapping given by $\mathfrak{g} \ni x \mapsto [f,x] \in \mathfrak{g}$. Due to Jacobi identity we have $\operatorname{ad}_f([x,y]) = [\operatorname{ad}_f x, y] + [x, \operatorname{ad}_f y]$. Therefore ad_f is a derivation of the Lie algebra $\mathfrak{g}$. We will call it the **inner derivation** determined by f. A derivation which is not inner is called **outer derivation**.

The mapping $\operatorname{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ given by $\mathfrak{g} \ni f \mapsto \operatorname{ad}_f \in \mathrm{Der}(\mathfrak{g})$ clearly satisfies $\operatorname{ad}_{[x,y]} = \operatorname{ad}_x \circ \operatorname{ad}_y - \operatorname{ad}_y \circ \operatorname{ad}_x$. Hence ad is a homomorphism of Lie algebras. This homomorphism is called the **adjoint representation** of $\mathfrak{g}$.

Definition 1.23. The **central extension** of an arbitrary Lie algebra $\mathfrak{g}$ by an abelian Lie algebra $\mathfrak{c}$ is the Lie algebra that is the direct sum $\mathfrak{g} \oplus \mathfrak{c}$ endowed with the Lie algebra bracket defined by $[\mathfrak{g}, \mathfrak{c}] = 0$.

Note that a Lie algebra is in general a non-unital non-associative algebra.

Proposition 1.24. *Let $\mathfrak{g}$ be a Lie algebra. Then the following conditions are equivalent:*

- $\mathfrak{g}$ *is associative,*
- $[x,[y,z]] = 0$ *for all* $x, y, z \in \mathfrak{g}$,

- $\mathfrak{g}$ *is a central extension of an abelian Lie algebra.*

Definition 1.25. Let $\mathfrak{g}$ be a Lie algebra. The **derived series** is the sequence of ideals of $\mathfrak{g}$ defined recursively by $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(1)} = [\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}]$, $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}]$, ..., $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$. Clearly,

$$\mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \ldots \supseteq \mathfrak{g}^{(k)} \supseteq \ldots.$$

The Lie algebra $\mathfrak{g}$ is called **solvable** if $\mathfrak{g}^{(n)} = 0$ for some $n \in \mathbb{N}$.

The **lower central series** is the sequence of ideals of $\mathfrak{g}$ defined recursively by $\mathfrak{g}_{(0)} = \mathfrak{g}$, $\mathfrak{g}_{(1)} = [\mathfrak{g}, \mathfrak{g}_{(0)}]$, $\mathfrak{g}_{(2)} = [\mathfrak{g}, \mathfrak{g}_{(1)}]$, ..., $\mathfrak{g}_{(i+1)} = [\mathfrak{g}, \mathfrak{g}_{(i)}]$. Clearly,

$$\mathfrak{g}_{(0)} \supseteq \mathfrak{g}_{(1)} \supseteq \ldots \supseteq \mathfrak{g}_{(k)} \supseteq \ldots.$$

The Lie algebra $\mathfrak{g}$ is called **nilpotent** if $\mathfrak{g}_{(n)} = 0$ for some $n \in \mathbb{N}$. The smallest value of n for which holds $\mathfrak{g}_{(n)} = 0$ is called the **degree of nilpotency** of the nilpotent Lie algebra $\mathfrak{g}$.

Clearly, a nilpotent Lie algebra is also solvable. An abelian Lie algebra is nilpotent of degree 1.

Let $\mathfrak{g}$ be an arbitrary Lie algebra, then there exists a unique maximal solvable ideal of $\mathfrak{g}$, called the **radical** of $\mathfrak{g}$ and denoted $\operatorname{Rad}\mathfrak{g}$.

A *subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is said to be *nilpotent* or *solvable* if $\mathfrak{h}$ is nilpotent or solvable when considered as a Lie algebra in its own right. The terms may also be applied to ideals of $\mathfrak{g}$, since every ideal of $\mathfrak{g}$ is also a subalgebra.

The first property of the Lie bracket saying $[x, x] = 0$ implies that $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$. Therefore all ideals in a Lie algebra are two-sided.

Definition 1.26. The Lie algebra $\mathfrak{g}$ is called **simple** if $\mathfrak{g}$ is non-abelian and has no proper ideals.

The Lie algebra $\mathfrak{g}$ is called **semisimple** if $\mathfrak{g}$ is non-abelian and has no proper solvable ideals.

There is a one-to-one correspondence between simple Lie groups and simple Lie algebras of dimension greater than 1: the Lie algebra of a simple Lie group is a simple Lie algebra.

Remark 1.27. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The following conditions are equivalent:

- $\mathfrak{g}$ is semisimple,
- $\mathfrak{g}$ is a direct sum of simple Lie algebras,
- the Killing form, $\kappa(x, y) = \operatorname{tr}(\operatorname{ad}_x \operatorname{ad}_y)$, is nondegenerate,

- $\mathfrak{g}$ has no non-zero abelian ideals,
- $\mathfrak{g}$ has no non-zero solvable ideals,
- the radical of $\mathfrak{g}$ is 0.

Example 1.28. Every $\Bbbk$-algebra gives rise to a Lie algebra as follows. Let $\mathcal{A}$ be a $\Bbbk$-algebra with the multiplication $*$. For $x, y \in \mathcal{A}$, define the Lie product of x and y by $[x, y] := x * y - y * x$. One checks immediately that this product satisfies conditions (1) and (2) of definition 1.18. This gives $\mathcal{A}$ the structure of a Lie algebra. We denote this Lie algebra by $\mathcal{A}^{Lie}$.

Moreover, it is easy to check that in this case the bracket also has the following property:

$$[z, x * y] = [z, x] * y + x * [z, y], \tag{1.6}$$

for all $x, y, z \in \mathcal{A}$.

Because of equation (1.6) the kernel of the inner derivation $\operatorname{Ker}\operatorname{ad}_f = \{x \in \mathcal{A} : [f, x] = 0\}$ is a subalgebra of $\mathcal{A}$ for any $f \in \mathcal{A}$.

Definition 1.29. A **restricted Lie algebra** $\mathfrak{g}$ of characteristic $p \neq 0$ is a Lie algebra over a field $\Bbbk$ of characteristic p endowed with a mapping $a \mapsto a^{[p]}$ such that

1. $(a + b)^{[p]} = a^{[p]} + b^{[p]} + \Lambda(a, b)$,
2. $(\alpha a)^{[p]} = \alpha^p a^{[p]}, \quad \forall \alpha \in \Bbbk$,
3. $\operatorname{ad}_{b^{[p]}} = (\operatorname{ad}_b)^p$,

where $\mathfrak{g} \ni \Lambda(a, b) := \sum_{k=1}^{p-1} s_k(a, b)$ and $k \cdot s_k(a, b)$ is the coefficient of λ^{k-1} in $(\operatorname{ad}_{\lambda a+b})^{p-1} a$.

Example 1.30. Let A be an associative algebra over a field of characteristic p. Then A^{Lie} endowed with the map $a^{[p]} := a^p$ becomes a restricted Lie algebra.

Example 1.31. Let $\mathcal{A}$ be an arbitrary algebra over a field of characteristic p. Then $\operatorname{Der}(\mathcal{A})$ becomes a restricted Lie algebra when $\mathrm{D}^{[p]} := \mathrm{D}^p$ due to remark 1.11.

Example 1.32. Let V be a vector space. Then $End(V)$ is a $\Bbbk$-algebra with respect to composition. Moreover, it becomes a Lie algebra if we define the bracket as in example 1.28. This Lie algebra is called the **general linear algebra** and denoted by $\mathfrak{gl}(V)$. Any subalgebra of the Lie algebra $\mathfrak{gl}(V)$ is called a **linear Lie algebra**.

Suppose that V is n dimensional. Choosing a basis for V we can identify $\mathfrak{gl}(V)$ with the space of all $n \times n$ matrices over $\Bbbk$. In this case it is denoted $\mathfrak{gl}_n(\Bbbk)$ or simply $\mathfrak{gl}_n$.

Note that with respect to the matrix multiplication $\mathfrak{gl}_n(\Bbbk)$ is a finitely generated $\Bbbk$-algebra.

The standard basis of $\mathfrak{gl}_n$ consists of matrices e_{ij} having 1 in the (i,j)-th position and 0 elsewhere. Since $e_{ij}e_{kl} = \delta_{jk}e_{il}$, it follows that:

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj} \tag{1.7}$$

Notice that in the right side of formula (1.7) all coefficients are ± 1 or 0.

The Lie algebra $\mathfrak{gl}_{l+1}$ ($l \geqslant 1$) has the following Lie subalgebra:

$$\mathfrak{sl}_{l+1} = \{a \in \mathfrak{gl}_{l+1} \mid Tr(a) = 0\},$$

here $Tr(a) = \sum a_{ii}$ is the trace of a. This is a Lie algebra since $Tr([a,b]) = Tr(a*b) - Tr(b*a) = 0$. This subalgebra is called **special linear algebra**.

It is convenient to choose the following basis of $\mathfrak{sl}_{l+1}$:

$$\{e_{ij}, 1 \leqslant i \neq j \leqslant l+1; h_i = e_{ii} - e_{i+1,i+1}, 1 \leqslant i \leqslant l\}. \tag{1.8}$$

In particular, for $\mathfrak{sl}_2$ we have the following basis:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the bracket in $\mathfrak{sl}_2$ is given by the formulas

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f.$$

1.3.2 Root systems

In this section we give a short overview of root systems and their use in Lie theory due to [16] and [15].

Let $\mathbf{E}$ be a fixed euclidean space, i.e., a finite dimensional vector space over $\mathbb{R}$ endowed with a positive definite symmetric form $(\cdot,\cdot)$. A subspace of $\mathbf{E}$ of codimension one is called a **hyperplane**. A **reflection** in $\mathbf{E}$ is an invertible linear transformation leaving some hyperplane pointwise fixed and sending any vector orthogonal to that hyperplane into its negative. Clearly, a reflection preserves the inner product on $\mathbf{E}$, i.e., is *orthogonal*.

Any non-zero vector α determines a reflection σ_α, with **reflecting hyperplane** $P_\alpha = \{\beta \in \mathbf{E} \mid (\beta,\alpha) = 0\}$. Of course, non-zero vectors proportional to α yield the same reflection. One can easily write down an explicit formula: $\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$. The number $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ is usually denoted by $\langle\beta,\alpha\rangle$. Notice that $\langle\beta,\alpha\rangle$ is linear only in the first variable.

Definition 1.33. A subset $\mathbf{\Phi}$ of the euclidean space $\mathbf{E}$ is called a **root system** in $\mathbf{E}$ if $\mathbf{\Phi}$ satisfies the following axioms:

(R1) $\mathbf{\Phi}$ is finite, spans $\mathbf{E}$ and does not contain 0.

(R2) If $\alpha \in \mathbf{\Phi}$, the only multiples of α in $\mathbf{\Phi}$ are $\pm\alpha$.

(R3) If $\alpha \in \mathbf{\Phi}$, the reflection σ_α leaves $\mathbf{\Phi}$ invariant.

(R4) If $\alpha, \beta \in \mathbf{\Phi}$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

Let $\mathbf{\Phi}$ be a root system in $\mathbf{E}$. Let us denote the subgroup of $GL(\mathbf{E})$ generated by the reflections $\sigma_\alpha, \alpha \in \mathbf{\Phi}$ by $\mathcal{W}$. This subgroup is called the **Weyl group** of $\mathbf{\Phi}$. Because of axioms (R1) and (R3) we can identify $\mathcal{W}$ with a subgroup of the symmetric group on $\mathbf{\Phi}$; in particular, $\mathcal{W}$ is finite.

Definition 1.34. Let $\mathbf{\Phi}$ and $\mathbf{\Phi}'$ be root systems in respective euclidean spaces $\mathbf{E}$, $\mathbf{E}'$. We call $(\mathbf{\Phi}, \mathbf{E})$ and $(\mathbf{\Phi}', \mathbf{E}')$ **isomorphic** if there exists a vector space isomorphism $\phi : \mathbf{E} \to \mathbf{E}'$ sending $\mathbf{\Phi}$ onto $\mathbf{\Phi}'$ such that $\langle \phi(\beta), \phi(\alpha) \rangle = \langle \beta, \alpha \rangle$ for all $\beta, \alpha \in \mathbf{\Phi}$.

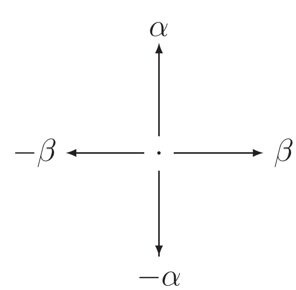
We call $l = \dim \mathbf{E}$ the **rank** of the root system $\mathbf{\Phi}$.

Example 1.35. We can draw a picture of $\mathbf{\Phi}$ when its rank $\leqslant 2$.

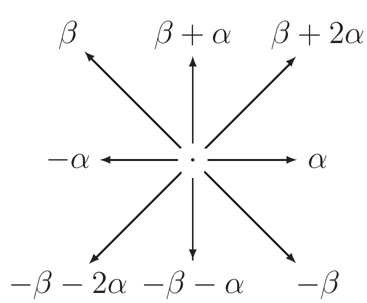
Due to (R2), there exists only one possible root system of rank 1, labeled A_1:

There exists 4 possibilities in rank 2:

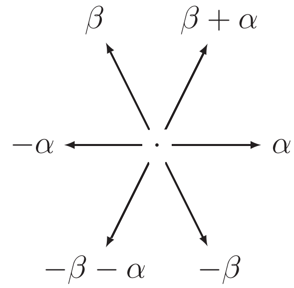
- $A_1 \times A_2$:



- B_2:



- A_2:



- G_2 (cf. [16, Chapter III, Figure 1]).

For the construction of other root systems see [16, Section 12].

Definition 1.36. A subset Δ of $\mathbf{\Phi}$ is called a **base** if:

(B1) Δ is a basis of $\mathbf{E}$,

(B2) each root $\beta \in \mathbf{\Phi}$ can be written as $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ with integral coefficients k_α all nonnegative or all nonpositive.

The roots in Δ are called **simple**. Due to (B1), $\operatorname{Card} \Delta = l$, and the expression for β in (B2) is unique. This allows us to define the **height** of a root (wrt. Δ) by $\operatorname{ht} \beta := \sum_{\alpha \in \Delta} k_\alpha$. The root β is called **positive** (resp., **negative**) if all $k_\alpha \geqslant 0$ (resp., all $k_\alpha \leqslant 0$) and write in this case $\beta \succ 0$ (resp., $\beta \prec 0$).

The set of all positive (resp., negative) roots is denoted by $\mathbf{\Phi}^+$ (resp., $\mathbf{\Phi}^-$). Clearly, $\mathbf{\Phi} = \mathbf{\Phi}^+ \cup \mathbf{\Phi}^-$.

Remark 1.37. If α and β are positive roots and $\alpha + \beta$ is a root, then clearly $\alpha + \beta$ is also positive. In fact, Δ defines a **partial order** on $\mathbf{E}$, which extends the notation $\alpha \succ 0$: define $\beta \prec \alpha$ iff $\alpha - \beta$ is a sum of positive roots (equivalently, of simple roots) or $\alpha = \beta$.

Theorem 1.38 (cf. Section 10.1 in [16]). $\mathbf{\Phi}$ *has a base.*

Definition 1.39. Let us fix an ordering $(\alpha_1, \dots, \alpha_l)$ of the simple roots. The matrix $(\langle \alpha_i, \alpha_j \rangle)$ is called the **Cartan matrix** of $\mathbf{\Phi}$. Its entries are called **Cartan integers**.

Remark 1.40. The Cartan matrix is independent of the choice of Δ. Since Δ is a basis of $\mathbf{E}$ it follows that the Cartan matrix is nonsingular. It turns out that the Cartan matrix of $\mathbf{\Phi}$ determines $\mathbf{\Phi}$ up to an isomorphism.

Definition 1.41. $\mathbf{\Phi}$ is called **irreducible** if it cannot be partitioned into the union of two proper subsets such that each root in one set is orthogonal to each root in the other.

Theorem 1.42 ([16] and [15]). $\mathbf{\Phi}$ *decomposes (uniquely) as the union of irreducible root systems* $\mathbf{\Phi}_i$ *(in subspace* $\mathbf{E}_i$ *of* $\mathbf{E}$*) such that* $\mathbf{E} = \mathbf{E}_1 \oplus \dots \mathbf{E}_t$ *(orthogonal direct sum). Moreover, every irreducible root system is isomorphic to precisely one root system from the following list:*

1. *The classical root system* $\mathbf{A}_l, l \geqslant 1$,
2. *The classical root system* $\mathbf{B}_l, l \geqslant 2$,
3. *The classical root system* $\mathbf{C}_l, l \geqslant 3$,
4. *The classical root system* $\mathbf{D}_l, l \geqslant 4$,
5. *The exceptional root system* $\mathbf{G_2}, \mathbf{F_4}, \mathbf{E_6}, \mathbf{E_7}$ *and* $\mathbf{E_8}$.

1.3.3 Classification of semisimple complex Lie algebras

Irreducible root systems classify a number of related objects in Lie theory, notably:

- Simple complex Lie algebras,
- Simple complex Lie groups,
- Simple compact Lie groups.

Let us consider the one-to-one correspondence between root systems and semisimple complex Lie algebras in more details.

Definition 1.43. Let $\mathfrak{g}$ be a semisimple complex Lie algebra. A **Cartan subalgebra** of $\mathfrak{g}$ is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ with the following properties:

1. For all $H_1, H_2 \in \mathfrak{h} : [H_1, H_2] = 0$.
2. Let $X \in \mathfrak{g}$. If for all $H \in \mathfrak{h} : [H, X] = 0$, then $X \in \mathfrak{h}$.
3. For all $H \in \mathfrak{h}$, ad_H is diagonalizable.

Condition 1 says that $\mathfrak{h}$ is a commutative subalgebra of $\mathfrak{g}$. Condition 2 says that $\mathfrak{h}$ is a *maximal* commutative subalgebra (i.e., not contained in any larger commutative subalgebra).

Proposition 1.44. *Every semisimple complex Lie algebra has a Cartan subalgebra.*

A Cartan subalgebra of not necessarily semisimple Lie algebra is defined as follows:

Definition 1.45. Let $\mathfrak{g}$ be a Lie algebra. A **Cartan subalgebra** of $\mathfrak{g}$ is a maximal subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which is **self-normalizing**, that is, if $[g, h] \in \mathfrak{h}$ for all $h \in \mathfrak{h}$, then $g \in \mathfrak{h}$ as well.

Any Cartan subalgebra $\mathfrak{h}$ is nilpotent, and if $\mathfrak{g}$ is semisimple, it is abelian.

All Cartan subalgebras of a Lie algebra are conjugate by the adjoint action of any Lie group with the Lie algebra $\mathfrak{g}$.

Remark 1.46. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. One can show that all Cartan subalgebras have the same dimension. This dimension is called the **rank** of $\mathfrak{g}$.

One passes from a Lie algebra to a root system as follows:

Let $\mathfrak{g}$ be a semisimple complex Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. The subalgebra $\mathfrak{h}$ is abelian and acts on $\mathfrak{g}$, via the adjoint representation, by commuting simultaneously diagonalizable linear maps. The simultaneous eigenspaces of this $\mathfrak{h}$ action are called **root spaces**. The decomposition of $\mathfrak{g}$ into $\mathfrak{h}$ and the root spaces is called a **root decomposition** of $\mathfrak{g}$.

For $\lambda \in \mathfrak{h}^*$ we set

$$\mathfrak{g}_\lambda := \{a \in \mathfrak{g} \colon [h, a] = \lambda(h)a \text{ for all } h \in \mathfrak{h}\}.$$

We call a non-zero $\lambda \in \mathfrak{h}^*$ a **root** if $\mathfrak{g}_\lambda$ is non-trivial, in which case $\mathfrak{g}_\lambda$ is called a **root space**. One can show that that $\mathfrak{g}_0 = \mathfrak{h}$ and $\dim \mathfrak{g}_\lambda = 1$ for each root λ. Let us denote the set of all roots by $\mathbf{\Phi}$. Thus we obtain the following root decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \in \mathbf{\Phi}} \mathfrak{g}_\lambda.$$

The Cartan subalgebra $\mathfrak{h}$ has a natural inner product, called the **Killing form**, which in turn induces an inner product on $\mathfrak{h}^*$. One can show that with respect to this inner product $\mathbf{\Phi}$ is a root system.

Conversely, let $\mathbf{\Phi} \subset \mathbf{E}$ be a root system. Let $\mathbf{\Phi} = \mathbf{\Phi}^+ \cup \mathbf{\Phi}^-$ be a decomposition of $\mathbf{\Phi}$ into subsets of positive roots and negative roots. Clearly, the negation in $\mathbf{E}$ acts as a bijection between $\mathbf{\Phi}^+$ and $\mathbf{\Phi}^-$.

Consider the vector space

$$\mathfrak{g} = \mathbf{E} \oplus \mathbb{C}[\mathbf{\Phi}],$$

where $\mathbb{C}[\mathbf{\Phi}]$ denotes the complex finite dimensional vector space generated by the elements of $\mathbf{\Phi}$. Denote the basis elements of $\mathbb{C}[\mathbf{\Phi}]$ by $X_\lambda,\ \lambda \in \mathbf{\Phi}$. For each $\lambda \in \mathbf{\Phi}$ we set

$$H_\lambda := 2\frac{\lambda}{(\lambda, \lambda)} \in \mathbf{E}.$$

Next, we define a skew-symmetric bilinear bracket on $\mathfrak{g}$ by imposing so called **Chevalley-Serre relations**:

$$\begin{aligned}
[H_1, H_2] &= 0, \quad H_1, H_2 \in \mathbf{E};\\
[H, X_\lambda] &= (\lambda, H)X_\lambda, \quad H \in \mathbf{E},\ \lambda \in \mathbf{\Phi};\\
[X_\lambda, X_\mu] &= 0, \quad \lambda + \mu \neq 0;\\
[X_\lambda, X_\mu] &= H_\lambda, \quad \lambda + \mu = 0,\ \lambda \in \mathbf{\Phi}^+.
\end{aligned}$$

The resulting bracket satisfies the Jacobi identity, and thus endows $\mathfrak{g}$ with a structure of a Lie algebra. This Lie algebra turns out to be semisimple, with a root system isomorphic to $\mathbf{\Phi}$.

Let us recall the classical complex Lie algebras ($\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{so}_n(\mathbb{C})$ and $\mathfrak{sp}_n(\mathbb{C})$) and their root systems:

- The root system A_n is the root system of $\mathfrak{sl}_{n+1}(\mathbb{C})$, which has the rank n.
- The root system B_n is the root system of $\mathfrak{so}_{2n+1}(\mathbb{C})$, which has the rank n.
- The root system C_n is the root system of $\mathfrak{sp}_n(\mathbb{C})$, which has the rank n.
- The root system D_n is the root system of $\mathfrak{so}_{2n}(\mathbb{C})$, which has the rank n.

In rank one, there is only one isomorphism class of complex semisimple Lie algebras. The Lie algebra $\mathfrak{so}_2(\mathbb{C})$ is not semisimple and the remaining three, $\mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{so}_3(\mathbb{C})$, and $\mathfrak{sp}_1(\mathbb{C})$, are isomorphic. In rank two the root system D_2 is not irreducible, reflecting that $\mathfrak{so}_4(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. Also, the root systems B_2 and C_2 are isomorphic, reflecting that $\mathfrak{so}_5(\mathbb{C}) \cong \mathfrak{sp}_2(\mathbb{C})$. In rank three, the root systems A_3 and D_3 are isomorphic, reflecting that $\mathfrak{so}_6(\mathbb{C}) \cong \mathfrak{sl}_4(\mathbb{C})$

The classification of semisimple Lie algebras is equivalent to the classification of root systems, as the following theorem explains.

Theorem 1.47.

1. *If R_1 and R_2 are the root systems for two different Cartan subalgebras of the same complex semisimple Lie algebra, then R_1 and R_2 are isomorphic.*
2. *A semisimple Lie algebra is simple if and only if its root system is irreducible.*
3. *If two complex semisimple Lie algebras have isomorphic root systems, then they are isomorphic.*
4. *Every root system arises as the root system of some complex semisimple Lie algebra.*

The theorems 1.42 and 1.47 lead to the following classification of complex simple Lie algebras:

Theorem 1.48. *Every complex simple Lie algebra is isomorphic to precisely one algebra from the following list:*

1. $\mathfrak{sl}_{n+1}(\mathbb{C}), n \geqslant 1$
2. $\mathfrak{so}_{2n+1}(\mathbb{C}), n \geqslant 2$
3. $\mathfrak{sp}_n(\mathbb{C}), n \geqslant 3$
4. $\mathfrak{so}_{2n}(\mathbb{C}), n \geqslant 4$
5. *The exceptional Lie algebras* $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7$ *and* $\mathfrak{e}_8$.

1.3.4 Chevalley basis of a semisimple Lie algebra

In this section we closely follow [16].

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over an algebraically closed field $\mathbf{F}$ of characteristic 0, then $\mathfrak{g}$ has a canonical basis (called **Chevalley basis**) with to which the structure constants are integers.

Though the construction of a Chevalley basis depends on the choice of a basis of the root system $\mathbf{\Phi}$ the $\mathbb{Z}$-span $\mathrm{L}(\mathbb{Z})$ of Chevalley basis $\{x_\alpha, h_i\}_{\alpha\in\mathbf{\Phi}, i=1,\ldots,r}$ is a lattice in $\mathfrak{g}$, independent of it. It is even a Lie algebra over $\mathbb{Z}$ (in the obvious sense) under the bracket operation inherited from $\mathfrak{g}$.

Chevalley basis enables one to replace the scalars from $\mathbf{F}$ by members of an arbitrary field.

If $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the prime field of characteristic p, then the tensor product $\mathrm{L}(\mathbb{F}_p) = \mathrm{L}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p$ is defined: $\mathrm{L}(\mathbb{F}_p)$ is a vector space over $\mathbb{F}_p$ with the basis $\{x_\alpha \otimes 1, h_i \otimes 1\}$. Moreover, the bracket operation in $\mathrm{L}(\mathbb{Z})$ induces a natural Lie algebra structure on $\mathrm{L}(\mathbb{F}_p)$. The multiplication table is essentially the same with integers reduced mod p.

If $\mathbb{k}$ is any field extension of $\mathbb{F}_p$ then $\mathrm{L}(\mathbb{k}) = \mathrm{L}(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathbb{k} = \mathrm{L}(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{k}$ inherits both the basis and the Lie algebra structure from $\mathrm{L}(\mathbb{F}_p)$. In this way we associate with a pair $(\mathfrak{g}, \mathbb{k})$ the Lie algebra $\mathrm{L}(\mathbb{k})$ whose structure resembles that of $\mathfrak{g}$. $\mathrm{L}(\mathbb{k})$ is called the **Chevalley algebra**. Even though $\mathrm{L}(\mathbb{Z})$ depends on how the root vectors are chosen, it is defined up to an isomorphism (over $\mathbb{Z}$) by $\mathfrak{g}$ alone; similarly, the algebra $\mathrm{L}(\mathbb{k})$ depends (up to isomorphism) on the pair $(\mathfrak{g}, \mathbb{k})$.

To illustrate these remarks, we consider $\mathfrak{g} = \mathfrak{sl}_{l+1}(\mathbf{F})$. It is clear that $\mathrm{L}(\mathbb{k})$ has precisely the same multiplication table as $\mathfrak{sl}_{l+1}(\mathbf{F})$, relative to the standard basis (1.8). So $\mathrm{L}(\mathbb{k}) \cong \mathfrak{sl}_{l+1}(\mathbb{k})$. The only real change that takes place in passing from $\mathbf{F}$ to $\mathbb{k}$ is that $\mathrm{L}(\mathbb{k})$ *may fail to be simple*.

1.3.5 Enveloping algebras of Lie algebras

Enveloping algebras are important for us since they provide us with a number of interesting and computable examples.

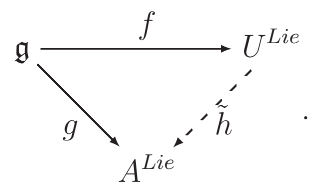
For any Lie algebra $\mathfrak{g}$ over $\mathbb{k}$ we can construct its enveloping algebra $\mathcal{U}(\mathfrak{g})$. This construction passes from the non-associative structure of $\mathfrak{g}$ to the "most general" $\mathbb{k}$-algebra A such that the Lie algebra A^{Lie} contains $\mathfrak{g}$; this algebra A is $\mathcal{U}(\mathfrak{g})$.

The important constraint is to preserve the representation theory: the representations of $\mathfrak{g}$ correspond in a one-to-one manner to the modules over $\mathcal{U}(\mathfrak{g})$. In a typical context where $\mathfrak{g}$ is acting by infinitesimal transformations, the elements of $\mathcal{U}(\mathfrak{g})$ act like differential operators of all orders.

Definition 1.49. Let $\mathfrak{g}$ be a Lie algebra over a field $\Bbbk$. The **Enveloping algebra** of $\mathfrak{g}$ is a $\Bbbk$-algebra $U = \mathcal{U}(\mathfrak{g})$ endowed with a Lie algebra homomorphism

$$f : \mathfrak{g} \to U^{Lie}$$

such that U and f satisfy the following *universal property*: for any $\Bbbk$-algebra A and a Lie algebra homomorphism $g : \mathfrak{g} \to A^{Lie}$ there exists a unique $\Bbbk$-algebra homomorphism $h : U \to A$ inducing a Lie algebra homomorphism $\tilde{h} : U^{Lie} \to A^{Lie}$ such that the following diagram commutes:



The enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ (if it exists) is unique up to a unique isomorphism (because of its universal property).

Remark 1.50. We can construct the enveloping algebra U of a Lie algebra $\mathfrak{g}$ in the following way: Let $\mathcal{T}(\mathfrak{g})$ be the free associative algebra of $\mathfrak{g}$ with the canonical imbedding $i : \mathfrak{g} \to \mathcal{T}(\mathfrak{g})$.

Let I be the (two-sided) ideal generated by the element $i([x, y]) - i(x) * i(y) + i(y) * i(x) \in \mathcal{T}(\mathfrak{g})$, for all $x, y \in \mathfrak{g}$. Put $U = \mathcal{U}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/I$.

Example 1.51. Let $\mathfrak{g}$ be a commutative Lie algebra. Then $\mathcal{U}(\mathfrak{g})$ is the factor of $\mathcal{T}(\mathfrak{g})$ by the ideal generated by elements $i(x) * i(y) - i(y) * i(x)$, for all $x, y \in \mathfrak{g}$. If we choose a basis $x_1, \ldots, x_n$ of $\mathfrak{g}$, $\mathcal{T}(\mathfrak{g})$ identifies with the algebra of non-commutative polynomials in $x_1, \ldots, x_n$ while the enveloping algebra is the algebra of commutative polynomials $\Bbbk[x_1, \ldots, x_n]$

Example 1.52. Using our presentation of $\mathfrak{sl}_2$ and the commutator formulas, we see that $\mathcal{U}(\mathfrak{sl}_2)$ is isomorphic to the factor of $\Bbbk\langle e, f, h\rangle$ by the two sided ideal generated by $e * f - f * e - h, h * e - e * h - 2e, h * f - f * h + 2f$.

Theorem 1.53 (Poincaré-Birkhoff-Witt). *Let $\mathfrak{g}$ be a Lie algebra over a field $\Bbbk$, finitely generated with basis $x_1, x_2, \ldots, x_n$. Furthermore let $U = \mathcal{U}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$ with the canonical imbedding $f : \mathfrak{g} \to U^{Lie}$. Then linear combinations of elements $x_{i_1} \ldots x_{i_m} = f(x_{i_1} \otimes \cdots \otimes x_{i_m})$ with $m > 0$ and $1 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_m$ along with 1 form a basis of U.*

Notice that if $\mathfrak{g}$ is a finite dimensional Lie algebra then its enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a finitely generated $\Bbbk$-algebra.

Remark 1.54. There exists a filtration by degree on $\mathcal{U}(\mathfrak{g})$ coming from $\mathcal{T}(\mathfrak{g})$.

1.4 Algebras of Solvable Type

Another fruitful source of our examples is the theory of algebras of solvable type introduced in [21]. These algebras are also considered in [26] (they are called GR-algebras there) and in [4] under the name of **PBW-algebras**. Although the theory of algebras of solvable type can be generalized further(see [24]), the algebras arising there usually are not $\Bbbk$-algebras.

First we give some computer algebra notions following [4], [13] and [26].

Definition 1.55. Let $\preceq$ be a partial ordering, i.e., a reflexive, antisymmetric and transitive relation, on a non-empty set M.

A partial ordering $\preceq$ on a set M satisfies the **descending chain condition** if there exists no infinite strictly descending chain

$$\gamma_1 \succ \gamma_2 \succ \ldots \succ \gamma_n \succ \ldots$$

Proposition 1.56. *A partial ordering $\preceq$ on a set M satisfies the descending chain condition iff every non-empty subset of M has a minimal element.*

Recall that a monoid $(M, \cdot)$ with neutral element $e \in M$ is a set M endowed with a binary operation $\cdot$ which is associative and satisfies the following property:

$$e \cdot m = m \cdot e = m \quad \forall m \in M.$$

Example 1.57. Let X be a non-empty set, called alphabet. A word or a term over X is an ordered finite sequence $x_1 \cdots x_s$ of elements $x_i \in X$. Adding the empty sequence, denoted by 1, to the set of words over X, we obtain the **free monoid** on X, denoted by $\langle X \rangle$. The multiplication in $\langle X \rangle$ is just the concatenation of words and 1 acts as neutral element. The characteristic property of X is that it is a free object, i.e., any mapping $X \to M$, where M is a monoid, extends uniquely to a homomorphism of monodies $\langle X \rangle \to M$.

We are especially interested in the case when X is finite, say $X = \{x_1, \ldots, x_n\}$. In this case we use the notation $\langle X \rangle = \langle x_1, \ldots, x_n \rangle$.

Example 1.58. Let n be a positive integer and let

$$\mathbb{N}^n = \{\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) : \alpha_1, \ldots, \alpha_n \in \mathbb{N}\}.$$

We will consider the commutative monoid $(\mathbb{N}^n, +)$ with sum defined componentwise. The neutral element is then given by $\underline{0} = (0, \ldots, 0)$.

Definition 1.59. Let $(M, \cdot)$ be a monoid. A partial ordering $\preceq$ on M is called **monoid ordering** if

$$\forall m_1, m_2, a, b \in M : m_1 \preceq m_2 \Rightarrow am_1b \preceq am_2b. \tag{1.9}$$

Remark 1.60. When the monoid M is cancelative i.e., when $am = bm$ or $ma = mb$ implies $a = b$, then condition (1.9) in definition 1.59 can be replaced by

$$\forall m_1, m_2, a, b \in M : m_1 \prec m_2 \Rightarrow am_1b \prec am_2b. \tag{1.10}$$

The examples 1.57 and 1.58 are clearly cancelative.

Definition 1.61. A non-empty subset E of $\mathbb{N}^n$ is said to be a **monoideal** if $E + \mathbb{N}^n = E$. If B is a subset of $\mathbb{N}^n$, then we define the *monoideal* generated by B to be

$$B + \mathbb{N}^n = \bigcup_{\underline{\beta} \in B} (\underline{\beta} + \mathbb{N}^n) = \left\{\underline{\beta} + \underline{\gamma}; \underline{\beta} \in B, \underline{\gamma} \in \mathbb{N}^n\right\}.$$

If $E = B + \mathbb{N}^n$, then we call the elements of B *generators* of E.

Definition 1.62. The partial ordering $\preceq^n$ in $\mathbb{N}^n$ is defined by

$$\underline{\alpha} \preceq^n \underline{\beta} \Leftrightarrow \underline{\beta} \in \underline{\alpha} + \mathbb{N}^n.$$

In other words, $\underline{\alpha} \preceq^n \underline{\beta}$ if $\alpha_i \leqslant \beta_i$ for all $1 \leqslant i \leqslant n$.

Clearly, partial ordering $\preceq^n$ satisfies the descending chain condition.

Lemma 1.63 (Dickson). *For any non-empty $E \subseteq \mathbb{N}^n$, there exists a finite subset $B = \left\{\underline{\alpha_1}, \ldots, \underline{\alpha_m}\right\}$ of E such that*

$$E \subseteq \bigcup_{i=1}^{m} (\underline{\alpha_i} + \mathbb{N}^n).$$

Observe that every monoideal has a set of generators (for example the whole monoideal).

Proposition 1.64. *Every monoideal E of $\mathbb{N}^n$ possesses a unique finite minimal set of generators B of E.*

Definition 1.65. An **admissible ordering** on $(\mathbb{N}^n, +)$ is a total monoid ordering $\preceq$ such that $\underline{0} \preceq \underline{\alpha}$ for every $\underline{\alpha} \in \mathbb{N}^n$. By remark 1.60 the total ordering $\preceq$ is admissible iff it satisfies the following two conditions:

(1) $\underline{0} \prec \underline{\alpha}$ for every $\underline{0} \neq \underline{\alpha} \in \mathbb{N}^n$;

(2) $\underline{\alpha} + \underline{\gamma} \prec \underline{\beta} + \underline{\gamma}$ for all $\underline{\alpha}, \underline{\beta}, \underline{\gamma} \in \mathbb{N}^n$ with $\underline{\alpha} \prec \underline{\beta}$.

The **total degree** of the element $\underline{\alpha} \in \mathbb{N}^n$ is

$$|\underline{\alpha}| = \alpha_1 + \cdots + \alpha_n.$$

Example 1.66. The **total degree ordering** $\preceq_{tot}$ on $\mathbb{N}^n$ is defined by

$$\underline{\beta} \prec_{tot} \underline{\alpha} \Leftrightarrow |\underline{\beta}| < |\underline{\alpha}|.$$

The ordering $\preceq_{tot}$ is only partial ordering, and hence not an admissible ordering.

For any $1 \leqslant i \leqslant n$ we denote by $\underline{\epsilon_i}$ the element $(0, \ldots, 1, \ldots, 0) \in \mathbb{N}^n$ whose all entries are 0 except for the value 1 in the i-th component.

Example 1.67. The **reverse lexicographical ordering** $\preceq_{revlex}$ on $\mathbb{N}^n$ with $\underline{\epsilon_1} \prec \underline{\epsilon_2} \prec \cdots \prec \underline{\epsilon_n}$ is defined by

$$\underline{\alpha} \prec_{revlex} \underline{\beta} \Leftrightarrow \exists j \in \{1, 2, \ldots, n\} \text{ such that } \alpha_i = \beta_i \ \forall i < j \text{ and } \alpha_j > \beta_j.$$

$\preceq_{revlex}$ is a total ordering which is compatible with the monoid structure, but is not admissible.

Let us give now some examples of standard admissible orderings.

Example 1.68. The **lexicographical ordering** $\preceq_{lex}$ on $\mathbb{N}^n$ with $\underline{\epsilon_1} \prec \underline{\epsilon_2} \prec \cdots \prec \underline{\epsilon_n}$ is defined by

$$\underline{\alpha} \prec_{lex} \underline{\beta} \Leftrightarrow \exists j \in \{1, 2, \ldots, n\} \text{ such that } \alpha_i = \beta_i \ \forall i > j \text{ and } \alpha_j < \beta_j.$$

Example 1.69. The **degree lexicographical ordering** $\preceq_{deglex}$ on $\mathbb{N}^n$ with $\underline{\epsilon_1} \prec \underline{\epsilon_2} \prec \cdots \prec \underline{\epsilon_n}$ is defined by

$$\underline{\alpha} \prec_{deglex} \underline{\beta} \text{ iff } \Leftrightarrow |\underline{\alpha}| < |\underline{\beta}| \text{ or } \left(|\underline{\alpha}| = |\underline{\beta}| \text{ and } \underline{\alpha} \prec_{lex} \underline{\beta}\right).$$

Example 1.70. The **degree reverse lexicographical ordering** $\preceq_{degrevlex}$ on $\mathbb{N}^n$ with $\underline{\epsilon_1} \prec \underline{\epsilon_2} \prec \cdots \prec \underline{\epsilon_n}$ is defined by

$$\underline{\alpha} \prec_{degrevlex} \underline{\beta} \Leftrightarrow |\underline{\alpha}| < |\underline{\beta}| \text{ or } \left(|\underline{\alpha}| = |\underline{\beta}| \text{ and } \underline{\alpha} \prec_{revlex} \underline{\beta}\right).$$

Example 1.71. Let $\underline{\omega} = (\omega_1, \ldots, \omega_n) \in \mathbb{N}^n$. The **weighted total degree** with respect to $\underline{\omega}$ of the element $\underline{\alpha} \in \mathbb{N}^n$ is

$$|\underline{\alpha}|_{\underline{\omega}} = \langle \underline{\omega}, \underline{\alpha} \rangle = \sum_{i=1}^{n} \omega_i \alpha_i.$$

The **$\underline{\omega}$-weighted degree lexicographical ordering** $\preceq_{\underline{\omega}}$ on $\mathbb{N}^n$ with $\underline{\epsilon_1} \prec \underline{\epsilon_2} \prec \cdots \prec \underline{\epsilon_n}$ is defined by

$$\underline{\alpha} \prec_{\underline{\omega}} \underline{\beta} \text{ iff } \Leftrightarrow |\underline{\alpha}|_{\underline{\omega}} < |\underline{\beta}|_{\underline{\omega}} \text{ or } \left(|\underline{\alpha}|_{\underline{\omega}} = |\underline{\beta}|_{\underline{\omega}} \text{ and } \underline{\alpha} \prec_{lex} \underline{\beta}\right).$$

Proposition 1.72. *Any admissible ordering $\preceq$ on $\mathbb{N}^n$ is a refinement of the partial ordering $\preceq^n$ (defined in 1.62), that is, $\underline{\alpha} \preceq^n \underline{\beta}$ implies $\underline{\alpha} \preceq \underline{\beta}$.*

Proposition 1.73. *Any admissible ordering on $\mathbb{N}^n$ is a well-ordering, that is, every non-empty subset of $\mathbb{N}^n$ has a least element.*

Let $R = \Bbbk[x_1, \ldots, x_n]$ be the (commutative) polynomial ring generated by variables x_i over $\Bbbk$. We let

$$M = M(x_1, \ldots x_n) = \left\{ x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_m}^{\alpha_m} | 1 \leqslant i_1 < i_2 < \ldots < i_m \leqslant n, 0 \leqslant \alpha_k \right\}$$

denote the set of monomials in variables $x_1, \ldots x_n$. For any polynomial $f \in R$, $T(f)$ denotes the set of terms occurring in f with non-zero coefficient.

In what follows, for any $\underline{\alpha} \in \mathbb{N}^n$ and any $x_1, \ldots, x_n \in R$, we denote by $\mathbf{x}^{\underline{\alpha}}$ the standard term $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.

Recall, that every element $f \in R$ has a unique **standard representation**

$$f = \sum_{\underline{\alpha} \in \mathbb{N}^n} c_{\underline{\alpha}} \mathbf{x}^{\underline{\alpha}}.$$

For $0 \neq f \in R$ we define the **Newton diagram** of f by

$$\mathcal{N}(f) = \{ \underline{\alpha} \in \mathbb{N}^n : c_{\underline{\alpha}} \neq 0 \}.$$

For any admissible order on $\mathbb{N}^n$, let us introduce the following notions, which obviously depend on the choice of $\preceq$.

Notation. The **exponent** of $f \neq 0$ is defined by $\exp(f) = \max \mathcal{N}(f)$.

The standard representation of any $0 \neq f \in R$ thus becomes

$$f = c_{\exp(f)} \mathbf{x}^{\exp(f)} + \sum_{\underline{\alpha} \prec \exp(f)} c_{\underline{\alpha}} \mathbf{x}^{\underline{\alpha}}.$$

We call $\operatorname{lm}(f) := \mathbf{x}^{\exp(f)}$ the **leading monomial** of f and $\operatorname{lc}(f) := c_{\exp(f)}$ its **leading coefficient**. Finally, the **leading term** of f is defined by

$$\operatorname{lt}(f) := \operatorname{lc}(f) \operatorname{lm}(f) = c_{\exp(f)} \mathbf{x}^{\exp(f)}.$$

The polynomial rings of solvable type (or solvable polynomial rings for short) R are intermediate between the commutative and the most general non-commutative case. These rings R will be described briefly as follows: the elements of R are commutative polynomials over field $\Bbbk$, but the multiplication $*$ may be non-commutative. The decisive restriction on $*$ is that the difference between $f * g$ and a suitable scalar multiple of corresponding commutative product $f \cdot g$ is smaller than $f \cdot g$ in the sense of an arbitrary but fixed admissible term ordering on R. This can be guaranteed by a few, simple axioms on $*$.

An **admissible ordering** on $M = M(x_1, \ldots, x_n)$ induces in a natural way an ordering "$<$" on R: $f < g$ iff there exists $t \in T(g) \setminus T(f)$ such that for all $t' \in M$ with $t' > t, t' \in T(f)$ iff $t' \in T(g)$. The induced quasi ordering on R admit no infinite strictly decreasing chain.

The **non-commutative polynomial rings of solvable type** will be obtained from R by introducing a new multiplication on R subject to certain conditions: Fix an admissible ordering $<$ on M, and let $* : R \times R \to R$ be a new binary operation on R. Then we call $(R, *)$ a polynomial rings of solvable type, if the operation $*$ satisfies the following axioms:

Axioms 1.74.

(1) $(R, 0, 1, +, -, *)$ is an associative ring with 1.

(2) For all $a, b \in \mathbb{k}, 1 \leqslant h \leqslant i \leqslant j \leqslant k \leqslant n, t \in M(x_i, \ldots x_j)$,

  (i) $a * bt = bt * a = abt$,

  (ii) $x_h * bt = bx_h t$,

  (iii) $bt * x_h = btx_h$.

(3) For all $1 \leqslant i \leqslant j \leqslant n$ there exists $0 \neq c_{ij} \in \mathbb{k}$ and $p_{ij} \in R$ such that $x_j * x_i = c_{ij}x_ix_j + p_{ij}$ and $p_{ij} < x_ix_j$.

We denote this solvable polynomial ring by

$$\mathbb{k}\left\langle x_1, \ldots, x_n; x_j * x_i = c_{ij}x_ix_j + p_{ij} \quad \forall 1 \leqslant i \leqslant j \leqslant n, < \right\rangle .$$

The class of polynomial rings of solvable type introduced in this way is quite comprehensive: It includes commutative polynomial rings; iterated Ore extensions of the ground field $\mathbb{k}$; factors of a general non-commutative polynomial ring over $\mathbb{k}$ by fairly general commutation relations; and enveloping algebras of finite dimensional Lie algebras over $\mathbb{k}$, in particular the Weyl algebras arising in quantum physics.

Now we will show how enveloping algebras of finite dimensional Lie algebras over $\mathbb{k}$ turn into solvable polynomial rings: Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a ground field $\mathbb{k}$ and let $x_1, \ldots, x_n$ be a basis of $\mathfrak{g}$ over $\mathbb{k}$. It has been above shown that there is a canonical construction of a finitely generated $\mathbb{k}$-algebra $\mathcal{U}(\mathfrak{g})$ from $\mathfrak{g}$ such that $\mathfrak{g}$ embeds into $\mathcal{U}(\mathfrak{g})$, when the Lie product in $\mathcal{U}(\mathfrak{g})$ is taken as the commutator $[a, b] = a * b - b * a$. By the Poincaré-Birkhoff-Witt theorem, the elements of $\mathcal{U}(\mathfrak{g})$ can be represented uniquely as commutative polynomials in $\mathbb{k}[x_1, \ldots, x_n]$. Then for $1 \leqslant i < j \leqslant n$ holds $p_{ij} := x_j * x_i - x_i * x_j = [x_j, x_i] \in \mathfrak{g}$, and so $[x_j, x_i]$ is a linear form in $x_1, \ldots, x_n$ with coefficients in $\mathbb{k}$. Moreover, $*$ satisfies axioms 1.74 (1) and (2).

Let now $<$ be any **degree-compatible** admissible ordering on $M = M(x_1, \ldots, x_n)$ (i.e., $deg(s) < deg(t)$ implies $s < t$ for $s, t \in M$). Then by the above, $x_j * x_i = x_ix_j + p_{ij}$ with $deg(p_{ij}) \leqslant 1 < deg(x_ix_j) = 2$; consequently, $p_{ij} < x_ix_j$ for $1 \leqslant i < j \leqslant n$, and so all the axioms of solvable polynomial rings are satisfied. If $\mathfrak{g}$ is a solvable Lie algebra then $p_{ij} \in \mathbb{k}[x_1, \ldots, x_{j-1}]$ for a suitable choice of the basis $x_1, \ldots, x_n$ of $\mathfrak{g}$, and so the axioms 1.74 are also satisfied for the pure lexicographical order.

We can now define the commutator relations for a solvable polynomial ring as:

$$x_j * x_i = x_i x_j + p_{ij},$$

that is, $c_{ij} = 1, p_{ij} = \sum_{k=1}^{n} a_{ijk} x_k = [x_j, x_i], a_{ijk} \in \Bbbk, 1 \leqslant i \leqslant j \leqslant n, 1 \leqslant k \leqslant n.$

Theorem 1.75. *The enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ over $\Bbbk$ is a solvable polynomial ring with respect to any degree-compatible admissible ordering $<$ on M. Moreover, if $\mathfrak{g}$ is solvable, then we can choose the basis $x_1, \ldots, x_n$ of $\mathfrak{g}$ such that $<$ may also be taken to be the pure lexicographical ordering.*

Let R be a solvable polynomial ring over $\Bbbk$ and I be a two-sided ideal in R, let $A = R/I$. Then we call the finitely generated $\Bbbk$-algebra A an **algebra of solvable type over** $\Bbbk$. These include all Clifford algebras and hence all Grassmann algebras.

Notice that in order to guarantee associativity we must require that the multiplication of variables is associative, that is: $x_i * (x_j * x_k) = (x_i * x_j) * x_k$, for all $1 \leqslant i, j, k \leqslant n$. This leads to the non-degeneracy conditions(cf. [26]).

Though the original names for the algebras defined above are "solvable polynomial ring" and "algebra of solvable type" it appears that these name can lead to some misunderstanding. Therefore, in what follows, we will call them "G-Algebra" and "GR-Algebra" (cf. [26]).

1.5 Centralizer and the center

Definition 1.76. Let F and S be two subsets of $\mathcal{A}$, then the **centralizer of** F **in** S is a subset of S defined by:

$$\operatorname{Cen}(F, S) := \{s \in S : [f, s] = 0 \quad \forall f \in F\}.$$

We write $\operatorname{Cen}(f, S)$ instead of $\operatorname{Cen}(\{f\}, S)$.

Obviously $\operatorname{Cen}(f, \mathcal{A}) = \operatorname{Ker} \operatorname{ad}_f$ and therefore it is a subalgebra of $\mathcal{A}$.

It is clear from the above definition that: $\operatorname{Cen}(F, S) = \operatorname{Cen}(F, \mathcal{A}) \cap S$. Since $\operatorname{Cen}(F, S) = \cap_{f \in F} \operatorname{Cen}(f, S)$, we obtain:

$$\operatorname{Cen}(F, S) = S \cap (\cap_{f \in F} \operatorname{Cen}(f, \mathcal{A})) = S \cap (\cap_{f \in F} \operatorname{Ker} \operatorname{ad}_f).$$

Therefore, if S is a vector subspace of $\mathcal{A}$ then $\operatorname{Cen}(F, S)$ is a vector subspace of S and if moreover S is a subalgebra of $\mathcal{A}$ then $\operatorname{Cen}(F, S)$ is a subalgebra of S.

Definition 1.77. The **center** of a $\Bbbk$-algebra $\mathcal{A}$ is the set

$$\mathcal{Z}(\mathcal{A}) := \{y \in \mathcal{A} : x * y = y * x \quad \text{for all} \quad x \in \mathcal{A}\}$$

The mapping $a \mapsto 1_{\mathcal{A}} * a$ imbeds $\Bbbk$ in $\mathcal{A}$, provided only that $\mathcal{A}$ is non-trivial, that is, $1_{\mathcal{A}} \neq 0$. Therefore $\Bbbk$ can be identified with a subring of the center of $\mathcal{A}$. In particular, $1_{\mathcal{A}} = 1_{\Bbbk}$ and $\Bbbk \subset \mathcal{Z}(\mathcal{A})$.

Remark 1.78. Obviously, $\mathcal{Z}(\mathcal{A})$ is a commutative subalgebra of $\mathcal{A}$, but it is not the biggest one in general, we are going to explore the topic concerning maximal commutative subalgebras in chapter 6.

Clearly $\mathcal{Z}(\mathcal{A}) = \operatorname{Ker} \operatorname{ad}$ and

$$\Bbbk \subset \mathcal{Z}(\mathcal{A}) \subset \operatorname{Cen}(F, \mathcal{A}),$$

for all subsets F of $\mathcal{A}$.

Obviously, $\mathcal{Z}(\mathcal{A}) = \operatorname{Cen}(\mathcal{A}, \mathcal{A})$, but it would be nice if we could find such a finite subset $X \subset \mathcal{A}$ that $\mathcal{Z}(\mathcal{A}) = \operatorname{Cen}(X, \mathcal{A})$ and to be able to compute $\operatorname{Cen}(f, V)$.

Chapter 2

Computation of the center and a centralizer

Throughout this paper, the letter $\mathcal{A}$ will stand for a finitely generated unital associative algebra over a field $\Bbbk$. We assume, unless explicitly specified, that every **$\Bbbk$-algebra** is of this type.

2.1 Theoretical background

Proposition 2.1. *Let V_1 and V_2 be vector subspaces of $\mathcal{A}$, F and G be subsets of $\mathcal{A}$, then*

$$\mathrm{Cen}(F, V_1) \cap \mathrm{Cen}(G, V_2) = \mathrm{Cen}(F, \mathrm{Cen}(G, V_1 \cap V_2)) = \mathrm{Cen}(F \cup G, V_1 \cap V_2)$$

is a vector subspace of $V_1 \cap V_2$.

Proof. Because of definition 1.76 we have the following:

$$\begin{aligned}
&\mathrm{Cen}(F, V_1) \cap \mathrm{Cen}(G, V_2) = (\mathrm{Cen}(F, \mathcal{A}) \cap V_1) \cap (\mathrm{Cen}(G, \mathcal{A}) \cap V_2) \\
&= \mathrm{Cen}(F, \mathcal{A}) \cap (\mathrm{Cen}(G, \mathcal{A}) \cap (V_1 \cap V_2)) = \mathrm{Cen}(F, \mathcal{A}) \cap \mathrm{Cen}(G, V_1 \cap V_2) = \mathrm{Cen}(F, \mathrm{Cen}(G, V_1 \cap V_2)) \\
&= (\mathrm{Cen}(F, \mathcal{A}) \cap \mathrm{Cen}(G, \mathcal{A})) \cap (V_1 \cap V_2) = \mathrm{Cen}(F \cup G, \mathcal{A}) \cap (V_1 \cap V_2) = \mathrm{Cen}(F \cup G, V_1 \cap V_2).
\end{aligned}$$

This accomplishes the proof. ■

Corollary 2.2. *If $F = \{f_1, \ldots, f_k\}$, then*

$$\mathrm{Cen}(F, V) = \cap_{1 \le i \le k} \mathrm{Cen}(f_i, V) = \mathrm{Cen}(f_1, \mathrm{Cen}(f_2, \ldots, \mathrm{Cen}(f_k, V) \ldots))$$

Remark 2.3. If S_1 and S_2 are subalgebras of $\mathcal{A}$, then $\mathrm{Cen}(F, S_1) \cap \mathrm{Cen}(G, S_2) = \mathrm{Cen}(F \cup G, S_1 \cap S_2)$ is a subalgebra of $S_1 \cap S_2$.

It turns out that in order to compute the center it is enough to compute certain centralizer:

Lemma 2.4. *Let $\mathcal{A}$ be a $\Bbbk$-algebra generated by $x_1, x_2, \ldots, x_n$ (we will call these generators variables). Then $\mathcal{Z}(\mathcal{A}) = \mathrm{Cen}(\{x_1, \ldots, x_n\}, \mathcal{A})$.*

Proof. Clearly $\mathcal{Z}(\mathcal{A}) \subset \mathrm{Cen}(\{x_1, \ldots, x_n\}, \mathcal{A})$.

To show the another inclusion we choose any $f \in \mathrm{Cen}(\{x_1, \ldots, x_n\}, \mathcal{A})$, that is, f commutes with all variables.

We can represent any element of $\mathcal{A}$ as a linear combination (over $\Bbbk$) of products of variables. It is clear that if f commutes with $a \in \mathcal{A}$ and $b \in \mathcal{A}$ then f commutes with $c_1 \cdot a + c_2 \cdot b$, $c_1, c_2 \in \Bbbk$.

Hence, it suffices to show that f commutes with any product. The proof is by induction on the length of product.

By the choice of f we know that f commutes with variables, this gives us the base of induction.

Now, assuming that f commutes with n and m, we can simply check that f commutes with $n * m$:

$$f * (n * m) = (f * n) * m = (n * f) * m = n * (f * m) = (n * m) * f$$

■

By theorem 2.4 and corollary 2.2 we can compute the center in an iterative way:

$$\mathcal{Z}(\mathcal{A}) = \mathrm{Cen}(x_1, \mathrm{Cen}(x_2, \ldots, \mathrm{Cen}(x_n, \mathcal{A}) \ldots)).$$

Therefore, in order to compute the center we should be able to compute $\mathrm{Cen}(f, S)$, where S is a subalgebra of $\mathcal{A}$. If S is an infinite dimensional as a vector space it cannot be done in general, but as soon as S is finite dimensional we can compute $\mathrm{Cen}(f, S)$ by solving certain linear algebra problem.

That is why we proceed by intersecting the algebra with finite dimensional vector subspaces and compute therefore only parts of a centralizer.

From now on we assume moreover that $\mathcal{A}$ is filtered with a filtration $\{\mathcal{A}_i\}$.

We denote the corresponding vector space filtration of the center by $\mathcal{Z}_i(\mathcal{A})$. That is:

$$\mathcal{Z}_i(\mathcal{A}) := \mathcal{Z}(\mathcal{A}) \cap \mathcal{A}_i.$$

Obviously:

$$\bigcup_{i=0}^{\infty} \mathcal{Z}_i(\mathcal{A}) = \mathcal{Z}(\mathcal{A}).$$

Due to lemma 2.4 and by the properties of centralizers we can compute $\mathcal{Z}_i(\mathcal{A})$ as follows:

$$\mathcal{Z}_i(\mathcal{A}) = \mathcal{Z}(\mathcal{A}) \cap \mathcal{A}_i = \mathrm{Cen}(\{x_1, \ldots, x_n\}, \mathcal{A}) \cap \mathcal{A}_i = \mathrm{Cen}(\{x_1, \ldots, x_n\}, \mathcal{A}_i). \tag{2.1}$$

This shows us that in order to compute $Z_d(\mathcal{A})$ we should be able to compute $\operatorname{Cen}(f, V)$, where V is a finite dimensional vector subspace of $\mathcal{A}$. This can be done by means of linear algebra due to the following proposition:

Proposition 2.5. *Let V be a finite dimensional vector subspace of $\mathcal{A}$, let us consider $\operatorname{Cen}(f, V) = V \cap \operatorname{Ker} \operatorname{ad}_f$ as a vector subspace of V. This is exactly the kernel of the linear map $\operatorname{ad}_f |_V : V \to \mathcal{A}$. Since V is finite dimensional, the image vector space $\operatorname{Im} \operatorname{ad}_f |_V$ is also finite dimensional. Therefore, we can compute $\operatorname{Cen}(f, V)$ for any finite dimensional vector space V as the kernel of the linear map $\operatorname{ad}_f |_V$ between finite dimensional vector spaces. In terms of linear algebra this means: compute matrix of this linear map and compute the base of its kernel.*

In the case when $\mathcal{A}$ is itself finite dimensional vector space we can compute $\operatorname{Cen}(\{x_1, \dots, x_n\}, \mathcal{A}) = \mathcal{Z}(\mathcal{A})$ directly. But for finite dimensional $\mathbb{k}$-algebras over finite fields our general approach could be not so efficient as the probabilistic approach discussed in [9]. In fact, they arrived at a similar system of equations and using the probabilistic approach they can almost avoid solving it.

2.2 Computation of center and centralizer

In this section we describe our algorithms for computing centralizers of sets of elements.

INPUT: Sets of vectors: $Basis = \{b_1, \dots, b_m\}; Images = \{w_1, \dots, w_m\}$, where vectors from $Basis$ are linearly independent.
OUTPUT: vector space basis of the kernel of a linear map given by $b_i \mapsto w_i$.
let Q be the matrix of the linear map given by $b_i \mapsto w_i$;
compute a vector space basis Ω of the kernel of Q;
RETURN: Ω;

Algorithm 2.1: LINEARMAPKERNEL(list $Basis$, list $Images$)

ASSUME: $\mathcal{A}$ is a $\mathbb{k}$-algebra
INPUT: $f \in \mathcal{A}$; a vector subspace V of $\mathcal{A}$, given by its basis $\{v_1, \dots, v_s\}$.
OUTPUT: vector space basis of $\operatorname{Cen}(f, V)$
let $Images = \{w_1, \dots, w_s\}$ be the set of vectors: $w_i = \operatorname{ad}_f v_i$;
RETURN: LINEARMAPKERNEL($\{v_1, \dots, v_s\}, \{w_1, \dots, w_s\}$); // using algorithm 2.1

Algorithm 2.2: CENTRALIZEPOLY(poly f, list V)

After fixing a filtration $\mathcal{A}_i \subset \mathcal{A}$ we can give algorithms for the computation of vector space bases of centralizers $\operatorname{Cen}(F, \mathcal{A}_d)$ (algorithm 2.4) and the center $\mathcal{Z}_d(\mathcal{A})$ (algorithm 2.5) for any non negative integer d.

ASSUME: $\mathcal{A}$ is a $\mathbb{k}$-algebra
INPUT: $F = \{f_1, \dots f_m\} \subset \mathcal{A}$ and vector subspace V of $\mathcal{A}$, given by its basis $\{v_1, \dots, v_s\}$.
OUTPUT: vector space basis of $\mathrm{Cen}(F, V)$
  let $W = \{v_1, \dots, v_s\}$;
  for $i = 1$ to $i = m$ **do**
    $W = $ CENTRALIZEPOLY(f_i, W); // using algorithm 2.2
  end for
RETURN: W

Algorithm 2.3: CENTRALIZESET(set F, list V)

ASSUME: $\mathcal{A}$ is a filtered $\mathbb{k}$-algebra with a filtration $\{\mathcal{A}_i\}$
INPUT: integer $d \geq 0$, non-empty finite set $F \subset \mathcal{A}$.
OUTPUT: vector space basis of $\mathrm{Cen}(F, \mathcal{A}_d)$
RETURN: CENTRALIZESET$(F, \mathcal{A}_d)$; // using algorithm 2.3

Algorithm 2.4: CENTRALIZERVS(set F, integer d)

ASSUME: $\mathcal{A}$ is a filtered $\mathbb{k}$-algebra generated by variables $x_1, \dots, x_n$, with a filtration $\{\mathcal{A}_i\}$
INPUT: integer $d \geq 0$
OUTPUT: vector space basis of $\mathcal{Z}_d(\mathcal{A}) = \mathcal{Z}(\mathcal{A}) \cap \mathcal{A}_d$
RETURN: CENTRALIZERVS$(\{x_1, \dots, x_n\}, \mathcal{A}_d)$; // using algorithm 2.4

Algorithm 2.5: CENTERVS(integer d)

Theorem 2.6. *Algorithms 2.2, 2.4, 2.3 and 2.5 terminate and are correct.*

Proof. Obviously these algorithms terminate. Algorithm 2.2 is correct because of remark 2.5. Other algorithms are correct because of the properties of centralizers and the center considered in the previous section, namely:

- algorithm 2.3 is correct because of corollary 2.2.
- the correctness of algorithm 2.4 follows from the correctness of algorithm 2.3.
- finally, algorithm 2.5 is correct due to lemma 2.4 as formula (2.1) shows.

■

```
ASSUME: A is a filtered k-algebra with a filtration {A_i}
OUTPUT: vector space basis of Z(A)
  if ∃d : A_d = A then
    let Z = CENTERVS(d); //  using algorithm 2.5
  else
    for  i = 0, 1, ..., ∞  do
      let Z = CENTERVS(i); //  using algorithm 2.5
    end for
  end if
RETURN: Z;
```

Algorithm 2.6: CENTER()

Using algorithm 2.5 we can compute the whole center $\mathcal{Z}(\mathcal{A})$ as in algorithm 2.6. Algorithm 2.6 is correct and terminates whenever the filtration $\{\mathcal{A}_i\}$ of $\mathcal{A}$ is finite.

2.3 Implementation for algebras of PBW type

In order to be able to compute efficiently we require $\mathcal{A}$ to be of PBW type. This means that we may use (a subset of) the set of monomials $Mon(x_1, \ldots, x_n)$ as a vector space basis of $\mathcal{A}$.

Similar approach was also described in [24].

We fix the standard vector space filtration of $\mathcal{A}$ by degree, that is, we define $\mathcal{A}_d$ to be a vector space of polynomials of degree less or equal to d with the base M_d consisting of monomials of degree less or equal to d, specifically: $\mathcal{A}_d = \{a \in \mathcal{A} : deg(a) \leq d\}$ and $M_d = \{m \in Mon(x_1, \ldots, x_n) : deg(m) \leq d\}$.

Let V be a vector subspace of $\mathcal{A}_d$ with the basis vectors $\{v_1, \ldots, v_s\}$. Let $f \in \mathcal{A}$. Let us compute the basis of the vector subspace $\text{Cen}(f, V)$ of V:

Firstly we consider $W := \operatorname{Im} \operatorname{ad}_f |_V$ as a vector subspace of $\mathcal{A}$. Let $m_1, \ldots, m_p$ be the monomials occurring in the image polynomials $\{\operatorname{ad}_f v_i\}_{1 \leqslant i \leqslant s}$. Clearly they are linearly independent and span a vector space containing W.

Next we compute the matrix D of the linear map $\operatorname{ad}_f |_V \colon \langle v_1, \ldots, v_s \rangle_{\Bbbk} \to \langle m_1, \ldots, m_p \rangle_{\Bbbk}$ by decomposing $\{\operatorname{ad}_f v_i\}, 1 \leqslant i \leqslant s$ into linear combinations of m_j: $\operatorname{ad}_f v_i = \sum_{j=1}^{p} a_{j,i} \cdot m_j$, where the coefficients $a_{j,i} \in \Bbbk$ form the matrix D.

Let $\{\omega_1, \ldots, \omega_k\} \subset \Bbbk^s$ be a basis of the solution system of homogeneous equations with the matrix D. Then the following vectors form a basis of $\operatorname{Cen}(f, V)$: $\{\sum_{i=1}^{s} (\omega_j[i] \cdot v_i)\}_{1 \leq j \leq k}$, where $\omega_j[i]$ denotes the i^{th} coordinate of the vector $\omega_j \in \Bbbk^s$.

This is the description of our implementation of LINEARMAPKERNELalgorithm (cf. algorithm 2.7) for algebras of PBW type.

ASSUME: $\mathcal{A}$ is a filtered finitely generated $\Bbbk$-algebra of PBW type.
INPUT: Sets of vectors: $Basis = \{b_1, \ldots, b_k\}, Images = \{w_1, \ldots, w_k\}$, where vectors from $Basis$ are linearly independent.
OUTPUT: vector space basis of the kernel of a linear map given by $b_i \mapsto w_i$.
find all monomials $\{m_1, \ldots, m_p\}$ occurring in the polynomials w_i;
let Q be a module generated by $\sum_{j=1}^{k} Coeff(w_i, m_j) \cdot e_j, 1 \leqslant i \leqslant k$;
compute the syzygy module of Q and let $\Omega \subset \Bbbk^m$ be its basis;
RETURN: $\{\sum_{i=1}^{m} \omega_i \cdot b_i\}_{\omega \in \Omega}$;

Algorithm 2.7: LINEARMAPKERNEL(list $Basis$, list $Images$)

In algorithm 2.7 we use the following notation: $\operatorname{Coeff}(w, m)$ denotes the coefficient of the monomial m in the polynomial w and e_i denotes the i^{th} generators of the free module of rank p.

2.4 Center of a factor algebra

Let $\mathcal{A}$ be a G-Algebra and I be a two-sided ideal in $\mathcal{A}$. In this section we consider a factor algebra $\mathcal{A}/I$, which is a GR-algebra (see [21] and [26]). The SINGULAR can deal with factor algebras in the following way: polynomial data are stored internally in the same manner(as in the case of $\mathcal{A}$), the only difference is that this polynomial representation is in general not unique, therefore when we need a normal form of a polynomial p in a factor algebra we compute it by the command `NF(p, std(0))`.

Therefore we need to modify our algorithms to work with "factors": we add normal form computation after polynomial multiplications and change PBW basis computation (we throw away from PBW basis all monomials which can be reduced w.r.t the ideal I).

Having implemented all these modifications in our library we are able to compute correct results within factors.

In particular, this was done in order to check the following conjecture:

Conjecture 2.7. Let $\mathcal{A}$ be a G-Algebra and $I \subset \mathcal{A}$ be a two-sided ideal in $\mathcal{A}$, then:

$$\mathcal{Z}(\mathcal{A}/I) = \mathcal{Z}(\mathcal{A})/(I \cap \mathcal{Z}(\mathcal{A})) \tag{2.2}$$

The motivation for this conjecture is the following proposition (cf. [6, Prop. 4.2.5, p. 134]):

Proposition 2.8. *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic 0, I a two-sided ideal of $\mathcal{U}(\mathfrak{g})$ and φ the canonical mapping of $\mathcal{U}(\mathfrak{g})$ onto $\mathcal{U}(\mathfrak{g})/I$. Then $\varphi(\mathcal{Z}(\mathcal{U}(\mathfrak{g})))$ is the center of $\mathcal{U}(\mathfrak{g})/I$.*

Example [26, p. 110] shows that conjecture 2.7 fails to be true for the first Heisenberg algebra over a field $\Bbbk$ of characteristic 0 (cf. section 5.5):

$$\mathcal{H}_1 = \Bbbk \langle x, y, h \mid [x, y] = h, [h, x] = [h, y] = 0 \rangle .$$

It is easy to see that its center is $\Bbbk[h]$. Let us consider the two-sided ideal I generated by h, then $I \cap \mathcal{Z}(\mathcal{H}_1) = \langle h \rangle_{\Bbbk[h]}$ and $\mathcal{Z}(\mathcal{H}_1)/(I \cap \mathcal{Z}(\mathcal{H}_1)) = \Bbbk$. On the other hand, $\mathcal{H}_1/I \cong \Bbbk[x, y]$, hence:

$$\mathcal{Z}(\mathcal{H}_1/I) = \mathcal{H}_1/I = \Bbbk[x, y] \ncong \Bbbk = \Bbbk[h]/ \langle h \rangle_{\Bbbk[h]} = \mathcal{Z}(\mathcal{H}_1)/(I \cap \mathcal{Z}(\mathcal{H}_1))$$

It would be interesting to know, under which conditions on $\mathcal{A}$ formula (2.2) holds true.

Chapter 3

Examples of computation

In this chapter we assume the ground filed $\Bbbk$ to be of characteristic 0.

In the following examples we consider enveloping algebras of certain Lie algebras. These non-commutative $\Bbbk$-algebras are supported by Singular. All computations were done with the help of Singular and our library.

3.1 Enveloping algebra of sl_2

We have already seen that $\mathcal{U}(\mathfrak{sl}_2)$ is given by

$$\Bbbk \langle e, f, h | f * e = ef - h, h * e = eh + 2e, h * f = fh - 2f \rangle .$$

Let us compute vector space basis of $\mathcal{Z}_2(\mathcal{A})$ using the iterative approach, that is, by the following formula $\mathcal{Z}_2(\mathcal{A}) = \mathrm{Cen}(h, \mathrm{Cen}(f, \mathrm{Cen}(e, \mathcal{A}_2)))$.

We choose the PBW basis M_2 of $\mathcal{A}_2$, specifically: $M_2 = \{e^2, ef, eh, f^2, fh, h^2, e, f, h, 1\}$

To compute $\mathrm{Cen}(e, \mathcal{A}_2)$ we need images (under ad_e) of basis vectors of $\mathcal{A}_2$, that is, we compute $\mathrm{ad}_e(v), v \in M_2$:

$$\begin{cases} \operatorname{ad}_e(e^2) & = 0, \\ \operatorname{ad}_e(ef) & = eh, \\ \operatorname{ad}_e(eh) & = -2e^2, \\ \operatorname{ad}_e(f^2) & = 2fh - 2f, \\ \operatorname{ad}_e(fh) & = -2ef + h^2 + 2h, \\ \operatorname{ad}_e(h^2) & = -4eh - 4e, \\ \operatorname{ad}_e(e) & = 0, \\ \operatorname{ad}_e(f) & = h, \\ \operatorname{ad}_e(h) & = -2e, \\ \operatorname{ad}_e(1) & = 0. \end{cases}$$

Next we consider the vector space $\operatorname{Im}\operatorname{ad}_e|_{\mathcal{A}_2}$ as a subspace of $\mathcal{A}_2$ and compute the matrix (3.1) of the map $\operatorname{ad}_e|_{\mathcal{A}_2}$ by decomposing elements $\operatorname{ad}_e(v), v \in M_2$ into columns of coefficients in front of the corresponding basis monomials:

$$\left(\begin{array}{c|cccccccccc} e^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ ef & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ eh & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 1 & 0 \\ f^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ fh & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ h^2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ e & 0 & -2 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ f & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ h & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & 1 & h & f & e & h^2 & fh & f^2 & eh & ef & e^2 \end{array}\right) \tag{3.1}$$

To find the kernel of the operator ad_e we compute the Hermite form of matrix (3.1):

$$\left(\begin{array}{cccccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & h & f & e & h^2 & fh & f^2 & eh & ef & e^2
\end{array}\right)$$

Therefore the kernel of matrix (3.1) has the following basis:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{4} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Thus, $\left\{1, e, -\frac{1}{2}h + \frac{1}{4}h^2 + ef, e^2\right\}$ is a vector space basis of $\mathrm{Cen}(e, \mathcal{A}_2)$.

Next we compute $\mathrm{Cen}(f, \mathrm{Cen}(e, \mathcal{A}_2))$. We proceed, as before, by computing images of basis vectors:

$$\begin{cases}
\mathrm{ad}_f(1) & = 0, \\
\mathrm{ad}_f(e) & = -h, \\
\mathrm{ad}_f(-\frac{1}{2}h + \frac{1}{4}h^2 + ef) & = 0, \\
\mathrm{ad}_f(e^2) & = -2eh - 2e.
\end{cases}$$

We choose monomials eh, e, h to be the basis of this image space. Then the operator $\mathrm{ad}_f\,|_{\mathrm{Cen}(e,\mathcal{A}_2)}\colon \mathrm{Cen}(e,\mathcal{A}_2) \to \langle eh, e, h\rangle_{\Bbbk}$ has the following matrix:

$$\left(\begin{array}{c|cccc} eh & 0 & 0 & 0 & -2 \\ e & 0 & 0 & 0 & -2 \\ h & 0 & -1 & 0 & 0 \\ \hline & 1 & e & -\frac{1}{2}h+\frac{1}{4}h^2+ef & e^2 \end{array}\right) \tag{3.2}$$

The Hermite form of matrix (3.2) is:

$$\left(\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 1 & e & -\frac{1}{2}h+\frac{1}{4}h^2+ef & e^2 \end{array}\right)$$

Hence, the kernel of matrix (3.2) has the following basis:

$$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$

Thus, $\left\{1, -\frac{1}{2}h+\frac{1}{4}h^2+ef\right\}$ is a basis of $\mathrm{Cen}(f, \mathrm{Cen}(e,\mathcal{A}_2))$.
Finally:

$$\begin{cases} \mathrm{ad}_h(1) & = 0, \\ \mathrm{ad}_h(-\frac{1}{2}h+\frac{1}{4}h^2+ef) & = 0. \end{cases}$$

Result.

$$\mathcal{Z}_2(\mathcal{U}(\mathfrak{sl}_2)) = \mathrm{Cen}(h, \mathrm{Cen}(f, \mathrm{Cen}(e,\mathcal{A}_2))) = \left\langle 1, -\frac{1}{2}h+\frac{1}{4}h^2+ef\right\rangle_{\Bbbk}$$

Remark 3.1. From general theory (cf. chapter 5) it is known that over a field of characteristic 0 the center of $\mathcal{U}(\mathfrak{sl}_2)$ is generated by a single element of degree 2. Thus we have computed already essentially the whole center of this algebra. Computing $\mathcal{Z}_d(\mathcal{U}(\mathfrak{sl}_2))$ for $d > 2$ over this field we can only find linear combinations of powers of already found central element of degree 2.

With the use of Singular and our library the center of $\mathcal{U}(\mathfrak{sl}_2)$ can be computed as follows:

```
> // definitions of some non-commutative algebra
> LIB "ncalg.lib"; // for makeUsl(l, p)
>
> // our library
> LIB "center.lib";
>
> int d = 2; // Upper degree bound of the center polynomials
> int p = 0; // The characteristic of the ground field
>
> // Let us set U(sl2(Q)) be our current algebra
> def A = makeUsl(2, p); // U(sl2) over Fp or Q if p==0
>
> setring(A); A;
//   characteristic : 0
//   number of vars : 3
//        block   1 : ordering dp
//                  : names    e f h
//        block   2 : ordering C
//   noncommutative relations:
//    fe=ef-h
//    he=eh+2e
//    hf=fh-2f
>
> // basically Z(A_d) can be computed as follows:
>
> // Set of all variables of the current algebra
> ideal X = variablesSorted(); X;
X[1]=h
X[2]=f
X[3]=e
> // Compute the PBW Basis of A_d:
> ideal V = PBW_maxDeg(d); V;
V[1]=e
V[2]=f
V[3]=h
V[4]=h2
V[5]=fh
V[6]=f2
V[7]=eh
V[8]=ef
V[9]=e2
>  // Apply the CENTRALIZE_SET algorithm:
> ideal Z1 = centralizeSet(X, V);
> Z1;
Z1[1]=ef+1/4h2-1/2h
> // check whether elements of Z1:
> inCenter(Z1); // 1 if they are in the center:
1
>
> // One can also use a shortcut function (it does the same thing):
```

```
>  ideal Z2 = centerVS(d); Z2;
Z2[1]=ef+1/4h2-1/2h
>  inCenter(Z2);
1
```

Conclusions 3.2.

0) Since the ground field $\mathbb{k}$ belongs to the center of any $\mathbb{k}$-algebra we will omit the unit generator in PBW bases and results.

1) Since all matrices are sparse, we can trivially optimize some cases:

   1.1) We can exclude columns of zeroes, that is, we take the corresponding basis vector into the output.

   1.2) In the case when there is only one non-zero coefficient in a row we throw away a basis vector corresponding to that coefficient.

2) We can decompose systems (matrices) into the smaller ones, which can be processed independently.

3.2 Enveloping algebra of sl_3

Due to sections 1.3.5 and 1.4, the enveloping algebra $\mathcal{U}(\mathfrak{gl}_n)$ is given by:

$$\mathbb{k}\left\langle e_{ij},\ 1 \leqslant i,j \leqslant n \mid e_{ij} * e_{kl} = e_{kl}e_{ij} + \delta_{jk}e_{il} - \delta_{li}e_{kj} \right\rangle .$$

A vector space basis of $\mathcal{Z}_2(\mathcal{U}(\mathfrak{gl}_3))$ can be computed as follows:

$$\mathrm{Cen}(e_{33}, \mathrm{Cen}(e_{32}, \mathrm{Cen}(e_{31}, \mathrm{Cen}(e_{23}, \mathrm{Cen}(e_{22}, \mathrm{Cen}(e_{21}, \mathrm{Cen}(e_{13}, \mathrm{Cen}(e_{12}, \mathrm{Cen}(e_{11}, \mathcal{A}_2))))))))).$$

In what follows we compute "by hands" only $\mathrm{Cen}(e_{11}, \mathcal{A}_2)$.

As before we need PBW base of $\mathcal{A}_2$, but this time we do not include 1 into it (see conclusions 3.2). Therefore M_2 consists of the following monomials: $e_{33}, e_{32}, e_{31}, e_{23}, e_{22}, e_{21}, e_{13}, e_{12}, e_{11}, e_{33}^2, e_{32}e_{33}, e_{32}^2, e_{31}e_{33}, e_{31}e_{32}, e_{31}^2, e_{23}e_{33}, e_{23}e_{32}, e_{23}e_{31}, e_{23}^2, e_{22}e_{33}, e_{22}e_{32}, e_{22}e_{31}, e_{22}e_{23}, e_{22}^2, e_{21}e_{33}, e_{21}e_{32}, e_{21}e_{31}, e_{21}e_{23}, e_{21}e_{22}, e_{21}^2, e_{13}e_{33}, e_{13}e_{32}, e_{13}e_{31}, e_{13}e_{23}, e_{13}e_{22}, e_{13}e_{21}, e_{13}^2, e_{12}e_{33}, e_{12}e_{32}, e_{12}e_{31}, e_{12}e_{23}, e_{12}e_{22}, e_{12}e_{21}, e_{12}e_{13}, e_{12}^2, e_{11}e_{33}, e_{11}e_{32}, e_{11}e_{31}, e_{11}e_{23}, e_{11}e_{22}, e_{11}e_{21}, e_{11}e_{13}, e_{11}e_{12}, e_{11}^2$.

First, we compute the images $\{\mathrm{ad}_{e_{11}}(v) : v \in M_2\}$ as follows:

$$\begin{cases} \mathrm{ad}_{e_{11}}(e_{33}) &= 0, \\ \mathrm{ad}_{e_{11}}(e_{32}) &= 0, \end{cases} \qquad \begin{cases} \mathrm{ad}_{e_{11}}(e_{31}) &= -e_{31}, \\ \mathrm{ad}_{e_{11}}(e_{23}) &= 0, \end{cases}$$

$$\begin{cases}
\mathrm{ad}_{e_{11}}(e_{22}) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{21}) &= -e_{21},\\
\mathrm{ad}_{e_{11}}(e_{13}) &= e_{13},\\
\mathrm{ad}_{e_{11}}(e_{12}) &= e_{12},\\
\mathrm{ad}_{e_{11}}(e_{11}) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{33}^2) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{32}e_{33}) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{32}^2) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{31}e_{33}) &= -e_{31}e_{33},\\
\mathrm{ad}_{e_{11}}(e_{31}e_{32}) &= -e_{31}e_{32},\\
\mathrm{ad}_{e_{11}}(e_{31}^2) &= -2e_{31}^2,\\
\mathrm{ad}_{e_{11}}(e_{23}e_{33}) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{23}e_{32}) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{23}e_{31}) &= -e_{23}e_{31},\\
\mathrm{ad}_{e_{11}}(e_{23}^2) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{22}e_{33}) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{22}e_{32}) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{22}e_{31}) &= -e_{22}e_{31},\\
\mathrm{ad}_{e_{11}}(e_{22}e_{23}) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{22}^2) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{21}e_{33}) &= -e_{21}e_{33},\\
\mathrm{ad}_{e_{11}}(e_{21}e_{32}) &= -e_{21}e_{32},\\
\mathrm{ad}_{e_{11}}(e_{21}e_{31}) &= -2e_{21}e_{31},\\
\mathrm{ad}_{e_{11}}(e_{21}e_{23}) &= -e_{21}e_{23},\\
\mathrm{ad}_{e_{11}}(e_{21}e_{22}) &= -e_{21}e_{22},
\end{cases}
\qquad
\begin{cases}
\mathrm{ad}_{e_{11}}(e_{21}^2) &= -2e_{21}^2,\\
\mathrm{ad}_{e_{11}}(e_{13}e_{33}) &= e_{13}e_{33},\\
\mathrm{ad}_{e_{11}}(e_{13}e_{32}) &= e_{13}e_{32},\\
\mathrm{ad}_{e_{11}}(e_{13}e_{31}) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{13}e_{23}) &= e_{13}e_{23},\\
\mathrm{ad}_{e_{11}}(e_{13}e_{22}) &= e_{13}e_{22},\\
\mathrm{ad}_{e_{11}}(e_{13}e_{21}) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{13}^2) &= 2e_{13}^2,\\
\mathrm{ad}_{e_{11}}(e_{12}e_{33}) &= e_{12}e_{33},\\
\mathrm{ad}_{e_{11}}(e_{12}e_{32}) &= e_{12}e_{32},\\
\mathrm{ad}_{e_{11}}(e_{12}e_{31}) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{12}e_{23}) &= e_{12}e_{23},\\
\mathrm{ad}_{e_{11}}(e_{12}e_{22}) &= e_{12}e_{22},\\
\mathrm{ad}_{e_{11}}(e_{12}e_{21}) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{12}e_{13}) &= 2e_{12}e_{13},\\
\mathrm{ad}_{e_{11}}(e_{12}^2) &= 2e_{12}^2,\\
\mathrm{ad}_{e_{11}}(e_{11}e_{33}) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{11}e_{32}) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{11}e_{31}) &= -e_{11}e_{31},\\
\mathrm{ad}_{e_{11}}(e_{11}e_{23}) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{11}e_{22}) &= 0,\\
\mathrm{ad}_{e_{11}}(e_{11}e_{21}) &= -e_{11}e_{21},\\
\mathrm{ad}_{e_{11}}(e_{11}e_{13}) &= e_{11}e_{13},\\
\mathrm{ad}_{e_{11}}(e_{11}e_{12}) &= e_{11}e_{12},\\
\mathrm{ad}_{e_{11}}(e_{11}^2) &= 0.
\end{cases}$$

Remark 3.3. Observer that every monomial appears in the previous images at most once. Namely, $\mathrm{ad}_{e_{11}} m = \alpha_m \cdot m, \forall m \in M_2$ for some $\alpha_m \in \Bbbk$. This implies that if $m \in M_2$ commutes with e_{11} (that is, $\alpha = 0$), then m gives rise to a column of zeroes in the matrix of the map $\mathrm{ad}_{e_{11}}$ and because of conclusions 3.2 we can take m directly into the resulting vector space basis, otherwise if the image of m is non-zero (that is, $\alpha \neq 0$), it gives rise to a row with a single non-zero element in the matrix of the map $\mathrm{ad}_{e_{11}}$ and because of conclusions 3.2 we can simply throw it away. This observation gives rise to the definition of Cartan elements (definition 5.5) and further optimization (cf. procedure VARIABLESSORTED in `center.lib`).

Thus, monomials from M_2 commuting with e_{11} constitute a basis of $\mathrm{Cen}(e_{11}, \mathcal{A}_2)$:
$\mathrm{Cen}(e_{11}, \mathcal{A}_2) = \langle e_{33}, e_{32}, e_{23}, e_{22}, e_{11}, e_{33}^2, e_{32}e_{33}, e_{32}^2, e_{23}e_{33}, e_{23}e_{32}, e_{23}^2, e_{22}e_{33}, e_{22}e_{32}, e_{22}e_{23}, e_{22}^2, e_{13}e_{31}, e_{13}e_{21}, e_{12}e_{31}, e_{12}e_{21}, e_{11}e_{33}, e_{11}e_{32}, e_{11}e_{23}, e_{11}e_{22}, e_{11}^2\rangle_{\Bbbk}$.
We proceed further with SINGULAR and our library:

```
>  // definitions of some non-commutative algebra
>  LIB "ncalg.lib"; // for makeUgl(n, p)
>
>  // our library
>  LIB "center.lib";
>
>  int d = 2; // Upper degree bound of the center polynomials
>  int p = 0; // The characteristic of the ground field
>
>  // Let us set U(gl_2(Q)) to be our current algebra:
>  def A = makeUgl(2, p); // U(gl_2) over F_p or Q if p==0
>
>  setring(A); A;
//   characteristic : 0
//   number of vars : 9
//        block   1 : ordering dp
//                  : names    e_1_1 e_1_2 e_1_3 e_2_1 e_2_2
e_2_3 e_3_1 e_3_2 e_3_3
//        block   2 : ordering C
//   noncommutative relations:
//    e_1_2e_1_1=e_1_1*e_1_2-e_1_2
//    e_1_3e_1_1=e_1_1*e_1_3-e_1_3
//    e_2_1e_1_1=e_1_1*e_2_1+e_2_1
//    e_3_1e_1_1=e_1_1*e_3_1+e_3_1
//    e_2_1e_1_2=e_1_2*e_2_1-e_1_1+e_2_2
//    e_2_2e_1_2=e_1_2*e_2_2-e_1_2
//    e_2_3e_1_2=e_1_2*e_2_3-e_1_3
//    e_3_1e_1_2=e_1_2*e_3_1+e_3_2
//    e_2_1e_1_3=e_1_3*e_2_1+e_2_3
//    e_3_1e_1_3=e_1_3*e_3_1-e_1_1+e_3_3
//    e_3_2e_1_3=e_1_3*e_3_2-e_1_2
//    e_3_3e_1_3=e_1_3*e_3_3-e_1_3
//    e_2_2e_2_1=e_2_1*e_2_2+e_2_1
//    e_3_2e_2_1=e_2_1*e_3_2+e_3_1
//    e_2_3e_2_2=e_2_2*e_2_3-e_2_3
//    e_3_2e_2_2=e_2_2*e_3_2+e_3_2
//    e_3_1e_2_3=e_2_3*e_3_1-e_2_1
//    e_3_2e_2_3=e_2_3*e_3_2-e_2_2+e_3_3
//    e_3_3e_2_3=e_2_3*e_3_3-e_2_3
//    e_3_3e_3_1=e_3_1*e_3_3+e_3_1
//    e_3_3e_3_2=e_3_2*e_3_3+e_3_2
>
>  // basically Z(A_d) can be computed as follows:
>  // Set of all variables of the current algebra
>  ideal X = e_1_1; X;
```

```
X[1]=e_1_1
>  // Compute the PBW Basis of A_d
>  ideal V = PBW_maxDeg(d); V;
... (an ideal of all 54 PBW monomials)
>   // Apply the centralizeSet algorithm:
>  ideal Y = centralizeSet(X, V); Y; // note that we got the same result
Y[1]=e_1_1
Y[2]=e_2_2
Y[3]=e_2_3
Y[4]=e_3_2
Y[5]=e_3_3
Y[6]=e_3_3^2
Y[7]=e_3_2*e_3_3
Y[8]=e_3_2^2
Y[9]=e_2_3*e_3_3
Y[10]=e_2_3*e_3_2
Y[11]=e_2_3^2
Y[12]=e_2_2*e_3_3
Y[13]=e_2_2*e_3_2
Y[14]=e_2_2*e_2_3
Y[15]=e_2_2^2
Y[16]=e_1_3*e_3_1
Y[17]=e_1_3*e_2_1
Y[18]=e_1_2*e_3_1
Y[19]=e_1_2*e_2_1
Y[20]=e_1_1*e_3_3
Y[21]=e_1_1*e_3_2
Y[22]=e_1_1*e_2_3
Y[23]=e_1_1*e_2_2
Y[24]=e_1_1^2
>  // check whether elements of Y are in the centralizer of X
>  inCentralizer( Y, X );
1 // Yes! They are! We were right!
>
>  // Let's compute the basis of Z(U(gl_3)) ∩ A_2:
>  ideal Z = centerVS(d); Z;
Z[1]=e_1_1+e_2_2+e_3_3
Z[2]=-e_1_2*e_2_1+e_1_1*e_2_2-e_1_3*e_3_1-e_2_3*e_3_2+e_1_1*e_3_3+
e_2_2*e_3_3+2*e_1_1+e_2_2
Z[3]=e_1_1^2+2*e_1_2*e_2_1+e_2_2^2+2*e_1_3*e_3_1+2*e_2_3*e_3_2+e_3_3^2
-4*e_1_1-2*e_2_2
>  inCenter(Z);
1
```

Result. From the previous computations we conclude:

$\mathcal{Z}_2(\mathcal{U}(\mathfrak{gl}_3)) = \langle 1, e_{11} + e_{22} + e_{33}, -e_{12}e_{21} + e_{11}e_{22} - e_{13}e_{31} - e_{23}e_{32} + e_{11}e_{33} + e_{22}e_{33} + 2e_{11} + e_{22}, e_{11}^2 + 2e_{12}e_{21} + e_{22}^2 + 2e_{13}e_{31} + 2e_{23}e_{32} + e_{33}^2 - 4e_{11} - 2e_{22}\rangle_{\Bbbk}$.

Chapter 4

Subalgebra reduction

In this chapter we consider a GR-Algebra $\mathcal{A}$ and discuss possible approaches to the computation of reduced subalgebra bases of vector space bases of $\mathcal{Z}_d(\mathcal{A})$ and $\mathrm{Cen}(F, \mathcal{A}_d)$.

We fix the standard filtration of $\mathcal{A}$ by degree $\{\mathcal{A}_i\}$.

4.1 Introduction

We already know how to compute a vector space basis of $\mathrm{Cen}(F, \mathcal{A}_d)$ but the whole centralizer itself is a subalgebra of $\mathcal{A}$, this means, in particular, that besides computing reduced subalgebra generators of it, we also compute linear combinations of their products. For example the vector space basis of $\mathcal{Z}_2(\mathcal{U}(\mathfrak{gl}_3))$ computed in section 3.1 contains the elements $e_{11}^2 + 2e_{12}e_{21} + e_{22}^2 + 2e_{13}e_{31} + 2e_{23}e_{32} + e_{33}^2 - 4e_{11} - 2e_{22}$ which is equal to $(e_{11} + e_{22} + e_{33})^2 + 2(e_{11} + e_{22} + e_{33})$.

In general, it is not so easy to compute a reduced subalgebra base for an algebra generated by an arbitrary finite set of elements. Computation of a reduced subalgebra basis is a hard computational problem even in a commutative case. Resulting bases may sometimes fail to be finite.

For the general method for the computation of canonical subalgebra bases in the noncommutative case one may see [24, sec. 5.8]. This subject is far aside from our work, and we are not going to describe it in details here. Instead, we propose two approaches for the computation of reduced subalgebra bases for the center and centralizers which avoid canonical subalgebra reduction.

4.2 Iterative approach

Recall that a set of polynomials P is **autoreduced** if all $f \in P$ are subalgebra irreducible with respect to $P \setminus \{f\}$.

We restrict ourselves to the computation of an autoreduced subalgebra base of $\mathrm{Cen}(F, \mathcal{A}_d)$.

We can omit the computation of canonical subalgebra bases since instead of performing general subalgebra reduction of a vector space basis of $\mathcal{Z}_d(\mathcal{A})$ or $\mathrm{Cen}(F, \mathcal{A}_d)$ we can proceed by consequently computing $\mathcal{Z}_i(\mathcal{A})$ (resp., $\mathrm{Cen}(F, \mathcal{A}_i)$), for $1 \leqslant i \leqslant d$ simultaneously removing from the PBW base (for the next step) products of leading monomials of already computed elements (cf. algorithms 4.1 and 4.2).

ASSUME: $\mathcal{A}$ is a GR-Algebra with the filtration $\{\mathcal{A}_i\}$ by degree. The ordering on $\mathcal{A}$ is degree-compatible.
INPUT: integer $d \geq 0$, non-empty finite set F.
OUTPUT: autoreduced subalgebra basis of $\mathrm{Cen}(F, \mathcal{A}_d)$
  let $M = \emptyset$;
  for $i = 1$ to $i = d$ **do**
    let V be a PBW basis of $\mathcal{A}_i$ without leading monomials of products of elements from M;
    let $S = \textsc{centralizeSet}(F, V)$; // using algorithm 2.3
    let $S = \textsc{interred}(S)$; // using commutative interreduction
    let $M = M \cup S$;
  end for
RETURN: M;

Algorithm 4.1: $\textsc{centralizerRed}$(set F, integer d)

Note that since the ordering on $\mathcal{A}$ is degree compatible the leading monomials of elements computed on the step i are always of degree i and these elements are linearly independent. In the interreduction we cannot obtain polynomials of lower degree (since the ordering is degree compatible it would mean that we have found an element of lower degree having unknown previously leading monomial, which is impossible). Thus there will be performed no polynomial multiplication during the interreduction. Therefore we can use the standard commutative interreduction.

After the interreduction we get another basis of the same vector space with the property that every leading monomial occurs in a single element.

Obviously this interreduction terminates. Thus the algorithm 4.1 terminates.

The result of this algorithm is a set of elements from $\mathrm{Cen}(F, \mathcal{A}_d)$ such that every leading monomial occurs in a single element and no element contains products of leading monomials of other elements, that is, this set is autoreduced. Moreover, it generates $\mathrm{Cen}(F, \mathcal{A}_d)$ by construction.

Thus the algorithm 4.1 is correct. Hence, we obtain the following proposition:

Proposition 4.1. *Algorithm 4.1 terminates and is correct.*

Now it is easy to give an algorithm 4.2 for the computation of subalgebra basis of the center of an algebra up to a given degree.

ASSUME: current algebra $\mathcal{A}$ is a filtered$\Bbbk$-algebra generated by variables: $x_1, \ldots, x_n$ with a filtration $\{\mathcal{A}_i\}$
INPUT: integer $d \geq 0$
OUTPUT: subalgebra basis of $\mathcal{Z}_d(\mathcal{A}) = \mathcal{Z}(\mathcal{A}) \cap \mathcal{A}_d$
RETURN: CENTRALIZERRED$(\{x_1, \ldots, x_n\}, \mathcal{A}_d)$; // using algorithm 4.1

Algorithm 4.2: CENTERRED(integer d)

Remark 4.2. The only difference between the computation of canonical subalgebra base of $\mathcal{Z}_d(\mathcal{A})$ and $\mathrm{Cen}(F, \mathcal{A}_d)$ is the fact that in the first case we can, a priori, use standard commutative canonical subalgebra base computation, while it is clearly not always possible in the second case.

4.3 Alternative approach

Let $F := \{f_1, \ldots, f_r\} \subset \mathcal{A}$ be an autoreduced subalgebra basis of a subalgebra $\mathcal{B}$ of $\mathcal{A}$ (that is, F generates $\mathcal{B}$ and is autoreduced). Then a vector space basis of $\mathcal{B}_d := \mathcal{B} \cap \mathcal{A}_d$ consists of linear combinations of the products of elements from $F \cap \mathcal{A}_d$. Let $G = \{g_1, \ldots, g_k\}$ be a vector space basis of $\mathcal{B}_d$, then we can express all $f_j \in F \cap \mathcal{A}_d$ as linear combinations of g_i over $\Bbbk$!

Furthermore, after reducing common leading monomials (e.g., using the interreduction algorithm 4.3) we obtain a basis G' of $\mathcal{B}_d$ such that for all different $g_i, g_j \in G'$ holds: $\mathrm{lm}(g_i) \neq \mathrm{lm}(g_j)$. Clearly, leading monomial of any linear combination of elements from G' belongs to $\mathrm{L}(G') := \{\mathrm{lm}(g) \mid g \in G'\}$. Thus $\mathrm{L}(F \cap \mathcal{A}_d) \subset \mathrm{L}(G)$. Clearly, all monomials $\mathrm{L}(G) \setminus \mathrm{L}(F \cap \mathcal{A}_d)$ are leading monomials of products of elements from $\mathrm{L}(F \cap \mathcal{A}_d)$. This is exactly the idea behind our "subalgebra reduction" algorithm 4.4.

INPUT: finite set $S = \{f_1, \ldots, f_k\}$ of linearly independent polynomials.
OUTPUT: interreduced set of polynomials $g_1, \ldots, g_k$ with $\langle f_1, \ldots, f_k \rangle_\Bbbk = \langle g_1, \ldots, g_k \rangle_\Bbbk$

```
let M = S;
if M ≠ ∅ then
  while ∃p, q ∈ M : lm(p) = lm(q) do
    let M = M \ {p};
    let p = p − (lc(p)/ lc(q)) ∗ q; // reduce common leading monomial
    let M = M ∪ {p};
  end while
end if
```

RETURN: M;

Algorithm 4.3: INTERRED(set S)

Based on subalgebra reduction algorithm 4.4 we can also perform the subalgebra reduction of a polynomial $p \in \mathcal{A}$ with respect to a subalgebra generated by the vector space $\mathcal{B}_d$.

INPUT: vector space basis $G = \{g_1, \ldots, g_k\}$ of some $\mathcal{B}_d$
OUTPUT: $F^{\leqslant d}$
let $M = G$;
let $F = \emptyset$;
while $M \neq \emptyset$ **do**
  for all $m \in M$ **do**
    reduce from m all monomial which are leading monomials of products of elements from F by subtracting a corresponding product;
  end for
  let $M = \text{INTERRED}(M)$; // using algorithm 4.3
  choose a minimal element g from M;
  let $M = M \setminus \{g\}$;
  let $F = F \cup g$;
end while
RETURN: F;

Algorithm 4.4: SA_REDUCE(set G)

We reduce those monomials from p which are leading monomials of products of elements from a reduced subalgebra base of $\mathcal{B}_d$ (e.g., computed by algorithm 4.4) by subtracting corresponding products. This gives us algorithm 4.5.

INPUT: polynomial p and vector space basis $G = \{g_1, \ldots, g_k\}$ of $\mathcal{B}_d$
OUTPUT: polynomial q which is subalgebra reduction of p w.r.t. $\mathcal{B}$.
let $F = \text{SA\_REDUCE}(G)$; // using algorithm 4.4
let $q = p$;
reduce from q all monomial which are leading monomials of products of elements from F by subtracting corresponding products;
RETURN: q;

Algorithm 4.5: SA_POLY_REDUCE(poly p, set G)

Clearly, algorithms 4.3, 4.4 and 4.5 terminate and are correct due to the discussion above.

Thus, we can compute reduced subalgebra bases of $\mathcal{Z}_d(\mathcal{A})$ and $\text{Cen}(X, \mathcal{A}_d)$ by applying algorithm 4.4 to a corresponding vector space basis..

Chapter 5

Small Atlas of Important Algebras

Using our methods we can compute central elements only up to a given degree. Thus for the computation of the whole center we need to know the number of generators of the center or/and theirs degrees. In this chapter we list some known results which provide us with this information in many cases. Moreover, we will use this knowledge in section 8.4 for estimating the vector space dimension of the center.

5.1 Enveloping algebras of Lie algebras

In this section we are interested only in the center $\mathcal{Z}(U)$ of an enveloping algebra $U = \mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. By definition, $\mathcal{Z}$ consists of all elements $z \in U$ which commute with all elements of U. These elements are called *central elements* or *Casimir operators*.

5.1.1 The center of $U(gl_n)$

Due to [49, Chapter IX] all Casimir operators of $\mathcal{U}(\mathfrak{gl}_n(\mathbb{C}))$ can be found as follows:

Recall that $\mathfrak{gl}_n$ has the standard basis consisting of the unit matrices $e_{ij},\ i,j = 1,\dots,n$.

First of all, one can immediately see that $c_1 = e_{11} + \ldots + e_{nn}$ is a Casimir operator. In this case $c_1 \in \mathfrak{gl}_n$ and the operator c_1 is easily shown to be the only (up to a multiple factor) Casimir operator in the algebra $\mathfrak{gl}_n$ itself.

Concerning the other Casimir operators, we first assume that $U = \mathcal{U}(\mathfrak{gl}_n)$ contains a system of elements $x_{ij},\ i,j = 1,\dots,n$, which transform like the elements e_{ij}:

$$[x_{ij}, x_{kl}] = \delta_{jk} x_{il} - \delta_{il} x_{kj}.$$

It is then obvious that the sum of the diagonal elements x_{ii} is a Casimir operator in U. Furthermore, let $x_{ij}, y_{ij}\ i,j = 1,\dots,n$ be two such systems. Set $z_{ij} = \sum_k x_{ik} y_{kj}$. It is not difficult to see that z_{ij} again form a system of this type, and we can thus construct a family of such systems, starting from the basis system e_{ij}:

$$e_{ij}^{(m)} = e_{ii_2} e_{i_2 i_3} \cdots e_{i_m j}.$$

Contracting over the indices i, j, we get a family of Casimir operators

$$c_m = \sum_{1 \leqslant i_1, \ldots, i_m \leqslant n} e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_m i_1}. \tag{5.1}$$

Thus, we obtain the following proposition:

Proposition 5.1. *The center $\mathcal{Z}(\mathcal{U}(\mathfrak{gl}_n(\mathbb{C})))$ is generated by n elements $c_1, \ldots, c_n$ of degrees $1, 2, \ldots, n$.*

5.1.2 The center of the enveloping algebra of a semisimple complex Lie algebra

Let $\mathfrak{g}$ be a semisimple complex Lie algebra and $U = \mathcal{U}(\mathfrak{g})$ its universal enveloping algebra. We will consider a fixed Cartan subalgebra H of $\mathfrak{g}$ and denote by r its dimension (it is also the **rank** of $\mathfrak{g}$ and it coincides with the number of positive roots in the root system of $\mathfrak{g}$).

The center $Z = \mathcal{Z}(U)$ of U is identified with the algebra $\mathrm{I}(\mathfrak{g})$ of all polynomials over the algebra $\mathfrak{g}$ which are invariant under the adjoint representation (cf. [6]). By Chevalley's theorem (cf. [49, Chapter XVII, §125, Theorem 6]) we may also identify Z with the algebra $\mathrm{I}(H)$ of all polynomials over H invariant under the Weyl group. Both these correspondences are linear but not multiplicative.

According to another theorem of Chevalley (cf. [49, Chapter XV, §107, Theorem 19]), the algebra $\mathrm{I}(H)$ has exactly r independent generators. All generators may be assumed homogeneous. Let $p_1, \ldots, p_r$ be the total degrees of these generators, then according to Chevalley (cf. [5]) $p_1 \cdots p_r = w$, where w is the number of elements in the Weyl group. In other words, the product of degrees of all generators is equal to the order of the Weyl group. The numbers p_i are intimately related to the most important topological properties of the corresponding Lie groups (Betti numbers, Poincaré polynomial).

The following important theorem may be found in [3, Ch.VIII, §8, no.3, Corollary 1 and no.5, Theorem 2]:

Theorem 5.2 (Chevalley). *Let $\mathfrak{g}$ be a complex simple Lie algebra, $r = \mathrm{rank}(\mathfrak{g})$, m_k the exponents of $\mathfrak{g}$. Then one can choose elements $I_k \in \mathcal{Z}_{m_k+1}(\mathcal{U}(\mathfrak{g}))$, $\quad 1 \leqslant k \leqslant r$ such that $\mathcal{Z}(\mathcal{U}(\mathfrak{g})) = \mathbb{C}\,[I_1, \ldots, I_r]$ is a polynomial algebra in r generators.*

In the following table we collect some information about all known (cf. theorem 1.48) simple complex Lie algebras:

Root system	h	Lie algebra	Rank	Dimension	$p_1; \dots; p_r$
$A_r, r \geqslant 1$	$(r+1)$	$\mathfrak{sl}_{r+1}$	r	$r(r+2)$	$2; \dots; (r+1)$
$B_r, r \geqslant 2$	$2r$	$\mathfrak{so}_{2r+1}$	r	$r(2r+1)$	$2; 4; \dots; 2r$
$C_r, r \geqslant 3$	$2r$	$\mathfrak{sp}_r$	r	$r(2r+1)$	$2; 4; \dots; 2r$
$D_r, r \geqslant 4$	$2(r-1)$	$\mathfrak{so}_{2r}$	r	$r(2r-1)$	$2; 4; \dots; 2(r-1); r$
G_2	6	$\mathfrak{g}_2$	2	14	$2; 6$
F_4	12	$\mathfrak{f}_4$	4	52	$2; 6; 8; 12$
E_6	12	$\mathfrak{e}_6$	6	78	$2; 5; 6; 8; 9; 12$
E_7	18	$\mathfrak{e}_7$	7	133	$2; 6; 8; 10; 12; 14; 18$
E_8	30	$\mathfrak{e}_8$	8	248	$2; 6; 12; 14; 18; 20; 24; 30$

Remark 5.3. In the previous table the number h denotes the Coxeter number of the corresponding root system, computed by the formula $h = (\dim - \operatorname{rank}) / \operatorname{rank}$.

Recall that the **Coxeter number** is the order of a Coxeter element (the product of the simple reflections, taken in any order) in the Weyl group of the corresponding root system.

5.1.3 The center of the enveloping algebra of a Lie algebra over a field of prime characteristic

Notice that our algorithms do not depend on the base field. We can use our library to compute the center and centralizers over any field supported by SINGULAR.

In this section we study the dependence of the center on the base field.

Let $\mathcal{A}$ be a GR-Algebra generated by $x_1, x_2, \dots, x_n$ over a field $\Bbbk$ such that all commutators $[x_i, x_j]$ are linear combinations of variables. For example, the enveloping algebras of finite dimensional Lie algebras and their tensor products over a field are of this kind by construction (see section 1.3). Below we consider only algebras of this kind.

Let us denote $V = \langle x_1, \dots, x_n \rangle_{\Bbbk}$ the vector space of linear combinations of variables.

Because of the decomposition of commutators $[x_i, x_j]$ into linear combinations of variables, for every variable x_i we can compute the matrix $A^{(x_i)} \in Mat_{n \times n}(\Bbbk)$ of the linear map $\operatorname{ad}_{x_i} \mid V : V \to V$, as follows:

$$\operatorname{ad}_{x_i} x_j = \sum_{k=1}^{n} (A^{(x_i)})_{k,j} \cdot x_k, \qquad 1 \leqslant i, j \leqslant n. \tag{5.2}$$

In fact, this gives us the **adjoint representation** of $\mathcal{A}$.

Example 5.4. Let us consider the $\mathcal{U}(\mathfrak{sl}_2)$ as in 3.1. We fix the basis $\{e, f, h\}$ of V. Then we have the following associated matrices:

$$A^{(h)} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; A^{(f)} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \; A^{(e)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & 0 \end{pmatrix}.$$

Definition 5.5. We call an element $f \in \mathcal{A}$ a **Cartan element** if there exist constants $C_j \in \Bbbk$ such that $\operatorname{ad}_f x_j = C_j \cdot x_j, 1 \leqslant j \leqslant n$.

Remark 5.6. If a variable x_i is a Cartan element then its associated matrix $A^{(x_i)}$ is diagonal. Notice that, the definition of a Cartan element is equivalent to the following one:

$$\forall g \in \mathcal{A}\, \exists \alpha_g \in \Bbbk : \operatorname{ad}_f g = \alpha_g g.$$

That is, the linear map between infinite dimensional vector spaces $\operatorname{ad}_f : \mathcal{A} \to \mathcal{A}$ has a diagonal matrix.

Example 5.7. Example 5.4 shows that the variable h is a Cartan element in $\mathcal{U}(\mathfrak{sl}_2)$.

Example 5.8. Due to section 3.2 the variable e_{11} is a Cartan element in $\mathcal{U}(\mathfrak{gl}_3)$.

Lemma 5.9. *Let $x \in \mathcal{A}$ be any variable, let $A := A^{(x)}$. Then for all $m \in \mathbb{N}$ and variables x_ν holds:*

$$[x^m, x_\nu] = \sum_{k=1}^{m} \binom{m}{k} \cdot \left(\sum_{i=1}^{n} (A^k)_{i,\nu} \cdot x_i \right) * x^{m-k} \tag{5.3}$$

Proof. Induction on m.

■

Since in a field of characteristic p holds $\binom{p}{k} = 0$, for all $1 \leqslant k \leqslant p-1$ we obtain the following corollary from lemma 5.9:

Corollary 5.10. *Let the ground field be characteristic p, then*

$$[x^p, x_\nu] = \sum_{i=1}^{n} (A^p)_{i,\nu} \cdot x_i. \tag{5.4}$$

Moreover, since the associated matrix of the Cartan element h is diagonal, we obtain $\left(A^{(h)}\right)^p = A^{(h)}$. Hence, $[h^p, x_\nu] = [h, x_\nu]$ for all variables x_ν. In particular $[h^p - h, x_\nu] = 0$. Thus we obtain the following proposition:

Proposition 5.11. *Let $\mathcal{A}$ be an algebra as above over a field of characteristic p. Let h be a Cartan element in $\mathcal{A}$, then $h^p - h \in \mathcal{Z}(\mathcal{A})$.*

Now, we list some classical results about Lie algebras of algebraic groups over algebraically closed field of characteristic $p > 0$.

As we have already seen, in characteristic 0 the center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ is comparatively small: it is isomorphic to the Weyl group of invariants on $\mathcal{U}(H)$, a polynomial algebra in r variables. But in characteristic p the center is much larger.

It is easy to see that for each $x \in \mathfrak{g}$, the element $x^p - x^{[p]}$ of $\mathcal{U}(\mathfrak{g})$ lies in $\mathcal{Z}(\mathfrak{g})$. Methods of Zassenhaus [48] (in the more general context of arbitrary modular Lie algebras) show that these elements generate the algebra isomorphic to the polynomial algebra in n indeterminates, where $n = \dim \mathfrak{g}$. This subalgebra of $\mathcal{Z}(\mathfrak{g})$ is denoted $\mathcal{O}$ and called the **p-center**. Moreover, $\mathcal{U}(\mathfrak{g})$ is the free $\mathcal{O}$-module of rank p^n.

A precise description of $\mathcal{Z}(\mathfrak{g})$ relative to $\mathcal{O}$ is given by Veldkamp [45]. This description involves the Weyl group invariants (generators of the center in characteristic 0), but requires some restrictions on p.

Veldkamp considers the center $\mathcal{Z}$ of the universal enveloping algebra $\mathcal{U}$ of a Lie algebra $\mathfrak{g}$, which is the Lie algebra of a semisimple algebraic group G over a field of characteristic $p > 0$.

Let $H_1, \ldots, H_l$ and $X_\alpha, \alpha \in \mathbf{\Phi}$ be a basis of $\mathfrak{g}$ derived from a Chevalley basis in characteristic 0 (cf. 1.3.4). Let $\mathcal{L}$ be the subspace of $\mathcal{U}$ spanned by all p^i-th powers (in $\mathcal{U}$) of elements of $\mathfrak{g}$, $i = 0, 1, 2, \ldots$, and $\mathcal{M} = \mathcal{L} \cap \mathcal{Z}$. Let $\mathcal{O}$ be the subalgebra of $\mathcal{Z}$ generated by 1 and $\mathcal{M}$.

From the binomial formula it follows that $\mathrm{ad}_{x^p} = (\mathrm{ad}_x)^p$ (cf. [48, formula (1) on p.4]). Therefore, $\mathfrak{g}$ has a structure of restricted Lie algebra such that

$$H_i^{[p]} = H_i, \quad X_\alpha^{[p]} = 0.$$

For $X \in \mathfrak{g}, \mathrm{ad}_{X^p} = (\mathrm{ad}_X)^p = \mathrm{ad}_{X^p}$. It follows that the elements $H_i^p - H_i$ and X_α^p belong to $\mathcal{M}$. Moreover, $\mathcal{M}$ has a basis over $\Bbbk$ consisting of all monomials of positive total degree in $H_i^p - H_i$ and X_α^p. From the Poincaré-Birkhoff-Witt theorem one deduces that $\mathcal{O} = \Bbbk\,[H_i^p - H_i, X_\alpha^p \mid 1 \leqslant i \leqslant l, \alpha \in \mathbf{\Phi}]$ since $H_i^p - H_i$ and X_α^p are algebraically independent over $\Bbbk$.

The following theorem (cf. [45, theorem (3.1)]) describes the structure of $\mathcal{Z}$ over $\mathcal{O}$:

Theorem 5.12. *Let G, $\mathfrak{g}$ and $\mathcal{U}$ be as above. Let h be the Coxeter number of G and $p =$ $\mathrm{char}(\Bbbk) > h$. Let $I_1, \ldots, I_r$ denote algebraically independent generators of the invariants in $\mathcal{U}$ under the adjoint action of G (roughly speaking, they are generators of the center in characteristic 0). Then $\mathcal{Z}(\mathcal{U}(\mathfrak{g})) = \mathcal{O}\,[I_1, \ldots, I_r]$, and the products $I_1^{j_1} \cdots I_r^{j_r}$, with $0 \leqslant j_i <$ p form a basis of $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ as an $\mathcal{O}$-module.*

5.2 A class of algebras similar to $U(sl_2)$

This example is taken from [43] and [24]. The author studies a class of algebras which are similar to the enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$ over the complex numbers $\mathbb{C}$. He considers the

Lie algebra $\mathfrak{sl}_2$ over $\mathbb{C}$ to be generated by x, y, h with the Lie product

$$[x, y] = h, \quad [h, x] = x, \quad [h, y] = -y.$$

Then the enveloping algebra of $\mathfrak{sl}_2$ is given by

$$S = \mathbb{C} \langle h, x, y \mid y * x = xy - h, x * h = hx - x, y * h = hy + y \rangle .$$

As one can see from this representation, we need not take a degree-compatible ordering as ordering. So an inverse lexicographical ordering is suitable, which suggests, that S could be considered as some Ore extension. It is indeed shown by Smith, that

$$S \cong \mathcal{U}(b) [y, \sigma, \delta] ,$$

where

1. $\mathcal{U}(b)$ denotes the enveloping algebra of the 2-dimensional non-abelian Lie algebra b, generated by h and x with the commutator relation $[h, x] = x$,
2. σ is defined by $\sigma(x) = x$ and $\sigma(h) = h - 1$
3. δ is defined by $\delta(x) = h$ and $\delta(h) = 0$

Now Smith observes that by this representation of S, the definition of $\delta(x)$ can be deliberately replaced by any univariate polynomial in h without losing the property of being an Ore extension: $\delta(x) = f(h)$.

The resulting algebra, denoted by $S = \mathcal{U}(\mathfrak{sl}_2, f)$, is given by

$$\mathbb{C} \langle h, x, y \mid y * x = xy - f(h), x * h = hx - x, y * h = hy + y \rangle .$$

Now Smith observes, that the center of R is generated by a unique polynomial which is determined by f

$$C = x * y + y * x + g(h) \in \mathcal{Z}(R)$$

In the following example we take $f = \frac{3}{2}h(h + 1)$ from [43, example 2.4]. We compute $C \in \mathcal{Z}(S)$ as $C = 2xy + h^3 - h$:

```
> ring Usl2f = 0,(x,y,h),lp;
> matrix D[3][3]=0;
> poly f = 3*h*(h+1)/2; // U(sl_2, f)
//   poly f = h; // one can use identity for the standard U(sl2)
> D[1,2] = -f;
> D[1,3] =  x;
> D[2,3] = -y;
> ncalgebra(1,D);
> Usl2f;
```

```
//   characteristic : 0
//   number of vars : 3
//        block   1 : ordering lp
//                  : names    x y h
//        block   2 : ordering C
//   noncommutative relations:
//    yx=xy-3/2h2-3/2h
//    hx=xh+x
//    hy=yh-y
> LIB "center.lib";

> ideal Z = centerVS(3); Z;
Z[1]=2xy+h3-h
> inCenter( Z );
1
```

5.3 Quantum enveloping algebras

The invention of quantum groups is one of the outstanding achievements of the mathematical physics and mathematics in the late twentieth century. Quantum groups arose in the work of L. D. Faddeev and the Leningrad school on the inverse scattering method in order to solve integrable models. The algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ appeared first in 1981 in a paper by P. P. Kulish and N. Yu. Reshetikhin on the study of integrable XYZ models with highest spin. Later its Hopf algebra structure was discovered. A major event was the discovery by V. G. Drinfeld and M. Jimbo of a class of Hopf algebras which can be considered as one-parameter deformations of enveloping algebras of semisimple complex Lie algebras. There Hopf algebras are called Drinfeld–Jimbo algebras.

A striking feature of quantum group theory is the surprising connections with many branches of mathematics and physics. These are links with mathematical fields such as Lie groups, Lie algebras and their representations, special functions, knot theory, low-dimensional topology, operator algebras, noncommutative geometry and combinatorics. On the physical side there are interrelations with the quantum inverse scattering method, the theory of integrable models, elementary particle physics, conformal and quantum field theories and others. It is expected that quantum groups will lead to a deeper understanding of the concept of symmetry in physics.

5.3.1 $\mathcal{U}_q'(\mathfrak{so}_3)$

In this subsection we closely follow [18].

The algebra $\mathcal{U}_q'(\mathfrak{so}_m)$ is a non-standard q-deformation of the enveloping algebra $\mathcal{U}(\mathfrak{so}_m)$ of the Lie algebra $\mathfrak{so}_m$.

It is known that the Fairlie–Odesskii algebra $\mathcal{U}_q'(\mathfrak{so}_3)$ appears as algebra of observables in quantum gravity in (2+1)-dimensional de Sitter space with space being torus. The parameter q is related to the Plank constant and the curvature of the de Sitter space. Thus it is important, from the point of view of physics, to study the structure (in particular, the center) of this algebra.

The algebra $\mathcal{U}'_q(\mathfrak{so}_3)$ is an associative unital algebra generated by the elements I_1, I_2, I_3 which satisfy the following relations:

$$q^{1/2}I_1I_2 - q^{-1/2}I_2I_1 = I_3, \quad q^{1/2}I_2I_3 - q^{-1/2}I_3I_2 = I_1, \quad q^{1/2}I_3I_1 - q^{-1/2}I_1I_3 = I_2,$$

where $q \neq 0, \pm 1$, is a complex number called deformation parameter. In the limit $q \to 1$, the algebra $\mathcal{U}'_q(\mathfrak{so}_3)$ reduces to the algebra $\mathcal{U}(\mathfrak{so}_3)$. It is easy to check that for any value of q the algebra $\mathcal{U}'_q(\mathfrak{so}_3)$ has the following central element: $C = -q^{1/2}(q - q^{-1})I_1I_2I_3 + qI_1^2 + q^{-1}I_2^2 + qI_3^2$, which is the deformation of the Casimir element of the algebra $\mathcal{U}(\mathfrak{so}_3)$.

Proposition 5.13. *The element C generates the center of $\mathcal{U}'_q(\mathfrak{so}_3)$ when q is not a root of 1.*

As in the case of quantum algebras (cf. [22, Chapter 6]) this algebra has additional central elements if q is a root of unity:

Let us fix q to be a primitive root of 1 of order $p > 2$, that is, $q^p = 1, q^{p'} \neq 1$ for all $1 \leqslant p' < p$. Then the following elements are also central in $\mathcal{U}'_q(\mathfrak{so}_3)$:

$$C_k = 2T_p(I_k(q - q^{-1})/2), \quad k = 1, 2, 3,$$

where $T_p(x)$ is the p-th Chebyshev polynomial of the first kind.

Let us recall that the p-th **Chebyshev polynomial** of the first kind $T_p(x)$ is uniquely defined by $T_p(cos\theta) = cos(p\theta)$. Its explicit form is given by:

$$T_p(x) = \frac{p}{2}\sum_{k=0}^{\lfloor p/2 \rfloor} \frac{(-1)^k(p-k-1)!}{k!(p-2k)!}(2x)^{p-2k},$$

where $\lfloor p/2 \rfloor$ is the integral part of $p/2$.

The elements C, C_1, C_2 and C_3 are algebraically dependent. Let us use the following element $\partial = (q + q^{-1})\mathbf{1} - (q - q^{-1})^2 C$ instead of C in the following proposition which is due to [18, Prop. 2]:

Proposition 5.14. *Let q be a primitive root of unity of order $p > 2$. Then the algebraic dependence between the central elements ∂, C_1, C_2, C_3 has the following form:*

- $-q^{p/2}C_1C_2C_3 + C_1^2 + C_2^2 + C_3^2 + 2T_p(\partial/2) - 2 = 0$, *if* $p = 2k + 1$;
- $-C_1C_2C_3 + C_1^2 + C_2^2 + C_3^2 + 2T_p(\partial/2) + 16T_{p/2}(\partial/2) + 4(T_{p/2}(\partial/2) + 1)(C_1 + C_2 + C_3) + 10 = 0$, *if* $p = 4k$;

- $-C_1C_2C_3+C_1^2+C_2^2+C_3^2+2T_p(\partial/2)-16T_{p/2}(\partial/2)-4(T_{p/2}(\partial/2)-1)(C_1+C_2+C_3)+10 = 0$, *if* $p = 4k+2$.

Conjecture 5.15. Let q be a root of unity. Then the elements C, C_1, C_2, C_3 generate the center of the algebra $\mathcal{U}_q'(\mathfrak{so}_3)$. All algebraic dependences among them follow from the dependences described in proposition 5.14.

5.3.2 $\mathcal{U}_q(\mathfrak{sl}_2)$

The Hopf algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ can be considered as a one-parameter deformation of the enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$. This algebra is the simplest example of the quantized enveloping algebras $\mathcal{U}_q(\mathfrak{g})$. Following common terminology in physics, we call $\mathcal{U}_q(\mathfrak{sl}_2)$ a **quantum algebra**.

Let q be a fixed complex number such that $q \neq 0$ and $q^2 \neq 1$. We denote by $\mathcal{U}_q(\mathfrak{sl}_2)$ the $\mathbb{C}$-algebra generated by E, F, K, K^{-1} subject to the following relations:

$$KK^{-1} = K^{-1}K = 1, KEK^{-1} = q^2E, KFK^{-1} = q^{-2}F, [E,F] = \frac{K-K^{-1}}{q-q^{-1}}.$$

Proposition 5.16. *The* ***quantum Casimir element***

$$C_q := EF + \frac{Kq^{-1}+K^{-1}q}{(q-q^{-1})^2} = FE + \frac{Kq+K^{-1}q^{-1}}{(q-q^{-1})^2}$$

lies in the center of the algebra $\mathcal{U}_q(\mathfrak{sl}_2)$*. If* q *is not a root of unity, then the center of* $\mathcal{U}_q(\mathfrak{sl}_2)$ *is generated by* C_q.

Proposition 5.17. *Let* q *be a primitive* p*-th root of unity, with* $p > 3$*. Let* $p' = p$ *if* p *is odd and* $p' = p/2$ *if* p *is even. Then:*

(i) The elements $E^{p'}, F^{p'}, K^{p'}, K^{-p'}$ *belong to the center of* $\mathcal{U}_q(\mathfrak{sl}_2)$*.*

(ii) The center of $\mathcal{U}_q(\mathfrak{sl}_2)$ *is generated by the elements* $E^{p'}, F^{p'}, K^{p'}, K^{-p'}$ *and the Casimir element* C_q*.*

Note that in the case when q is a root of unity, the algebraic dependence of central elements of the algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ is expressed (an in the case of $\mathcal{U}_q'(\mathfrak{so}_3)$) in terms of Chebyshev polynomials (cf. [1]).

5.4 Weyl algebras

In this section we closely follow [39] and [46].

Abstract definition

Definition 5.18. Let F be a field and let V be an F-vector space with a basis $\{P_i\}_{i\in I} \cup \{Q_i\}_{i\in I}$, where I is some non-empty index set. Let $T = \mathcal{T}(V)$ be the tensor algebra(cf. 1.12) of V and let J be the two-sided ideal in T generated by elements $P_i \otimes Q_j - Q_j \otimes P_i - \delta_{i,j}, i, j \in I$. Then the factor algebra T/J is the $|I|$-th **Weyl algebra**.

A more concrete definition

If the field F has characteristic zero we have the following more concrete definition. Let $R := F[\{X_i\}_{i\in I}]$ be the polynomial ring over F in indeterminates X_i labeled by $i \in I$. For any $i \in I$, let ∂_i denote the partial differential operator with respect to X_i. Then the $|I|$-th **Weyl algebra** is the set W of all differential operators of the form $D = \sum_{|\alpha|\leq n} f_\alpha \partial^\alpha$ where the summation variable α is a multi-index with $|I|$ entries, n is the degree of D, and $f_\alpha \in R$. The algebra structure is defined by the usual operator multiplication, where the coefficients $f_\alpha \in R$ are identified with the operators of left multiplication with them for conciseness of notation. Since the derivative of a polynomial is again a polynomial, it is clear that W is closed under the multiplication.

The equivalence of these definitions can be seen by replacing the generators Q_i with the left multiplication by the indeterminates X_i, the generators P_i with the partial differential operators ∂_i, and the tensor product with operator multiplication, and observing that $\partial_i X_j - X_j \partial_i = \delta_{ij}$. If, however, the characteristic p of F is positive, the resulting homomorphism to W is not injective, since for example the expressions ∂_i^p and X_i^n commute, while $P_i^{\otimes p}$ and $Q_i^{\otimes n}$ do not.

Remark 5.19. The first Weyl algebra is an example of a simple ring that is not a matrix ring over a division ring. It is also a non-commutative example of a domain, and an example of an Ore extension.

The n-th Weyl algebra $\mathcal{W}_n$ is given by

$$\mathcal{W}_n = \Bbbk \langle x_1, \dots, x_n, D_1, \dots, D_n \mid D_i x_i = x_i D_i + 1,\ i = 1, \dots, n \rangle .$$

It is know (cf. [26, Example 1.3]) that if $\operatorname{char} \Bbbk = 0$ then its center is trivial: $\mathcal{Z}(\mathcal{W}_n) = \Bbbk$, and if $\operatorname{char} \Bbbk = p$ then we only get p-center: $\mathcal{Z}(\mathcal{W}_n) = \Bbbk [x_1^p, \dots, x_n^p, D_1^p, \dots, D_n^p]$.

5.5 Heisenberg algebras

This section is mainly due to [47, The Heisenberg Algebra] and [39].

In classical mechanics the state of a particle at a given time t is determined by its position vector $\mathbf{q} \in \mathbb{R}^3$ and its momentum vector $\mathbf{p} \in \mathbb{R}^3$. Heisenberg's crucial idea that lead to quantum mechanics was to take the components of these vectors to be operators on a Hilbert space H, satisfying the commutation relations

$$[Q_i, Q_j] = 0, \qquad [P_i, P_j] = 0, \qquad [P_i, Q_j] = -\imath\hbar\delta_{i,j}$$

for $i, j = 1, 2, 3$.

One can think of the Heisenberg commutation relations as the defining relations for a $(2n+1)$-dimensional Lie algebra, so one can use the following definition:

Definition 5.20. The **Heisenberg Lie algebra** $\mathfrak{h}_n$ is the $2n+1$ dimensional real Lie algebra with the basis elements

$$\{P_1, \ldots, P_n, Q_1, \ldots, Q_n, C\}$$

and the Lie bracket defined by

$$[Q_i, Q_j] = [P_i, P_j] = [Q_i, C] = [P_i, C] = [C, C] = 0, [P_i, Q_j] = \delta_{i,j}C$$

One can also use more abstract definition:

Definition 5.21. Let R be a commutative ring. Let M be a module over R freely generated by sets $\{P_i\}_{i\in I}$, $\{Q_i\}_{i\in I}$ and an element C, where I is an index set.

Let us define $[Q_i, Q_j] = [P_i, P_j] = [Q_i, C] = [C, Q_i] = [P_i, C] = [C, P_i] = [C, C] = 0, [P_i, Q_j] = -[Q_j, P_i] = \delta_{i,j}C$ for all $i, j \in I$. This operation $[\cdot, \cdot]$ extends by bilinearity to the map $M \times M \to M$.

The module M together with this product is called a **Heisenberg algebra**. The element C is called the central element.

It is easy to see that the product $[\cdot, \cdot]$ also fulfills the Jacobi identity, so a Heisenberg algebra is actually a Lie algebra of rank $|I| + 1$ (opposed to the rank of M as free module, which is $2|I| + 1$) with one-dimensional center generated by C.

Heisenberg algebras arise in quantum mechanics with $R = \mathbb{R}, \mathbb{C}$ and typically $I = \{1, 2, 3\}$, but also in the theory of vertex algebras with $I = \mathbb{Z}$.

In the case where R is a field, the Heisenberg algebra is related to a Weyl algebra: let U be the enveloping algebra of M, then the factor algebra $U/\langle C - 1\rangle$ is isomorphic to the $|I|$-th Weyl algebra over R.

Clearly, the enveloping algebra $\mathcal{H}_n$ of the Heisenberg Lie algebra with $I = 1, \ldots, n$ over a field $\Bbbk$ is given by

$$\mathcal{H}_n = \Bbbk\langle x_1, \ldots, x_n, y_1, \ldots, y_n, h \mid y_i x_i = x_i y_i + h,\ i = 1, \ldots, n\rangle .$$

It is know (cf. [26]) that if $\operatorname{char} \Bbbk = 0$ then its center is almost trivial: $\mathcal{Z}(\mathcal{H}_n) = \Bbbk[h]$, and if $\operatorname{char} \Bbbk = p$ then we additionally get p-center: $\mathcal{Z}(\mathcal{W}_n) = \Bbbk[h, x_1^p, \ldots, x_n^p, y_1^p, \ldots, y_n^p]$.

Chapter 6

Commutative subalgebras

In this chapter we consider commutative subalgebras of a GR-Algebra $\mathcal{A}$ over an algebraically closed field $\Bbbk$ of characteristic 0. The treatment of this subject was inspired by V.Levandovskyy.

Whenever we need a filtration we choose the standard filtration by degree.

Obviously, the center $\mathcal{Z}(\mathcal{A})$ is a commutative subalgebra of $\mathcal{A}$.

From the representation theory of enveloping algebras we recall some more commutative subalgebras (cf. [6]):

- **Cartan subalgebra** $\mathcal{H}(\mathcal{A}) = \mathcal{U}(\mathfrak{h})$ (if exists).
- if there exists Cartan subalgebra, we can construct a bigger subalgebra

  $$\mathcal{CZ}(\mathcal{A}) := \mathcal{H}(\mathcal{A}) \otimes_{\Bbbk} \mathcal{Z}(\mathcal{A}),$$

  which is also commutative.
- **Gel'fand-Zetlin subalgebra** $\Gamma(\mathcal{A})$ (cf. [8]).

If all these subalgebras exist then $\mathcal{Z}(\mathcal{A}) \subseteq \mathcal{CZ}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$. Moreover, in several cases, if $\Gamma(\mathcal{A})$ exists, it is the maximal one (cf. [34]).

Unfortunately, the construction of Gel'fand-Zetlin subalgebra does not have yet a complete algorithmic solution and is known only for a few cases.

In this chapter we show how to use Singular to compute Gel'fand-Zetlin subalgebras and check the following conjecture for some algebras with known Gel'fand-Zetlin subalgebras.

Conjecture 6.1. Let $S = \{g_1, \ldots, g_k\}$ be an autoreduced subalgebra base of $\Gamma(\mathcal{A})$. Let us choose the greatest element $g \in S$ with respect to a degree-compatible ordering on $\mathcal{A}$. Then

$$\Gamma(\mathcal{A}) = \mathrm{Cen}(g, \mathcal{A}).$$

We prove that conjecture 6.1 holds true for $\mathcal{A} = \mathcal{U}(\mathfrak{gl}_2)$.

6.1 Gel'fand-Zetlin modules

Explicit formulas which effectively define all simple finite dimensional modules over the groups of unimodular and orthogonal matrices were obtained by Gel'fand and Zetlin in [11]. Using these formulas one can define and investigate big families of modules over the corresponding Lie algebras. For example, using these formulas for the unimodular group, Drozd, Futorny and Ovsienko constructed a large family of simple modules over $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ in [8, 7]. Roughly speaking this is an $n(n+1)/2$-parameter family of simple $\mathfrak{g}$-modules and each module is presented in a convenient basis for computations.

Gel'fand-Zetlin modules are defined as $\mathcal{U}(\mathfrak{g})$-modules which can be decomposed into a direct sum of finite-dimensional modules with respect to the so-called Gel'fand-Zetlin subalgebra, which is a big commutative subalgebra in $\mathcal{U}(\mathfrak{g})$.

It was shown in [34] that Gel'fand-Zetlin subalgebra is a maximal commutative subalgebra in $\mathcal{U}(\mathfrak{gl}_n)$. This is also the case for a quantum situation $\mathcal{U}_q(\mathfrak{gl}_n)$ (cf. [31]).

Roughly speaking, Gel'fand and Zetlin show that Gel'fand-Zetlin subalgebra has a simple spectrum on all finite dimensional modules. Simple modules constructed in [8, 7] inherit this property.

Further properties of Gel'fand-Zetlin modules were obtained in [28, 29, 32, 30, 34]. For example, a huge family of Verma and generalized Verma modules were realized as Gel'fand-Zetlin modules, which allows one to describe the structure of these modules.

There are many Gel'fand-Zetlin subalgebras, but all of them are built with the same recipe: for an algebra $\mathcal{A}$, find a sequence of inclusions

$$A(1) \hookrightarrow A(2) \hookrightarrow \ldots \hookrightarrow A(n) = \mathcal{A}$$

such that $A(i+1)$ has exactly one central generator more than $A(i)$. Then take Cartan elements for $A(1)$ (one can define $A(0)$ to be the Cartan subalgebra of $A(1)$) and add on each step generators of the center of $A(i)$, embedded under inclusions above.

Each sequence of inclusions gives rise to a different subalgebra; however the number of its reduced generators is invariant.

In the following sections we will construct Gel'fand-Zetlin subalgebras for $\mathcal{U}(\mathfrak{gl}_n)$ and $\mathcal{U}(\mathfrak{sl}_n)$.

6.2 Gel'fand-Zetlin subalgebra of the enveloping algebra of gl_n

Let us denote by $\{e_{ij}\}, 1 \leqslant i, j \leqslant n$ the standard basis of $U_n := \mathcal{U}(\mathfrak{gl}_n)$, and $Z_n := \mathcal{Z}(U_n)$. For $m \leqslant n$ we consider U_m as a subalgebra in U_n by an inclusion $U_m \hookrightarrow U_n : e_{ij} \mapsto e_{ij}, 1 \leqslant i, j \leqslant m$. Hence we obtain the following sequence of inclusions:

$$U_1 \hookrightarrow U_2 \hookrightarrow \ldots \hookrightarrow U_n = \mathfrak{gl}_n.$$

Then the **Gel'fand-Zetlin subalgebra** Γ in $\mathfrak{gl}_n$ is a commutative algebra, generated by the centers $Z_1, \dots, Z_n \subset \mathfrak{gl}_n$.

Due to [49] or chapter 5 we know that Z_m is the polynomial algebra in m variables $\{c_i^m \mid i = 1, \dots, m\}$ given by:

$$c_i^m = \sum_{1 \leqslant k_1, \dots, k_i \leqslant m} e_{k_1 k_2} e_{k_2 k_3} \cdots e_{k_{i-1} k_i} e_{k_i k_1}.$$

Hence the algebra Γ is also the polynomial algebra in $n(n+1)/2$ variables $\{c_i^m \mid 1 \leqslant i \leqslant m \leqslant n\}$.

Theorem 6.2 (Ovsienko). *Γ is a maximal commutative subalgebra in $\mathcal{U}(\mathfrak{gl}_n)$.*

Clearly $\mathcal{H}(\mathcal{U}(\mathfrak{gl}_n)) = \Bbbk[e_{ii} \mid i = 1, \dots, n]$, and $\mathcal{CZ}(\mathcal{U}(\mathfrak{gl}_n)) = \Bbbk[e_{ii}, c_i^m \mid i = 1, \dots, n]$.

Let us use Singular to compute generators of $\Gamma(\mathcal{U}(\mathfrak{gl}_n))$ for some small n:

```
> LIB "center.lib";
> LIB "ncalg.lib";
> // trivial case:
> def GL1 = makeUgl(1, p); setring GL1; GL1; // U(gl_1) = K[e_1_1]
//   characteristic : 0
//   number of vars : 1
//        block   1 : ordering dp
//                  : names    e_1_1
//        block   2 : ordering C
//   noncommutative relations:
> ideal Z = centerRed(1, 1); Z;
Z[1]=e_1_1
> ideal GZ = sa_reduce(Z); GZ;
GZ[1]=e_1_1
> centralizerRed(GZ[size(GZ)], 5);
_[1]=e_1_1
>
> def GL2 = makeUgl(2, p); setring GL2; GL2; // U(gl_2)
//   characteristic : 0
//   number of vars : 4
//        block   1 : ordering dp
//                  : names    e_1_1 e_1_2 e_2_1 e_2_2
//        block   2 : ordering C
//   noncommutative relations:
//    e_1_2e_1_1=e_1_1*e_1_2-e_1_2
//    e_2_1e_1_1=e_1_1*e_2_1+e_2_1
//    e_2_1e_1_2=e_1_2*e_2_1-e_1_1+e_2_2
//    e_2_2e_1_2=e_1_2*e_2_2-e_1_2
//    e_2_2e_2_1=e_2_1*e_2_2+e_2_1
> ideal Z = centerRed(2, 2); Z;
Z[1]=e_1_1+e_2_2
Z[2]=e_1_2*e_2_1-e_1_1*e_2_2+e_2_2
```

```
> ideal GZ = sa_reduce(imap(GL1, Z) + Z); GZ;
GZ[1]=e_2_2
GZ[2]=e_1_1
GZ[3]=e_1_2*e_2_1
> centralizerRed(GZ[size(GZ)], 5); // Cen(e_1_2*e_2_1, U(gl_2)_5) == GZ!
_[1]=e_2_2
_[2]=e_1_1
_[3]=e_1_2*e_2_1
>
> def GL3 = makeUgl(3, p); setring GL3; GL3; // U(gl_3)
//   characteristic : 0
//   number of vars : 9
//        block   1 : ordering dp
//                  : names    e_1_1 e_1_2 e_1_3 e_2_1 e_2_2 e_2_3 e_3_1
e_3_2 e_3_3
//        block   2 : ordering C
//   noncommutative relations:
//    e_1_2e_1_1=e_1_1*e_1_2-e_1_2
//    e_1_3e_1_1=e_1_1*e_1_3-e_1_3
//    e_2_1e_1_1=e_1_1*e_2_1+e_2_1
//    e_3_1e_1_1=e_1_1*e_3_1+e_3_1
//    e_2_1e_1_2=e_1_2*e_2_1-e_1_1+e_2_2
//    e_2_2e_1_2=e_1_2*e_2_2-e_1_2
//    e_2_3e_1_2=e_1_2*e_2_3-e_1_3
//    e_3_1e_1_2=e_1_2*e_3_1+e_3_2
//    e_2_1e_1_3=e_1_3*e_2_1+e_2_3
//    e_3_1e_1_3=e_1_3*e_3_1-e_1_1+e_3_3
//    e_3_2e_1_3=e_1_3*e_3_2-e_1_2
//    e_3_3e_1_3=e_1_3*e_3_3-e_1_3
//    e_2_2e_2_1=e_2_1*e_2_2+e_2_1
//    e_3_2e_2_1=e_2_1*e_3_2+e_3_1
//    e_2_3e_2_2=e_2_2*e_2_3-e_2_3
//    e_3_2e_2_2=e_2_2*e_3_2+e_3_2
//    e_3_1e_2_3=e_2_3*e_3_1-e_2_1
//    e_3_2e_2_3=e_2_3*e_3_2-e_2_2+e_3_3
//    e_3_3e_2_3=e_2_3*e_3_3-e_2_3
//    e_3_3e_3_1=e_3_1*e_3_3+e_3_1
//    e_3_3e_3_2=e_3_2*e_3_3+e_3_2
> ideal Z = centerRed(3, 3); Z;
Z[1]=e_1_1+e_2_2+e_3_3
Z[2]=e_1_2*e_2_1-e_1_1*e_2_2+e_1_3*e_3_1+e_2_3*e_3_2-e_1_1*e_3_3-e_2_2*e_3_3
+e_2_2+2*e_3_3
Z[3]=e_1_3*e_2_2*e_3_1-e_1_2*e_2_3*e_3_1-e_1_3*e_2_1*e_3_2+e_1_1*e_2_3*e_3_2
+e_1_2*e_2_1*e_3_3-e_1_1*e_2_2*e_3_3-e_1_3*e_3_1-2*e_2_3*e_3_2+e_1_1*e_3_3
+2*e_2_2*e_3_3-2*e_3_3
> ideal GZ = sa_reduce(imap(GL2, Z) + imap(GL1, Z) + Z); GZ;
GZ[1]=e_3_3
GZ[2]=e_2_2
GZ[3]=e_1_1
GZ[4]=e_1_3*e_3_1+e_2_3*e_3_2
GZ[5]=e_1_2*e_2_1
```

```
GZ[6]=e_1_2*e_2_3*e_3_1+e_1_3*e_2_1*e_3_2-e_1_1*e_2_3*e_3_2+e_2_2*e_2_3*e_3_2
+e_2_3*e_3_2
> centralizerRed(GZ[size(GZ)], 5); // Cen(GZ[6], U(gl_3)_5) == GZ!
_[1]=e_3_3
_[2]=e_2_2
_[3]=e_1_1
_[4]=e_1_3*e_3_1+e_2_3*e_3_2
_[5]=e_1_2*e_2_1
_[6]=e_1_2*e_2_3*e_3_1+e_1_3*e_2_1*e_3_2-e_1_1*e_2_3*e_3_2+e_2_2*e_2_3*e_3_2
+e_2_3*e_3_2
```

Result 6.3. Our computations show that:

1. $\mathcal{U}(\mathfrak{gl}_1)$:
   (a) $\mathcal{Z}(\mathcal{U}(\mathfrak{gl}_1)) = \mathbb{k}\langle e_{11}\rangle$,
   (b) $\Gamma(\mathcal{U}(\mathfrak{gl}_1)) = \mathbb{k}\langle e_{11}\rangle$.
2. $\mathcal{U}(\mathfrak{gl}_2)$:
   (a) $\mathcal{Z}(\mathcal{U}(\mathfrak{gl}_2)) = \mathbb{k}\langle e_{11} + e_{22}, e_{12} * e_{21} - e_{11} * e_{22} + e_{22}\rangle$.
   (b) $\Gamma(\mathcal{U}(\mathfrak{gl}_2)) = \mathbb{k}\langle e_{22}, e_{11}, e_{12} * e_{21}\rangle$.
3. $\mathcal{U}(\mathfrak{gl}_3)$:
   (a) $\mathcal{Z}(\mathcal{U}(\mathfrak{gl}_3))$ is generated by the following 3 elements:
      - $e_{11} + e_{22} + e_{33}$,
      - $e_{12} * e_{21} - e_{11} * e_{22} + e_{13} * e_{31} + e_{23} * e_{32} - e_{11} * e_{33} - e_{22} * e_{33} + e_{22} + 2 * e_{33}$,
      - $e_{13} * e_{22} * e_{31} - e_{12} * e_{23} * e_{31} - e_{13} * e_{21} * e_{32} + e_{11} * e_{23} * e_{32} + e_{12} * e_{21} * e_{33} - e_{11} * e_{22} * e_{33} - e_{13} * e_{31} - 2 * e_{23} * e_{32} + e_{11} * e_{33} + 2 * e_{22} * e_{33} - 2 * e_{33}$.
   (b) $\Gamma(\mathcal{U}(\mathfrak{gl}_3))$ is generated by the following elements:
      - Cartan elements: e_{33}, e_{22}, e_{11},
      - $e_{12} * e_{21}, e_{13} * e_{31} + e_{23} * e_{32}$,
      - $e_{12} * e_{23} * e_{31} + e_{13} * e_{21} * e_{32} - e_{11} * e_{23} * e_{32} + e_{22} * e_{23} * e_{32} + e_{23} * e_{32}$.

Moreover, let $g = e_{12} * e_{23} * e_{31} + e_{13} * e_{21} * e_{32} - e_{11} * e_{23} * e_{32} + e_{22} * e_{23} * e_{32} + e_{23} * e_{32}$. Our computations show that $\mathrm{Cen}(g, \mathcal{U}(\mathfrak{gl}_n)_5)$ has the same generators as $\Gamma(\mathcal{U}(\mathfrak{gl}_n))$, for $n = 1, 2, 3$.

Proposition 6.4. $\mathrm{Cen}(e_{12} * e_{21}, \mathcal{U}(\mathfrak{gl}_2)) = \Gamma(\mathcal{U}(\mathfrak{gl}_2))$.

Proof. Clearly $\Gamma(\mathcal{U}(\mathfrak{gl}_2)) \subset \mathrm{Cen}(e_{12} * e_{21}, \mathcal{U}(\mathfrak{gl}_2))$.

In order to show the other inclusion we choose any non-zero $p \in \mathrm{Cen}(e_{12} * e_{21}, \mathcal{U}(\mathfrak{gl}_2))$.

Let us observe that e_{11} and e_{22} are Cartan elements of $\mathcal{U}(\mathfrak{gl}_2)$. Therefore, there exist constants (depending on p) $\alpha, \beta \in \Bbbk$ such that $[p, e_{11}] = \alpha \cdot p$ and $[p, e_{22}] = \beta \cdot p$. Moreover, $[p, e_{11} * e_{22}] = e_{11} * [p, e_{22}] + [p, e_{11}] * e_{22} = \beta e_{11} * p + \alpha \cdot p * e_{22} = \beta \cdot e_{11} * p + \alpha(\beta \cdot p + e_{22} * p) = (\beta \cdot e_{11} + \alpha\beta + \alpha \cdot e_{22}) * p$.

We recall that $\mathcal{Z}(\mathcal{U}(\mathfrak{gl}_2)) = \Bbbk \langle e_{11} + e_{22}, e_{12} * e_{21} - e_{11} * e_{22} + e_{22} \rangle$. Thus: $0 = [p, e_{11} + e_{22}] = [p, e_{11}] + [p, e_{22}] = (\alpha + \beta) \cdot p$ and $0 = [p, e_{12} * e_{21} - e_{11} * e_{22} + e_{22}] = [p, e_{12} * e_{21}] - [p, e_{11} * e_{22}] + [p, e_{22}] = -(\alpha\beta + \beta \cdot e_{11} + \alpha \cdot e_{22}) * p + \beta \cdot p$.

Hence, we obtain the system of equations on α and β:

$$\begin{cases} (\alpha + \beta) \cdot p = 0, \\ (\beta - \alpha\beta - \alpha \cdot e_{22} - \beta \cdot e_{11}) * p = 0. \end{cases}$$

Since $\mathcal{U}(\mathfrak{gl}_2)$ is an integral domain and $p \neq 0$, we deduce that $\alpha = -\beta$ and $\beta + \beta^2 + \beta \cdot e_{22} - \beta \cdot e_{11} = 0$. The last equation is equivalent to the following system:

$$\begin{cases} \beta^2 + \beta = 0, \\ \beta \cdot (e_{22} - e_{11}) = 0. \end{cases}$$

Thus: $\beta = 0$ and $\alpha = -\beta = 0$. This shows that p commutes with all generators of Gel'fand-Zetlin subalgebra. But we know that $\Gamma(\mathcal{U}(\mathfrak{gl}_2))$ is a maximal commutative subalgebra. Therefore $p \in \Gamma(\mathcal{U}(\mathfrak{gl}_2))$. ■

6.3 Gel'fand-Zetlin subalgebra of the enveloping algebra of sl_n

The enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$ is generated by $X(1), Y(1), H(1)$ subject to the following relations:

$$\begin{cases} Y(1)X(1) = X(1) * Y(1) - H(1), \\ H(1)X(1) = X(1) * H(1) + 2 * X(1), \\ H(1)Y(1) = Y(1) * H(1) - 2 * Y(1). \end{cases}$$

The Cartan subalgebra $\mathcal{H}(\mathcal{U}(\mathfrak{sl}_2))$ is generated by Cartan element $H(1)$. The center $\mathcal{Z}(\mathcal{U}(\mathfrak{sl}_2))$ is generated by Casimir element $4 * X(1) * Y(1) + H(1)^2 - 2 * H(1)$.

Both Gel'fand-Zetlin subalgebra $\Gamma(\mathcal{U}(\mathfrak{sl}_2))$ and $\mathcal{CZ}(\mathcal{U}(\mathfrak{sl}_2))$ are generated by Cartan element and Casimir element of this algebra:

$$\Gamma(\mathcal{U}(\mathfrak{sl}_2)) = \mathcal{CZ}(\mathcal{U}(\mathfrak{sl}_2)) = \Bbbk \left[H(1), 4 * X(1) * Y(1) + H(1)^2 - 2 * H(1) \right].$$

Clearly $\Gamma(\mathcal{U}(\mathfrak{sl}_2)) = \mathcal{CZ}(\mathcal{U}(\mathfrak{sl}_2)) = \Bbbk [H(1), X(1) * Y(1)]$.

Proposition 6.5. *Let us assume that* $\Gamma(\mathcal{U}(\mathfrak{sl}_2))$ *is a maximal commutative subalgebra. Then* $\mathrm{Cen}(X(1)*Y(1), \mathcal{U}(\mathfrak{sl}_2)) = \Gamma(\mathcal{U}(\mathfrak{sl}_2))$.

Proof. Similarly to proposition 6.4. ∎

Let us consider the enveloping algebra $\mathcal{U}(\mathfrak{sl}_3)$. It is generated by $X(1), X(2), X(3), Y(1), Y(2), Y(3), H(1), H(2)$ subject to the following relations:

$$\begin{cases} X(2)X(1) = X(1)*X(2)+X(3), Y(1)X(1) = X(1)*Y(1)-H(1), \\ Y(3)X(1) = X(1)*Y(3)-Y(2), H(1)X(1) = X(1)*H(1)+2*X(1), \\ H(2)X(1) = X(1)*H(2)-X(1), Y(2)X(2) = X(2)*Y(2)-H(2), \\ Y(3)X(2) = X(2)*Y(3)+Y(1), H(1)X(2) = X(2)*H(1)-X(2), \\ H(2)X(2) = X(2)*H(2)+2*X(2), Y(1)X(3) = X(3)*Y(1)-X(2), \\ Y(2)X(3) = X(3)*Y(2)+X(1), Y(3)X(3) = X(3)*Y(3)-H(1)-H(2), \\ H(1)X(3) = X(3)*H(1)+X(3), H(2)X(3) = X(3)*H(2)+X(3), \\ Y(2)Y(1) = Y(1)*Y(2)-Y(3), H(1)Y(1) = Y(1)*H(1)-2*Y(1), \\ H(2)Y(1) = Y(1)*H(2)+Y(1), H(1)Y(2) = Y(2)*H(1)+Y(2), \\ H(2)Y(2) = Y(2)*H(2)-2*Y(2), H(1)Y(3) = Y(3)*H(1)-Y(3), \\ H(2)Y(3) = Y(3)*H(2)-Y(3). \end{cases}$$

The Cartan subalgebra $\mathcal{H}(\mathcal{U}(\mathfrak{sl}_3))$ is generated by Cartan elements $H(1)$ and $H(2)$. The center $\mathcal{Z}(\mathcal{U}(\mathfrak{sl}_3))$ is generated by Casimir elements:

- $3*X(1)*Y(1)+3*X(2)*Y(2)+3*X(3)*Y(3)+H(1)^2+H(1)*H(2)+H(2)^2-3*H(1)-3*H(2)$,
- $27*X(3)*Y(1)*Y(2)+27*X(1)*X(2)*Y(3)-9*X(1)*Y(1)*H(1)+18*X(2)*Y(2)*H(1)-9*X(3)*Y(3)*H(1)-2*H(1)^3-18*X(1)*Y(1)*H(2)+9*X(2)*Y(2)*H(2)+9*X(3)*Y(3)*H(2)-3*H(1)^2*H(2)+3*H(1)*H(2)^2+2*H(2)^3-54*X(2)*Y(2)-27*X(3)*Y(3)-9*H(1)*H(2)-18*H(2)^2+18*H(1)+36*H(2)$.

Clearly, $\mathcal{CZ}(\mathcal{U}(\mathfrak{sl}_3))$ is generated by the following elements:

- $H(1), H(2)$,
- $X(1)*Y(1)+X(2)*Y(2)+X(3)*Y(3)$,
- $X(3)*Y(1)*Y(2)+X(1)*X(2)*Y(3)+X(2)*Y(2)*H(1)+X(2)*Y(2)*H(2)+X(3)*Y(3)*H(2)-2*X(2)*Y(2)-X(3)*Y(3)$.

A Gel'fand-Zetlin subalgebra $\Gamma(\mathcal{U}(\mathfrak{sl}_3))$ is generated by Cartan elements $H(1), H(2)$, Casimir elements of $\mathcal{U}(\mathfrak{sl}_3)$ and the image of Casimir element $C = 4*X(1)*Y(1)+H(1)^2-2*H(1)$ of $\mathcal{U}(\mathfrak{sl}_2)$ under an imbedding $i : \mathcal{U}(\mathfrak{sl}_2) \hookrightarrow \mathcal{U}(\mathfrak{sl}_3)$ from the following list:

1. $i_1 : \mathcal{U}(\mathfrak{sl}_2) \hookrightarrow \mathcal{U}(\mathfrak{sl}_3)$ given by:

$$X(1) \mapsto X(1), \quad Y(1) \mapsto Y(1), \quad H(1) \mapsto H(1).$$

2. $i_2 : \mathcal{U}(\mathfrak{sl}_2) \hookrightarrow \mathcal{U}(\mathfrak{sl}_3)$ given by:

$$X(1) \mapsto X(2), \quad Y(1) \mapsto Y(2), \quad H(1) \mapsto H(2).$$

3. $i_3 : \mathcal{U}(\mathfrak{sl}_2) \hookrightarrow \mathcal{U}(\mathfrak{sl}_3)$ given by:

$$X(1) \mapsto X(3), \quad Y(1) \mapsto Y(3), \quad H(1) \mapsto H(1) + H(2).$$

Let us use Singular to compute corresponding Gel'fand-Zetlin subalgebras:

```
> LIB "center.lib";
> LIB "algebras.lib";
> def A1 = makeUsl2(0); setring A1; A1; // U(sl_2):
//   characteristic : 0
//   number of vars : 3
//        block   1 : ordering dp
//                  : names    X(1) Y(1) H(1)
//        block   2 : ordering C
//   noncommutative relations:
//    Y(1)X(1)=X(1)*Y(1)-H(1)
//    H(1)X(1)=X(1)*H(1)+2*X(1)
//    H(1)Y(1)=Y(1)*H(1)-2*Y(1)
> ideal Z1 = centerRed(2, 1); Z1;
Z1[1]=4*X(1)*Y(1)+H(1)^2-2*H(1) // =: C
> ideal HH = H(1);
> ideal GZ1= HH + Z1;
> ideal GZ = sa_reduce(GZ1); GZ;
GZ[1]=H(1)
GZ[2]=X(1)*Y(1)
> centralizerRed(GZ[size(GZ)], 6); // Check conjecture:
_[1]=H(1)
_[2]=X(1)*Y(1) // Ok!

> def A2 = makeUsl3(p); setring A2; A2; // U(sl_3):
//   characteristic : 0
//   number of vars : 8
//        block   1 : ordering dp
//                  : names    X(1) X(2) X(3) Y(1) Y(2) Y(3) H(1) H(2)
//        block   2 : ordering C
//   noncommutative relations:
//    X(2)X(1)=X(1)*X(2)+X(3)
//    Y(1)X(1)=X(1)*Y(1)-H(1)
//    Y(3)X(1)=X(1)*Y(3)-Y(2)
//    H(1)X(1)=X(1)*H(1)+2*X(1)
//    H(2)X(1)=X(1)*H(2)-X(1)
```

```
//    Y(2)X(2)=X(2)*Y(2)-H(2)
//    Y(3)X(2)=X(2)*Y(3)+Y(1)
//    H(1)X(2)=X(2)*H(1)-X(2)
//    H(2)X(2)=X(2)*H(2)+2*X(2)
//    Y(1)X(3)=X(3)*Y(1)-X(2)
//    Y(2)X(3)=X(3)*Y(2)+X(1)
//    Y(3)X(3)=X(3)*Y(3)-H(1)-H(2)
//    H(1)X(3)=X(3)*H(1)+X(3)
//    H(2)X(3)=X(3)*H(2)+X(3)
//    Y(2)Y(1)=Y(1)*Y(2)-Y(3)
//    H(1)Y(1)=Y(1)*H(1)-2*Y(1)
//    H(2)Y(1)=Y(1)*H(2)+Y(1)
//    H(1)Y(2)=Y(2)*H(1)+Y(2)
//    H(2)Y(2)=Y(2)*H(2)-2*Y(2)
//    H(1)Y(3)=Y(3)*H(1)-Y(3)
//    H(2)Y(3)=Y(3)*H(2)-Y(3)
> ideal Z2 = centerRed(3, 2); Z2;
Z2[1]=3*X(1)*Y(1)+3*X(2)*Y(2)+3*X(3)*Y(3)+H(1)^2+H(1)*H(2)+H(2)^2-3*H(1)-3*H(2)
Z2[2]=27*X(3)*Y(1)*Y(2)+27*X(1)*X(2)*Y(3)-9*X(1)*Y(1)*H(1)+18*X(2)*Y(2)*H(1)
-9*X(3)*Y(3)*H(1)-2*H(1)^3-18*X(1)*Y(1)*H(2)+9*X(2)*Y(2)*H(2)+9*X(3)*Y(3)*H(2)
-3*H(1)^2*H(2)+3*H(1)*H(2)^2+2*H(2)^3-54*X(2)*Y(2)-27*X(3)*Y(3)-9*H(1)*H(2)-
18*H(2)^2+18*H(1)+36*H(2)
> ideal HH = H(1), H(2);

// i_1:
> map IdMap1 = A1, X(1), Y(1), H(1);
> ideal Z1 = IdMap1(Z1); Z1;
Z1[1]=4*X(1)*Y(1)+H(1)^2-2*H(1) // == i_1(C)
> ideal GZ2 = HH + Z1 + Z2;
> ideal GZ = sa_reduce(GZ2); GZ; // Gel'fand-Zetlin 1:
GZ[1]=H(2)
GZ[2]=H(1)
GZ[3]=X(2)*Y(2)+X(3)*Y(3)
GZ[4]=X(1)*Y(1)
GZ[5]=X(3)*Y(1)*Y(2)+X(1)*X(2)*Y(3)-X(3)*Y(3)*H(1)+X(3)*Y(3)
> centralizerRed(GZ[size(GZ)], 5); // Check conjecture:
_[1]=H(2)
_[2]=H(1)
_[3]=X(2)*Y(2)+X(3)*Y(3)
_[4]=X(1)*Y(1)
_[5]=X(3)*Y(1)*Y(2)+X(1)*X(2)*Y(3)-X(3)*Y(3)*H(1)+X(3)*Y(3) // Ok!

// i_2:
> map IdMap2 = A1, X(2), Y(2), H(2);
> Z1 = IdMap2(Z1); Z1;
Z1[1]=4*X(2)*Y(2)+H(2)^2-2*H(2) // == i_2(C)
> GZ2 = HH + Z1 + Z2;
> GZ = sa_reduce(GZ2); GZ; // Gel'fand-Zetlin 2:
GZ[1]=H(2)
GZ[2]=H(1)
GZ[3]=X(2)*Y(2)
```

```
GZ[4]=X(1)*Y(1)+X(3)*Y(3)
GZ[5]=X(3)*Y(1)*Y(2)+X(1)*X(2)*Y(3)+X(3)*Y(3)*H(2)-X(3)*Y(3)
> centralizerRed(GZ[size(GZ)], 5); // Check conjecture:
_[1]=H(2)
_[2]=H(1)
_[3]=X(2)*Y(2)
_[4]=X(1)*Y(1)+X(3)*Y(3)
_[5]=X(3)*Y(1)*Y(2)+X(1)*X(2)*Y(3)+X(3)*Y(3)*H(2)-X(3)*Y(3) // Ok!

// i_3:
> map IdMap3 = A1, X(3), Y(3), H(1) + H(2);
> Z1 = IdMap3(Z1); Z1;
Z1[1]=4*X(3)*Y(3)+H(1)^2+2*H(1)*H(2)+H(2)^2-2*H(1)-2*H(2) // == i_3(C)
> GZ2 = HH + Z1 + Z2;
> GZ = sa_reduce(GZ2); GZ; // Gel'fand-Zetlin 3:
GZ[1]=H(2)
GZ[2]=H(1)
GZ[3]=X(3)*Y(3)
GZ[4]=X(1)*Y(1)+X(2)*Y(2)
GZ[5]=X(3)*Y(1)*Y(2)+X(1)*X(2)*Y(3)+X(2)*Y(2)*H(1)+X(2)*Y(2)*H(2)-2*X(2)*Y(2)
> centralizerRed(GZ[size(GZ)], 5); // Check conjecture:
_[1]=H(2)
_[2]=H(1)
_[3]=X(3)*Y(3)
_[4]=X(1)*Y(1)+X(2)*Y(2)
_[5]=X(3)*Y(1)*Y(2)+X(1)*X(2)*Y(3)+X(2)*Y(2)*H(1)+X(2)*Y(2)*H(2)-2*X(2)*Y(2) // Ok!
```

Result 6.6. Our computations show that $\Gamma(\mathcal{U}(\mathfrak{sl}_3))$ is generated:

1. in the case of imbedding i_1 by
   - $H(1), H(2)$,
   - $X(1) * Y(1)$,
   - $X(2) * Y(2) + X(3) * Y(3)$,
   - $X(3) * Y(1) * Y(2) + X(1) * X(2) * Y(3) - X(3) * Y(3) * H(1) + X(3) * Y(3)$.
2. in the case of imbedding i_2 by
   - $H(1), H(2)$,
   - $X(2) * Y(2)$,
   - $X(1) * Y(1) + X(3) * Y(3)$,
   - $X(3) * Y(1) * Y(2) + X(1) * X(2) * Y(3) + X(3) * Y(3) * H(2) - X(3) * Y(3)$.
3. in the case of imbedding i_3 by
   - $H(1), H(2)$,
   - $X(3) * Y(3)$,

- $X(1) * Y(1) + X(2) * Y(2)$,
- $X(3) * Y(1) * Y(2) + X(1) * X(2) * Y(3) + X(2) * Y(2) * H(1) + X(2) * Y(2) * H(2) - 2 * X(2) * Y(2)$.

Moreover, let g be a computed generator of $\Gamma(\mathcal{U}(\mathfrak{sl}_n))$ of degree 3. Our computations show that $\mathrm{Cen}(g, \mathcal{U}(\mathfrak{sl}_n)_5)$ has the same generators as $\Gamma(\mathcal{U}(\mathfrak{sl}_n))$, for $n = 2, 3$ for every imbedding.

Chapter 7

Algebraic dependence of polynomials

In this chapter our aim is to compute the algebraic dependence between the subalgebra reduced generators of the center. Since they pairwise commute, we are almost in commutative case.

The algebraic relation between the generators of the center of $\mathcal{U}(\mathfrak{sl}_2)$ over a field of characteristic $p > 2$ is known due to [41]:

Example 7.1. Let us consider the algebra $\mathcal{U}(\mathfrak{sl}_2)$ as in section 3.1, then the algebra $Z = \mathcal{Z}(\mathcal{U}(\mathfrak{sl}_2))$ is generated by the elements $x = e^p, y = f^p, z = h^p - h, t = (h+1)^2 + 4f * e$, which satisfy the following relation:

$$\prod_{k=0}^{p-1}(t - k^2) = z^2 + 4xy,$$

which defines the algebra Z.

7.1 General setting

In the case of polynomial ring we can compute the algebraic relations by means of elimination: let $f_1, \ldots, f_m \in \Bbbk[x_1, \ldots, x_n]$, and let $I = \langle Y_1 - f_1, \ldots, Y_m - f_m \rangle \subseteq \Bbbk[x_1, \ldots, x_n, Y_1, \ldots, Y_m]$. Then $I \cap \Bbbk[Y_1, \ldots, Y_m]$ are the algebraic relations between $f_1, \ldots, f_m$ (cf. [42])

7.2 Perron's Theorem

This section is due to [40]. Throughout this section, $\Bbbk$ will stand either for a field of characteristic 0 or for an infinite field of prime characteristic.

Another way to compute the algebraic dependence provides the following theorem (due to Oskar Perron, see [37, Satz 57] or [36]):

Theorem 7.2. *Let $F_1, \dots, F_{n+1} \in \Bbbk[X]$ be a sequence of $n+1$ non-constant polynomials in n variables $X = (X_1, \dots, X_n)$ and let $deg(F_i) = d_i$ for $i = 1, \dots, n+1$. Then there exists a non-zero polynomial $P = P(Y) \in \Bbbk[Y]$ in $n+1$ variables $Y = (Y_1, \dots, Y_{n+1})$ such that*

(a) $P(F_1, \dots, F_{n+1}) = 0$,

(b) if $weight(Y_i) = d_i$ for $i = 1, \dots, n+1$ then $weight(P) \leqslant d_1 \cdots d_{n+1}$.

A polynomial P satisfying the conditions (a) and (b) will be called a **Perron's relation** between $F_1, \dots, F_{n+1}$.

Note that

$$\deg(P) \leqslant \frac{\prod_{i=1}^{n+1} d_i}{\min_{i=1}^{n+1} d_i} \leqslant (\max_{i=1}^{n+1} d_i)^n.$$

In general the Perron's relation is not uniquely determined by the given sequence of polynomials. But if the sequence $F_1, \dots, F_{n+1}$ contains n algebraically independent polynomials then there exists a unique irreducible polynomial (up to a constant factor in $\Bbbk^*$) $P_0 = P_0(Y)$ such that $P_0(F_1, \dots, F_{n+1}) = 0$. The polynomial P_0 is called the **minimal polynomial** of $F_1, \dots, F_{n+1}$.

Proposition 7.3. *Suppose that the sequence $F_1, \dots, F_{n+1}$ contains n algebraically independent polynomials. Then the minimal polynomial of $F_1, \dots, F_{n+1}$ is a Perron's relation between $F_1, \dots, F_{n+1}$.*

7.3 Our approach

Let us consider the following ring homomorphism:

$$\Psi : \Bbbk[t_1, \dots, t_{n+1}] \ni t_i \mapsto F_i \in \mathcal{A}. \tag{7.1}$$

Clearly $\operatorname{Ker}\Psi$ is the set of relations between F_i.

Therefore, in order to find relations between F_i we intersect $\operatorname{Ker}\Psi$ with the vector space of polynomials in $T = (t_1, \dots, t_{n+1})$ of degree less or equal to $D := \frac{\prod_{i=1}^{n+1} d_i}{min_{i=1}^{n+1} d_i}$.

This gives us an algorithm 7.1 for the computation of relations between a set of pairwise commuting polynomials.

ASSUME: $\mathcal{A}$ is a $\mathbb{k}$-algebra
INPUT: A set of polynomials $F = \{F_1, \ldots, F_{n+1}\} \subset \mathcal{A}$, and optional bound D
OUTPUT: relations between $F_1, \ldots, F_{n+1}$
  if no bound D was specified **then**
    let $D = \frac{\prod_{i=1}^{n+1} deg(F_i)}{min_{i=1}^{n+1} deg(F_i)}$;
  end if
  let Ψ be the map given by equation (7.1);
  let $Basis = \{v_1, \ldots, v_k\}$ be a set of monomials in $T = (t_1, \ldots, t_{n+1})$ of degree less or equal to D;
  let $Images = \{w_1, \ldots, w_k\}$, where $w_i = \Psi v_i$;
RETURN: LINEARMAPKERNEL($\{v_1, \ldots, v_k\}, \{w_1, \ldots, w_k\}$); // using algorithm 2.1

Algorithm 7.1: PERRON(set F[, int D])

7.4 Examples

Now we will use `perron.lib` in order to check the result of [41]:

```
> LIB "perron.lib";
> proc Test( int p )
>
>     ///////////////////////////////////////////////////////////////
>     // char = p
>     def g = makeUsl2(p); setring g;      // only 'p' will be shown
>
>     ideal L = e^p, f^p, h^p-h, 4*e*f+h^2-2*h; // the Center
>
>     def R = perron( L, p ); // !
>     setring R;
>
>     Relations;      // This will be shown
>     poly P = Relations[1];
>     poly Q = P+4*F(1)*F(2)+F(3)^2;
>     Q;              // Q will be shown
>     factorize( Q ); // This will be also shown
>
> intvec Prims = 3, 5, 7, 11, 13;

> // Our computations:
> for( int i = 1; i <= size(Prims); i++ )
> Test( Prims[i] );

///////////////////////////////////////////////////////////////
// char = 3:

// Relations:
Relations[1]=F(4)^3-F(1)*F(2)-F(3)^2+F(4)^2
// Q:
```

```
F(4)^3+F(4)^2
// factorize(Q):
[1]:
   _[1]=1
   _[2]=F(4)+1
   _[3]=F(4)
[2]:
   1,1,2

//////////////////////////////////////////////////////////////
// char = 5:

// Relations:
Relations[1]=F(4)^5-2*F(4)^3+F(1)*F(2)-F(3)^2-F(4)^2
// Q:
F(4)^5-2*F(4)^3-F(4)^2
// factorize(Q):
[1]:
   _[1]=1
   _[2]=F(4)+1
   _[3]=F(4)
   _[4]=F(4)+2
[2]:
   1,1,2,2

//////////////////////////////////////////////////////////////
// char = 7:

// Relations:
Relations[1]=F(4)^7-2*F(4)^4-F(4)^3+3*F(1)*F(2)-F(3)^2+2*F(4)^2
// Q:
F(4)^7-2*F(4)^4-F(4)^3+2*F(4)^2
// factorize(Q):
[1]:
   _[1]=1
   _[2]=F(4)+1
   _[3]=F(4)
   _[4]=F(4)-3
   _[5]=F(4)-1
[2]:
   1,1,2,2,2

//////////////////////////////////////////////////////////////
// char = 11:

// Relations:
Relations[1]=F(4)^11-2*F(4)^6-F(4)^5+3*F(4)^4+4*F(4)^3
-4*F(1)*F(2)-F(3)^2+3*F(4)^2
// Q:
F(4)^11-2*F(4)^6-F(4)^5+3*F(4)^4+4*F(4)^3+3*F(4)^2
// factorize(Q):
```

```
[1]:
   _[1]=1
   _[2]=F(4)+1
   _[3]=F(4)-2
   _[4]=F(4)
   _[5]=F(4)+3
   _[6]=F(4)-4
   _[7]=F(4)-3
[2]:
   1,1,2,2,2,2,2

/////////////////////////////////////////////////////////////
// char = 13:

// Relations:
Relations[1]=F(4)^13-2*F(4)^7-F(4)^6-3*F(4)^5-5*F(4)^4-5*F(4)^3
-4*F(1)*F(2)-F(3)^2-3*F(4)^2
// Q:
F(4)^13-2*F(4)^7-F(4)^6-3*F(4)^5-5*F(4)^4-5*F(4)^3-3*F(4)^2
// factorize(Q):
[1]:
   _[1]=1
   _[2]=F(4)+1
   _[3]=F(4)+4
   _[4]=F(4)-2
   _[5]=F(4)
   _[6]=F(4)+2
   _[7]=F(4)-3
   _[8]=F(4)+5
[2]:
   1,1,2,2,2,2,2,2
```

The only difference to [41] is caused by the minus in their $\mathcal{U}(\mathfrak{sl}_2)$ relations and by our choice of t equals $4ef + h^2 - 2h$ instead of t equals $4ef + h2 - 2h + 1$. Now one can observe that our computations produce the same relations.

Obviously, this method can be used in commutative case, as the following example shows:

```
> LIB "perron.lib";
> ring r=0,(x,y,z),dp;
>
> ideal J = xy+z2, z2+y2, x2y2-2xy3+y4;
> def P2 = perron(J);
> // algebraicDep was taken from "http://www.singular.uni-kl.de/Manual/3-0-0/sing_484.htm"
> algebraicDep(J,0);
_[1]=Y(1)^2-2*Y(1)*Y(2)+Y(2)^2-Y(3)
>
> setring P2; Relations;
Relations[1]=F(1)^2-2*F(1)*F(2)+F(2)^2-F(3)
```

Chapter 8

Appendix

In this chapter we provide User's manual for the described libraries `center.lib` and `perron.lib`.

Additionally, we provide benchmarks for the computation of central elements of various $\Bbbk$-algebras with the procedure CENTERVSfrom `center.lib`.

8.1 Debugging `center.lib`

The library `center.lib` has the following imbedded debugging mechanism:

- if `printlevel` $\geqslant 2$ then some progress information will be produced.
- if `printlevel` $\geqslant 3$ and there was defined a variable `@@@DEBUG` then almost every internal procedure (except trivial ones) will print its calling parameters and the computed result.

8.2 center.lib

Library:

center.lib

Purpose:

computation of central elements of GR-algebras

Author:

Oleksandr Motsak, motsak@mathematik.uni-kl.de.

Overview:

A library for computing elements of the center and centralizers of elements in various non-commutative algebras.

Support:

Forschungsschwerpunkt 'Mathematik und Praxis', University of Kaiserslautern

8.2.1 centralizeSet

Procedure from library `center.lib` (section 8.2 [center_lib] on this page).

Input:

a finite set of elements F, vector space basis V

Return:

ideal, generated by base elements

Purpose:

computes the vector space basis of the centralizer of F in the vector space spanned by V, that is, Cen(F[N],Cen(F[N-1],...,Cen(F[1],V)...))

Example:

```
LIB "center.lib";
ring A_4_1 = 0,(e(1..4)),dp;
matrix D[4][4]=0;
D[2,4] = -e(1);
D[3,4] = -e(2);
// This is $A_{41}$ - the first real Lie algebra of dimension $4$.
ncalgebra(1,D);
ideal F = variablesSorted(); F;
==> F[1]=e(1)
==> F[2]=e(4)
==> F[3]=e(3)
==> F[4]=e(2)
// the center of $A_{41}$ is generated by
// $e(1)$ and $-1/2*e(2)^2+e(1)*e(3)$
// therefore one may consider computing it in the following way:
// 1. Compute PBW basis consisting of
//    monomials of exponent <= (1,2,1,0)
ideal V = PBW_maxMonom( e(1) * e(2)^ 2 * e(3) );
// 2. Compute the centralizer of F within vector space
//    spanned by these monomials:
ideal C = centralizeSet( F, V ); C;
==> C[1]=e(1)
==> C[2]=e(2)^2-2*e(1)*e(3)
inCenter(C);
==> 1
```

8.2.2 centralizerVS

Procedure from library `center.lib` (section 8.2 [center_lib] on the preceding page).

Usage:

centralizerVS(F, D); ideal F, int D

Return:

ideal, generated by elements of degree <= D

Purpose:

computes a vector space basis of the centralizer of F up to degree D.

Note:

D must be non-negative

Example:

```
LIB "center.lib";
ring A = 0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z;  D[1,3]=2*x;  D[2,3]=-2*y;
ncalgebra(1,D); // this algebra is U(sl_2)
ideal F = x, y;
// find all elements commuting with x and y of degree <= 4:
ideal C = centralizerVS(F, 4);
C;
==> C[1]=4xy+z2-2z
==> C[2]=16x2y2+8xyz2+z4-32xyz-4z3-4z2+16z
inCentralizer(C, F);
==> 1
```

8.2.3 centralizerRed

Procedure from library `center.lib` (section 8.2 [center_lib] on page 78).

Usage:

centralizerRed(F, D[, N]); ideal F, int D[, int N]

Return:

ideal, generated by computed generators

Purpose:

if N is absent and D >= 0 computes a subalgebra generators of the centralizer of F up to degree D, otherwise if N is present computes N(at least) first generators of the centralizer, if moreover D > 0 it will be used as the first maximal degree estimation.

Note:

Current ordering must be a degree compatible well-ordering.

Example:

```
LIB "center.lib";
ring A = 0,(x,y,z),dp;
matrix D[3][3]=0;
```

```
D[1,2]=-z;  D[1,3]=2*x;  D[2,3]=-2*y;
ncalgebra(1,D); // this algebra is U(sl_2)
ideal F = x, y;
// find subalgebra generators degree <= 4 of an algebra of
// all elements commuting with x and y:
ideal C = centralizerRed(F, 4);
C;
==> C[1]=4xy+z2-2z
inCentralizer(C, F);
==> 1
```

8.2.4 centerVS

Procedure from library `center.lib` (section 8.2 [center_lib] on page 78).

Usage:

centerVS(D); int D

Return:

ideal, generated by elements of degree <= D

Purpose:

computes a vector space basis of the center of the current algebra up to degree D.

Note:

D must be non-negative

Example:

```
LIB "center.lib";
ring A = 0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z;  D[1,3]=2*x;  D[2,3]=-2*y;
ncalgebra(1,D); // this algebra is U(sl_2)
// find all central elements of degree <= 4
ideal Z = centerVS(4);
Z;
==> Z[1]=4xy+z2-2z
==> Z[2]=16x2y2+8xyz2+z4-32xyz-4z3-4z2+16z
```

```
// note that the second element is the square of the first
// plus the multiple of the first:
Z[2] - Z[1]^2 + 8*Z[1];
==> 0
inCenter(Z);
==> 1
```

8.2.5 centerRed

Procedure from library `center.lib` (section 8.2 [center_lib] on page 78).

Usage:

centerRed(D[, k]); int D[, int k]

Return:

ideal, generated by computed generators

Purpose:

if N is absent and D >= 0 computes a subalgebra generators of the center up to degree D, otherwise if N is present computes N(at least) first generators of the center, if moreover D > 0 it will be used as the first maximal degree estimation.

Note:

Current ordering must be a degree compatible well-ordering.

Example:

```
LIB "center.lib";
ring A = 0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=z;
ncalgebra(1,D); // it is a Heisenberg algebra
// find vector space basis of center of degree <= 3
ideal VSZ = centerVS(3);
// There should be 3 degrees of z.
VSZ;
==> VSZ[1]=z
==> VSZ[2]=z2
==> VSZ[3]=z3
```

```
inCenter(VSZ);
==> 1
// find "minimal" central elements of degree <= 3
ideal SAZ = centerRed(3);
// Only 'z' must be computed
SAZ;
==> SAZ[1]=z
inCenter(SAZ);
==> 1
```

8.2.6 center

Procedure from library `center.lib` (section 8.2 [center_lib] on page 78).

Return:

ideal, generated by elements of degree at most D

Purpose:

computes a minimal set of central elements up to degree D.

Note:

In general, one cannot predict the number or the highest degree of central elements. Hence, one has to specify a termination condition via arguments D and/or N.
If D is positive, the computation stops after all central elements of degree at most D has been found.
If D is negative, the termination is determined by N only.
If N is given, the computation stops if at least N central elements has been found.
Warning: if N is given and bigger than the actual number of generators, the procedure may not terminate.

Example:

```
LIB "center.lib";
ring A = 0,(x,y,z,t),dp;
matrix D[4][4]=0;
D[1,2]=-z;  D[1,3]=2*x;  D[2,3]=-2*y;
ncalgebra(1,D); // this algebra is U(sl_2) tensored with K[t]
ideal Z = center(3); // find all central elements of degree <= 3
Z;
==> Z[1]=t
```

```
==> Z[2]=4xy+z2-2z
inCenter(Z);
==> 1
ideal ZZ = center(-1, 1); // find one central element of the lowest degree
ZZ;
==> ZZ[1]=t
inCenter(ZZ);
==> 1
```

8.2.7 centralizer

Procedure from library `center.lib` (section 8.2 [center_lib] on page 78).

Return:

ideal, generated by elements of degree <= MaxDeg

Purpose:

computes a minimal set of elements centralizer(S) up to degree MaxDeg.

Note:

In general, one cannot predict the number or the highest degree of centralizing elements.
Hence, one has to specify a termination condition via arguments MaxDeg and/or N.
If MaxDeg is positive, the computation stops after all centralizing elements of degree at most MaxDeg has been found.
If MaxDeg is negative, the termination is determined by N only.
If N is given, the computation stops if at least N centralizing elements has been found.
Warning: if N is given and bigger than the actual number of generators, the procedure may not terminate.

Example:

```
LIB "center.lib";
ring A = 0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z; D[1,3]=2*x; D[2,3]=-2*y;
ncalgebra(1,D); // this algebra is U(sl_2)
poly f = 4*x*y+z^2-2*z; // a central polynomial
f;
==> 4xy+z2-2z
ideal c = centralizer(f, 2); // find all elements of the centralizer of f
```

```
// of degree <= 2
c;  // since f is central, the answer consists of generators of A
==> c[1]=z
==> c[2]=y
==> c[3]=x
inCentralizer(c, f);
==> 1
ideal cc = centralizer(f,-1,2); // find at least two elements of the centralizer of f
cc;
==> cc[1]=z
==> cc[2]=y
==> cc[3]=x
inCentralizer(cc, f);
==> 1
poly g = z^2-2*z; // some non-central polynomial
c = centralizer(g, 2); // find all elements of the centralizer of g
// of degree <= 2
c;
==> c[1]=z
==> c[2]=xy
inCentralizer(c, g);
==> 1
centralizer(g,-1,1); // find the element of the lowest degree in the centralizer
==> _[1]=z
cc = centralizer(g,-1,2); // find at least two elements of the centralizer of g
cc;
==> cc[1]=z
==> cc[2]=xy
inCentralizer(cc, g);
==> 1
```

8.2.8 sa_reduce

Procedure from library `center.lib` (section 8.2 [center_lib] on page 78).

Return:

ideal, generated by found elements

Purpose:

compute a subalgebra basis of an algebra generated by polynomial from V

Note:

May produce wrong result in quotient algebras.

Example:

```
LIB "center.lib";
ring A = 0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z; D[1,3]=2*x; D[2,3]=-2*y;
ncalgebra(1,D); // this algebra is U(sl_2)
poly f = 4*x*y+z^2-2*z; // a central polynomial
ideal I = f, f*f, f*f*f - 10*f*f, f+3*z^3; I;
==> I[1]=4xy+z2-2z
==> I[2]=16x2y2+8xyz2+z4-32xyz-4z3+32xy+4z2
==> I[3]=64x3y3+48x2y2z2+12xyz4+z6-288x2y2z-96xyz3-6z5+352x2y2+224xyz2+2z4-12\
   8xyz+32z3-64xy-40z2
==> I[4]=3z3+4xy+z2-2z
sa_reduce(I); // should be just f and z^3
==> _[1]=4xy+z2-2z
==> _[2]=z3
```

8.2.9 sa_poly_reduce

Procedure from library `center.lib` (section 8.2 [center_lib] on page 78).

Return:

polynomial, a reduction of p wrt V

Purpose:

computes a reduction of a given polynomial p wrt a set of polynomials V

Note:

May produce wrong result in quotient algebras.

Example:

```
LIB "center.lib";
ring A = 0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z; D[1,3]=2*x; D[2,3]=-2*y;
ncalgebra(1,D); // this algebra is U(sl_2)
```

```
poly f = 4*x*y+z^2-2*z; // a central polynomial
sa_poly_reduce(f + 3*f*f + x, ideal(f) ); // should be just 'x'
==> x
```

8.2.10 inCenter

Procedure from library `center.lib` (section 8.2 [center_lib] on page 78).

Return:

integer, 1 if a in the center, 0 otherwise

Purpose:

check whether a given element is central

Example:

```
LIB "center.lib";
ring r=0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z;
D[1,3]=2*x;
D[2,3]=-2*y;
ncalgebra(1,D); // this is U(sl_2)
poly p=4*x*y+z^2-2*z;
inCenter(p);
==> 1
poly f=4*x*y;
inCenter(f);
==> 0
list l= list( 1, p, p^2, p^3);
inCenter(l);
==> 1
ideal I= p, f;
inCenter(I);
==> 0
```

8.2.11 inCentralizer

Procedure from library `center.lib` (section 8.2 [center_lib] on page 78).

Return:

integer, 1 if a in the centralizer(S), 0 otherwise

Purpose:

check whether a given element is centralizing with respect to elements of S

Example:

```
LIB "center.lib";
ring r=0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z;
ncalgebra(1,D); // the Heisenberg algebra
poly f = x^2;
poly a = z; // we know this element if central
poly b = y^2;
inCentralizer(a, f);
==> 1
inCentralizer(b, f);
==> 0
list  l = list(1, a);
inCentralizer(l, f);
==> 1
ideal I = a, b;
inCentralizer(I, f);
==> 0
printlevel = 2;
inCentralizer(a, f); // yes
==> 1
inCentralizer(b, f); // no
==> [1]:
==>    POLY: y2 is NOT in the centralizer of poly {x2}
==> 0
```

8.2.12 isCartan

Procedure from library `center.lib` (section 8.2 [center_lib] on page 78).

Purpose:

check whether f is a Cartan element.

Return:

1 if f is a Cartan element and 0 otherwise.

Note:

f is a Cartan element iff for all g in A there exists C in K such that [f, g] = C * g iff for all variables v_i of A there exist C in K such that [f, v_i] = C * v_i.

Example:

```
LIB "center.lib";
ring r=0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z;
D[1,3]=2*x;
D[2,3]=-2*y;
ncalgebra(1,D); // this is U(sl_2) with cartan - z
isCartan(z); // yes!
==> 1
poly p=4*x*y+z^2-2*z;
isCartan(p); // central elements are Cartan elements!
==> 1
poly f=4*x*y;
isCartan(f); // no way!
==> 0
isCartan( 10 + p + z ); // scalar + central + cartan
==> 1
```

8.2.13 applyAdF

Procedure from library `center.lib` (section 8.2 [center_lib] on page 78).

Usage:

applyAdF(Basis, f); ideal Basis, poly f

Purpose:

Apply Ad_{f} to every element of Basis

Return:

ideal, Ad_{f}(Basis)

Example:

```
LIB "center.lib";
ring A = 0,(e,f,h),dp;
matrix D[3][3]=0;
D[1,2]=-h;  D[1,3]=2*e;  D[2,3]=-2*f;
ncalgebra(1,D); // this algebra is U(sl_2)
// Let us consider the linear map Ad_{e} from A_2 into A.
// Compute the PBW basis of A_2:
ideal Basis = PBW_maxDeg( 2 ); Basis;
==> Basis[1]=e
==> Basis[2]=f
==> Basis[3]=h
==> Basis[4]=h2
==> Basis[5]=fh
==> Basis[6]=f2
==> Basis[7]=eh
==> Basis[8]=ef
==> Basis[9]=e2
// Compute images of basis elements under the linear map Ad_e:
ideal Image = applyAdF( Basis, e ); Image;
==> Image[1]=0
==> Image[2]=h
==> Image[3]=-2e
==> Image[4]=-4eh-4e
==> Image[5]=-2ef+h2+2h
==> Image[6]=2fh-2f
==> Image[7]=-2e2
==> Image[8]=eh
==> Image[9]=0
// Now we have a linear map given by: Basis_i --> Image_i
// Let's compute its kernel:
module C = linearMapKernel( Image ); C;
==> C[1]=gen(1)
==> C[2]=gen(8)+1/4*gen(4)-1/2*gen(3)
==> C[3]=gen(9)
// Now we can compute the kernel of Ad_e by means of basis vectors:
ideal K = linearCombinations(Basis, C); K;
==> K[1]=e
==> K[2]=ef+1/4h2-1/2h
==> K[3]=e2
// Let's check that Ad_e(K) is zero:
applyAdF( K, e );
==> _[1]=0
==> _[2]=0
==> _[3]=0
```

8.2.14 linearMapKernel

Procedure from library `center.lib` (section 8.2 [center_lib] on page 78).

Usage:

linearMapKernel(Images); ideal Images

Purpose:

Computes the kernel of a linear map given by its images on certain basis vectors

Return:

syzygy module, or 0 if all images are zeroes

Example:

```
LIB "center.lib";
ring A = 0,(e,f,h),dp;
matrix D[3][3]=0;
D[1,2]=-h;  D[1,3]=2*e;  D[2,3]=-2*f;
ncalgebra(1,D); // this algebra is U(sl_2)
// Let us consider the linear map Ad_{e} from A_2 into A.
// Compute the PBW basis of A_2:
ideal Basis = PBW_maxDeg( 2 ); Basis;
==> Basis[1]=e
==> Basis[2]=f
==> Basis[3]=h
==> Basis[4]=h2
==> Basis[5]=fh
==> Basis[6]=f2
==> Basis[7]=eh
==> Basis[8]=ef
==> Basis[9]=e2
// Compute images of basis elements under the linear map Ad_e:
ideal Image = applyAdF( Basis, e ); Image;
==> Image[1]=0
==> Image[2]=h
==> Image[3]=-2e
==> Image[4]=-4eh-4e
==> Image[5]=-2ef+h2+2h
```

```
==> Image[6]=2fh-2f
==> Image[7]=-2e2
==> Image[8]=eh
==> Image[9]=0
// Now we have a linear map given by: Basis_i --> Image_i
// Let's compute its kernel:
module C = linearMapKernel( Image ); C;
==> C[1]=gen(1)
==> C[2]=gen(8)+1/4*gen(4)-1/2*gen(3)
==> C[3]=gen(9)
// Now we can compute the kernel of Ad_e by means of basis vectors:
ideal K = linearCombinations(Basis, C); K;
==> K[1]=e
==> K[2]=ef+1/4h2-1/2h
==> K[3]=e2
// Let's check that Ad_e(K) is zero:
ideal Z = applyAdF( K, e ); Z;
==> Z[1]=0
==> Z[2]=0
==> Z[3]=0
// Now linearMapKernel will return a single integer 0:
def CC  = linearMapKernel(Z); typeof(CC); CC;
==> int
==> 0
```

8.2.15 linearCombinations

Procedure from library `center.lib` (section 8.2 [center_lib] on page 78).

Usage:

linearCombinations(Basis, C); ideal Basis, module C

Purpose:

computes linear combinations of Basis vectors with the coefficients from C.

Return:

ideal of linear combinations of Basis vectors with the coefficients from C.

Example:

```
LIB "center.lib";
ring A = 0,(e,f,h),dp;
matrix D[3][3]=0;
D[1,2]=-h;  D[1,3]=2*e;  D[2,3]=-2*f;
ncalgebra(1,D); // this algebra is U(sl_2)
// Let us consider the linear map Ad_{e} from A_2 into A.
// Compute the PBW basis of A_2:
ideal Basis = PBW_maxDeg( 2 ); Basis;
==> Basis[1]=e
==> Basis[2]=f
==> Basis[3]=h
==> Basis[4]=h2
==> Basis[5]=fh
==> Basis[6]=f2
==> Basis[7]=eh
==> Basis[8]=ef
==> Basis[9]=e2
// Compute images of basis elements under the linear map Ad_e:
ideal Image = applyAdF( Basis, e ); Image;
==> Image[1]=0
==> Image[2]=h
==> Image[3]=-2e
==> Image[4]=-4eh-4e
==> Image[5]=-2ef+h2+2h
==> Image[6]=2fh-2f
==> Image[7]=-2e2
==> Image[8]=eh
==> Image[9]=0
// Now we have a linear map given by: Basis_i --> Image_i
// Let's compute its kernel:
module C = linearMapKernel( Image ); C;
==> C[1]=gen(1)
==> C[2]=gen(8)+1/4*gen(4)-1/2*gen(3)
==> C[3]=gen(9)
// Now we can compute the kernel of Ad_e by means of basis vectors:
ideal K = linearCombinations(Basis, C); K;
==> K[1]=e
==> K[2]=ef+1/4h2-1/2h
==> K[3]=e2
// Let's check that Ad_e(K) is zero:
applyAdF( K, e );
==> _[1]=0
==> _[2]=0
==> _[3]=0
```

8.2.16 variablesStandard

Procedure from library `center.lib` (section 8.2 [center_lib] on page 78).

Return:

ideal, generated by algebra variables

Purpose:

computes the ideal generated by algebra variables taken in their natural order

Example:

```
LIB "center.lib";
ring A = 0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z;  D[1,3]=2*x;  D[2,3]=-2*y;
ncalgebra(1,D); // this algebra is U(sl_2)
// Variables in their natural order:
variablesStandard();
==> _[1]=x
==> _[2]=y
==> _[3]=z
```

8.2.17 variablesSorted

Procedure from library `center.lib` (section 8.2 [center_lib] on page 78).

Return:

ideal, generated by sorted algebra variables

Purpose:

computes the ideal generated by algebra variables sorted so that Cartan variables are first and all other variables are behind.

Note:

This is a heuristics for the computation of center: it is better to compute centralizers of Cartan variables first since we can omit solving the system of equations.

Example:

```
LIB "center.lib";
ring A = 0,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z;  D[1,3]=2*x;  D[2,3]=-2*y;
ncalgebra(1,D); // this algebra is U(sl_2)
// There is only one Cartan variable - z in U(sl_2),
// it must go 1st:
variablesSorted();
==> _[1]=z
==> _[2]=y
==> _[3]=x
```

8.2.18 PBW_eqDeg

Procedure from library `center.lib` (section 8.2 [center_lib] on page 78).

Usage:

PBW_eqDeg(Deg); int Deg

Purpose:

Compute the PBW basis (of a given degree) of a current algebra.

Return:

ideal consisting of PBW elements.

Note:

Unit is omitted. Weights are ignored!

Example:

```
LIB "center.lib";
ring A = 0,(e,f,h),dp;
matrix D[3][3]=0;
D[1,2]=-h;  D[1,3]=2*e;  D[2,3]=-2*f;
ncalgebra(1,D); // this algebra is U(sl_2)
// PBW Basis of A_2 \ A_1 - monomials of degree == 2:
PBW_eqDeg( 2 );
==> _[1]=h2
==> _[2]=fh
```

```
==> _[3]=f2
==> _[4]=eh
==> _[5]=ef
==> _[6]=e2
```

8.2.19 PBW_maxDeg

Procedure from library `center.lib` (section 8.2 [center_lib] on page 78).

Usage:

PBW_maxDeg(MaxDeg); int MaxDeg

Purpose:

Compute the PBW basis (up to a given maximal degree) of a current algebra.

Return:

ideal consisting of PBW elements.

Note:

unit is omitted. Weights are ignored!

Example:

```
LIB "center.lib";
ring A = 0,(e,f,h),dp;
matrix D[3][3]=0;
D[1,2]=-h;  D[1,3]=2*e;  D[2,3]=-2*f;
ncalgebra(1,D); // this algebra is U(sl_2)
// PBW Basis of A_2 - monomials of degree <= 2, without unit:
PBW_maxDeg( 2 );
==> _[1]=e
==> _[2]=f
==> _[3]=h
==> _[4]=h2
==> _[5]=fh
==> _[6]=f2
==> _[7]=eh
==> _[8]=ef
==> _[9]=e2
```

8.2.20 PBW_maxMonom

Procedure from library `center.lib` (section 8.2 [center_lib] on page 78).

Usage:

PBW_maxMonom(m); poly m

Purpose:

Compute the PBW basis, up to a given maximal exponent, of a current algebra.

Input:

Maximal exponent is given by the corresponding monomial.

Return:

ideal consisting of PBW elements.

Note:

Unit is omitted. Weights are ignored!

Example:

```
LIB "center.lib";
ring A = 0,(e,f,h),dp;
matrix D[3][3]=0;
D[1,2]=-h;  D[1,3]=2*e;  D[2,3]=-2*f;
ncalgebra(1,D); // this algebra is U(sl_2)
// At most 1st degree in e, h and at most 2nd degree in f, unit is omitted:
PBW_maxMonom( e*(f^2)* h );
==> _[1]=e
==> _[2]=f
==> _[3]=ef
==> _[4]=f2
==> _[5]=ef2
==> _[6]=h
==> _[7]=eh
==> _[8]=fh
==> _[9]=efh
==> _[10]=f2h
==> _[11]=ef2h
```

8.3 perron.lib

Library:

perron.lib

Purpose:

computation of algebraic dependences

Authors:

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8.3.1 perron

Procedure from library `perron.lib` (section 8.3 [perron_lib] on this page).

Usage:

perron(L [, D])

Return:

a commutative ring, containing an exported ideal ‘Relations‘ with found polynomial relations.

Purpose:

computes relations between pairwise commuting polynomials of L[, up to a given degree bound D]

Note:

the implementation was partially inspired by the Perron’s theorem.

Example:

```
LIB "perron.lib";
int p = 3;
ring A = p,(x,y,z),dp;
matrix D[3][3]=0;
D[1,2]=-z; D[1,3]=2*x; D[2,3]=-2*y;
```

```
ncalgebra(1,D); // this algebra is U(sl_2)
ideal L = x^p, y^p, z^p-z, 4*x*y+z^2-2*z; // the center
def R = perron( L, p );
setring R;
R;
==> //   characteristic : 3
==> //   number of vars : 4
==> //        block   1 : ordering dp
==> //                  : names    F(1) F(2) F(3) F(4)
==> //        block   2 : ordering C
Relations; // it was exported from perron to be in the returned ring.
==> Relations[1]=F(4)^3-F(1)*F(2)-F(3)^2+F(4)^2
kill R;
// perron can be also used in a commutative case, for example:
ring r=0,(x,y,z),dp;
ideal J = xy+z2, z2+y2, x2y2-2xy3+y4;
def R = perron(J);
setring R;
Relations;
==> Relations[1]=F(1)^2-2*F(1)*F(2)+F(2)^2-F(3)
```

8.4 Benchmarks

To be able to compute in enveloping algebras over "computational" fields (say $\mathbb{Q}$, $\mathbb{F}_p$) we have taken Chevalley bases (cf. 1.3.4) of some simple Lie algebras from [10] and put them into the additional Singular library `algebras.lib`.

For automated benchmarks computation we use the system *SymbolicData* (cf. [44]). In order to make this system work for us we have added definitions of many GR-Algebras to it, taught it how to generate GR-algebra definitions for Singular:Plural and how to ask Singular to compute the center etc.

We have generated the *SymbolicData* definition for the following $\Bbbk$-algebras: $\mathcal{U}(\mathfrak{sl}_n)$ ($n = 2, \ldots, 9$), $\mathcal{U}(\mathfrak{so}_n)$ ($n = 5, \ldots, 12$), $\mathcal{U}(\mathfrak{sp}_n)$ ($n = 1, \ldots, 5$), $\mathcal{U}(\mathfrak{g}_2)$, $\mathcal{U}(\mathfrak{f}_4)$, $\mathcal{U}(\mathfrak{e}_n)$ ($n = 6, 7, 8$), $\mathcal{U}(\mathfrak{gl}_n)$ ($n = 2, \ldots, 6$), Heisenberg algebras: $\mathcal{H}_n$ ($n = 1, \ldots, 5$) and Weyl algebras: $\mathcal{W}_n$ ($n = 1, \ldots, 5$) over several "computational" fields: $\mathbb{Q}$, $\mathbb{F}_p$ ($p = 3, 5, 7, 11, 13$), and $\mathcal{U}_q(\mathfrak{sl}_n)$ ($n = 2, 3$), $\mathcal{U}'_q(\mathfrak{so}_3)$ over $\mathbb{Q}(q)$ with q being free parameter (this case is analogous to the case of characteristics 0) and q being p-th prime complex root of unity (this case is analogous to the case of characteristics p).

We provide benchmarks, computed in Singular 3–0–0 under GNU/Linux, running on AMD Athlon(TM) MP 2000+ (1666MHz) with 3Gb RAM, for the computation of vector space basis of $\mathcal{Z}_d$ with the procedure centerVSfrom the library **center.lib**.

In the following table the column "`Time`" contains the time (in seconds, up to 1/100 sec.) for the computation of a vector space basis of $\mathcal{Z}_d(\mathcal{A})$. Moreover for every degree d the column "`PBW`" contains the size of the first PBW base (it equals to $\dim \mathcal{A}_d - 1$); the column entitled "`DIM`" contains before slash "/" the dimensions of the actually computed center and after the slash the expected dimension because of remark 8.1. The computations are grouped by a $\Bbbk$-algebra: `Algebra` ($\mathcal{A}$). The ground field is specified by the parameter `p` (see above).

Note that, in degree 1 the first PBW basis consists of all variables and therefore we do not include the number of algebra generators into this table.

Remark 8.1. Let $\mathcal{A}$ be a $\Bbbk$-algebra filtered by degree and let the center $\mathcal{Z}$ of $\mathcal{A}$ be generated by r elements $z_1, \ldots, z_r$ of degrees $p_1, \ldots, p_r$. Then the dimension of $\mathcal{Z}_d(\mathcal{A})$ is the number of non-negative integer solutions of the inequality $t_1 \cdot p_1 + \ldots + t_r \cdot p_r \leqslant d$.

Although our estimations on the dimension of computed vector space basis are almost always true, there are several cases when they are false. For example in the case of $\mathcal{Z}_4(\mathcal{U}(\mathfrak{f}_4(\mathbb{F}_3)))$ we have computed one unexpected polynomial.

Unfortunately, in several cases we run out of memory. These cases denoted by "-4-" in `Time` column and by "?" in `DIM` column.

It appears that 3Gb of memory is not enough to compute a vector space basis of the center in the case when the first PBW base has dimension more than 1000000. Therefore we did not try to compute a vector space basis of the center in these cases, these examples are denoted by "–".

A	p	Degree: 1 DIM	PBW	Time	Degree: 2 DIM	PBW	Time	Degree: 3 DIM	PBW	Time	Degree: 4 DIM	PBW	Time	Degree: 5 DIM	PBW	Time	Degree: 6 DIM	PBW	Time
	0	1/1		0.04	2/2		0.04	3/3		0.05	4/4		0.07	5/5		0.09	6/6		0.13
	3	1/1		0.04	2/2		0.04	5/6		0.05	8/10		0.08	11/14		0.10	17/24		0.15
$\mathcal{H}_1$	5	1/1	3	0.03	2/2	9	0.04	3/3	19	0.05	4/4	34	0.06	7/8	55	0.09	10/12	83	0.14
	7	1/1		0.04	2/2		0.04	3/3		0.05	4/4		0.06	5/5		0.09	6/6		0.13
	11	1/1		0.04	2/2		0.04	3/3		0.05	4/4		0.06	5/5		0.09	6/6		0.14
	13	1/1		0.04	2/2		0.04	3/3		0.05	4/4		0.07	5/5		0.09	6/6		0.13
	0	1/1		0.05	2/2		0.06	3/3		0.13	4/4		0.31	5/5		0.91	6/6		2.56
	3	1/1		0.04	2/2		0.07	7/8		0.13	12/14		0.36	17/20		1.08	32/41		3.05
$\mathcal{H}_2$	5	1/1	5	0.05	2/2	20	0.07	3/3	55	0.13	4/4	125	0.32	9/10	251	0.92	14/16	461	2.64
	7	1/1		0.05	2/2		0.07	3/3		0.13	4/4		0.31	5/5		0.90	6/6		2.52
	11	1/1		0.04	2/2		0.06	3/3		0.13	4/4		0.33	5/5		0.90	6/6		2.54
	13	1/1		0.04	2/2		0.06	3/3		0.12	4/4		0.32	5/5		0.90	6/6		2.54
	0	1/1		0.05	2/2		0.11	3/3		0.40	4/4		2.05	5/5		9.63	6/6		40.25
	3	1/1		0.06	2/2		0.12	9/10		0.44	16/18		2.20	23/26		10.71	51/62		45.52
$\mathcal{H}_3$	5	1/1	7	0.06	2/2	35	0.11	3/3	119	0.41	4/4	329	2.03	11/12	791	9.51	18/20	1715	40.08
	7	1/1		0.06	2/2		0.12	3/3		0.40	4/4		2.03	5/5		9.46	6/6		39.64
	11	1/1		0.06	2/2		0.11	3/3		0.41	4/4		2.02	5/5		8.81	6/6		39.64
	13	1/1		0.05	2/2		0.11	3/3		0.41	4/4		2.04	5/5		9.49	6/6		39.61
	0	1/1		0.07	2/2		0.21	3/3		1.42	4/4		10.84	5/5		71.09	6/6		403.31
	3	1/1		0.06	2/2		0.19	11/12		1.42	20/22		11.38	29/32		76.23	74/87		437.12
$\mathcal{H}_4$	5	1/1	9	0.07	2/2	54	0.21	3/3	219	1.40	4/4	714	10.68	13/14	2001	70.02	22/24	5004	397.63
	7	1/1		0.07	2/2		0.21	3/3		1.39	4/4		10.64	5/5		69.82	6/6		397.84
	11	1/1		0.07	2/2		0.20	3/3		1.37	4/4		10.53	5/5		69.78	6/6		397.78
	13	1/1		0.07	2/2		0.20	3/3		1.39	4/4		10.64	5/5		69.86	6/6		397.28
	0	1/1		0.08	2/2		0.39	3/3		4.21	4/4		44.80	5/5		392.46	6/6		2957.73
	3	1/1		0.08	2/2		0.37	13/14		4.24	24/26		46.39	35/38		411.09	101/116		3125.58
$\mathcal{H}_5$	5	1/1	11	0.09	2/2	77	0.38	3/3	363	4.13	4/4	1364	44.14	15/16	4367	386.47	26/28	12375	2923.49
	7	1/1		0.09	2/2		0.38	3/3		4.15	4/4		44.12	5/5		385.86	6/6		2911.19
	11	1/1		0.08	2/2		0.38	3/3		4.13	4/4		43.81	5/5		385.41	6/6		2912.30
	13	1/1		0.08	2/2		0.38	3/3		4.13	4/4		44.13	5/5		384.93	6/6		2913.72
	0	0/0		0.04	1/1		0.05	1/1		0.07	2/2		0.12	2/2		0.18	3/3		0.39
	3	0/0		0.04	1/1		0.05	5/5		0.07	6/6		0.11	10/10		0.20	19/21		0.34
$\mathcal{U}_q(\mathfrak{sl}_2)$	5	0/0	4	0.04	1/1	14	0.05	1/1	34	0.07	2/2	69	0.12	6/6	125	0.20	7/7	209	0.39
	7	0/0		0.04	1/1		0.06	1/1		0.09	2/2		0.15	2/2		0.27	3/3		0.54
	11	0/0		0.04	1/1		0.07	1/1		0.11	2/2		0.22	2/2		0.45	3/3		1.39
	13	0/0		0.05	1/1		0.07	1/1		0.13	2/2		0.26	2/2		0.59	3/3		6.52
	0	0/-		0.05	0/-		0.14	0/-		1.48	0/-		17.43	0/-		230.05	0/-		7093.53
	3	0/-		0.05	2/-		0.15	16/-		1.36	18/-		15.14	34/-		181.80	119/-		4386.93
$\mathcal{U}_q(\mathfrak{sl}_3)$	5	0/-	10	0.05	0/-	65	0.17	0/-	285	1.58	0/-	1000	17.90	10/-	3002	221.24	12/-	8007	6021.15
	7	0/-		0.06	0/-		0.21	0/-		1.89	0/-		19.53	0/-		238.75	0/-		6704.85
	11	0/-		0.06	0/-		0.30	0/-		2.79	0/-		26.17	0/-		287.79	0/-		6740.51
	13	0/-		0.07	0/-		0.36	0/-		5.50	0/-		67.77	0/-		639.91	0/-		9766.86
	0	0/0		0.04	0/0		0.04	1/1		0.06	1/1		0.10	1/1		1.96	2/2		21.28
	3	0/0		0.04	0/0		0.04	4/4		0.06	4/4		0.10	4/4		0.18	14/14		0.37
$\mathcal{U}'_q(\mathfrak{so}_3)$			3			9			19			34			55			83	

	5	0/0		0.04	0/0		0.04	1/1		0.06	1/1		0.11	4/4		0.33	5/5		0.95
	7	0/0		0.04	0/0		0.04	1/1		0.07	1/1		0.12	1/1		0.89	2/2		3.08
	11	0/0		0.04	0/0		0.04	1/1		0.07	1/1		0.39	1/1		75.58	2/2		342.62
	13	0/0		0.04	0/0		0.04	1/1		0.08	1/1		3.86	1/1		1371.11	2/2		5990.08
	0	0/0		0.30	1/1		2.30	1/1		1029.31	?/2		-4 -	-/3		-	-/5		-
	3	1/0		0.37	2/1		2.13	80/79		1027.67	-/80		-	-/159		-	-/3242		-
$\mathcal{U}(\mathfrak{e}_6)$	5	0/0	78	0.29	1/1	3159	2.25	1/1	85319	1019.33	-/2	1749059	-	-/81	-	-	-/83	-	-
	7	0/0		0.29	1/1		2.27	1/1		1021.28	-/2		-	-/3		-	-/5		-
	11	0/0		0.29	1/1		2.27	1/1		1021.84	-/2		-	-/3		-	-/5		-
	13	0/0		0.29	1/1		2.26	1/1		1031.42	-/2		-	-/3		-	-/5		-
	0	0/0		0.69	1/1		18.53	1/1		26735.82	-/2		-	-/2		-	-/4		-
	3	0/0		0.69	1/1		18.32	134/134		26749.10	-/135		-	-/268		-	-/9181		-
$\mathcal{U}(\mathfrak{e}_7)$	5	0/0	133	0.70	1/1	9044	18.36	1/1	410039	26730.78	-/2	14043869	-	-/135	-	-	-/137	-	-
	7	0/0		0.70	1/1		18.36	1/1		26696.97	-/2		-	-/2		-	-/4		-
	11	0/0		0.66	1/1		18.39	1/1		26720.99	-/2		-	-/2		-	-/4		-
	13	0/0		0.70	1/1		18.39	1/1		26728.28	-/2		-	-/2		-	-/4		-
	0	0/0		2.43	1/1		306.23	?/1		-4 -	-/2		-	-/2		-	-/4		-
	3	0/0		2.34	1/1		305.19	?/249		-4 -	-/250		-	-/498		-	-/31376		-
$\mathcal{U}(\mathfrak{e}_8)$	5	0/0	248	2.35	1/1	31124	306.18	?/1	2604124	-4 -	-/2	164059874	-	-/250	-	-	-/252	-	-
	7	0/0		2.33	1/1		307.72	?/1		-4 -	-/2		-	-/2		-	-/4		-
	11	0/0		2.41	1/1		305.60	?/1		-4 -	-/2		-	-/2		-	-/4		-
	13	0/0		2.42	1/1		306.29	?/1		-4 -	-/2		-	-/2		-	-/4		-
	0	0/0		0.15	1/1		0.58	1/1		86.62	2/2		15734.95	-/2		-	-/4		-
	3	0/0		0.15	1/1		0.60	53/53		90.98	55/54		17871.89	-/106		-	-/1486		-
$\mathcal{U}(\mathfrak{f}_4)$	5	0/0	52	0.15	1/1	1430	0.57	1/1	26234	86.54	2/2	367289	15722.89	-/54	4187105	-	-/56	40475357	-
	7	0/0		0.14	1/1		0.59	1/1		86.61	2/2		15602.10	-/2		-	-/4		-
	11	0/0		0.15	1/1		0.59	1/1		87.32	2/2		15766.81	-/2		-	-/4		-
	13	0/0		0.14	1/1		0.58	1/1		87.83	2/2		15697.26	-/2		-	-/4		-
	0	0/0		0.05	1/1		0.08	1/1		0.21	2/2		1.09	2/2		16.96	4/4		875.20
	3	0/0		0.05	1/1		0.09	15/15		0.41	16/16		3.36	30/30		51.07	136/137		923.34
$\mathcal{U}(\mathfrak{g}_2)$	5	0/0	14	0.05	1/1	119	0.08	1/1	679	0.20	2/2	3059	1.12	16/16	11627	14.67	18/18	38759	199.88
	7	0/0		0.05	1/1		0.09	1/1		0.21	2/2		1.01	2/2		11.92	4/4		158.41
	11	0/0		0.05	1/1		0.09	1/1		0.20	2/2		1.01	2/2		11.73	4/4		155.40
	13	0/0		0.05	1/1		0.09	1/1		0.22	2/2		1.00	2/2		11.77	4/4		155.37
	0	1/1		0.04	3/3		0.04	5/5		0.05	8/8		0.08	11/11		0.11	15/15		0.18
	3	1/1		0.03	3/3		0.04	8/9		0.06	14/16		0.08	23/27		0.13	38/49		0.24
$\mathcal{U}(\mathfrak{gl}_2)$	5	1/1	4	0.04	3/3	14	0.04	5/5	34	0.05	8/8	69	0.07	14/15	125	0.10	21/23	209	0.17
	7	1/1		0.04	3/3		0.04	5/5		0.06	8/8		0.07	11/11		0.11	15/15		0.16
	11	1/1		0.04	3/3		0.04	5/5		0.05	8/8		0.07	11/11		0.11	15/15		0.15
	13	1/1		0.04	3/3		0.04	5/5		0.05	8/8		0.07	11/11		0.11	15/15		0.17
	0	1/1		0.05	3/3		0.07	6/6		0.14	10/10		0.40	15/15		1.29	22/22		4.55
	3	1/1		0.05	3/3		0.07	14/15		0.16	26/28		0.55	47/51		2.44	113/130		11.64
$\mathcal{U}(\mathfrak{gl}_3)$	5	1/1	9	0.05	3/3	54	0.07	6/6	219	0.13	10/10	714	0.39	23/24	2001	1.22	38/40	5004	4.28
	7	1/1		0.05	3/3		0.07	6/6		0.14	10/10		0.39	15/15		1.19	22/22		3.88
	11	1/1		0.05	3/3		0.07	6/6		0.13	10/10		0.28	15/15		1.20	22/22		3.93
	13	1/1		0.05	3/3		0.07	6/6		0.14	10/10		0.38	15/15		1.21	22/22		3.97

$\mathcal{U}(\mathfrak{gl}_4)$	0	1/1	16	0.08	3/3	152	0.12	6/6	968	0.47	11/11	4844	2.69	17/17	20348	44.50	26/26	74612	561.89
	3	1/1		0.07	3/3		0.12	21/22		0.53	41/43		3.70	77/81		61.36	250/274		841.57
	5	1/1		0.07	3/3		0.13	6/6		0.46	11/11		2.60	32/33		43.77	56/58		558.14
	7	1/1		0.07	3/3		0.13	6/6		0.45	11/11		2.63	17/17		43.65	26/26		552.77
	11	1/1		0.08	3/3		0.13	6/6		0.45	11/11		2.61	17/17		43.91	26/26		553.45
	13	1/1		0.07	3/3		0.13	6/6		0.45	11/11		2.62	17/17		43.67	26/26		552.26
$\mathcal{U}(\mathfrak{gl}_5)$	0	1/1	25	0.11	3/3	350	0.24	6/6	3275	1.50	11/11	23750	65.29	18/18	142505	2095.78	28/28	736280	54615.34
	3	1/1		0.11	3/3		0.24	30/31		1.72	59/61		69.20	114/118		2253.54	495/528		105871.91
	5	1/1		0.11	3/3		0.24	6/6		1.51	11/11		64.35	42/43		2081.64	76/78		54567.68
	7	1/1		0.11	3/3		0.23	6/6		1.49	11/11		64.50	18/18		2077.50	28/28		54306.00
	11	1/1		0.10	3/3		0.23	6/6		1.05	11/11		64.47	18/18		2093.32	28/28		54352.19
	13	1/1		0.11	3/3		0.23	6/6		1.54	11/11		64.37	18/18		2084.34	28/28		54430.49
$\mathcal{U}(\mathfrak{gl}_6)$	0	1/1	36	0.15	3/3	702	0.44	6/6	9138	11.79	11/11	91389	938.52	18/18	749397	60927.11	-/29	5245785	-
	3	1/1		0.16	3/3		0.44	41/42		12.21	81/83		965.84	158/162		63255.49	-/947		-
	5	1/1		0.16	3/3		0.44	6/6		11.53	11/11		940.75	53/54		60474.62	-/101		-
	7	1/1		0.17	3/3		0.44	6/6		11.51	11/11		953.49	18/18		61467.05	-/29		-
	11	1/1		0.16	3/3		0.43	6/6		11.67	11/11		939.31	18/18		60465.92	-/29		-
	13	1/1		0.17	3/3		0.43	6/6		11.55	11/11		963.30	18/18		60565.07	-/29		-
$\mathcal{U}(\mathfrak{sl}_2)$	0	0/0	3	0.03	1/1	9	0.03	1/1	19	0.04	2/2	34	0.05	2/2	55	0.05	3/3	83	0.06
	3	0/0		0.03	1/1		0.04	4/4		0.04	5/5		0.04	8/8		0.06	14/15		0.08
	5	0/0		0.04	1/1		0.04	1/1		0.04	2/2		0.05	5/5		0.06	6/6		0.06
	7	0/0		0.03	1/1		0.04	1/1		0.04	2/2		0.04	2/2		0.05	3/3		0.05
	11	0/0		0.04	1/1		0.04	1/1		0.04	2/2		0.05	2/2		0.05	3/3		0.06
	13	0/0		0.03	1/1		0.04	1/1		0.04	2/2		0.04	2/2		0.05	3/3		0.05
$\mathcal{U}(\mathfrak{sl}_3)$	0	0/0	8	0.04	1/1	44	0.06	2/2	164	0.10	3/3	494	0.21	4/4	1286	0.56	6/6	3002	1.94
	3	1/0		0.05	2/1		0.07	10/10		0.18	19/11		0.77	28/20		4.42	65/66		24.30
	5	0/0		0.04	1/1		0.06	2/2		0.10	3/3		0.20	12/12		0.52	14/14		1.57
	7	0/0		0.04	1/1		0.06	2/2		0.09	3/3		0.20	4/4		0.49	6/6		1.34
	11	0/0		0.04	1/1		0.06	2/2		0.10	3/3		0.20	4/4		0.49	6/6		1.37
	13	0/0		0.04	1/1		0.06	2/2		0.09	3/3		0.20	4/4		0.48	6/6		1.35
$\mathcal{U}(\mathfrak{sl}_4)$	0	0/0	15	0.06	1/1	135	0.11	2/2	815	0.32	4/4	3875	1.80	5/5	15503	27.42	8/8	54263	614.68
	3	0/0		0.05	1/1		0.10	17/17		0.40	19/19		2.31	35/35		33.05	172/173		434.93
	5	0/0		0.05	1/1		0.10	2/2		0.31	4/4		1.59	20/20		22.50	23/23		280.75
	7	0/0		0.06	1/1		0.10	2/2		0.30	4/4		1.60	5/5		22.64	8/8		278.79
	11	0/0		0.05	1/1		0.10	2/2		0.30	4/4		1.60	5/5		22.60	8/8		281.57
	13	0/0		0.06	1/1		0.10	2/2		0.30	4/4		1.60	5/5		22.68	8/8		282.53
$\mathcal{U}(\mathfrak{sl}_5)$	0	0/0	24	0.08	1/1	324	0.20	2/2	2924	1.16	4/4	20474	50.59	6/6	118754	1680.47	9/9	593774	67748.35
	3	0/0		0.07	1/1		0.18	26/26		1.34	28/28		52.71	54/54		1699.55	380/381		73005.67
	5	1/0		0.10	2/1		0.18	3/2		1.02	5/4		49.04	30/30		1587.43	57/33		43924.35
	7	0/0		0.08	1/1		0.18	2/2		1.07	4/4		49.01	6/6		1572.18	9/9		40979.38
	11	0/0		0.08	1/1		0.18	2/2		1.09	4/4		49.29	6/6		1576.02	9/9		41309.04
	13	0/0		0.08	1/1		0.19	2/2		1.11	4/4		49.24	6/6		1576.86	9/9		41408.18
$\mathcal{U}(\mathfrak{sl}_6)$	0	0/0	35	0.11	1/1	665	0.35	2/2	8435	9.18	4/4	82250	739.67	6/6	658007	45745.37	-/10	4496387	-
	3	1/0		0.14	2/1		0.33	37/37		9.53	74/39		764.26	111/76		70284.51	-/745		-
	5	0/0		0.11	1/1		0.33	2/2		9.03	4/4		735.36	41/41		44891.48	-/45		-

	7	0/0		0.11	1/1		0.35	2/2		9.15	4/4		734.99	6/6		44891.90	-/10		-
	11	0/0		0.10	1/1		0.33	2/2		8.96	4/4		744.94	6/6		45059.84	-/10		-
	13	0/0		0.11	1/1		0.34	2/2		9.01	4/4		738.05	6/6		45031.22	-/10		-
	0	0/0		0.16	1/1		0.66	2/2		56.20	4/4		8477.50	-/6		-	-/10		-
	3	0/0		0.14	1/1		0.60	50/50		57.16	52/52		8578.75	-/102		-	-/1330		-
$\mathcal{U}(\mathfrak{sl}_7)$	5	0/0	48	0.15	1/1	1224	0.62	2/2	20824	55.65	4/4	270724	8474.07	-/54	2869684	-	-/58	25827164	-
	7	1/0		0.22	2/1		0.61	3/2		56.82	5/4		8507.52	-/6		-	-/10		-
	11	0/0		0.15	1/1		0.61	2/2		55.74	4/4		8536.04	-/6		-	-/10		-
	13	0/0		0.16	1/1		0.62	2/2		55.80	4/4		8542.72	-/6		-	-/10		-
	0	0/0		0.24	1/1		1.18	2/2		286.67	4/4		72665.91	-/6		-	-/10		-
	3	0/0		0.23	1/1		1.14	65/65		288.14	67/67		73109.57	-/132		-	-/2215		-
$\mathcal{U}(\mathfrak{sl}_8)$	5	0/0	63	0.24	1/1	2079	1.16	2/2	45759	285.25	4/4	766479	72651.91	-/69	10424127	-	-/73	119877471	-
	7	0/0		0.24	1/1		1.15	2/2		285.34	4/4		72771.39	-/6		-	-/10		-
	11	0/0		0.24	1/1		1.20	2/2		285.97	4/4		74284.07	-/6		-	-/10		-
	13	0/0		0.23	1/1		1.14	2/2		298.15	4/4		74327.57	-/6		-	-/10		-
	0	0/0		0.34	1/1		2.90	2/2		1217.13	-/4		-	-/6		-	-/10		-
	3	1/0		0.43	2/1		2.68	82/82		1195.52	-/84		-	-/166		-	-/3490		-
$\mathcal{U}(\mathfrak{sl}_9)$	5	0/0	80	0.36	1/1	3320	2.80	2/2	91880	1216.21	-/4	1929500	-	-/86	32801516	-	-/90	470155076	-
	7	0/0		0.34	1/1		2.82	2/2		1218.43	-/4		-	-/6		-	-/10		-
	11	0/0		0.32	1/1		2.81	2/2		1226.30	-/4		-	-/6		-	-/10		-
	13	0/0		0.33	1/1		2.84	2/2		1198.08	-/4		-	-/6		-	-/10		-
	0	0/0		0.15	1/1		0.52	1/1		38.16	3/3		5100.56	-/4		-	-/7		-
	3	0/0		0.14	1/1		0.49	46/46		39.12	48/48		5123.78	-/94		-	-/1132		-
$\mathcal{U}(\mathfrak{so}_{10})$	5	0/0	45	0.14	1/1	1080	0.52	1/1	17295	37.98	3/3	211875	5138.79	-/49	2118759	-	-/52	18009459	-
	7	0/0		0.14	1/1		0.53	1/1		38.00	3/3		5157.70	-/4		-	-/7		-
	11	0/0		0.15	1/1		0.49	1/1		38.05	3/3		5109.31	-/4		-	-/7		-
	13	0/0		0.15	1/1		0.49	1/1		38.07	3/3		5100.74	-/4		-	-/7		-
	0	0/0		0.18	1/1		0.76	1/1		138.70	3/3		26144.39	-/3		-	-/6		-
	3	0/0		0.18	1/1		0.72	56/56		133.79	58/58		26148.06	-/113		-	-/1656		-
$\mathcal{U}(\mathfrak{so}_{11})$	5	0/0	55	0.18	1/1	1595	0.73	1/1	30855	131.29	3/3	455125	26297.31	-/58	5461511	-	-/61	55525371	-
	7	0/0		0.18	1/1		0.72	1/1		131.14	3/3		25903.82	-/3		-	-/6		-
	11	0/0		0.18	1/1		0.72	1/1		131.27	3/3		26018.36	-/3		-	-/6		-
	13	0/0		0.18	1/1		0.73	1/1		143.23	3/3		25823.24	-/3		-	-/6		-
	0	0/0		0.25	1/1		1.17	1/1		366.69	3/3		103977.92	-/3		-	-/7		-
	3	0/0		0.24	1/1		1.14	67/67		368.25	69/69		104449.41	-/135		-	-/2350		-
$\mathcal{U}(\mathfrak{so}_{12})$	5	0/0	66	0.24	1/1	2277	1.14	1/1	52393	365.53	3/3	916894	103648.79	-/69	13019908	-	-/73	156238907	-
	7	0/0		0.23	1/1		1.13	1/1		363.64	3/3		104006.62	-/3		-	-/7		-
	11	0/0		0.24	1/1		1.14	1/1		365.29	3/3		104341.95	-/3		-	-/7		-
	13	0/0		0.24	1/1		1.14	1/1		372.05	3/3		103943.08	-/3		-	-/7		-
	0	0/0		0.05	1/1		0.07	1/1		0.12	3/3		0.36	3/3		1.13	5/5		6.44
	3	0/0		0.04	1/1		0.06	11/11		0.16	13/13		0.61	23/23		3.36	79/80		22.00
$\mathcal{U}(\mathfrak{so}_5)$	5	0/0	10	0.05	1/1	65	0.07	1/1	285	0.11	3/3	1000	0.34	13/13	3002	1.17	15/15	8007	5.59
	7	0/0		0.04	1/1		0.07	1/1		0.11	3/3		0.34	3/3		1.06	5/5		4.70
	11	0/0		0.05	1/1		0.07	1/1		0.11	3/3		0.33	3/3		1.02	5/5		4.66
	13	0/0		0.05	1/1		0.06	1/1		0.11	3/3		0.33	3/3		1.05	5/5		4.36
	0	0/0		0.06	1/1		0.07	2/2		0.31	4/4		1.80	5/5		26.60	8/8		545.53
$\mathcal{U}(\mathfrak{so}_6)$			15			135			815			3875			15503			54263	

	3	0/0		0.05	1/1		0.10	17/17		0.39	19/19		2.32	35/35		33.22	172/173		435.91
	5	0/0		0.06	1/1		0.11	2/2		0.30	4/4		1.62	20/20		22.80	23/23		281.89
	7	0/0		0.06	1/1		0.10	2/2		0.31	4/4		1.62	5/5		22.62	8/8		279.29
	11	0/0		0.05	1/1		0.10	2/2		0.30	4/4		1.58	5/5		22.60	8/8		281.42
	13	0/0		0.06	1/1		0.11	2/2		0.30	4/4		1.59	5/5		22.66	8/8		282.02
	0	0/0		0.06	1/1		0.14	1/1		0.64	3/3		17.20	3/3		617.03	6/6		216276.33
	3	0/0		0.07	1/1		0.15	22/22		0.86	24/24		21.38	45/45		628.52	278/279		81944.37
$\mathcal{U}(\mathfrak{so}_7)$	5	0/0	21	0.06	1/1	252	0.15	1/1	2023	0.59	3/3	12649	16.58	24/24	65779	435.32	27/27	296009	9741.75
	7	0/0		0.07	1/1		0.15	1/1		0.62	3/3		16.61	3/3		433.63	6/6		9610.87
	11	0/0		0.06	1/1		0.15	1/1		0.62	3/3		16.50	3/3		435.20	6/6		9811.77
	13	0/0		0.06	1/1		0.15	1/1		0.61	3/3		16.58	3/3		435.75	6/6		9851.68
	0	0/0		0.10	1/1		0.22	1/1		1.92	4/4		150.37	4/4		6996.81	-/8		-
	3	0/0		0.09	1/1		0.22	29/29		2.11	32/32		153.88	60/60		6860.80	-/470		-
$\mathcal{U}(\mathfrak{so}_8)$	5	0/0	28	0.08	1/1	434	0.22	1/1	4494	1.91	4/4	35959	147.21	32/32	237335	6344.62	-/36	1344903	-
	7	0/0		0.09	1/1		0.22	1/1		1.89	4/4		147.63	4/4		6379.73	-/8		-
	11	0/0		0.09	1/1		0.22	1/1		1.80	4/4		147.04	4/4		6378.76	-/8		-
	13	0/0		0.09	1/1		0.21	1/1		2.09	4/4		147.06	4/4		6405.32	-/8		-
	0	0/0		0.11	1/1		0.33	1/1		10.28	3/3		963.00	3/3		67426.05	-/6		-
	3	0/0		0.11	1/1		0.33	37/37		11.21	39/39		1012.48	75/75		121960.85	-/744		-
$\mathcal{U}(\mathfrak{so}_9)$	5	0/0	36	0.11	1/1	702	0.32	1/1	9138	10.23	3/3	91389	962.83	39/39	749397	63448.66	-/42	5245785	-
	7	0/0		0.11	1/1		0.33	1/1		10.08	3/3		943.95	3/3		61728.95	-/6		-
	11	0/0		0.11	1/1		0.32	1/1		10.19	3/3		960.06	3/3		63802.26	-/6		-
	13	0/0		0.11	1/1		0.32	1/1		10.20	3/3		958.23	3/3		63816.45	-/6		-
	0	0/0		0.04	1/1		0.03	1/1		0.04	2/2		0.04	2/2		0.05	3/3		0.06
	3	0/0		0.04	1/1		0.04	4/4		0.05	5/5		0.04	8/8		0.06	14/15		0.07
$\mathcal{U}(\mathfrak{sp}_1)$	5	0/0	3	0.04	1/1	9	0.04	1/1	19	0.04	2/2	34	0.04	5/5	55	0.05	6/6	83	0.06
	7	0/0		0.04	1/1		0.04	1/1		0.04	2/2		0.04	2/2		0.04	3/3		0.05
	11	0/0		0.03	1/1		0.04	1/1		0.04	2/2		0.04	2/2		0.05	3/3		0.05
	13	0/0		0.03	1/1		0.04	1/1		0.04	2/2		0.04	2/2		0.06	3/3		0.06
	0	0/-		0.04	1/-		0.07	1/-		0.12	3/-		0.35	3/-		1.19	5/-		6.25
	3	0/-		0.04	1/-		0.06	11/-		0.15	13/-		0.59	23/-		3.37	79/-		22.36
$\mathcal{U}(\mathfrak{sp}_2)$	5	0/-	10	0.04	1/-	65	0.06	1/-	285	0.12	3/-	1000	0.34	13/-	3002	1.16	15/-	8007	5.53
	7	0/-		0.04	1/-		0.07	1/-		0.11	3/-		0.34	3/-		1.06	5/-		4.32
	11	0/-		0.04	1/-		0.06	1/-		0.12	3/-		0.34	3/-		1.07	5/-		4.30
	13	0/-		0.04	1/-		0.07	1/-		0.11	3/-		0.35	3/-		1.07	5/-		4.40
	0	0/0		0.07	1/1		0.14	1/1		0.60	3/3		16.66	3/3		452.07	6/6		14355.36
	3	0/0		0.07	1/1		0.14	22/22		0.85	24/24		20.93	45/45		594.05	278/279		60858.94
$\mathcal{U}(\mathfrak{sp}_3)$	5	0/0	21	0.07	1/1	252	0.14	1/1	2023	0.60	3/3	12649	16.19	24/24	65779	426.08	27/27	296009	8735.26
	7	0/0		0.07	1/1		0.13	1/1		0.59	3/3		16.09	3/3		422.58	6/6		8641.62
	11	0/0		0.07	1/1		0.14	1/1		0.57	3/3		16.03	3/3		423.00	6/6		8725.50
	13	0/0		0.07	1/1		0.14	1/1		0.58	3/3		16.32	3/3		424.28	6/6		8746.83
	0	0/0		0.11	1/1		0.33	1/1		10.16	3/3		957.52	3/3		64053.85	-/6		-
	3	0/0		0.11	1/1		0.32	37/37		10.68	39/39		1000.26	75/75		101861.28	-/744		-
$\mathcal{U}(\mathfrak{sp}_4)$	5	0/0	36	0.11	1/1	702	0.31	1/1	9138	10.11	3/3	91389	954.54	39/39	749397	62646.95	-/42	5245785	-
	7	0/0		0.11	1/1		0.32	1/1		9.99	3/3		933.00	3/3		60743.22	-/6		-
	11	0/0		0.11	1/1		0.33	1/1		10.09	3/3		961.40	3/3		62659.04	-/6		-

	13	0/0		0.11	1/1		0.32	1/1		10.15	3/3		955.22	3/3		62779.30	-/6		-
	0	0/0		0.19	1/1		0.73	1/1		131.42	3/3		26300.92	-/3		-	-/6		-
	3	0/0		0.20	1/1		0.71	56/56		136.42	58/58		26643.81	-/113		-	-/1656		-
$\mathcal{U}(\mathfrak{sp}_5)$	5	0/0	55	0.19	1/1	1595	0.70	1/1	30855	146.29	3/3	455125	26190.44	-/58	5461511	-	-/61	55525371	-
	7	0/0		0.18	1/1		0.70	1/1		132.35	3/3		26398.86	-/3		-	-/6		-
	11	0/0		0.19	1/1		0.71	1/1		146.12	3/3		26050.13	-/3		-	-/6		-
	13	0/0		0.18	1/1		0.72	1/1		131.06	3/3		26291.42	-/3		-	-/6		-
	0	0/0		0.04	0/0		0.03	0/0		0.04	0/0		0.04	0/0		0.04	0/0		0.05
	3	0/0		0.04	0/0		0.03	2/2		0.04	2/2		0.04	2/2		0.05	5/5		0.06
$\mathcal{W}_1$	5	0/0	2	0.04	0/0	5	0.04	0/0	9	0.04	0/0	14	0.04	2/2	20	0.05	2/2	27	0.06
	7	0/0		0.04	0/0		0.03	0/0		0.04	0/0		0.04	0/0		0.04	0/0		0.05
	11	0/0		0.04	0/0		0.04	0/0		0.04	0/0		0.04	0/0		0.05	0/0		0.05
	13	0/0		0.04	0/0		0.04	0/0		0.04	0/0		0.04	0/0		0.04	0/0		0.05
	0	0/0		0.04	0/0		0.05	0/0		0.07	0/0		0.13	0/0		0.26	0/0		0.57
	3	0/0		0.04	0/0		0.05	4/4		0.07	4/4		0.15	4/4		0.31	14/14		0.71
$\mathcal{W}_2$	5	0/0	4	0.04	0/0	14	0.05	0/0	34	0.07	0/0	69	0.12	4/4	125	0.27	4/4	209	0.58
	7	0/0		0.04	0/0		0.05	0/0		0.07	0/0		0.12	0/0		0.27	0/0		0.58
	11	0/0		0.05	0/0		0.05	0/0		0.07	0/0		0.13	0/0		0.26	0/0		0.56
	13	0/0		0.04	0/0		0.05	0/0		0.07	0/0		0.13	0/0		0.27	0/0		0.56
	0	0/0		0.05	0/0		0.08	0/0		0.22	0/0		0.81	0/0		3.10	0/0		10.87
	3	0/0		0.05	0/0		0.09	6/6		0.23	6/6		0.90	6/6		3.51	27/27		12.63
$\mathcal{W}_3$	5	0/0	6	0.05	0/0	27	0.08	0/0	83	0.22	0/0	209	0.80	6/6	461	3.11	6/6	923	10.96
	7	0/0		0.05	0/0		0.08	0/0		0.22	0/0		0.80	0/0		3.06	0/0		10.73
	11	0/0		0.05	0/0		0.07	0/0		0.21	0/0		0.80	0/0		3.08	0/0		10.70
	13	0/0		0.05	0/0		0.08	0/0		0.22	0/0		0.80	0/0		3.07	0/0		10.72
	0	0/0		0.06	0/0		0.15	0/0		0.77	0/0		4.89	0/0		27.45	0/0		134.97
	3	0/0		0.06	0/0		0.14	8/8		0.81	8/8		5.18	8/8		29.95	44/44		148.27
$\mathcal{W}_4$	5	0/0	8	0.06	0/0	44	0.15	0/0	164	0.77	0/0	494	4.82	8/8	1286	27.10	8/8	3002	133.63
	7	0/0		0.06	0/0		0.13	0/0		0.76	0/0		4.81	0/0		27.09	0/0		132.87
	11	0/0		0.06	0/0		0.14	0/0		0.76	0/0		4.81	0/0		27.04	0/0		132.71
	13	0/0		0.07	0/0		0.15	0/0		0.76	0/0		4.80	0/0		26.56	0/0		133.14
	0	0/0		0.08	0/0		0.28	0/0		2.46	0/0		22.79	0/0		173.80	0/0		1135.61
	3	0/0		0.07	0/0		0.28	10/10		2.51	10/10		23.51	10/10		183.72	65/65		1218.36
$\mathcal{W}_5$	5	0/0	10	0.07	0/0	65	0.27	0/0	285	2.43	0/0	1000	22.43	10/10	3002	171.21	10/10	8007	1124.26
	7	0/0		0.07	0/0		0.27	0/0		2.44	0/0		22.34	0/0		170.74	0/0		1119.23
	11	0/0		0.07	0/0		0.29	0/0		2.38	0/0		22.32	0/0		169.51	0/0		1118.92
	13	0/0		0.08	0/0		0.27	0/0		2.44	0/0		22.41	0/0		170.44	0/0		1116.61

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Eidesstattliche Erklärung

Hiermit erkläre ich an Eides Statt, dass ich die vorliegende Arbeit alleine angefertigt habe und keine anderen als die angegebenen Hilfsmittel benutzt habe.

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