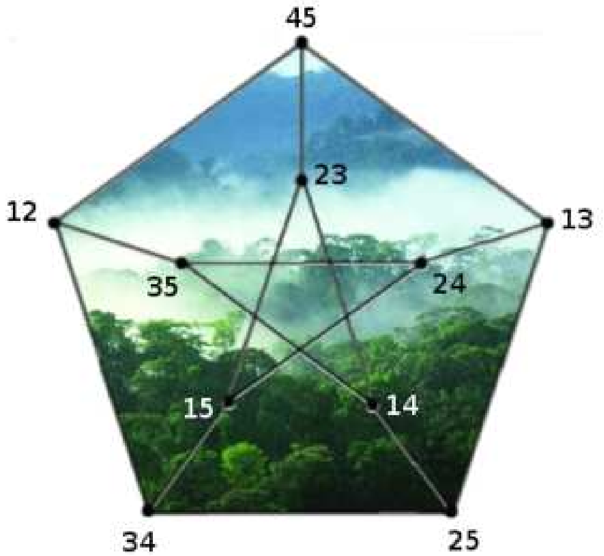
Intersection Theory of the Tropical Moduli Spaces of Curves



Diplomarbeit
bei Prof. Dr. Gathmann

vorgelegt von Matthias Herold
am Fachbereich Mathematik der
Technischen Universität Kaiserslautern

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Contents

Introduction

Tropical geometry is a very new mathematical domain. The appearance of tropical geometry was motivated by its deep relations to other mathematical branches. These include algebraic geometry, symplectic geometry, complex analysis, combinatorics and mathematical biology.

In this work we see some more relations between algebraic geometry and tropical geometry. Our aim is to prove a one-to-one correspondence between the divisor classes on the moduli space of n-pointed rational stable curves and the divisors of the moduli space of n-pointed abstract tropical curves. Thus we state some results of the algebraic case first. In algebraic geometry these moduli spaces are well understood. In particular, the group of divisor classes is calculated by S. Keel. We recall the needed results in chapter one.

For the proof of the correspondence we use some results of toric geometry. Further we want to show an equality of the Chow groups of a special toric variety and the algebraic moduli space. Thus we state some results of the toric geometry as well.

This thesis tries to discover some connection between algebraic and tropical geometry. Thus we also need the corresponding tropical objects to the algebraic objects. Therefore we give some necessary definitions such as fan, tropical fan, morphisms between tropical fans, divisors or the topical moduli space of all n-marked tropical curves. Since we need it, we show that the tropical moduli space can be embedded as a tropical fan.

After this preparatory work we prove that the group of divisor classes in classical algebraic geometry has it equivalence in tropical geometry. For this it is useful to give a map from the group of divisor classes of the algebraic moduli space to the group of divisors of the tropical moduli space. Our aim is to prove the bijectivity of this map in chapter three. On the way we discover a deep connection between the algebraic moduli space and the toric variety given by the tropical fan of the tropical moduli space.

Chapter 1

Classical $M_{0,n}$ and Toric Geometry

In this chapter we will recall some facts about the classical moduli space of n-pointed rational stable curves ($M_{0,n}$) as well as about toric geometry. The results about the classical moduli space have motivated this work, and some of the algebro-geometric results will be proved to hold for its tropical counterpart as well.
In the first section we will recapitulate some facts of S. Keel's paper "Intersection Theory of moduli space of stable n-pointed curves of genus zero" [Kee92]. His results about the Chow ring of $M_{0,n}$ motivate the relations in the tropical case.
In the second section we will give some basic definitions and results of Toric Varieties which can be found in "Introduction to Toric Varieties" by W. Fulton [Ful93].

1.1 The Classical Moduli Space $M_{0,n}$

The classical $M_{0,n}$ is the moduli space of n-pointed stable curves of genus 0. Moduli spaces are parameter spaces for families of algebraic objects (in this case curves).
Let $M_{0,n}$ be the contravariant functor which sends a scheme S to the collection of n-pointed stable curves of genus 0 over S modulo isomorphisms. Let us first of all give a definition of an n-pointed stable curve of genus 0.

Definition 1.1. Let k be an algebraically closed field. Let S be a scheme and assume that all schemes are defined over k, in particular S. We say that a flat proper morphism $\mathcal{C} \xrightarrow{\Pi} S$ with n distinct sections $s_1, s_2, ..., s_n$ is an $n-$*pointed stable curve of genus* 0 if the following conditions hold:

1. The geometric fibres $\mathcal{C}_s$ of Π are reduced connected curves, with at worst ordinary double points, each irreducible component isomorphic to $\mathbb{P}^1$.
2. With $P_i = s_i(s)$, $P_i \neq P_j$ for $i \neq j$.
3. P_i is a smooth point of $\mathcal{C}_s$.
4. For each irreducible component of $\mathcal{C}_s$, the number of singular points of $\mathcal{C}_s$ which lie on it plus the number of P_i on it is at least three.
5. $\dim H^1(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}) = 0$.

Example 1.2.

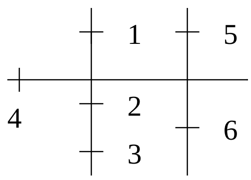


Figure 1.1: A 6-pointed stable curve of genus zero (over Spec k).

The three long lines illustrate the irreducible components of the curve and the six labeled dashes denote the marked points.

Before describing the main idea of S. Keel we have to define two maps: contraction and stabilization.

Definition 1.3. Let $\mathcal{C} \xrightarrow{\pi} S$ be an $(n+1)$-pointed curve with sections $s_1, s_2, ..., s_{n+1}$. A *contraction* of $\mathcal{C} \xrightarrow{\pi} S$ is an (n)-pointed curve $\mathcal{C}' \xrightarrow{\pi} S$ with sections $s'_1, s'_2, ..., s'_n$ provided there is a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "c"] \arrow[d, "\pi"] & \mathcal{C}' \arrow[d, "\pi'"] \\ S \arrow[r, equal] & S \end{tikzcd}$$

satisfying
(1) $c \circ s_i = s'_i$ for $i \leq n$.

(2) Let c_s be the induced morphism on a geometric fiber $\mathcal{C}_s$ and $P = s_{n+1}(s)$. Denote the irreducible component containing P by E. If the number of sections $s_i(s)$ other than P, plus the number of other components which E meets, is at least three, then c_s is an isomorphism. Otherwise, c_s restricted to $\mathcal{C}_s \backslash E$ is an isomorphism and $c_s(E)$ is a point.

Example 1.4.

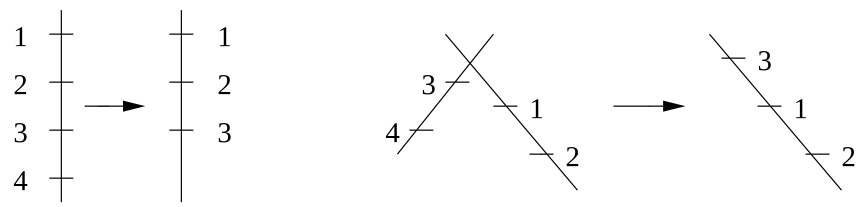


Figure 1.2: Contraction of two different 4-pointed curve (over Spec k).

Definition 1.5. Let $\mathcal{C} \xrightarrow{\pi} S$ with $s_1, s_2, ..., s_n$ be an n-pointed curve with an additional section s. There exists a unique (up to isomorphism) $(n+1)$-pointed curve $\mathcal{C}^s \xrightarrow{\pi'} S$ with sections $s'_1, s'_2, ..., s'_{n+1}$ such that $\mathcal{C}$ is the contraction of $\mathcal{C}^s$ along s'_{n+1} and s'_{n+1} is sent to s (demonstrated by F. Knudsen [Knu83]). This $(n+1)$-pointed curve will be called the *stabilization* of $\mathcal{C} \xrightarrow{\pi} S$ with section $s_1, s_2, ..., s_n$ and additional section s.

F. Knudsen demonstrates [Knu83], that $M_{0,n}$ is represented by a smooth complete variety X_n together with a universal curve $U_n \xrightarrow{\pi} X_n$, and universal sections $\sigma_1, \sigma_2, ..., \sigma_n$.
The idea is recursive construction. We take $U_n \xrightarrow{\pi} X_n$ with sections $\sigma_1,\sigma_2,...,\sigma_n$, representing $M_{0,n}$. Now we want to construct the universal $(n+1)$-pointed curve over the n-pointed universal curve with additional section. We get, that $U_n \times_{X_n} U_n \to U_n$ with the pulled back sections $\sigma^1, ..., \sigma^n$ and the additional section Δ (the diagonal map) is the universal n-pointed curve with an additional section. Its stabilization $(U_n \times_{X_n} U_n)^s \to U_n$ is the universal $(n+1)$-pointed curve. In particular, $X_{n+1} = U_n$, and U_{n+1} is a blowup of $X_{n+1} \times_{X_n} X_{n+1}$.
In order to present this blow-up description S. Keel introduces various "*vital*" divisors on X_n. Let S be a subset of $\{1, 2, ..., n\}$ and denote by $S^c := \{1, 2, ..., n\} \backslash S$ its complement. For each subset $T \subset \{1, 2, ..., n\}$ with $|T| \geq 2$ and $|T^c| \geq 2$ he lets $D^T \hookrightarrow X_n$ be the divisor whose generic element is a curve with two components. The points of T lie on one branch and the points of

T^c on the other.

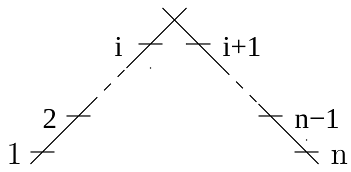


Figure 1.3: A generic curve with $T = \{1, ..., i\}$ on one branch.

In addition he defines a map $\Pi_{1,2,3,n+1} : X_{n+1} \to X_4$ which is obtained by composing contractions, in such a way that every section but the first, second, third, and $(n+1)$st is contracted. Let $B_1 = X_n \times X_4$ and $\pi_1 : X_{n+1} \to X_n \times X_4$ induced by π and $\Pi_{1,2,3,n+1}$. The universal sections $\sigma^1, \sigma^2, ..., \sigma^n$ of π induce sections

$$\sigma_1^1 = \Pi_1 \circ \sigma^1, ..., \sigma_1^n = \Pi_1 \circ \sigma^n$$

of B_1 (Π_1 denotes the first projection). Now he embeds $D^T \hookrightarrow B_1$ as $\sigma_1^i \left(D^T\right)$ for all $i \in T$ and let B_2 be the blowup of B_1 along the union of D^T with $|T^c| = 2$. Inductively he defines

$$B_k \to B_{k-1} \to ... \to B_1$$

by letting B_{k+1} be the blowup of B_k along the union of the strict transforms of the $D^T \hookrightarrow B_1$, under $B_k \to B_1$, for which $|T^c| = k + 1$. His key result is that $X_{n+1} \xrightarrow{p} B_1$ is isomorphic to $B_{n-2} \to B_1$. With this X_{n+1} is a blow-up of $X_n \times X_4 = X_n \times \mathbb{P}^1$. Thus the Chow ring can be calculated from X_n.

With this S. Keel could prove some statements. Among these are the following, which are the motivation for this work. By $A^*(X_n)$ we denote the Chow ring, which gives some structure to the set of divisors. $D^S D^T$ denote the intersection product of D^S and D^T which we obtain by taking the class of $D^S \cap D^T$ if we associate certain multiplicities.

Theorem 1.6. (about the Chow ring)

- The Chow groups $A^k(X_n)$ are free abelian groups and the divisor group has the rank
$$rk(n) = 2^{n-1} - \binom{n}{2} - 1$$
- $$A^*(X_n) = \frac{\mathbb{Z}\left[D^S | S \subset \{1, 2, ..., n\} \text{ and } |S|, |S^c| \geq 2\right]}{\text{the following relations}}$$

1. $D^S = D^{S^c}$.
2. For any four distinct elements $i, j, k, l \in \{1, 2, \dots, n\}$:

$$\sum_{\substack{i,j \in S \\ k,l \notin S}} D^S = \sum_{\substack{i,k \in S \\ j,l \notin S}} D^S = \sum_{\substack{i,l \in S \\ j,k \notin S}} D^S.$$

3. $D^S D^T = 0$ unless one of the following holds:

$$S \subset T, \quad T \subset S, \quad S \subset T^c, \quad T^c \subset S.$$

Proof. Can be found in [Kee92], Theorem 1. □

1.2 Toric Geometry

Next let us have a look at some results of toric geometry. It will be seen later that there is a deep connection between the divisors of the toric variety given by the fan $M_{0,n,trop}$ (see chapter 2 for the definition) and the divisors of $M_{0,n}$. Because of this we can use a result about the divisors of a toric variety which helps us to prove a connection between $\mathrm{Div}(M_{0,n})$ and $\mathrm{Div}(M_{0,n,trop})$. Throughout this paper, for a given space X, we denote by $Div(X)$ the group of divisor classes.

The idea of toric geometry is to associate a variety to a given fan. So let us define a fan and show, how to associate a variety.

Definition 1.7. Let N be a lattice isomorphic to $\mathbb{Z}^n$. A fan Δ in N is a set of rational strongly convex polyhedral cones σ in $N_{\mathbb{R}}$ such that

(1) Each face of a cone in Δ is also a cone in Δ;

(2) The intersection of two cones in Δ is a face of each.

For a more precise definition see Chapter 2.

We define the affine toric variety U_σ for a cone σ in the following way. Let $S_\sigma = \sigma^\vee \cap Hom(N, \mathbb{Z})$, $U_\sigma = \mathrm{Spec}\,(\mathbb{C}\,[S_\sigma])$. From a fan Δ the *toric variety* X(Δ) is constructed by taking the affine toric varieties U_σ, one for each σ in Δ, and gluing as follows: for cones σ and τ, the intersection $\sigma \cap \tau$ is a face of each. Now glue U_σ and U_τ along this face.

Let $\tau_1, ..., \tau_d$ be the edges of the fan, and let v_i be the first lattice point met along τ_i. It is known that these edges correspond to divisors

$$D_i \leftrightarrow \tau_i.$$

For more details see [Ful93], section 1.4 and 3.3.
Further the Weil divisors of this toric variety are the sums $\sum a_i D_i$ for integers a_i.

Next we recall the relations between divisors of a nonsingular projective toric variety. But before this we will give a characterization of a nonsingular variety.

Proposition 1.8. An affine toric variety U_σ is nonsingular if and only if σ is generated by part of a basis for the lattice N, in which case

$$U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}, \quad k = \dim(\sigma).$$

Proof. A proof can be found in [Ful93], chapter 2. $\square$

Therefore we will call a cone nonsingular if it's generated by part of a lattice basis, and a fan nonsingular if all its cones are nonsingular.
Let $M = Hom(N, \mathbb{Z})$ denote the dual lattice. The main result of toric varieties for us will be the following proposition:

Proposition 1.9. For a nonsingular simplicial toric variety X, $A^*X = H^*X = \mathbb{Z}\left[D_1, ..., D_d\right]/I$, where I is the ideal generated by all

(i) $D_{i_1} \cdot ... \cdot D_{i_k}$ for $v_{i_1}, ..., v_{i_k}$ not in a cone of Δ;

(ii) $\sum\limits_{i=1}^{d} < u, v_i > D_i$ for u in M.

Proof. A proof is given in [Ful93], chapter 5. $\square$

Now we have the necessary results about the divisors in the $M_{0,n}$ as well as for toric varieties. They are useful for the tropical case and will be mentioned if necessary.

Chapter 2

Tropical $M_{0,n}$

In this chapter we will give some facts of tropical geometry and the tropical $M_{0,n}$ which can be found in [GKM] and [AGR07].
In the first part we will define $M_{0,n,trop}$ (the tropical counterpart to $M_{0,n}$) and prove some properties of it. In the second part we will have a closer look at divisors in tropical geometry, which we consider in chapter 3 for the $M_{0,n,trop}$.
So let us start with the moduli space, which also can be found in [GKM].

2.1 Fans and $M_{0,n,trop}$

In the previous chapter we have given a definition of fans which we now will make more precise in connection with the tropical fan.
In this chapter we will denote a finitely generated free abelian group by Λ and the corresponding real vector space by $V := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. So we can consider Λ to be a lattice in V. The dual lattice with corresponding vector space will be denoted by $\Lambda^\vee \subset V^\vee$.

Definition 2.1. (cone) A *cone* is a subset σ of V given by a finite number of linear integral equalities and finitely many non-strict inequalities. I.e. a set of the form

$$\sigma = \{x \in V; f_i(x) = 0 \ \forall i = 1, ..., n \text{ and } g_j(x) \geq 0 \ \forall j = 1, ..., m\}$$

for some $f_1, ...f_n, g_1, ...g_m \in \Lambda^\vee \subset V^\vee$. By $V_\sigma \subset V$ we denote the sub-vector space of V corresponding to σ and by $\Lambda_\sigma := V_\sigma \cap \Lambda$ its *lattice*. The dimension of σ is defined by $\dim \sigma := \dim V_\sigma$.

Definition 2.2. A *face* of σ is defined to be a cone $\tau \subset \sigma$ which can be received from σ by changing some of the inequalities to equalities.
A cone generated by only one vector is called an *edge* and it is called *simplicial* if it can be generated by $\dim \sigma$ vectors.

Definition 2.3. (fan) A *fan* X in V is a set of cones σ in V such that

(1) Each face of a cone in X is also a cone in X;

(2) The intersection of two cones in X is a face of each.

Example 2.4.

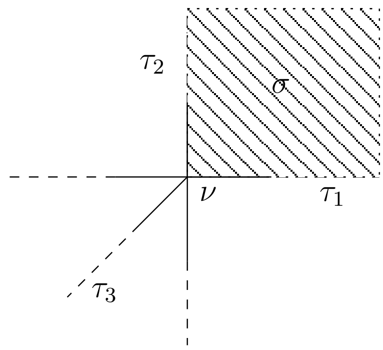


Figure 2.1: A fan with 5 cones of dimension 0, 1 and 2.

We will denote the cones of dimension k by $X^{(k)}$ and the maximal dimension is defined to be the dimension of the fan. If each inclusion-maximal cone is of the same dimension, we call the fan pure-dimensional. A fan is called *simplicial* if this holds for each cone. The set-theoretical union of all cones is denoted by $|X|$.
If X is a pure-dimensional fan of dimension k we will call the cones $\sigma \in X^{(k)}$ *facets* of X.

Definition 2.5. (subfan) Let X be a fan in V. A *subfan* Y of X is a fan of V with the property that each cone of Y is contained in a cone of X. In this case we define a map $C_{Y,X} : Y \to X$ that sends each cone $\tau \in Y$ to the inclusion-minimal cone σ of X with $\tau \subset \sigma$.

After having given the definitions of a fan we will define a tropical fan, which should (as for tropical varieties) fulfill some balancing condition. To define a balancing condition we will use so called "normal" vectors.

Definition 2.6. (normal vector) If $\tau < \sigma$ (:=τ subcone of σ) are cones in V with $\dim \tau = \dim \sigma - 1$, then there is a non-negative, non-zero linear form $g \in \Lambda^\vee$ on σ, which is zero on τ. Then g induces a non-negative and not identically zero isomorphism $V_\sigma / V_\tau \cong \mathbb{R}$. There exists a unique generator $u_{\sigma/\tau} \in \Lambda_\sigma / \Lambda_\tau$, lying in the same half-space as σ and we call it the primitive *normal vector* of σ relative to τ.

Definition 2.7. (weighted and tropical fans) A *weighted fan* (X, ω_X) in V is a pure-dimensional fan X of dimension n with a map $\omega_X : X^{(n)} \to \mathbb{Z}$. The numbers $\omega_X(\sigma)$ are called the weights of the cones $\sigma \in X^{(n)}$. By abuse of notation we also write ω for the map and X for the weighted fan.
A *tropical fan* in V is a weighted fan (X, ω_X) fulfilling the balancing condition

$$\sum_{\sigma > \tau} \omega_X(\sigma) \cdot u_{\sigma/\tau} = 0 \quad \in V/V_\tau$$

$\forall \tau \in X^{(\dim X - 1)}$ ($u_{\sigma/\tau}$ denotes the primitive normal vector defined in 2.6).

Definition 2.8. (Irreducible fans) Let X be a tropical fan in V. We say that X is *irreducible* if there is no tropical fan Y of the same dimension in V with $|V| \subsetneq |X|$.

Definition 2.9. (Morphism of fans). Let X be a fan in $V = \Lambda \otimes \mathbb{R}$, and let Y be a fan in $V' = \Lambda \otimes \mathbb{R}$. A *morphism* $f : X \to Y$ is simply a $\mathbb{Z}$-linear map, i.e. a map from $|X| \subset V$ to $|Y| \subset V'$ induced by a linear map from Λ to Λ'. A morphism of weighted fans is a morphism of fans.

Now we can come to the definition of the $M_{0,n,trop}$, which is our object of interest in the following chapter. For this we have to define an abstract tropical curve.

Definition 2.10. (abstract tropical curve). An *abstract tropical curve* is a connected rational graph Γ. The vertices of this graph have valence at least 3 and the bounded edges are equipped with a positive length. If $x_1, ..., x_n$ are distinct unbounded edges of an abstract tropical curve Γ we define an n-marked abstract tropical curve to be the tuple $(\Gamma, x_1, ..., x_n)$. A more detailed definition is given in [GM].

With this we can define the moduli space of abstract tropical curves.

Definition 2.11. ($M_{0,n,trop}$). We define $M_{0,n,trop}$ to be the space of all n-marked abstract tropical curves (modulo isomorphism) with exactly n leaves.

To embed this space (as a tropical fan) in an $\mathbb{R}^m$ we need a linear map Φ_n which can be found in [SS04] for better understanding.

Definition 2.12. Let $\Phi_n : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{2}}$ be the linear map which sends a vector $(a_1, ..., a_n)$ to the vector (b_{ij}) with $b_{ij} = a_i + a_j,\ 1 \leq i < j \leq n$ and denote by s_n the map $s_n : \mathbb{R}^{\binom{n}{2}} \to \mathbb{R}^{\binom{n}{2}}/\mathrm{Im}\,(\Phi_n)$.

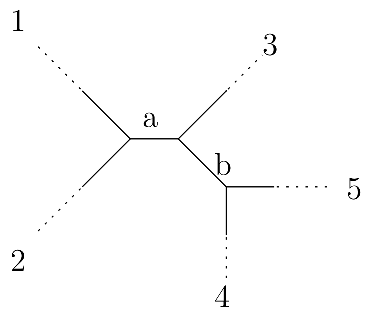
With this maps it is possible by [SS04] to embed the $M_{0,n,trop}$ as a simplicial tropical fan into the vector space $\mathbb{R}^{\binom{n}{2}}/\mathrm{Im}\,(\Phi_n)$ in the following way:

Definition 2.13. (Tropical $M_{0,n,trop}$ embedded). Let

$$\begin{array}{rcl} \phi_n : M_{0,n,trop} & \longrightarrow & \mathbb{R}^{\binom{n}{2}} \\ (\Gamma, x_1, ..., x_n) & \longmapsto & (\mathrm{dist}_\Gamma\,(x_i, x_j))_{(i,j)} \end{array}$$

where $\mathrm{dist}_\Gamma\,(x_i, x_j)$ denotes the distance between the unbounded edges (or leaves) x_i and x_j. It is the length of the unique path from x_i to x_j (The map $s_n(\phi_n)$ is denoted by φ_n in [GKM]).

Example 2.14. The maximal cones correspond to the marked tropical curves with only 3-valent vertices. For n=5 we have for example the following curve Γ:



which will be mapped on the vector

$$\begin{pmatrix} 0 \\ a \\ a+b \\ a+b \\ a \\ a+b \\ a+b \\ b \\ b \\ 0 \end{pmatrix} \begin{array}{l} = \mathrm{dist}_\Gamma(1,2) \\ = \mathrm{dist}_\Gamma(1,3) \\ = \mathrm{dist}_\Gamma(1,4) \\ = \mathrm{dist}_\Gamma(1,5) \\ = \mathrm{dist}_\Gamma(2,3) \\ = \mathrm{dist}_\Gamma(2,4) \\ = \mathrm{dist}_\Gamma(2,5) \\ = \mathrm{dist}_\Gamma(3,4) \\ = \mathrm{dist}_\Gamma(3,5) \\ = \mathrm{dist}_\Gamma(4,5) \end{array}$$

for arbitrary n:
Codim-0 faces correspond to marked tropical curves with only 3-valent vertices. Codim-1 faces correspond to curves with one 4-valent vertex and 3-valent vertices else. Codim-2 faces correspond to curves with one 5-valent vertex or two 4-valent vertices and 3-valent vertices else.

From this definitions A. Gathmann, M. Kerber and H. Markwig demonstrate the following theorem. The proof can be found in [GKM].

Theorem 2.15. The space $s_n\left(\phi_n\left(M_{0,n,trop}\right)\right)$ is a tropical fan of dimension $n-3$.

We will sketch the main idea of the proof, because the idea is helpful for a following proof.

We have to prove the balancing condition for each codimension one cell τ. Let C be a curve in the interior of τ. Because of the fact that C has dimension n-4 it follows, that C has exactly $n-4$ bounded edges. This means that there exists exactly one 4-valent vertex v and all other vertices have valence 3. If C_1, C_2, C_2 are the 3 curves obtained from C via resolving the 4-valent vertex (v) by inserting an edge E of length 1 we can associate the corresponding cones $\sigma_1, \sigma_2, \sigma_3 \in s_n\left(\phi_n\left(M_{0,n,trop}\right)\right)^{n-3}$.
Example for this resolution:

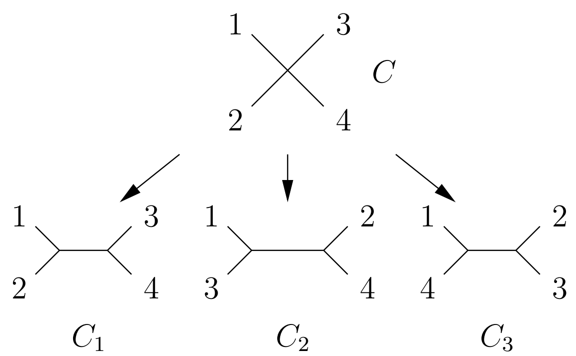


Figure 2.2: Resolution of C to $C_1, C_2,$ or C_3.

If we define $\tilde{C}$ to be the curve obtained form C by shrinking all bounded edges not containing v to 0 and $\tilde{C}_k, k=1,2,3$ to be the curve obtained from C_k by shrinking all edges expect E to length 0 then $v_{\sigma_k/\tau} = \phi_n\left(\tilde{C}_k\right)$ for $k=1,2,3$. One can show that

$$\sum_{k=1}^{3} v_{\sigma_k/\tau} = \phi_n\left(\tilde{C}\right) + \Phi_n\left(a\right) \in V_\tau \oplus Im\left(\Phi_n\right),$$

where $a \in \mathbb{R}^n$ has entry one at position i, if leaf i is adjacent to v and value zero else.

Theorem 2.16. The space $X = s_n(\Phi_n(M_{0,n,trop}))$ is an irreducible fan.

Proof. We have to show that there does not exist a subfan of the same dimension. So let us suppose Π is a tropical fan of same dimension as X with $|\Pi| \subsetneq |X|$.
Let σ be an $n-3$ dimensional subset of a cone $\tilde{\sigma}$ of X which itself is a cone. Π is a tropical fan, thus σ fulfills the balancing condition at each of its borders τ. Unless τ is a codim 1 face of X, $\tilde{\sigma}$ is the only facet adjacent to τ. Because of the balancing condition it follows that $\tilde{\sigma} \subset |\Pi|$. So we can assume that the $n-3$ dimensional cones of Π are also cones of X. Further we know that all the codim-1 faces τ of X have exactly 3 facets containing τ. The unique 4-valent vertex has three possible resolutions (see proof of theorem 2.15). So we can split our problem in 2 parts:

(1) Π has no codim-1 face with less than 3 associated facets.

(2) if $\exists \sigma \in X, \sigma$ a codim-1 face of X but not of Π then Π is empty.

(1) Suppose there is a codim-1 face τ with only two associated facets in Π fulfilling the balancing condition. We are looking at $v_{\sigma_1/\tau}, v_{\sigma_2/\tau}, v_{\sigma_3/\tau}$ which have to fulfill the balancing condition. This means $v_{\sigma_1/\tau} + v_{\sigma_2/\tau} + v_{\sigma_3/\tau} = 0 \bmod \tau$. By the proof of the theorem stated before the $v_{\sigma_i/\tau}$ correspond to vectors, which we get by resolving the four-valent vertex of τ. So we can restrict ourselves to regard at the 4-valent vertex, which by the following is the same as for the case $n = 4$.

If we denote the four parts of the 4-valent knot by A, B, C and D we have three possible edges to resolve the knot. To fulfill the balancing condition we have to look at these edges, which correspond to the normal vectors. For each of them we have all the marked edges of A (resp. B, C, D) on one side. Thus, all the marked edges of A (resp. B, C, D) have the same distance to an arbitrary marked edge for all normal vectors. So we can restrict ourselves to the case $|A| = |B| = |C| = |D| = 1$, which means we have to consider only the case $n = 4$.

Two edges e_1, e_2 are able to fulfill the balancing condition only if there exist integral vectors $v_1 \in e_1, v_2 \in e_2$ s.t. $v_1 + v_2 \in Im(\Phi_n)$, because $\tau = 0$

for $n = 4$(the weighted vectors have to fulfill the balancing condition which is given by the edge vectors times a weight). The two vectors are given by

$$v_1 = \begin{pmatrix} a \\ 0 \\ a \\ a \\ 0 \\ a \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 0 \\ b \\ b \\ b \\ b \\ 0 \end{pmatrix}.$$

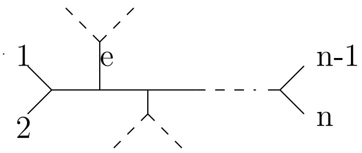
(The entries represent the distance of two marked edges: $e_1 = d(1,2)$, $e_2 = d(1,3)$, $e_3 = d(1,4)$, $e_4 = d(2,3)$, $e_5 = d(2,4)$, $e_6 = d(3,4)$. Because of symmetry it suffices to treat only these two vectors.)
$Im(\Phi_4)$ is equal to $< d_1, d_2, d_3, d_4 >$ with

$$d_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, d_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, d_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, d_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

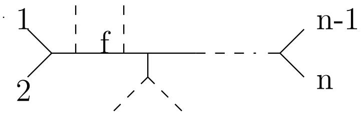
If v_1 and v_2 fulfill the balancing condition it follows that:
$v_1 + v_2 = a_1 d_1 + a_2 d_2 + a_3 d_3 + a_4 d_4$ with $a_1, ..., a_4 \in \mathbb{Q}$. Thus $a_1 + a_2 = a$; $a_1 + a_3 = b$; $a_1 + a_4 = a + b$; $a_2 + a_3 = a + b$; $a_2 + a_4 = b$; $a_3 + a_4 = a$. Hence it follows that $a_1 = a - a_2$. So we get $a_3 = b - a + a_2$. With this $a_2 + b - a + a_2 = a + b$ and thus $a_2 = a$. We can conclude that $a_1 = 0$ and from this that $a_4 = a + b$. Hence $a_2 + a_4 = a + a + b = b$ which means that $a = 0$. Analogously $b = 0$. In total we get that there do not exist $a, b \in \mathbb{Q}\backslash\{0\} : v_1 + v_2 \in Im(\Phi_4)$.
Thus each codim-1 face must have three adjacent maximal faces to fulfill the balancing condition.

(2) Let σ be a maximal face which does not lie in the reduced fan Π and let τ be a codim 1 face lying in σ. To resolve the 4-valent vertex of τ we have three possibilities: σ and two other, which we denote by σ_1, σ_2. By part (1) we know that σ_1, σ_2 can not fulfill the balancing condition on their own. Hence, σ_1, σ_2 do not lie in Π either. By doing this successively we can show that no maximal face can be in Π. We will see that it is possible to reach every facet $\tilde{\sigma}$ from any other facet σ by considering certain codim-1 faces τ

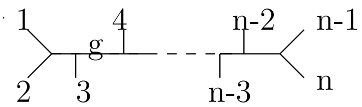
of σ and resolving the 4-valent vertex in another way.
By contracting successively edges like the edge e of



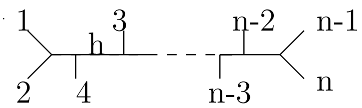
and resolve it to edges like the edge f



it suffices to consider facets of the form



(i.e the bounded edges are in a chain). But by contraction of g and resolving to h



we can show that each two neighboring marked edges can be changed. Thus, each facet can be constructed by each other with the help of contractions and resolutions.
So we conclude that X is irreducible. □

Now we will define divisors of tropical geometry and take our definitions and results from [AGR07].

2.2 Divisors

Let (X, ω_X) be a tropical fan of dimension k in V. Then we define X^* to be the fan

$$X^* := \{\tau \in X | \tau \leq \sigma \text{ for some facet } \sigma \in X \text{ with } \omega_X(\sigma) \neq 0\}.$$

Definition 2.17. (Refinements). Let (X, ω_X) and (Y, ω_Y) be weighted fans in V. We call (Y, ω_Y) a *refinement* of (X, ω_X) if the following holds:

(a) $Y^* \subseteq X^*$,

(b) $|Y^*| = |X^*|$ and

(c) $\omega_Y(\sigma) = \omega_X(C_{Y^*,X^*}(\sigma))$ for every $\sigma \in (Y^*)^{(\dim(Y))}$.

With this definition we can define an equivalence relation "$\sim$" in the following way. If (X, ω_X) and (Y, ω_Y) are weighted fans in V we call them equivalent if they have a common refinement. See [AGR07] for the proof. We call a fan (X, ω_X) reduced if $|X| = |X^*|$.

Having done this preparation we can define our first objects which are necessary to define tropical divisors.

Definition 2.18. (Affine cycles and affine tropical varieties) Let (X, ω_X) be a tropical fan of dimension k. We denote by $[(X, \omega_X)]$ its equivalence class under "$\sim$" and define

$$Z_k(V) := \{[(X, \omega_X)] \mid (X, \omega_X) \text{ tropical fan of dimension } k \text{ in } V\}.$$

With help of the refinements it is possible to define an addition by taking the union of the sets and a suitable refinement. This leads to the next lemma. For more details concerning the addition and the proof of the lemma see [AGR07].

Lemma 2.19. $Z_k(V)$ together with the operation "+" mentioned above forms an abelian group.

Proof. Can be found in [AGR07]. □

Next we want to define Weil divisors and rational functions which are necessary for our aim, to prove the relations of the divisors in the $M_{0,n,trop}$.

Definition 2.20. (Weil divisor) Let X be a fan in V. An affine k-cycle in X is an element $[(Y, \omega_Y)]$ of $Z_k(V)$ fulfilling $|Y^*| \subseteq |X|$. We write $Z_k(X)$ for the set of k-cycles in X and call the elements of $Z_{\dim X - 1}(X)$ *Weil divisor*.

Definition 2.21. (Rational function) Let C be an affine k-cycle. A (non-zero) *rational function* on C is a continuous piecewise linear function $\phi : |C| \to \mathbb{R}$ with the following property: Let (X, w_X) be a representative of C. For each cone σ of X we have that $\phi|_\sigma = \lambda + c,\ \lambda \in \Lambda_\sigma^\vee,\ c \in \mathbb{R}$ is an integer affine linear function. We define ϕ_σ to be λ.
We denote the set of (non-zero) rational functions by $\mathcal{K}^*(C)$.

To these rational functions we can associate a Weil divisor in the following way:

Definition 2.22. (Associated Weil divisors) Let C be an affine k-cycle in $V = \Lambda \otimes \mathbb{R}$ and $\phi \in \mathcal{K}^*(C)$ a rational function on C. If (X, ω) is a representative of C with ϕ affine linear on the cones of X we define $\operatorname{div}(\phi) := \phi \cdot C := \left[\left(\cup_{i=0}^{k-1} X^{(i)}, \omega_\phi\right)\right]$, where

$$\begin{aligned} \omega_\phi : X^{(k-1)} &\to \mathbb{Z}, \\ \tau &\mapsto \sum_{\substack{\sigma \in X^{(k)} \\ \tau < \sigma}} \phi_\sigma(w(\sigma) v_{\sigma/\tau}) - \phi_\tau\left(\sum_{\substack{\sigma \in X^{(k)} \\ \tau < \sigma}} w(\sigma) v_{\sigma/\tau}\right). \end{aligned}$$

The $v_{\sigma/\tau}$ are arbitrary representatives of the normal vectors $u_{\sigma/\tau}$.

Let (X, ω) be a representative of C on whose faces ϕ is affine linear. The idea of this definition is to consider the affine k-cycle $\tilde{C}$ in $V \times \mathbb{R}$ with cones $\tilde{\sigma} := \{(x, \phi_\sigma(x)) | x \in \sigma\}$ for each cone $\sigma \in X$ and $\tilde{\omega}(\tilde{\sigma}) := \omega(\sigma)$. To make this into a fan, fulfilling the balancing condition, we have to add cones $\vartheta := \tilde{\tau} + (\{0\} \times \mathbb{R}_{\leq 0})$ with weight $\tilde{\omega}(\vartheta) = \sum_{\sigma \in X^{(k)}: \tau < \sigma} \phi_\sigma(w(\sigma) v_{\sigma/\tau}) - \phi_\tau\left(\sum_{\sigma \in X^{(k)}: \tau < \sigma} w(\sigma) v_{\sigma/\tau}\right)$ for each face $\tilde{\tau}$ of dimension $k-1$.
We call $\tilde{C}$ an *expansion* of C and C a *contraction* of $\tilde{C}$. The expansion of X along h is a tropical fan denoted by $(\Gamma_{X,h}, \omega_{Gamma_{X,h}})$.
$\omega_{\Gamma_{X,h}}(\tilde{\sigma}) := \omega_X(\sigma)$ for all $\sigma \in X$,
$\omega_{\Gamma_{X,h}}(\tilde{\sigma} + \{0_V\} \times \mathbb{R}_{\leq 0}) := \omega_h(\sigma)$ for all $\sigma \in X^{(m-1)}$.
(see [All07], definition 1.1).

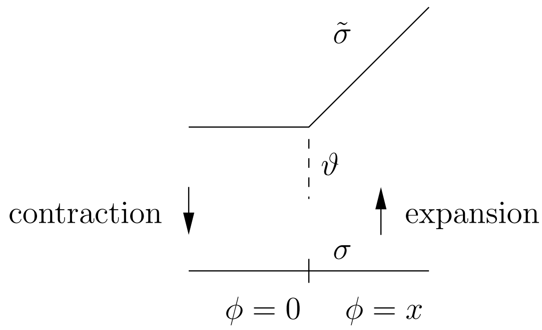


Figure 2.3: Expansion (resp. contraction) of C (resp. $\tilde{C}$).

Having defined the necessary objects we like to have some properties like commutativity or that the balancing condition holds. So let us state the following proposition:

Proposition 2.23. (Balancing Condition and Commutativity)

(a) Let C be an affine k-cycle in $V = \Lambda \otimes \mathbb{R}$ and $\phi \in \mathcal{K}^*(C)$ a rational function on C. Then $\mathrm{div}(\phi)$=$\phi \cdot C$ is an equivalence class of cycles, i.e. its representatives are balanced.

(b) Let $\varphi \in \mathcal{K}^*(C)$ be another rational function on C. Then it holds $\varphi \cdot (\phi \cdot C) = \phi \cdot (\varphi \cdot C)$.

Proof. Can be found in [AGR07]. □

Now we have mentioned all necessary, known objects and have stated some helpful tools. Thus, we can come to the main part and prove the connection between the divisors of $M_{0,n}$ and $M_{0,n,trop}$.

Chapter 3

Divisors on the tropical $M_{0,n}$

In this chapter we will define "vital" divisors for the $s_n(\Phi_n(M_{0,n,trop}))$. We will see that they fulfill the same relations as their corresponding vital divisors of classical algebraic geometry.
For this let us change the definition of divisors given in the previous chapter slightly and define it in the following way:

3.1 Preliminaries

Definition 3.1. Let Y be a tropical fan. A *divisor* D of Y is a pure-dimensional weighted subfan of codim-1 in Y with weights in $\mathbb{Z}$, fulfilling the balancing condition, such that all the faces of D are also faces of Y.

Remark 3.2. We have changed the definition of divisors, so we have to change the definition of $rational\ functions$ φ on Y as well. For this we have to require, that φ is linear on the cones of Y. Thus the associated Weil Divisor consists of faces of Y like the divisors just defined.
If a divisor $\tilde{D}$ differs from D only by faces with weight 0, we identify both with each other.

Because we have to define rational functions and will do this by defining the values on the dimension 1 faces we will introduce some notations for dimension 1 faces. Dimension 1 faces are given by the image of $s_n(\Phi_n)$ of an abstract tropical curve with only one bounded edge e. Thus we have a correspondence between dimension 1 faces (or cones) and edges of the moduli space. So we can introduce notations for this edges instead of dimension 1 faces.

Example 3.3.

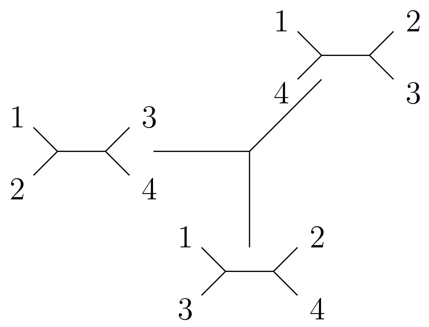


Figure 3.1: The tropical $M_{0,4}$ and the edges corresponding to its cones.

Definition 3.4. Let e be a bounded edge of an arbitrary curve C of $M_{0,n,trop}$ with the marked edges S on one side and $T := \{1,...,n\}\backslash S$ on the other side. We denote the edge e by $\langle S;T\rangle$. If v is a one-dimensional sub-cone of a cone σ or a non-zero vector of a one-dimensional cone we define e_v to be the bounded edge corresponding to the cone in the curve $(s_n(\Phi_n))^{-1}(\sigma)$. If $|\sigma| = 1$ we identify the whole curve with v as well. Let e be a bounded edge. To simplify notation we will denote the set of marked edges lying on the same side as 1 also by e. We denote by $v(e)$ the integral vector of the cone in the moduli space corresponding to the edge e.

Definition 3.5. Let S $\subset \{1,...,n\}$. Then we define φ_S to be the rational function defined by

$$\varphi_S : v(\langle I; I^c\rangle) \mapsto \begin{cases} 1 & \text{, if } I = S \text{ or } I^c = S \\ 0 & \text{, otherwise.} \end{cases}$$

By linearity the function is defined by this. In the following we will denote the integral vector as well as the edge by $\langle I; I^c\rangle$. From the context it will be clear wether we are talking about edges or the corresponding vector.
To facilitate the notation in the following proof we will set $\varphi_S(\langle I; I^c\rangle) = 0$ if $|I| = 1$ or $|I^c| = 1$. Normally this case is not necessary, because there do not exist edges with only one marked edge on one side. But with this we can avoid tedious distinctions of cases by summing up zeros. We will call the divisors induced by this function D^S and the set of these divisors vital divisors, motivated by classical geometry.
By Div($M_{0,n}$)$\rightarrow$ Div($M_{0,n,trop}$) we denote the map induced by the vital divisors.

Remark 3.6. In the same way as in the codim-1 case, we have a correspondence between the r-fold intersection of vital divisors and r-dimensional cones.
We will see, that the divisors of $s_n(\Phi_n(M_{0,n,trop}))$ are given by vital divisors. Thus an r-fold cut of divisors is given by the sum of cuts of vital divisors. But the cut of vital divisors D^T, D^S is unequal zero, only if one of the following condition holds: $S \subset T, \quad T \subset S, \quad S \subset T^c, \quad T^c \subset S$ (We will prove this later on). Or in other words, if there exist a cone with the edges e^T and e^S. If $r > 2$ we can construct a cone of dimension r in the same way (see example 3.2). Thus the r dimensional cone represents a cut of the corresponding r vital divisors.

Our next aim is to show that the divisors induced by the φ_S fulfill the same relations as their classical, corresponding divisors.
The relations are:

1. $D^S = D^{S^c}$.

2. For any four distinct elements $i, j, k, l \in \{1, 2, \dots, n\}$:

$$\sum_{\substack{i,j \in S \\ k,l \notin S}} D^S = \sum_{\substack{i,k \in S \\ j,l \notin S}} D^S = \sum_{\substack{i,l \in S \\ j,k \notin S}} D^S.$$

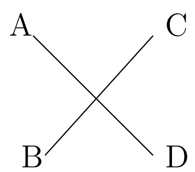
3. $D^S D^T = 0$ unless one of the following holds:

$$S \subset T, \quad T \subset S, \quad S \subset T^c, \quad T^c \subset S.$$

Theorem 3.7. Those relations hold also in tropical geometry.

Proof. 1. This is clear by definition. A vital divisor given by a rational function defined by a set S is the same for S^c.

2. Now we first want to have a closer look at the divisors in the moduli fan X. For this, we want to know the weights of the codim-1 faces of the vital divisors. As mentioned, these codim-1 faces have exactly one 4-valent vertex and the others are 3-valent. Thus, these faces correspond to



where A,B,C,D are parts of the graph, containing only 3-valent knots. This face we will call $DF_{A,B,C,D}$. Throughout this proof A, B, C, D will denote the four parts of a codim-1 face τ (DF=divisor facet). A,B,C,D will signify the parts of the graph as well as the set of the marked ends contained in this part of the graph. We will show that

$$\begin{aligned}
w_{\varphi_S} : X^{(n-4)} &\to \mathbb{Z}, \\
\tau &\mapsto \sum_{\substack{\sigma \in X^{(n-3)} \\ \tau < \sigma}} (\varphi_S)_\sigma (\underbrace{w(\sigma)}_{=1} v_{\sigma/\tau}) - (\varphi_S)_\tau \left(\sum_{\substack{\sigma \in X^{(n-3)} \\ \tau < \sigma}} \underbrace{w(\sigma)}_{=1} v_{\sigma/\tau} \right) \\
&= \varphi_S(v(\langle A \cup B; C \cup D \rangle)) + \varphi_S(v(\langle A \cup C; B \cup D \rangle)) \\
&\quad + \varphi_S(v(\langle A \cup D; B \cup C \rangle)) - \varphi_S(v(\langle A; B \cup C \cup D \rangle)) \\
&\quad - \varphi_S(v(\langle B; A \cup C \cup D \rangle)) - \varphi_S(v(\langle C; A \cup B \cup D \rangle)) \\
&\quad - \varphi_S(v(\langle D; A \cup B \cup C \rangle)) \\
&= \begin{cases} 1 & \text{, if } S = A \cup B \text{ , } A \cup C \text{ , } A \cup D \text{ , } B \cup C, \\ & \quad B \cup D \text{ , or } C \cup D \\ -1 & \text{, if } S = A, B, C \text{ or } D \text{ or if } S^c = A, B, C \text{ or } D \\ 0 & \text{, otherwise.} \end{cases} \quad (\star)
\end{aligned}$$

The first three summands of the first equation are given by the three possibilities of a facet associated to the face τ. By the proof of theorem 2.15 (there mentioned by C_1, C_2, C_3) the normal vectors $v_{\sigma/\tau}$ are given by resolving the knot. For example $\langle A \cup B; C \cup D \rangle$ denotes the edge with A and B on one side of the edge and C,D on the other. This edges are exactly the given ones. Further we have the weight one for each of the facets, thus we get the stated summands.
The other four summands are given analogously to the proof of theorem 2.15 and correspond to the curve $\tilde{C}$.

$$\sum_{\substack{\sigma \in X^{(n-3)} \\ \tau < \sigma}} w(\sigma) v_{\sigma/\tau} \simeq \sum_{k=1}^{3} v_{\sigma_k/\tau} = \phi_n \left(\tilde{C} \right) + \Phi_n \left(a \right).$$

$\tilde{C}$ contains the four edges adjacent to the 4-valent knot. By linearity we can split it into the stated sum. (Here we need that $\varphi_S(\langle I; I^c \rangle) = 0$ if $|I| = 1$ or $|I^c| = 1$, because only the bounded edges are considered. Thus we have to distinguish wether there exist an edge with A, B, C or D on one side. By setting $\varphi_S(\langle I; I^c \rangle) = 0$ for $|I| = 1$ resp. $|I^c| = 1$ this doesn't matter.)

With this preparatory work we can look at the equality

$$\sum_{\substack{i,j\in S\\k,l\notin S}} D^S = \sum_{\substack{i,k\in S\\j,l\notin S}} D^S = \sum_{\substack{i,l\in S\\j,k\notin S}} D^S$$

of the vital divisors. To show the equalities, we have to show that the equalities hold for each (n-4)-dimensional face $DF_{A,B,C,D}$ of a divisor. Up to symmetry we have 5 different cases:

(a) $i \in A$, $j \in B$, $k \in C$, $l \in D$

(b) $i, j \in A$, $k \in C$, $l \in D$

(c) $i, j \in A$, $k, l \in C$

(d) $i, j, k \in A$, $l \in C$

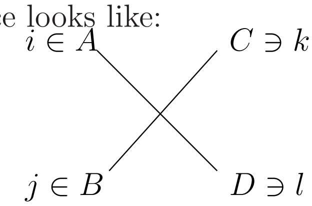
(e) $i, j, k, l \in A$

So let us prove them by showing that each of the sums have the same value.

(a) Claim:

$$\sum_{\substack{i,j\in S\\k,l\notin S}} w_{\varphi_S}(DF_{A,B,C,D}) = \sum_{\substack{i,k\in S\\j,l\notin S}} w_{\varphi_S}(DF_{A,B,C,D}) = \sum_{\substack{i,l\in S\\j,k\notin S}} w_{\varphi_S}(DF_{A,B,C,D}) = 1$$

Proof: The codim-1 face looks like:



Thus, in the first sum we have only a weight to be summed if $S = A \cup B$, in the second if $S = A \cup C$ and in the third if $S = A \cup D$. So we get a 1 for each sum. This follows by the representation of w_{φ_S} for a vital divisor D^S which is stated above $(\star)$. $w_{\varphi_S}(\tau) = 1$, if $S = A \cup B$,... or 0 otherwise. Thus a summand in the first sum will only be unequal to 0 if $S = A \cup B$.
Analogously we can prove the following equations:

(b)

$$\sum_{\substack{i,j\in S\\k,l\notin S}} w_{\varphi_S}(DF_{A,B,C,D}) = 1 - 1 = 0$$

$$\sum_{\substack{i,k\in S\\ j,l\notin S}} w_{\varphi_S}(DF_{A,B,C,D}) = \sum_{\substack{i,l\in S\\ j,k\notin S}} w_{\varphi_S}(DF_{A,B,C,D}) = 0$$

(c)

$$\sum_{\substack{i,j\in S\\ k,l\notin S}} w_{\varphi_S}(DF_{A,B,C,D}) = 2-2=0$$

$$\sum_{\substack{i,k\in S\\ j,l\notin S}} w_{\varphi_S}(DF_{A,B,C,D}) = \sum_{\substack{i,l\in S\\ j,k\notin S}} w_{\varphi_S}(DF_{A,B,C,D}) = 0$$

(d)

$$\sum_{\substack{i,j\in S\\ k,l\notin S}} w_{\varphi_S}(DF_{A,B,C,D}) = \sum_{\substack{i,k\in S\\ j,l\notin S}} w_{\varphi_S}(DF_{A,B,C,D}) = \sum_{\substack{i,l\in S\\ j,k\notin S}} w_{\varphi_S}(DF_{A,B,C,D}) = 0$$

(e)

$$\sum_{\substack{i,j\in S\\ k,l\notin S}} w_{\varphi_S}(DF_{A,B,C,D}) = \sum_{\substack{i,k\in S\\ j,l\notin S}} w_{\varphi_S}(DF_{A,B,C,D}) = \sum_{\substack{i,l\in S\\ j,k\notin S}} w_{\varphi_S}(DF_{A,B,C,D}) = 0$$

So we have shown the equality of the sums for each $n-3$ dimensional face. Thus we get that the equality holds for the divisors, which are defined by the weights on their maximal faces.
So we can come to part three of the proof, which is the last part.

3. Let $S,T \subset \{1,\dots,n\}$ with $2 \leq |S|,|T| \leq n-2$, $D^S D^T = \varphi_T \cdot E$ with $E = D^S \cdot X$. Given with the weight function this means

$$w_E(DF_{A,B,C,D}) = \begin{cases} 1 & \text{, if } S = A\bigcup B\ ,\ A\bigcup C\ ,\ A\bigcup D\ ,\ B\bigcup C, \\ & \quad B\bigcup D\ ,\ \text{or } C\bigcup D \\ -1 & \text{, if } S = A,B,C \text{ or } D \text{ or if } S^c = A,B,C \text{ or } D \\ 0 & \text{, otherwise.} \end{cases}$$

Let k=dim(E)=$n-4$:

$$\begin{aligned} w_{\varphi_T} : E^{(k-1)} &\to \mathbb{Z}, \\ \tau &\mapsto \sum_{\substack{\sigma\in E^{(k)}\\ \tau<\sigma}} \varphi_{T_\sigma}(w_E(\sigma)v_{\sigma/\tau}) - \varphi_{T_\tau}\left(\sum_{\substack{\sigma\in E^{(k)}\\ \tau<\sigma}} w_E(\sigma)v_{\sigma/\tau}\right). \end{aligned}$$

We have to prove, that $w_{\varphi_T}(\tau) = 0,\ \forall \tau \in E^{(k-1)}$. We will do this by demonstrating that each of the summands is zero. To get $w_{\phi_T}(\tau) \neq 0$ at least one of the summands above has to be unequal 0. For the first summand this could only happen if $w_E(\sigma) \neq 0$ (this only means that σ is indeed a face of E). But $v_{\sigma/\tau}$ is the vector of a codim-1 face of σ, such that S or S^c is contained in one side of the edge $e_{v_{\sigma/\tau}}$ if $w_E(\sigma) \neq 0$. By part two of this proof $\varphi_T(\langle R; R^c\rangle) \neq 0$ only if $R = T$ or $R = T^c$. So we get that $\varphi_{T_\sigma}(w(\sigma)v_{\sigma/\tau}) = 0$, if neither $S \subset T,\ T \subset S, S \subset T^c$ nor $T^c \subset S$.
$\forall$ codim-1 cones $v \in E$ with $w_E(v) \neq 0$ holds: S or S^c contained in one side of the corresponding edge. Thus, $\varphi_{T_\tau}\left(\sum_{\substack{\sigma\in E^{(k)}\\ \tau<\sigma}} w(\sigma)v_{\sigma/\tau}\right) = 0$, unless one of the conditions hold, because of the fact, that a rational function is defined by its values on the codim-1 cones and S or S^c is contained in one side for each edge corresponding to such a cone.
We conclude, that $D^S D^T = 0$ unless one of the following holds: $S \subset T, \quad T \subset S, \quad S \subset T^c, \quad T^c \subset S$.

□

3.2 Injectivity of the map Div($M_{0,n}$) $\longrightarrow$ Div($M_{0,n,trop}$)

We want to show the one-to-one correspondence to the classical case, i.e. there do not exist more relations between the given divisors and that all divisors are generated by the given ones. Beginning with the first part, let us state it as a theorem and prove it.

Theorem 3.8. There don't exist more relations between the vital divisors in $M_{0,n,trop}$ as for the vital divisors in $M_{0,n}$ ($\Leftrightarrow$ The map $\mathrm{Div}(M_{0,n}) \longrightarrow \mathrm{Div}(M_{0,n,trop})$ is injective).

Before we can prove this we will demonstrate some connections between toric geometry and tropical geometry as well as between toric geometry and the $M_{0,n}$.

For this we will treat our tropical fan as a toric variety To_n. Let $4 \leq n \in \mathbb{N}$. We choose the following base vectors of $\mathbb{R}^{\binom{n}{2}}$: $d_{r,s} = d_{s,r} : s \in \{4, \ldots, n\}, r \in \{2, \ldots, s-1\}, d_2, \ldots, d_n$ with $d_i = \Phi_n(e_i)$, and f, the vector with all entries equal to one (the $d_{r,s}$ are unit vectors).

To see that these vectors form a basis is straightforward induction in the following way: The vectors $\{d_{n,r} : 2 \leq r \leq n-1\}$ and d_n form a basis of $\langle d_{n,r} : 1 \leq r \leq n-1 \rangle$. Inductively the $\{d_{t,r}, d_t : 2 \leq r \leq t-1,\ s \leq t \leq n\}$ form a basis of $\langle d_{t,r} : 1 \leq r \leq t-1,\ s \leq t \leq n \rangle$, for $s \geq 4$. $\{d_{t,r} : 1 \leq r \leq 3,\ 4 \leq t \leq n\} \cup \{f,\ d_2,\ d_3\}$ are a basis of $\mathbb{R}^{\binom{n}{2}}$, so we are done.
Because we want to use some propositions from toric geometry we like to represent the edge vectors with a basis, which contains a basis of $\mathrm{Im}(\Phi_n)$. So we could take the basis given above. To simplify we change the basis by substituting f by d_1. It will not stay a basis of $\mathbb{R}^{\binom{n}{2}}$ nevertheless the vectors $d_{r,s} + \Phi(\mathbb{R}^n)$ will stay a basis of $\mathbb{R}^{\binom{n}{2}}/\Phi(\mathbb{R}^n)$ and they are easier to handle. If we represent the edge vectors e of the toric variety with this vectors we get the following:

$$e = \begin{cases} \sum\limits_{i \in S} \left(d_i - 2 \cdot \sum\limits_{i<j \in S} d_{j,i} \right) & \text{,if not cond and } 1 \notin S \\ d_1 + \sum\limits_{i \in S \setminus 1} \left(-d_i + 2 \cdot \sum\limits_{\substack{j \in \{1,\dots,n\} \setminus S \\ \text{or } i<j \in S}} d_{j,i} \right) & \text{,if cond and } 1 \in S \end{cases} \tag{$\star\star$}$$

cond means: 2,3 are on one side of the edge and 1 on the other.
Let us consider an example and after this we will prove the representation.

Example 3.9. Let n be 5 and e be the edge with 1 and 2 on one side. In our construction we are in the first case: 2 and 3 are on different sides of the edge. So we get $e = \sum_{i \in \{3,4,5\}} \left(d_i - 2 \cdot \sum_{i<j \in \{3,4,5\}} d_{j,i} \right)$ which is

$$e = \begin{pmatrix} 0\\0\\1\\1\\0\\1\\1\\1\\1\\0 \end{pmatrix} = \begin{pmatrix} 0\\1\\0\\0\\1\\0\\0\\1\\1\\0 \end{pmatrix} + \begin{pmatrix} 1\\0\\0\\0\\1\\1\\1\\0\\0\\0 \end{pmatrix} -2 \begin{pmatrix} 0\\0\\0\\0\\1\\0\\0\\0\\0\\0 \end{pmatrix} + \begin{pmatrix} 1\\1\\1\\1\\0\\0\\0\\0\\0\\0 \end{pmatrix} -2 \begin{pmatrix} 0\\1\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix} -2 \begin{pmatrix} 1\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix} \begin{matrix} d(5,4)\\d(5,3)\\d(5,2)\\d(5,1)\\d(4,3)\\d(4,2)\\d(4,1)\\d(3,2)\\d(3,1)\\d(2,1) \end{matrix}$$

Remark 3.10. The condition $1 \notin S$, resp. $1 \in S$ is only necessary to say to which side corresponds the S and is not a limitation.

To see the equality is a straightforward calculation.
In the first equation d_i gives the marked edge $i \in S$ the distance 1 to each other marked edge. $-2d_{i,j}$ subtracts this distance if $i, j \in S$. So we get the edge vector.
The idea of the second equation is similar. But because of the representing vectors we have to take -1 times the vectors $d_i,\ i \in S$ distinct from d_1. Thus we receive a distance $d_{i,j}$ of 0 for the marked edges $i \in S \setminus \{1\}$ and $j = 1$, as well as if $i, j \notin S$. Further we get the distance $d_{i,j} = -1$ for $i \in S$ and $j \in S^c$ and $d_{i,j} = -2$ for $i, j \in S\backslash\{1\}$. So we have to add $2d_{i,j}$ to change the -1 in 1 if $i \in S$ and $j \notin S$. Last, we have to add $2d_{i,j}$ for deleting the value -2 for $i, j \in S\backslash\{1\}$.

Note 3.11. These ideas will help us to prove proposition 3.13.

Now we have all what we need for a toric variety and want to apply theorem 1.9. For this, let us prove, that To_n is nonsingular by using proposition 1.8.
Let e be an edge of the toric variety. We are looking at the space $\mathbb{R}^{\binom{n}{2}}/\mathrm{Im}\,(\Phi_n)$, i.e. we can forget the d_i in our representation, which are lying in $\mathrm{Im}(\Phi_n)$. Thus all the basis vectors forming e have a common factor 2. Dividing out this factor 2 gives us a new generator of e, which is also lying in the lattice, namely:

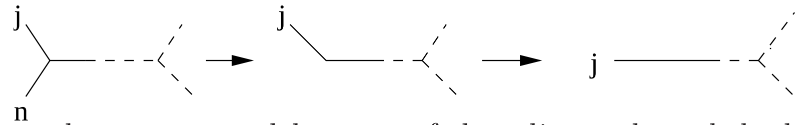
$$-\sum_{i\in S}\left(\sum_{i<j\in S} d_{j,i}\right) \text{ in the first and } \sum_{i\in S\backslash\{1\}}\left(\sum_{\substack{j\in\{1,\dots,n\}\backslash S\\ \text{or } i<j\in S}} d_{j,i}\right) \text{ in the second case.}$$

We know that To_n is simplicial and pure-dimensional. Thus, each maximal cone σ is generated by the one-dimensional faces $\tau \subset \sigma$. So we have to show that the vectors generating the edges given above, which correspond to the one-dimensional faces of σ, are a part of a lattice basis. For this we have to show that the vectors are part of a basis of $\langle d_{r,s} = d_{s,r} : s \in \{4, \dots, n\}, r \in \{2, \dots, s-1\}\rangle_{\mathbb{Z}}$. These vectors can be prolonged by a basis of $\mathrm{Im}(\Phi)$ to a basis of $\mathbb{R}^{\binom{n}{2}}$ and thus we are done.
Let us proof this by induction:
IS: $n = 4$: The three vectors corresponding to the edges (= 1-dimensional faces) are:
$e^{\{2,4\}} = -d_{2,4};\ e^{\{3,4\}} = -d_{3,4};\ e^{\{2,3\}} = d_{3,4} + d_{2,4}$.
To see that each of them could be enlarged to a basis of $U^4 := \langle d_{r,s} = d_{s,r} : s \in \{4, \dots, 4\}, r \in \{2, \dots, s-1\}\rangle_{\mathbb{Z}}$ it suffices to see that $\{e^{\{2,4\}}, e^{\{3,4\}}\}$ as well

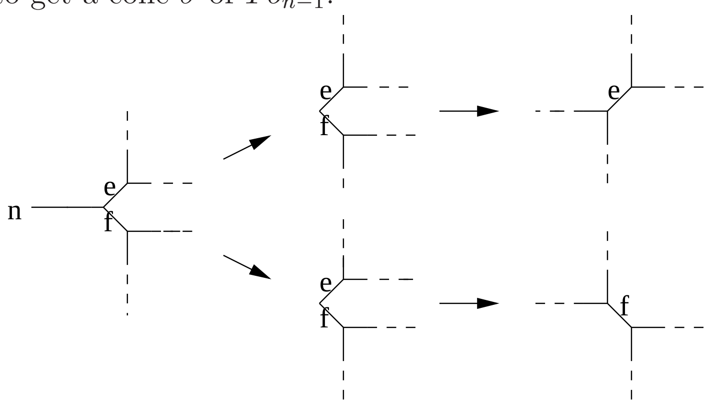
as $\{e^{\{3,4\}}, e^{\{2,3\}}\}$ are a basis of U^4. But $U^4 = \langle d_{2,4}, d_{3,4}\rangle$ thus it follows directly.

IA: Each set of vectors given by the edges of a codim-0 cone in To_{n-1} can be prolonged to a basis of $\langle d_{r,s} = d_{s,r} : s \in \{4, \dots, n-1\}, r \in \{2, \dots, s-1\}\rangle_{\mathbb{Z}}$.

IS: To show the statement for $To_n, n > 4$.
For the marked edge n we have two possibilities for each maximal cone σ. Either the edge n is adjacent to another marked edge j or not. In the first case we can delete the marked edge n and get a cone $\tilde{\sigma}$ of To_{n-1} by considering the edge $e = e^{\{n,j\}}$ and the marked edge j to be the marked edge j in To_{n-1}.



In the second case we can delete one of the adjacent bounded edges e and the edge n to get a cone $\tilde{\sigma}$ of To_{n-1}.



The corresponding vectors (of the bounded edges of $\tilde{\sigma}$) can by induction be enlarged to a basis of $\langle d_{r,s} = d_{s,r} : s \in \{4, \dots, n-1\}, r \in \{2, \dots, s-1\}\rangle_{\mathbb{Z}}$. The entries of the vectors are the same as the entries of the vectors in To_n, thus the sub matrix of the vectors of σ given by the vectors of the cone $\tilde{\sigma}$ by deleting rows can be enlarged to a basis of $\langle d_{r,s} = d_{s,r} : s \in \{4, \dots, n-1\}, r \in \{2, \dots, s-1\}\rangle_{\mathbb{Z}} \subset \langle d_{r,s} = d_{s,r} : s \in \{4, \dots, n\}, r \in \{2, \dots, s-1\}\rangle_{\mathbb{Z}}$. Thus it suffices to show that the vector v corresponding to the deleted edge e, subtracted with suitable vectors corresponding to the edges of the cone $\tilde{\sigma}$, can be enlarged to a basis of $\langle d_{r,n}, r \in \{2, \dots, n-1\}\rangle_{\mathbb{Z}}$.
In the first case

$$v = \begin{cases} -d_{j,n} & \text{,if } j \neq 1 \\ \sum\limits_{r=2}^{n-1} d_{r,n} & \text{,if } j = 1 \end{cases}$$

is a vector lying in $\langle d_{r,n}, r \in \{2, \dots, n-1\}\rangle_{\mathbb{Z}}$ with entries from 0, 1, or -1 and at least one entry different 0. Thus, v can be enlarged to a basis of

$\langle d_{r,n}, r \in \{2, \ldots, n-1\}\rangle_{\mathbb{Z}}$.
In the second case the edges e and f correspond to vectors e^S and $e^{S\cup\{n\}}$. So,

$$0 \neq e^S - e^{S\cup\{n\}} = \begin{cases} \sum\limits_{j\in S} d_{j,n} & \text{,if } 1 \notin S. \\ \sum\limits_{r=2}^{n-1} d_{r,n} & \text{,if } 1 \in S \end{cases} \in \langle d_{r,n}, r \in \{2, \ldots, n-1\}\rangle_{\mathbb{Z}}$$

with entries 0, 1, or -1. Thus $e^S - e^{S\cup\{n\}}$ can be enlarged to a basis of $\langle d_{r,n}, r \in \{2, \ldots, n-1\}\rangle_{\mathbb{Z}}$ and we are done.
So we can apply theorem 1.9 to receive the equations for the divisors. These relations will generate the same relations as in the classical $M_{0,n}$ thus they hold as well in the tropical case. Also they are easier to handle and will be helpful in showing that there do not exist more relations in the tropical case. So let us state the result:

Corollary 3.12. The weights of the facets in $M_{0,n,trop}$ are all 1.

Proof. By the proof of To_n being nonsingular we have seen that the edge vectors of a facet of $M_{0,n,trop}$ divided by 2 can be prolonged to a basis of the lattice $\mathbb{Z}^{\binom{n}{2}-n}$. Thus the edge vectors form a basis of the lattice $2\cdot\mathbb{Z}^{\binom{n}{2}-n}$. All vectors of the facets are represented by vectors in $2\cdot\mathbb{Z}^{\binom{n}{2}-n}$. Therefore the lattice spanned by them is a sublattice of $2\cdot\mathbb{Z}^{\binom{n}{2}-n}$. So we can conclude that the edge vectors of a facet can be prolonged to a lattice basis spanned by the edge vectors and thus the facet weights are all 1. $\square$

Proposition 3.13. The relations of the divisors in the toric variety To_n are given by the following equations:

1.
$$-\sum_{\substack{3,n\in S\\1,2\notin S}} D^S + \sum_{\substack{1,n\in S\\2,3\notin S}} D^S = 0, \; -\sum_{\substack{2,n\in S\\1,3\notin S}} D^S + \sum_{\substack{1,n\in S\\2,3\notin S}} D^S = 0$$

2.
$$-\sum_{\substack{r,n\in S\\1,2\notin S}} D^S + \sum_{\substack{r,1\in S\\2,n\notin S}} D^S - \sum_{\substack{2,n\in S\\1,3\notin S}} D^S + \sum_{\substack{2,3\in S\\1,n\notin S}} D^S = 0 \; \forall r \notin \{1,2,3,n\}$$

3.
$$\sum \pm D^S + \sum \pm D^{S\cup\{n\}} = 0$$

The sum $\sum \pm D^S$ in (3) is given inductively from the part $n-1$ beginning when $n=5$, i.e. for the case $n-1$ exist equations $\sum \pm D^S = 0$. For a set $S \subset \{1, ..., n-1\}$ we can associate also a vital divisor in $s_n(\Phi_n(M_{0,n,trop}))$. Thus, each sum of divisors in $s_{n-1}(\Phi_{n-1}(M_{0.n-1.trop}))$ can be seen as well as sum of divisors in $s_n(\Phi_n(M_{0,n,trop}))$.

Totally these are $\begin{pmatrix} n \\ 2 \end{pmatrix} - n$ equations.

Proof. We will denote e^S to be the edge vector with the marked edges S on one side and S^c on the other. S will denote the same side as in the representation above. The idea is that $\sum\limits_{e^S \text{ edge vector}} \langle d^t_{n,r}, v(e^S)\rangle D^S = 0, r \in \{2,3\}$ are the type 1, $\sum\limits_{e^S \text{ edge vector}} \langle d^t_{n,r}, v(e^S)\rangle D^S = 0, r \in \{4, \dots, n-1\}$ are the type 2 and $\sum\limits_{e^S \text{ edge vector}} \langle d^t_{s,r}, v(e^S)\rangle D^S = 0,\ s \in \{4, \dots, n-1\}, r \in \{2, \dots s-1\}$ are the type 3 equations. We will give an example and afterwards continue with the proof.

Example 3.14. Let $d(r,s)$ be the values of a vector represented with the basis vectors $d_{i,j}$ The representing vectors of $\mathbb{R}^{\binom{5}{2}}$ are:

$$\begin{array}{cc}
\begin{matrix} d_{5,4} & d_{5,3} & d_{5,2} & d_{4,3} & d_{4,2} & d_5 & d_4 & d_3 & d_2 & d_1 \end{matrix} & \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix} &
\begin{matrix} d(5,4) \\ d(5,3) \\ d(5,2) \\ d(5,1) \\ d(4,3) \\ d(4,2) \\ d(4,1) \\ d(3,2) \\ d(3,1) \\ d(2,1) \end{matrix}
\end{array}$$

With these we have the following presentation of the edge vectors: (see $(\star\star)$)

$$
\begin{array}{c}
\begin{array}{cccccccccc}
e^{\{5,4,3\}} & e^{\{5,4,2\}} & e^{\{5,4,1\}} & e^{\{4,1\}} & e^{\{4,2\}} & e^{\{4,3\}} & e^{\{5,4\}} & e^{\{5,3\}} & e^{\{5,2\}} & e^{\{5,1\}}
\end{array} \\
\begin{pmatrix}
-2 & -2 & 2 & 2 & 0 & 0 & -2 & 0 & 0 & 2 \\
-2 & 0 & 2 & 0 & 0 & 0 & 0 & -2 & 0 & 2 \\
0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & -2 & 2 \\
-2 & 0 & 2 & 2 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & -2 & 2 & 2 & -2 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

In this matrix the first row corresponds to the second type of equation, the following two to the type one equation and the fourth and fifth row to the inductive equations. So let us prove this first for the example, before we are coming to the general part.
If we divide by the common factor two we get the following equations:

$$
\begin{aligned}
\sum_{e^S \text{ edge vector}} \langle d_{5,4}^t, v(e^S)\rangle D^S &= -D^{\{5,4,3\}} - D^{\{5,4,2\}} + D^{\{5,4,1\}} + D^{\{4,1\}} - D^{\{5,4\}} + D^{\{5,1\}} \\
&= -\sum_{\substack{4,5\in S\\1,2\notin S}} D^S + \sum_{\substack{4,1\in S\\2,5\notin S}} D^S - \sum_{\substack{2,5\in S\\1,3\notin S}} D^S + \sum_{\substack{2,3\in S\\1,5\notin S}} D^S = 0 \\
\sum_{e^S \text{ edge vector}} \langle d_{5,3}^t, v(e^S)\rangle D^S &= -D^{\{5,4,3\}} + D^{\{5,4,1\}} - D^{\{5,3\}} + D^{\{5,1\}} \\
&= -\sum_{\substack{3,5\in S\\1,2\notin S}} D^S + \sum_{\substack{1,5\in S\\2,3\notin S}} D^S = 0 \\
\sum_{e^S \text{ edge vector}} \langle d_{5,2}^t, v(e^S)\rangle D^S &= -D^{\{5,4,2\}} + D^{\{5,4,1\}} - D^{\{5,2\}} + D^{\{5,1\}} \\
&= -\sum_{\substack{2,5\in S\\1,3\notin S}} D^S + \sum_{\substack{1,5\in S\\2,3\notin S}} D^S = 0
\end{aligned}
$$

$$
\begin{aligned}
\sum_{e^S \text{ edge vector}} \langle d^t_{4,3}, v(e^S)\rangle D^S &= -e^{\{5,4,3\}} + e^{\{5,4,1\}} + e^{\{4,1\}} - e^{\{4,3\}} \\
&= \left(- \sum_{\substack{3,4 \in S \subset \{1,..,4\} \\ 1,2 \notin S}} D^S + \sum_{\substack{1,4 \in S \subset \{1,..,4\} \\ 2,3 \notin S \subset \{1,..,4\}}} D^S \right) \\
&+ \left(- \sum_{\substack{3,4 \in S \subset \{1,..,4\} \\ 1,2 \notin S}} D^{S \cup \{5\}} + \sum_{\substack{1,4 \in S \subset \{1,..,4\} \\ 2,3 \notin S \subset \{1,..,4\}}} D^{S \cup \{5\}} \right) = 0
\end{aligned}
$$

Let us now prove the proposition by considering the three types separately.

1. We will show the second equation, the other is analogous. To show:

$$
\langle d^t_{n,2}, v(e^S)\rangle = \begin{cases} 2 & \text{,if } 1, n \in S,\ 2, 3 \notin S \\ -2 & \text{,if } 2, n \in S,\ 1, 3 \notin S \\ 0 & \text{,else.} \end{cases}
$$

   Let $1, n$ be in S, and $2, 3 \notin S$. So we are in case two of the representation of e^S and thus we get $\langle d^t_{n,2}, v(e^S)\rangle = 2$.
   If $2, n \in S,\ 1, 3 \notin S$ we are in case one and it follows that $\langle d^t_{n,2}, v(e^S)\rangle = -2$.
   Similarly we can argue that $\langle d^t_{n,2}, v(e^S)\rangle = 0$ else. Only we have to distinguish the following four remaining parts: $1, 2, n \in S$; $2, 3, n \in S$ and $1 \notin S$; $3, n \in S$ and $2 \notin S$; $1, 2, 3 \in S$. If we divide the obtained equation by the common factor 2 we receive the stated equation.

2. This type is similar to the first case. Only we have to take care because there are terms in the second and third sum which are cancelling out. Further we have to distinguish enough cases to be sure to be in case one or in case two of our representation. Then we can conclude whether $d^t_{n,r} \cdot e^S = 0,\ 2,$ or -2.

3. In the case $n = 4$ there do not exist equations of type 3.
   Let $n \geq 5$ and $i \in \{4, ...,\ n-1\}, 2 \leq j < i$. By the choice of the representing vectors $d_{i,j}$ can also be seen as a base vector of To_{n-1}.

Thus, we get $\langle d_{i,j}, v(e^S_{n-1})\rangle_{n-1} = \langle d_{i,j}, v(e^S_n)\rangle_n = \langle d_{i,j}, v(e_n^{S\cup\{n\}})\rangle_n \; \forall\, S \subset \{1, \dots , n-1\}$. So, it remains to show that $\langle d_{i,j}, e^S\rangle_n = 0 \;\forall\, S = \{r, n\} : r \in \{1, \dots , n-1\}$.
Only if $r = 1$ we are in case two of the representation. But neither i, nor j are in S, so we get $\langle d_{i,j}, v(e^{\{1,n\}})\rangle$=0. For all other choices of r we are in case one. But in this case at most one of the values i or j can be in S namely if $i = r$ of if $j = r$. Thus $\langle d_{i,j}, v(e^S)\rangle = 0$.

□

Remark 3.15. The group of divisor classes of the toric variety To_n has a one-to-one correspondence to the group of divisor classes of the classical moduli space $M_{0,n}$.

Proof. The edge vectors $e^{\{r,s\}}$ $2 \le r < s,\; s \ge 4$ written with the base vectors are $-d_{r,s} + d_r + d_s$. Thus $D^{\{r,s\}}$ only belongs to one of the equations given by the previous proposition. Therefore all given equations are independent and we get $\begin{pmatrix} n \\ 2 \end{pmatrix} - n$ independent equations.
The equations given by proposition 3.13 are given by equations of theorem 1.6:

$$\text{(eq. 1) e.g. } -\sum_{\substack{3,n\in S\\1,2\notin S}} D^S + \sum_{\substack{1,n\in S\\2,3\notin S}} D^S = 0 \Leftrightarrow \sum_{\substack{3,n\in S\\1,2\notin S}} D^S = \sum_{\substack{1,n\in S\\2,3\notin S}} D^S$$

$$\text{(eq. 2) } -\sum_{\substack{r,n\in S\\1,2\notin S}} D^S + \sum_{\substack{r,1\in S\\2,n\notin S}} D^S - \sum_{\substack{2,n\in S\\1,3\notin S}} D^S + \sum_{\substack{2,3\in S\\1,n\notin S}} D^S = 0$$

$$\Leftarrow \sum_{\substack{r,n\in S\\1,2\notin S}} D^S = \sum_{\substack{1,n\in S\\2,r\notin S}} D^S,\; \sum_{\substack{3,n\in S\\1,2\notin S}} D^S = \sum_{\substack{1,n\in S\\2,3\notin S}} D^S$$

(eq. 3) Let $i, j, k, l \neq n$ then $\sum_{\substack{i,j\in S\\k,l\notin S}} D^S = \sum_{\substack{i,j\in S\\k,l,n\notin S}} D^S + \sum_{\substack{i,j,n\in S\\k,l\notin S}} D^S$. Thus by induction the eq. 3 parts are given by the theorem.
Thus the equations hold also in the $M_{0,n}$. So we have $\binom{n}{2} - n$ independent equations in the $M_{0,n}$. By theorem 1.6 we know that the space of divisors of $M_{0,n}$ has dimension $2^{n-1} - \binom{n}{2} - 1$. There are exactly $2^{n-1} - n - 1$ vital divisors. The difference is $\binom{n}{2} - n$. We can conclude that the divisors of the toric variety fulfill the same relations as the divisors of M_n and vice versa. As a result there exists a one to one correspondence. □

Proof. (proof of theorem 3.8)
By remark 3.15 it suffices to show, that there are not more divisor relations in the tropical case than in the toric one. Let us suppose that there is an equality between the tropical vital divisors: $\sum a_S \cdot D^S = 0$. By the proof of remark 3.15 we have that the divisors $D^{\{r,s\}}$, $2 \leq r < s,\ s \geq 4$, occur in exactly one equation of proposition 3.13 and by theorem 3.7 they also hold tropically. These equations are different for each divisor. Thus we can add up them to get $a_{\{r,s\}} = 0$. So, w.l.o.g. $a_S = 0$, if $S = \{r, s\},\ 2 \leq r < s,\ s \geq 4$, in the sum $\sum a_S \cdot D^S = 0$.
Let us have a closer look at the face $DF_{\{4\}\{3\}\{5\}\{rest\}}$, where rest signifies $\{1, ..., n\}$ less the edges already mentioned (in this case $\{3, 4, 5\}$). We are looking for equations of divisors such that the result is the zero divisor. In particular the weight of each codim-1 face has to be zero. By the proof of Result 3.7, we have only four different divisors which have a non-zero weight on this face. They are for:
$S = \{4, 3\},\ S = \{4, 5\}, S = \{3, 5\}, S = \{4, 3, 5\}$. But $a_S = 0$ for the first three S, because of our assumption it follows that $a_{\{4,3,5\}} = 0$ as well.
Similarly we can show that $a_S = 0\ \forall S \subset \{1, ..., n\}$, $S \not\supset \{2, 3\},\ 1 \notin S, |S| = 3$. Inductively we can prove that $a_S = 0\ \forall S \subset \{1, ..., n\}$, $S \not\supset \{2, 3\},\ 1 \notin S$.
Next we want to treat the face $DF_{\{1\}\{3\}\{5\}\{rest\}}$. The possible sets S for a divisor leading to a weight on this face are the following ones: $S = \{rest, 5\},\ S = \{3, 5\}, S = \{rest\}, S = \{1, 5\}$. Since $a_S = 0$ for the first three S we get $a_{\{1,5\}} = 0$, too. Similar we can show that $a_{\{1,r\}} = 0\ \forall\ 2 \leq r \leq n$. To show that $a_{\{2,3\}} = 0$ we look at $DF_{\{1\}\{2\}\{3\}\{rest\}}$. By induction it follows, that $a_S = 0\ \forall S \subset \{1, ..., n\}$. □

Remark 3.16. The rational functions whose divisor is zero are exactly the globally affine functions.

Proof. $X = s_n(\Phi_n(M_{0,n,trop}))$ is embedded in a space of dimension $\binom{n}{2} - n$. Thus the space of global, affine functions has this dimension. By proposition 3.13, theorem 3.8 and remark 3.15 we have exactly $\binom{n}{2} - n$ independent equations of rational functions mapped to zero. So we get that the rational functions mapped to zero and the globally affine functions are the same. □

Theorem 3.17. The higher codimensional cycles of the classical moduli space $M_{0,n}$ correspond to those of the toric variety induced by the tropical fan To_n.

Proof. To prove this, we have to compare the equalities of theorem 1.6, 3 with those of proposition 1.9 (i). Thus we have to show the equivalence of

the following two statements:
1)$0 = D_{i_1} \cdot \ldots \cdot D_{i_r}$,if $v_{i_1}, \ldots, v_{i_r}$ not in a cone of To.
2) $D^S D^T = 0$,unless $S \subset T, T \subset S, S \subset T^c$ or $T^c \subset S$.

"$\Rightarrow$" Suppose that none of the inclusions hold for S and T, thus $v(\langle S; S^c\rangle)$ and $v(\langle T; T^c\rangle)$ are not together in a cone and we get that $D^S D^T = 0$.

"$\Leftarrow$" Let k be maximal, such that after reordering $v_{i_1}, \ldots,\ v_{i_k}$ lie in a cone but $v_{i_1}, \ldots,\ v_{i_k+1}$ not. Thus $v_{i_1}, \ldots,\ v_{i_k}, v_{i_j}$ are not in a cone $\forall j > k$. Let S be the minimal dimensional cone with the edges $v_{i_1}, \ldots, v_{i_k}$ (the cone spanned by the vectors $v_{i_1}, \ldots,\ v_{i_k}$) and S_c the corresponding curve with the length of the bounded edges equal to 1.
Suppose that $\forall i_l,\ l \leq k\ \exists$ cone $C_l : v_{i_{k+1}}$ and v_{i_l} lie in C_l. Each edge $e_{v_{i_j}}, j \leq k$ divides the n marked edges in two parts A, B. Because of the assumption $e_{v_{i_{k+1}}}$ divides one of these parts (w.l.o.g. A) in two parts (C, D) such that (w.l.o.g.) $B \cup C$ are on one side of the edge and D on the other.
Now we can assign an arrow to $e_{v_{i_j}}$ pointing to the side with the edges of A in S_c. By doing this successively with all edges $e_{v_{i_j}}$ we can assign a direction to each edge. All edges lying behind one edge (behind in the sense of the direction of the arrow) have to point to the same direction. Thus there exists a vertex all arrows are pointing to. All marked edges lying behind one of the bounded adjacent edges of the vertex are contained in $e_{v_{i_{k+1}}}$ or $e^c_{v_{i_{k+1}}}$ (Recall that e_v also denotes the unbounded edges lying on the same side as 1 for the edge corresponding to v). Thus we can insert $e_{v_{i_k+1}}$ to get a curve T_c with the edges $e_{v_{i_1}}, \ldots,\ e_{v_{i_{k+1}}}$ and thus a cone T with edges $v_{i_1}, \ldots,\ v_{i_{k+1}}$, contradiction.

Example:

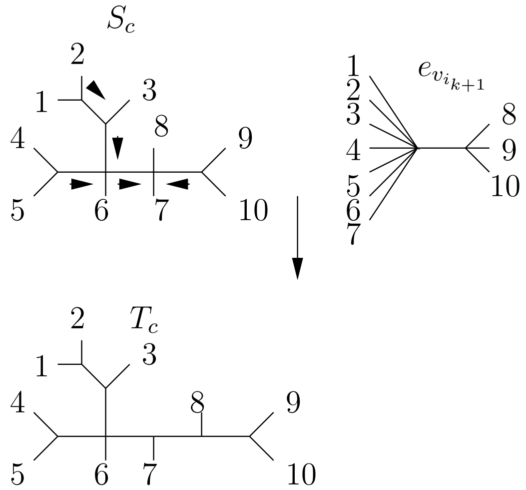


Figure 3.2: construction of the curve

We conclude, that $\exists v_{i_j}, j < k : v_{i_j}$ and v_{i_k} not together in a cone, i.e. $e_{v_{i_j}} \not\subset e_{v_{i_k}}$, $e_{v_{i_j}} \not\subset (e_{v_{i_k}})^c$, $e_{v_{i_j}} \not\supset e_{v_{i_k}}$, $(e_{v_{i_j}})^c \not\supset e_{v_{i_k}}$. So, $D_{i_j} D_{i_k} = D^{e_{v_{i_j}}} D^{e_{v_{i_k}}} = 0$. $\square$

3.3 Surjectivity of the map Div$(M_{0,n}) \to$ Div$(M_{0,n,trop})$

In the following passage we prove the second part of the correspondence of the divisors in $M_{0,n,trop}$ and in $M_{0,n}$.
We have to show that there do not exist divisors in $M_{0,n,trop}$ which cannot be generated by the vital divisors. The main idea of the proof is to show inductively, that the dimension of the space of divisors is $2^{n-1} - \binom{n}{2} - 1$. By theorem 3.8 and theorem 1.6 we know already that the space of divisors has dimension at least $2^{n-1} - \binom{n}{2} - 1$. To prove the equality it will be helpful to choose suitable subsets of $M_{0,n,trop}$ induced by $M_{0,n-1,trop}$. So let us state the theorem and prove it.

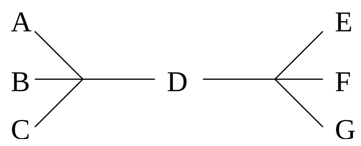
Theorem 3.18. The space of divisors of the tropical moduli space $M_{0,n,trop}$ is generated by the vital divisors ($\Leftrightarrow$ $\dim(\text{Div}(M_{0,n,trop})) = 2^{n-1} - \binom{n}{2} - 1$ $\Leftrightarrow$ The map $\text{Div}(M_{0,n}) \longrightarrow \text{Div}(M_{0,n,trop})$ is surjective).

Proof. To have a shorter notation we will set M_n to be $M_{0,n,trop}$. Further we denote by $M_n^{r,s}$, r, s marked edges, the subset of M_n of all the faces where r, s

are lying on the same side for each four or more valent knot. This means for example, if $DF_{A,B,C,D}$ is one of the codim-1 faces then $r, s \in A \vee ... \vee r, s \in D$. First of all we will simplify our problem by proving the following two remarks:

Remark 3.19. In any divisor of $M_{0,n,trop}$ all codim-1 faces $DF_{A,B,C,D}$, corresponding to the same partition A,B,C and D of the set $\{1, ..., n\}$, have the same weight.

Proof. For this we have to show that for a codim-2 face τ with two four-valent vertices



holds: the three resolutions on the left and on the right side each have to fulfill exactly one balancing condition. That means that the weights of the divisor facets are equal. Thus, by induction all codim-1 faces of a partition A, B, C, D have the same weight. Further we have to show that this two equations are independent.
This means, we have to consider the weighted normal vectors of the adjacent facets and have to show, that they sum up to zero modulo the vectors of τ. The normal vectors have the same entries for the distance of edges i or j in A (resp. B, C, D, E, F, G) to each other edge. Thus it suffices to consider the cases where $|A| = |B| = |C| = |E| = |F| = |G| = 1$. By adding the vector corresponding to $\langle A \cup B \cup C; D \cup E \cup F \cup G \rangle$ (resp. $\langle A \cup B \cup C \cup D; E \cup F \cup G \rangle$) which is lying in τ, we can delete the distance of D to another edge. Thus we can assume that $|D| = 0$. So it suffices to show the remark for the case $n = 6$:

Now it is a dimensional argument and follows by calculating some minors of the following matrix:

$$
\begin{array}{cccccccccc}
d(1,2) & d(1,3) & d(1,4) & d(1,5) & d(1,6) & d(2,3) & d(2,4) & d(2,5) & d(2,6) & d(3,4) \\
\end{array}
$$
$$
\left(\begin{array}{cccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1
\end{array}\right.
$$

$$
\begin{array}{ccccc}
d(3,5) & d(3,6) & d(4,5) & d(4,6) & d(5,6)
\end{array}
$$
$$
\left.\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0
\end{array}\right)
\begin{array}{l}
e^{1,2} \\ e^{1,3} \\ e^{2,3} \\ e^{4,5} \\ e^{4,6} \\ e^{5,6} \\
\left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array}\right\} Im(\Phi_n) \\
e^{1,2,3}
\end{array}
$$

$e^{1,2}, ..., e^{5,6}$ denote the 6 normal vectors of the resolutions.
Example:

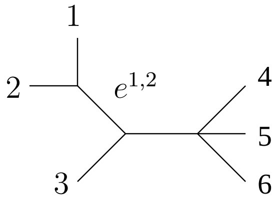


Figure 3.3: Resolution of the codim-2 face with the edge $e^{1,2}$.

The last 7 rows span a 7 dimensional vector space.
The first 3 with the last 7 rows a 9 dimensional.
The last 10 rows a 9 dimensional as well.
And altogether a 11 dimensional vector space.
Thus there exist exactly one linear independent equation for the first three rows and the second three rows together with the last seven rows. Because the total rank of the matrix is 11 it follows that there is not a relation between all the rows which is not a combination of the given ones. Thus the first three rows and the following three have to fulfill exactly one balancing condition. □

Because of this remark we will distinguish faces of higher codimension only by their knots of valence greater than 3. Thus we can define the following equivalence relation on the faces of M_n: $\sigma_1 \sim \sigma_2 :\Leftrightarrow \dim(\sigma_1)=\dim(\sigma_2)$ and the set of partitions of $\{1,...,n\}$ in r sets, given by each r-valent knot in the corresponding curve (of σ_1, resp. of σ_2), $r > 3$ are the same. With this, the following remark makes sense.

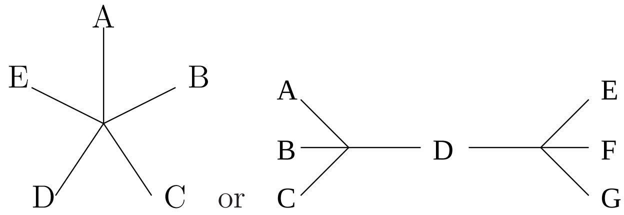
Remark 3.20. The elements of $(faces\ of\ M_n^{r,s})/\sim$, r,s marked edges correspond to elements of $(faces\ of\ M_{n-1})/\sim$ with the marked edges $1,...,\hat{r},...,\hat{s},...,n,\{r,s\}$. Thus the faces τ of $M_n^{r,s}$ belonging to an equivalence class of $M_n^{r,s}$ with $\dim(\tau)=n-5$ have a corresponding class in $(faces\ of\ M_{n-1})/\sim$. Let us choose a representative $\tilde{\tau} \in (faces\ of\ M_{n-1})$ of this class. If we resolve τ (resp. $\tilde{\tau}$) the classes of this resolutions also correspond to each other. Thus we will see that the weights of the faces of $M_n^{r,s}$ given by a divisor of M_n are in one-to-one correspondence to weights given by a divisor in M_{n-1}.

The correspondence of the weights of divisors comes from the remark before (remark 3.19). By this remark all faces which are equivalent under "$\sim$" have the same weight. Thus we only have to show that the balancing conditions on codim-2 faces with one 5-valent knot are the same. For the balancing condition we can, analogous to the proof of remark 3.19, reduce to the case $n = 5$. Thus we get the same balancing conditions for both cases.

This remark tells us, that the balancing condition imposes same equations for the divisors of the M_{n-1} as well as for the $M_n^{r,s}$. So we have the same degree of freedom to fix the faces corresponding to $M_n^{r,s}$ as for M_{n-1}.

Note 3.21. This could be continued such that $M_n^{r_1,...,r_i}$ correspond to M_{n-i+1}.

Example:
For a dim $n-5$ face are two possibilities: Either the face contains a 5-valent vertex or two 4-valent vertices. So it looks like:



If r and s are in the same end (A, B, C, D or E resp. A, B, C, D, E, F or G) all the possible resolutions to a codim-1 face also have r and s on the same end. By remark 3.19 the claim follows.

Let us now prove inductively the following statements:

1. $\dim(\mathrm{Div}(M_n)) = \dim(\mathrm{Div}(M_{0,n,trop})) = 2^{n-1} - \binom{n}{2} - 1$.
2. If we fix a divisor in M_n by giving first its weights in $M_n^{1,2}$, second $M_n^{1,3} ... M_n^{1,n}$ and then the weight of one other codim-1 face we get a degree of freedom of $df^n(n-r)$ for each $M_n^{1,r}$, after fixing the $M_n^{1,s}$, $s < r$ with:

   $df^n(s) = 2^s \ \forall s \leq n-5, df^n(n-4) = 2^{n-4} - 1,$
   $df^n(n-3) = 2^{n-3} - (n-2)$ and $df^n(n-2) = 2^{n-2} - \sum_{r=1}^{n-2} r - 1.$

IS: n=5: The codim-1 faces of the M_5 are edges corresponding to the rows of the following matrix:

$$\begin{array}{cc}
\begin{array}{cccccccccc} d(1,2) & d(1,3) & d(1,4) & d(1,5) & d(2,3) & d(2,4) & d(2,5) & d(3,4) & d(3,5) & d(4,5) \end{array} & \\
\left(\begin{array}{cccccccccc}
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1
\end{array}\right) &
\begin{array}{c} e^{4,5} \\ e^{3,5} \\ e^{3,4} \\ e^{2,5} \\ e^{2,4} \\ e^{2,3} \\ e^{1,2} \\ e^{1,3} \\ e^{1,4} \\ e^{1,5} \end{array}
\end{array}$$

The subspace $Im(\Phi_5)$ which we have to divide out, is given by the rows of the following matrix:

$$\begin{array}{c}
\begin{array}{cccccccccc} d(1,2) & d(1,3) & d(1,4) & d(1,5) & d(2,3) & d(2,4) & d(2,5) & d(3,4) & d(3,5) & d(4,5) \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1
\end{pmatrix}
\end{array}$$

An easy calculation of the dimension shows that these 15 vectors span the whole vector space and that the last 5 are linearly independent. Thus, there exist 5 conditions for the first 10 edges to fulfill the balancing condition. Or in other words, the space of divisors is generated by 5 vectors where the entries represent the edges. The five following vectors are linearly independent and fulfill the balancing condition, so they are a basis of the vector space of the group of divisors tensored by $\mathbb{Q}$:

$$d_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, d_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, d_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, d_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, d_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Every divisor D in M_5 is given by a linear combination of these five vectors and thus has the following representation:

$$D = a_1 d_1 + a_2 d_2 + a_3 d_3 + a_4 d_4 + a_5 d_5$$

Let the weight of the edges $e^{1,2}, ..., e^{1,5}$ corresponding to the faces $DF_{\{1,2\},\{3\},\{4\},\{5\}}, \dots\ , DF_{\{1,5\},\{2\},\{3\},\{4\}}$ be given (These faces correspond to four M_4). These four edges match with the last four entries of the vectors $d_1, ..., d_4$.
Now let e be another arbitrary edge of the undetermined edges. Because of

symmetry we can assume that this is $e^{2,3}$ matching to the 6th entry of the vectors. Only the vectors $d_3,\ d_4,\ d_5$ have entries in the 6th, 7th and 8th row. Furthermore the matrix of this three rows and vectors is a basis of $\mathbb{R}^3$. Thus we get, that $a_3,\ a_4,\ a_5$ are determined by the given weights. d_1 has a 0 in entry 9, but d_2 a 1, such that a_2 is determined by $a_3,\ a_4,\ a_5$ and the weight of $e^{1,4}$, analogously we can do the same for a_1.
Thus we get that a divisor of M_5 is determined by the weights for $M_4^{1,2}$, ..., $M_4^{1,5}$ and one further edge.
$df^5(0) = 1 = 2^0,\ df^5(1) = 1 = 2^1 - 1,\ df^5(2) = 1 = 2^2 - (5-3),\ df^n(3) = 1 = 2^3 - \sum_{r=1}^{5-2} r - 1$
$df^5(0) + df^5(1) + df^5(2) + df^5(3) + 1 = 5 = 2^4 - \binom{5}{2} - 1$
So we have proven the induction basis.

IA: $\dim\left(Div\left(M_p\right)\right) = 2^{p-1} - \binom{p}{2} - 1$ for $p \leq n$.
If we fix a divisor in M_p by giving first the weights for $M_p^{1,2}$, second $M_p^{1,3}$...$M_p^{1,p}$ and then one other codim-1 face we get a degree of freedom of $df^p(p-s)$ for each $M_p^{1,s}$, with:

$$df^p(s) = 2^s\ \forall s \leq p-5, df^p(p-4) = 2^{p-4} - 1,$$
$$df^p(p-3) = 2^{p-3} - (p-2) \text{ and } df^p(p-2) = 2^{p-2} - \sum_{i=1}^{p-2} i - 1$$

for all $p < n+1$. Now we have to conclude the case n+1.

We do this in such a way, that we start by proving:
A divisor in M_n is defined if the weights for $M_n^{1,2}, M_n^{1,3}, ..., M_n^{1,n}$ and one further weight of a codim-1 face is given.
If the weights for $M_n^{1,2}, M_n^{1,3}, ..., M_n^{1,n}$ are given, it follows, that all codim-1 faces with 1 not alone on one side are determined. If we fix one other weight for a codim-1 face σ we get a codim-2 face τ with this face associated and four other faces which are lying in $M_n^{1,2} \cup M_n^{1,3} \cup ... \cup M_n^{1,n}$.

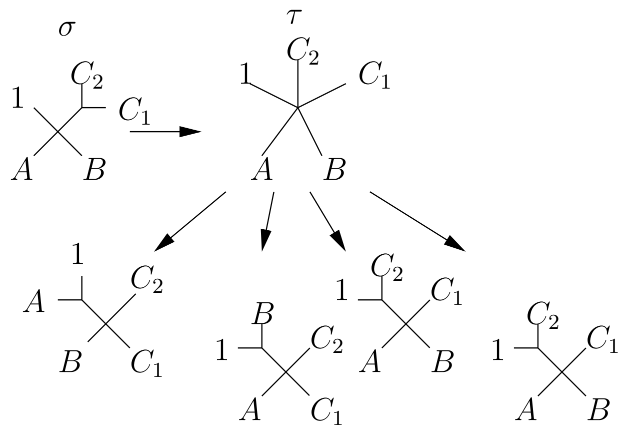


Figure 3.4: σ, τ and the other four faces.

As in the proof of remark 3.19 we can reduce to the case $|A| = |B| = |C_1| = |C_2| = 1$. Now we are in the same position as in the induction assumption and can determine the missing 5 weights. Recursively all weights are given.

So we have a divisor of M_{n+1} determined by the weights for $M^{1,2}_{n+1}, M^{1,3}_{n+1}$,..., $M^{1,n+1}_{n+1}$ and one further weight. Let $M^{1,r,s}_{n+1} = M^{1,s,r}_{n+1} \subset M^{1,r}_{n+1} \subset M_{n+1}$ be the subset given by the correspondence to $M^{\{1,r\},s}_{n}$. By induction and remark 3.20 the weights for $M^{1,r}_{n+1}, 2 \leq r \leq n+1$ are determined by the weights for $M^{\{1,r\},s}_{n+1}, 2 \leq s \leq n+1, r \neq s$ and one further edge $\forall\ 2 \leq s \leq r$. $M^{\{1,r\},s}_{n+1}$ is determined because the weights for $M^{1,s}_{n+1}$ are determined earlier and all faces of $M^{\{1,r\},s}_{n+1}$ are in $M^{1,s}_{n+1}$. Because we don't know if there are more relations between the $M^{1,r}_{n+1}$ we get the following inequality:
Degree of freedom of $M^{1,r}_{n+1} = df^{n+1}(n+1-r) \leq 1 + df^{n}(0) + ... + df^{n}(n-r)$. This follows, because we know by induction that dim(Div(M_n))=$1 + df^{n}(0) + ... + df^{n}(n-2)$. For $M^{1,r}_{n+1}$, we know the $M^{1,s}_{n+1}$, $s < r$ and thus we have to add the degree of freedom only until $df^{n}(n-r)$.

$\Rightarrow$

$$\begin{aligned}
df^{n+1}(s) \quad & \leq 1 + \sum_{i=0}^{s-1} 2^i = 2^s \forall s \leq n-4 \\
df^{n+1}(n-3) \quad & \leq 1 + \sum_{i=0}^{n-5} 2^i + 2^{n-4} - 1 = 2^{n-3} - 1 \\
df^{n+1}(n-2) \quad & \leq 1 + \sum_{i=0}^{n-5} 2^i + 2^{n-4} - 1 + 2^{n-3} - (n-2) = 2^{n-2} - (n-1) \\
df^{n+1}(n-1) \quad & \leq 1 + \sum_{i=0}^{n-5} 2^i + 2^{n-4} - 1 + 2^{n-3} - (n-2) + 2^{n-2} - \sum_{r=1}^{n-2} r - 1 \\
& = 2^{n-1} - \sum_{r=1}^{n-1} r - 1
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\dim\left(Div\left(M_{n+1}\right)\right) \quad & \leq 1 + \sum_{s=0}^{n-1} df^{n+1}(s) \\
& = 1 + \sum_{i=0}^{n-4} 2^i + 2^{n-3} - 1 + 2^{n-2} - (n-1) + 2^{n-1} - \sum_{r=1}^{n-1} r - 1 \\
& = 2^n - \sum_{r=1}^{n} r - 1 = 2^n - \binom{n+1}{2} - 1
\end{aligned}$$

We know already that $\dim\left(Div\left(M_n\right)\right) \geq 2^n - \binom{n+1}{2} - 1$. So we get equality in all the cases and we are done. $\square$

Now we want to give a shorter proof of Theorem 3.18, based on new facts and a better understanding of the relation between Cartier and Weil divisors proved by L. Allermann [All07].

Definition 3.22. Let Λ be a lattice and $V := \Lambda \otimes \mathbb{R}$ be the associated vector space. Let $C \in Z_m(V)$ be a cycle. C is called *locally irreducible* if for some reduced representative $[(X, \omega_X)]$ of X holds: For every cone $\tau \in X^{(m-1)}$ the equality

$$\sum_{\sigma > \tau} \lambda_\sigma \cdot u_{\sigma/\tau} = 0 \in \Lambda/\Lambda_\tau, \ \lambda_\sigma \in \mathbb{Z}$$

(where $u_{\sigma/\tau}$ denotes the primitive normal vector of σ relative to τ) implies that there exists $\lambda \in \mathbb{Q}$ such that $\lambda_\sigma = \lambda \cdot \omega_X(\lambda)$ for all $\sigma > \tau$.

L. Allermann was able to proof the following theorems which are necessary to give a shorter proof.

Theorem 3.23. Let $C \in Z_m(V)$ be a cycle such that $\text{div}(C) \xrightarrow{\sim} Z_{m-1}(C)$, let $h \in \mathcal{K}^*(C)$ be a rational function and let $D \in Z_{m-1}(\Gamma_{C_h})$. If $m \leq 2$ or

$\Gamma_{C,h}$ is locally irreducible then there exists a Cartier divisor $\varphi_D \in \mathrm{div}(\Gamma_{C,h})$ with $\varphi_D \cdot \Gamma_{C,h} = \mathrm{lcm}(\Gamma_{C,h}) \cdot D$.

Proof. see [All07] theorem 2.4. □

Theorem 3.24. Let $C \in Z_m(V)$ be a cycle such that $\mathrm{div}(C) \xrightarrow{\sim} Z_{m-1}(C)$, let $h \in \mathcal{K}^*(C)$ be a rational function and let $\Phi \in \mathrm{div}(\Gamma_{C,h})$ with $\Phi \cdot \Gamma_{C,h} = 0$. Then $\Phi = 0 \in \mathrm{div}(\Gamma_{C,h})$.

Proof. see [All07] theorem 2.5. □

Remark 3.25. The weights of the tropical moduli space $M_{0,n,trop}$ are all one (corollary 3.12), thus the work of L. Allermann told us, that Weil and Cartier divisors of $M_{0,n,trop}$ are the same, if it is a fan constructed by expansions of $\mathbb{R}^{(n-3)}$.

If the Cartier and the Weil divisors are the same it follows, that the Weil divisors are generated by rational functions and thus theorem 3.18 holds. So let us show the following proposition:

Proposition 3.26. $M_{0,n,trop}$ can be constructed from $\mathbb{R}^{(n-3)}$ by $\binom{n}{2} - 2n + 3$ expansions.

Proof. By abuse of notation we will denote $M_{0,n,trop}$ by M_n. Further $\mathbb{R}_{i,i+1} = \mathbb{R}$ will denote the real numbers. We will show the proposition by induction. For this we will show that the map

$$\phi = \phi' \times \phi'' \times \phi_4 \times ... \times \phi_{n-2} : M_n \to M_{n-1} \times M_4 \times \mathbb{R}_{4,5} \times \mathbb{R}_{5,6} \times \cdots \times \mathbb{R}_{n-2,n-1},$$

given by

$\phi' : M_n \to M_{n-1}, (C, x_1, \ldots, x_n) \to (C, x_1, \ldots, x_n)$,
$\phi'' : M_n \to M_4, (C, x_1, \ldots, x_n) \to (C, x_1, x_2, x_3, x_n)$ and
$\phi_i : M_n \to \mathbb{R}_{i,i+1}, (C, x_1, \ldots, x_n) \to (C, x_1, x_i, x_{i+1}, x_n) \overset{(\star)}{\to}$ contraction of the face with 1 and i on one side and $i+1$ and n on the other side, $\forall 4 \leq i \leq n-2$

is a contraction, as well as the projection (defined inductively from P_{n-1} down to P_5)

$$P_i : P_{i+1} \cdot ... \cdot P_{n-1}\phi(M_n) \to M_{n-1} \times M_4 \times \mathbb{R}_{4,5} \times \mathbb{R}_{5,6} \times \cdots \times \mathbb{R}_{i-2,i-1}.$$

The contraction ($\star$) looks like follows:

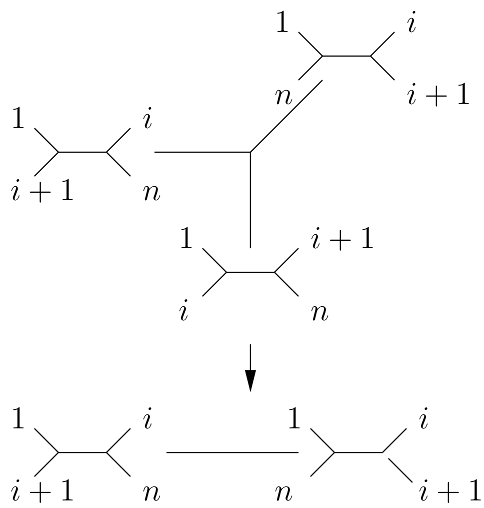


Figure 3.5: Contraction of an M_4.

The maps ϕ' and ϕ'' are forgetful maps, see [GM] definition 4.1 for the definition.
The forgetful maps are morphisms ([GKM], proposition 3.9), thus ϕ is a morphism and therefore $Im(\phi)$ is a tropical fan. This fan will be denoted by T. Further ϕ is surjective on the first component (M_{n-1}).

Now we consider the curves of M_n to be curves of M_{n-1} with the marked edge x_n inserted at one position. If we insert the edge on the marked edge i and move it on this edge, the value of $\mathbb{R}_{i,i+1}$ (resp. $\phi''(M_n)$ is the only changing value in the image, for $4 \leq i \leq n-2$ (resp. $i = 2$ or $i = 3$). Thus the image of ϕ spans the whole $M_{n-1} \times M_4 \times \mathbb{R}_{4,5} \times ... \times \mathbb{R}_{n-2,n-1}$. The ambient space has dimension $\binom{n-1}{2} - (n-1) + 2 + (n-5) = \binom{n-1}{2} - n - 1 + (n-1) = \binom{n}{2} - n - 1$ and thus has one dimension less than $\mathbb{R}^{\binom{n}{2}-n}$. A morphism is a $\mathbb{Z}$-linear map and thus $\phi : \mathbb{R}^{\binom{n}{2}-n} \to \mathbb{R}^{\binom{n}{2}-n-1}$ has a one dimensional kernel.

As mentioned, the curves in M_n are given by M_{n-1} and the position of x_n on it. If changing the position of x_n on one bounded edge i changes also the image, we will see, that the preimage of the image is only one point.
This depends on the fact, that a curve of M_n is given by a curve of M_{n-1} and the position of x_n. Each shifting of x_n on a bounded edge e changes $\mathbb{R}_{i,i+1}$ (resp. M_4) if 1 and i, for $4 \leq i \leq n-2$ (resp. 1 and 2 or 1 and 3) are on different sides of the meeting point of x_n. e is a bounded edge, thus at least two marked edges are on each side of the edge and therefore at least one of

the $\mathbb{R}_{i,i+1}$ or the M_4 will be changed (on the side which does not contain 1 must be an edge unequal $n-1$). So we can conclude that the image has only one preimage. In total this means, that each point of the image of M_n has one preimage or a half-line as preimage.

It remains to prove that M_n is an expansion of $M_{n-1} \times M_4 \times \mathbb{R}_{4,5} \times \mathbb{R}_{5,6} \times \cdots \times \mathbb{R}_{n-2,n-1}$. The forgetful maps (resp. forgetful maps with contraction) are projections, thus the map ϕ is a projection on a space with one dimension less. Each inner point of a facet of the image in $M_{n-1} \times M_4 \times \mathbb{R}_{4,5} \times \mathbb{R}_{5,6} \times \cdots \times \mathbb{R}_{n-2,n-1}$ has exactly one preimage and we can define a rational map $\varphi : \phi(M_n) \to M_n$ by taking ϕ^{-1} on the inner points and define the rational function on them. So let us define the rational function to be the projection of the preimage of one point onto the kernel of ϕ. We have to show that this is possible, i.e show that φ, which we have given only on the inner points, can be prolonged continuously.

Let $\varphi|_\sigma$ the rational function on σ by prolonging the function φ on the border of σ. At each cone σ the function $id \times \varphi|_\sigma$ is a map from cone to cone, thus $\varphi|_\sigma$ is continuous. The only thing which possibly happens is that φ is not defined at a codim-1 face τ. (the limit of the interior values of two adjacent cones σ_1, σ_2 are different,i.e. $\exists\ x \in \tau : \varphi|_{\sigma_1}(x) \neq \varphi|_{\sigma_2}(x)$). The cones σ_1, σ_2 are mapped by $id \times \varphi$ to cones of M_n. Thus there exist cones $\sigma_2(1), \sigma_2(2)$ such that $id \times \varphi|_{\sigma_2}(\sigma_2), \sigma_2(1), \sigma_2(2)$ fulfill the balancing condition at $id \times \varphi|_{\sigma_2}(\tau)$(recall that the codim-1 faces have exactly 3 adjacent facets). Suppose that $\sigma_2(1)$ and $\sigma_2(2)$ are unequal σ_1. Thus $\phi(\sigma_2(1)) = \phi(\sigma_2(2))) = \tau$ and the representatives for the normal vectors lie in $\tau \times \mathbb{R}$. But the first $\binom{n}{2} - n - 1$ values of $id \times \varphi(\sigma_2)$ are the same as for σ_2 thus no representative of the normal vector lies in $\tau \times \mathbb{R}$, contradiction. So the limit and thus the rational function are well defined. The corresponding expansion of the tropical fan T lying in $M_{n-1} \times M_4 \times \mathbb{R}_{4,5} \times \mathbb{R}_{5,6} \times \cdots \times \mathbb{R}_{n-2,n-1}$ lies in M_n. Further, the facets of T are mapped to facets of M_n, by irreducibility (theorem 2.16) it follows, that the expansion has to be the whole M_n.

Now we consider the map P_i.
For the $(n-3)$-dimensional parts we can, as before, look at the corresponding curve. This curve is given by a curve in M_{n-1} and the place of x_n on this curve. To see that the preimage of one point is either one point or one-dimensional we argue as before (i.e. we want to adapt the argument that the preimage of a point corresponding to the case where x_n lies on a bounded curve is one point).
With $P_{i+1} \cdot ... \cdot P_{n-1}\phi$ we have contracted some cones. Therefore we have to

consider only bounded edges whose corresponding cone does not lie in the same cone as a cone corresponding to an unbounded edge. Thus we change our definition of bounded. With "bounded" edges we mean the bounded edges which are also bounded edges of $ft_i \cdot ... \cdot ft_{n-1}(M_{n-1})$. By $P_{i+1} \cdot ... \cdot P_{n-1}\phi$ the corresponding values of the other bounded edges are in the same $(n-3)$-dimensional cone as the values of marked edges. Thus, as for ϕ, it follows that the points in $P_i \cdot ... \cdot P_{n-1}\phi(M_n)$ have one preimage under P_i or the preimage is 1-dimensional.
By induction $P_{i+1}, ..., P_{n-1}$ and ϕ are contractions. Thus all codim-1 faces of $P_{i+1} \cdot ... \cdot P_{n-1}\phi(M_n)$ have only 3 adjacent facets. Because of the correspondence to curves we can argue as in the proof of theorem 2.16 to show that $P_{i+1} \cdot ... \cdot P_{n-1}\phi(M_n)$ irreducible:
Maximal cones of $P_{i+1} \cdot ... \cdot P_{n-1}\phi(M_n)$ correspond to maximal curves in M_{n-1} and x_n inserted at one "proper" edge e ("proper" means moving x_n on e changes the value in $P_{i+1} \cdot ... \cdot P_{n-1}\phi(M_n)$). Thus changing the position of x_n has to change the M_4 or at least one of the $\mathbb{R}_{r,r+1},\ r < i$. Therefore we have that x_n lies between the edge x_1 and an edge $x_r, r < i$. Furthermore we have that adjacent cones correspond to curves which we get by contracting an edge and resolving it in another way such that x_n stays on a "proper" edge. By contracting and resolving cones in such a way we can move x_n to the edge x_1 (x_n will stay between x_1 and x_r the whole time). Further we can by the proof of theorem 2.16 change cones in M_{n-1} by contraction and resolving to every other cone (x_n stays on the edge x_1 an thus will be on a "proper" edge). Thus we can reach every cone by each other. Now we can conclude in the same way that $P_{i+1} \cdot ... \cdot P_{n-1}\phi(M_n)$ irreducible.
Thus we can argue as for φ that $M_{n-1} \times ... \times \mathbb{R}_{i-1,i}$ is an expansion of $M_{n-1} \times ... \times \mathbb{R}_{i-2,i-1}$. □

As conclusion of Theorem 3.8 and 3.18 we receive the main result of this work:

Theorem 3.27. There exists a one to one correspondence between the divisors of $M_{0,n,trop}$ and $M_{0,n}$ via the vital divisors.

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