Hierarchical Edge Colorings and Rehabilitation Therapy Planning in Germany*

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Abstract

In this paper we first give an overview on the system of rehabilitation clinics in Germany in general and the literature on patient scheduling applied to rehabilitation facilities in particular.

We apply a *class-teacher* model developed by [10] to this environment and then generalize it to meet some of the specific constraints of inpatient rehabilitation clinics. To this end we introduce a restricted edge coloring on undirected bipartite graphs which is called *group-wise balanced*. The problem considered is called *patient-therapist-timetable* problem with group-wise balanced constraints (PTTP_{gb}). In order to specify weekly schedules further such that they produce a reasonable allocation to morning/afternoon (second level decision) and to the single periods (third level decision) we introduce (*hierarchical* $PTTP_{gb}$). For the corresponding model, the *hierarchical edge coloring problem*, we present some first feasibility results.

Keywords: Timetabling; Scheduling; Graph coloring; Hierarchies; Rehabilitation Clinics

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1 Introduction

In this paper we consider Operations Research models for rehabilitation therapy. We focus on the German rehabilitation system, since it deviates partially in large areas from those in other countries. One reason for this discrepancy are the social law and the different cost objects [23], another one is the nonexistence of stationary rehabilitation clinics in those countries, for example in the Scandinavian countries or Great Britain [16]. These countries link rehabilitation measures more strongly with acute care [23, 16]. However, there are also countries, e.g. in Central Europe or in Eastern Europe, whose systems resemble the German system [16], such that the approach presented in this paper will be applicable in a larger context than the title of the paper may indicate.

In the following, we summarize some data which is extracted from [6] and which show that there is a large market for suitable Operations Research methods in rehabilitation. Germany had in 2010 1.237 facilities, which offered inpatient preventive or rehabilitation measures. This was a growth of approximately 5% in comparison to 1991. 26% of these facilities accommodated 200 and more beds. In total, there were 1,97 million patients treated, which corresponds to an increase of 500.000 patients in comparison to 1991. [6] assume that this increasing trend will continue due to the demographic development in Germany (see for comparison [23], p.5f.). The average residence time of a patient in such a facility was in 2010 about 25,4 days, which is a decrease of residence time of 18% in comparison to 1991. [6] suggest that a reason for this decrease could be seen in new increased extra payments for rehabilitation measures and a shortening of the general periods (see for comparison [6]).

The Rehab-Report 2012 of the German statutory pension insurance [13] gives an overview of the costs and achievements of the rehabilitation measures that they funded. In 2010, they had 1,6 million requests for medical and 412.966 requests for vocational rehabilitation. Only 12% of the measures in the area of medical rehabilitation were ambulant treatments, thus, 88% were inpatient treatments. On average, such an inpatient treatment costs 2.469 €, if it is caused by a physical illness. Addictive disorders cost on average 6.042 €. Patients which participate in inpatient medical rehabilitation measures because of a physical illness stay on average 23-25 days (without neurological diseases). Patients which suffer from an addictive disorder have to stay usually longer. In total, the German statutory pension insurance had to spend 5,38 billion after tax in 2010.

About 86% of the patients participating in a medical rehabilitation treatment are able to work again after at most two years following their treatment. [13] reports that, because of the demographic change, there will be more demand on rehabilitation measures in the future.

Rehabilitation clinics are economic enterprises, thus, they compete against each other for the restricted budgets of the cost objects. On top, they have to ensure meeting the legal rules which means increasing quality requirements, e.g. they have to participate in certification procedures (see [26]). Moreover, also the patients themselves can decide which clinic they prefer [24, 26]. Hence, rehabilitation clinics have to work competitively, economically, quality-oriented and patient-oriented.

Based on this data, it should be obvious that there is a big chance for Operations Research to help rehabilitation clinics to reach these goals.

Although (regular!) hospitals and rehabilitation clinics have several common issues which need to be tackled efficiently, like

- the treatment of both in- and outpatients [30, 32, 14, 17],
- the decentralized organizational structure [7, 2, 20],
- the self-management of wards and departments of therapists [20, 27, 33], or
- the pick-up and delivery service for immobile patients [31, 22, 3],

there are many differences between the two clinic types.

First, there are fewer uncertainties in rehabilitation hospitals than in regular hospitals with acute care [30]. Rehabilitation clinics house patients that already overcame the acute phase of their illness [7, 30]. Thus, the diagnoses are known when they arrive at a rehabilitation clinic. This also includes that there are in general very few emergencies in rehabilitation clinics [30]. Only some rehabilitation clinics have wards with acute care and sometimes even intensive care. But even then there is a difference between the two clinic types, as these rehabilitation clinics are then mostly specialized on certain kinds of illnesses [30], e.g. neurological sicknesses. In such clinics there can be emergencies disturbing the usual appointment planning. However, if this is not the case, it is simpler to estimate how long the stay of a patient will be. It also cannot happen in such a clinic, that a patient arrives without appointment as it can be the case in regular hospitals [15].

Secondly, human resources are at the center of all planning in rehabilitation clinics, while regular hospitals focus their planning attention on technical resources (operating theaters, MRTs, CTs, etc.), see for comparison [17, 30].

The success of individual therapies is depending on continuity - much more so than in regular hospitals. During his stay in a rehabilitation clinic, a patient participates in different types of therapies (e.g. physiotherapy, occupational therapy etc.) and occasional examinations, which all have to be coordinated. For the planning, the distinction between group and individual therapies is important. It is best that the therapist responsible for a certain type of individual therapy does not change during the time of stay (see [7, 30]), such that each therapist knows exactly where he finished the last time with his patient and can pick up working again from there on. This saves time [30].

Although the planning of the therapies takes place in teams, this does not transfer to the planning of appointments for therapies in general (see for comparison [7]). It is the decentralized structure which takes effect here. Each therapist department, or even each therapist, plans alone without knowing the appointment planning of the other departments (only the already fixed appointments of the patient are known at this point of time and this does not hold for changes later on). This procedure using pen and paper (see for comparison [30]) is inefficient, especially when appointments have to be canceled on short notice.

The remainder of this paper is organized as follows: Section 2 gives an overview of the literature on appointment planning in rehabilitation clinics and related topics. In Section 3 we present a model called *patient-therapist-timetable* which is based on restricting assumptions and derived from a model presented in [10]. Section 4 generalizes this model even further to *hierarchical edge colorings for bipartite graphs* and Section 5 gives a short summary of the discussed results as well as a formulation of open problems.

2 Literature

This section gives an overview of literature on the topic of patient scheduling for rehabilitation clinics. We focus on scheduling of appointments for inpatients. For a more thorough review we refer to the forthcoming dissertation [28].

[4] is a very early article investigating what happens in terms of efficiency and effectivity if computer based procedures are applied in a clinic offering acute care and rehabilitation measures. The authors only consider inpatients. The examined procedure has standardized treatment plans for specific disease patterns, which can be altered if need be. The single departments obtain the information what they have to do with a patient on a certain day. However, they do not investigate how the appointment planning really takes place. The authors are more interested in whether computer based scheduling does make sense in a clinical environment or if it doesn't.

In [29] the author works on the question how to design and implement a cost-efficient medical information system for rehabilitation clinics. He first focuses on rehabilitation patients with heart problems. The author explains why the environment of rehabilitation is interesting for system developers and what could be gained by such a system. He then describes the necessary ingredients for such an administration tool, e.g. guidelines in the field of rehabilitation and patient health records, and how they could be realized. In the end he explains why such systems were then not yet realized.

The authors of [25] develop a hierarchically built, integer optimization model to be able to conduct a weekly personnel scheduling for physical therapists. The environment of the case study is the physical therapy and rehabilitation department of a university medical center. Thus, here a physiotherapy department inside a regular hospital with acute care is considered. The goal of the presented model is to be able to take care of a maximum number of patients, to minimize their waiting times and to divide the load of work as equally as possible among the staff. The model works in three phases: selection of maximum number of patients not exceeding the capacity of staff, allocation of the patients to therapists, distribution of the patients through the day. The authors take into consideration that different patients can have different urgencies and that treatment time can vary. The model is weekly adapted and newly calculated. The authors do not directly consider emergencies, however, they say, if capacity is free, they can be squeezed in. Although the authors do not explicitly say whether they consider in- or outpatients, minimization of waiting time suggests that outpatients are considered. Preferences of the patients themselves are not considered.

Also [9] investigates scheduling of physiotherapy appointments in a rehabilitation department inside of a regular hospital. The goals are the minimization of the patients' waiting times and an optimal operating grade of existing resources to increase patient contentment while efficiently using the resources. The authors attach special importance to the partial precedence constraints. Because of the existence of these constraints, they argue, that the problem can be considered as a hybrid shop scheduling problem. Since this problem is strongly $\mathcal{NP}$-hard, they suggest a solution approach with a genetic algorithm. They compare this approach to a mixed integer problem formulation for small problem instances. Finally, the authors deduce a decision support system which has been applied in a hospital.

The same problem, scheduling of physiotherapy appointments in a rehabilitation department inside of a regular hospital, is considered in [8]. The authors' goal is to minimize the patient waiting times to increase service quality. They explain, that the difference between the considered scheduling problem and other scheduling problems is, that besides the therapists there is also need for therapeutic devices. As in the previous article, the problem is formulated as a hybrid shop scheduling problem. The mixed approach combines a genetic algorithm with data mining. They present two learning rules for the data mining and performed numerical tests.

The reference mostly related to this paper, is [30]. The authors present a mixed integer program to coordinate appointments of patients in an inpatient rehabilitation clinic. The German health system is considered and they also point out common and diverging features of rehabilitation clinics compared with regular hospitals with acute care. Moreover, they distinguish between individual therapy and group therapy and take into consideration that patients need to have time to recover between treatments. The authors describe the procedure which clinics apply so far and then build a monolithic optimization model which is replaced by a hierarchic model consisting of three phases. The model is presented as part of a potential decision support system that can offer a planner different alternatives of choice. They also present numeric results.

[14] investigates again the scheduling of physiotherapy appointments. They only consider the case of inpatients. The goal of this article is to show, that computer based scheduling is more efficient and effective than planning by hand. The problem is formulated as a multicriteria combinatorial optimization problem. As a solution approach, the authors suggest a local search procedure in three phases. They consider steepest descent, tabu search and simulated annealing. Different types of therapies are considered: individual therapy, small groups, groups, several therapists necessary, single therapist. The suggested approach is applied in a neurological rehabilitation facility and leads to a significant reduction in planning time.

The unpublished article [7] examines appointment planning for rehabilitation patients in an outpatient environment. The authors focus on the coordination of appointments in different departments of a certain rehabilitation outpatient clinic. For this problem they develop an integer programming formulation. Five different performance-indicators are developed in co-operation with a specific clinic and the procedure is tested.

3 Patient-Therapist-Timetable

In this section a *patient-therapist-timetable* model for an inpatient rehabilitation clinic is presented. It focuses solely on individual therapies based on the assumption that they will always have priority compared to group therapies. The reason for this prioritization is quite intuitive when thinking about the higher value of an individual supervision by a therapist for a patient. Except lunch breaks we do not consider any breaks for patients or therapists and we assume that each therapy session has for each patient the same duration.

In the design of the therapy plan it is required, that

- patients should, in general, not have the same therapy twice a day,
- if - for medical reasons - a double therapy is necessary on some days, the two sessions should have sufficient temporal distance for the patient to recover, and
- the sessions should be lead to an even distribution of work and therapy load for therapists and patients, respectively.

We start with the requirement that a patient should not have the same kind of therapy twice a day, have a single therapist per type of therapy and the goal of an evenly distributed daily workload for patients and therapists during a week. We adapt the class-teacher model of [10] to tackle this problem, and identify the set of classes and the set of teachers in the school environment with the set P of patients and the set T of therapists in the rehabilitation environment, respectively. The distinction between daily and weekly problems in [10] obviously makes sense in our case as well.

We denote with p_{ij} the number of therapy sessions of patient P_i with therapist T_j per week, i.e., we set $\tilde{P} := (p_{ij})_{\mathcal{N}\times\mathcal{M}}$, where $\mathcal{N} = \{1, ..., n\}$ is the index set of patients and $\mathcal{M} = \{1, ..., m\}$ is the index set of therapists. We denote by $|DpW|$ the number of working days considered per week and set

$$x_{ijk} = \begin{cases} y, & \text{patient } P_i \text{ and therapist } T_j \\ & \text{meet } y\text{-times on day } k; \\ 0, & \text{else.} \end{cases}$$

Using [10], the following model, coined *weekly problem* (WP), gives a solution for the allocation of therapy sessions during a week under the condition of an even distribution over the week: find x_{ijk}, such that

$$\sum_{k=1}^{|DpW|} x_{ijk} = p_{ij}, \text{ for all } i \in \mathcal{N}, j \in \mathcal{M} \tag{1}$$

$$\left\lfloor \sum_{i=1}^{n} \frac{p_{ij}}{|DpW|} \right\rfloor \leq \sum_{i=1}^{n} x_{ijk} \leq \left\lceil \sum_{i=1}^{n} \frac{p_{ij}}{|DpW|} \right\rceil, \text{ for all } j \in \mathcal{M}, k = 1, ..., |DpW|, \tag{2}$$

$$\left\lfloor \sum_{j=1}^{m} \frac{p_{ij}}{|DpW|} \right\rfloor \leq \sum_{j=1}^{m} x_{ijk} \leq \left\lceil \sum_{j=1}^{m} \frac{p_{ij}}{|DpW|} \right\rceil, \text{ for all } i \in \mathcal{N}, k = 1, ..., |DpW|, \tag{3}$$

$$\left\lfloor \frac{p_{ij}}{|DpW|} \right\rfloor \leq x_{ijk} \leq \left\lceil \frac{p_{ij}}{|DpW|} \right\rceil, \text{ for all } i \in \mathcal{N}, j \in \mathcal{M}, k = 1, ..., |DpW|, \tag{4}$$

$$x_{ijk} \in \mathbb{Z}_0^+, \text{ for all } i \in \mathcal{N}, j \in \mathcal{M}, k = 1, ..., |DpW|. \tag{5}$$

Constraints (2) and (3) ensure an evenly distributed daily workload for patients and therapists during a week. (4) guarantees a balanced distribution of therapy sessions of P_i and T_j during a week, especially satisfying that a patient will not have twice a day the same kind of therapy if $p_{ij} \leq |DpW|$.

(WP) can be interpreted as an edge coloring model on an undirected, bipartite graph $G = (V, E)$, where $V = P \dot{\cup} T$ and E contains p_{ij} parallel edges connecting P_i and T_j. A coloring of the edges using $\tilde{k}$ colors is a partition of $E = f(1) \dot{\cup} f(2) \dot{\cup} ... \dot{\cup} f(\tilde{k})$ (see, for instance,

[18]). In the following, we use the denotations $f(i)(v)$ for the set of edges incident with v having color i, and $f(i)(u,v)$ for the set of edges connecting u and v with color i. If we choose $\tilde{k} = |DpW|$, then the $\tilde{k}$ colors coincide with the $|DpW|$ working days of the week. Consequently, colorings with the property that none of the parallel edges have the same color are in one-to-one correspondence with (WP)s, in which no patient is treated by the same therapist twice on any single day of the week.

It can be shown [28] that (2) and (3) are equivalent to the constraints of an equitable coloring and (2), (3), and (4) are equivalent to the constraints of a balanced coloring, where the latter notions are defined according to [18, 19, 21] as follows:

Definition 3.1
An edge coloring of an undirected bipartite graph $G = (V, E)$ with $\tilde{k}$ colors is

1) *equitable*, if for all $v \in V$,

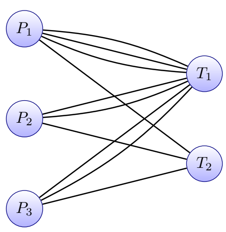
$$\max_{1 \le i < j \le \tilde{k}} ||f(i)(v)| - |f(j)(v)|| \le 1. \quad (6)$$

2) *balanced*, if it is equitable and for all $u, v \in V$, $u \neq v$,

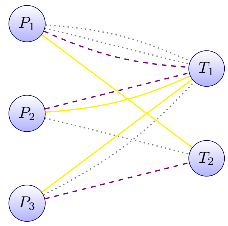
$$\max_{1 \le i < j \le \tilde{k}} ||f(i)(u,v)| - |f(j)(u,v)|| \le 1. \quad (7)$$

$\lhd$

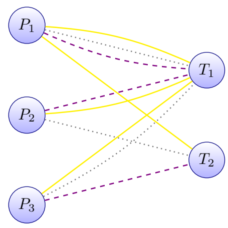
Example 3.2
Consider the following bipartite graph $G = (V, E)$, where $V = P \dot{\cup} T$ with $P = \{P_1, P_2, P_3\}$ and $T = \{T_1, T_2\}$:



The subsequent edge coloring satisfies (6) for all nodes, however, it violates (7) for node combination P_1, T_1, i.e., it is equitable, but not balanced.



In contrast to the above edge coloring, the next one satisfies both (6) and (7), so that it is balanced and thus also equitable.



$\lhd$

Notice that the extensions of edge-coloring general graphs equitably or balanced are $\mathcal{NP}$-complete in their decision versions [28].

Using the following result on the existence of a balanced coloring, which, according to [18] is due to de Werra, the weekly problem (WP) has for any given $|DpW|$ a feasible solution and can thus be solved as integer program. [10] even sketches a possible solution concept with network flow techniques to solve (WP) in polynomial time.

Proposition 3.3
For $\tilde{k} \ge 1$, each finite undirected bipartite graph has a balanced edge coloring with $\tilde{k}$ colors. $\lhd$

Based on the results of [10] and assuming some feasible solution x_{ijk} of (WP), we next tackle the *daily problem* (DP^k). Let

$$x^k_{ijs} = \begin{cases} 1, & \text{patient } P_i \text{ and therapist } T_j \\ & \text{meet in period } s \text{ on day } k; \\ 0, & \text{else;} \end{cases}$$

for $i \in \mathcal{N}, j \in \mathcal{M}$ and $k = 1, ..., |DpW|$. For each day $k = 1, ..., |DpW|$, the x_{ijk} define a new matrix $A_k = (a^k_{ij})$ which specifies how often T_j and P_i meet on day k. Then, (DP^k) is given as follows.

Find x^k_{ijs}, such that

$$\sum_{s=1}^{|PpD|} x^k_{ijs} = a^k_{ij}, \text{ for all } i \in \mathcal{N}, j \in \mathcal{M} \tag{8}$$

$$\sum_{i=1}^{n} x^k_{ijs} \leq 1, \text{ for all } j \in \mathcal{M}, s = 1, ..., |PpD|, \tag{9}$$

$$\sum_{j=1}^{m} x^k_{ijs} \leq 1, \text{ for all } i \in \mathcal{N}, s = 1, ..., |PpD|, \tag{10}$$

$$x^k_{ijs} \in \mathbb{B}, \text{ for all } i \in \mathcal{N}, j \in \mathcal{M}, s = 1, ..., |PpD|. \tag{11}$$

A day consists of a certain number of periods per day ($|PpD|$). Assuming usual working hours from 8am till 5pm, a lunch break of one hour, and periods of a half hour, this results in $2 \cdot 8 = 16$ periods each day. The total amount of periods per week is $|DpW| \cdot |PpD|$. (DP^k) can be formulated as an ordinary edge coloring problem, where no two adjacent edges are allowed to have the same color. [10] uses *König's Theorem* to obtain that the following proposition holds.

Proposition 3.4
(DP^k) is solvable for day k, if and only if

$$\sum_{i=1}^{n} a^k_{ij} \leq |PpD|, \text{ for all } j \in \mathcal{M},$$

$$\sum_{j=1}^{m} a^k_{ij} \leq |PpD|, \text{ for all } i \in \mathcal{N}.$$

$\triangleleft$

If the condition in Proposition 3.4 is satisfied, (DP^k) can be solved e.g. by an $\mathcal{O}(|E| \cdot log(|E|))$ algorithm (see [1]).

Based on ([10], [12]) a stronger statement is proved in [28].

Proposition 3.5
Each solution of (WP) implies a solvable system of (DP^k), for all $k = 1, ..., |DpW|$, if and only if

$$\sum_{i=1}^{n} p_{ij} \leq |DpW| \cdot |PpD|, \text{ for all } j \in \mathcal{M}, \text{ and}$$

$$\sum_{j=1}^{m} p_{ij} \leq |DpW| \cdot |PpD|, \text{ for all } i \in \mathcal{N}.$$

$\triangleleft$

Next, the problem is considered in which a patient needs to have for some specific kind of therapy two, but not more, therapy sessions on some days. Note that if the patient has a single therapist assigned to him, any solution of the weekly schedule (WP) still works fine, since the property of balancedness guarantees that it can never happen, assuming at most a double therapy, that a patient has a day without a certain planned therapy and another one with more than two sessions. This would contradict the coloring property (see Definition 3.1). If he has two therapists, one who does the major part and the other the minor, we can also apply (WP), by assigning the first therapist five times a week and the second one the rest. Their meetings will clearly all be balanced out.

A complicating condition is the one where none of the two therapists involved in this particular therapy has the full weekly load of, say, five meetings. In order to deal with this situation - and an even more general one of more than two therapists - we introduce a new type of edge coloring.

Definition 3.6
Given an undirected, bipartite graph $G = (V, E)$ with $V = P\dot{\cup}T$, where $T = \mathcal{T}_1\dot{\cup}\mathcal{T}_2\dot{\cup}...\dot{\cup}\mathcal{T}_q$, with edge coloring f. Let $(u, \mathcal{T}_s) := \{e \in E|e = (u, v), v \in \mathcal{T}_s\}$, for any $u \in P$, and let $f(i)(u, \mathcal{T}_s)$ be the set of all edges, that connect u with some vertex $v \in \mathcal{T}_s$ having color i.

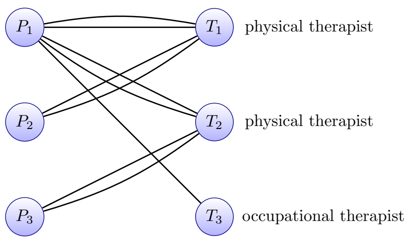
A balanced edge coloring of G with $\tilde{k}$ colors is called *group-wise balanced (with respect to T)*, if for all $u \in P$,

$$||f(i)(u, \mathcal{T}_s)| - |f(j)(u, \mathcal{T}_s)|| \le 1,$$
$$\text{for all } 1 \le i < j \le \tilde{k}, s = 1, ..., q.$$

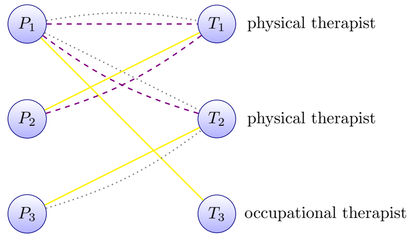
$\lhd$

This definition does, indeed, model, the modified patient-therapist planning problem, since it ensures an even distribution between a patient, corresponding to node u, and the group of therapists represented by the set $\mathcal{T}_s$. As decision version on general graphs, this problem is $\mathcal{NP}$-complete [28]. For a better understanding consider the following example.

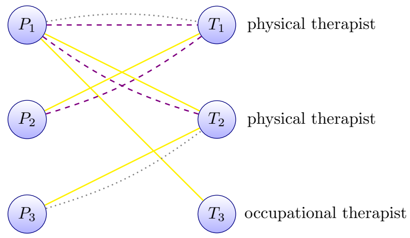
Example 3.7
In this example we have that $\mathcal{N} = \{1, 2, 3\}$, $\mathcal{M} = \{1, 2, 3\}$, the number of colors is $|DpW| = 3$ and $(p_{ij}) = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$.
Then we obtain the corresponding graph G as



The therapists T_1 and T_2 are supposed to be physical therapists ($\mathcal{T}_1$), whereas T_3 ($\mathcal{T}_2$) is an occupational therapist. Thus, by considering (p_{ij}), we see that patient P_1 should have four times physical therapy in three days where the workload is distributed between T_1 and T_2. Let us consider the following edge coloring



The above edge coloring is equitable and balanced, but patient P_1 would only have physical therapy sessions on two out of three days. However, the next edge coloring, which is equitable and balanced as well, does also satisfy the constraint of *physical therapy at least once a day*. It is group-wise balanced.



$\lhd$

Theorem 3.8
Let $\tilde{k} \ge 1$, $G = (V, E)$ be a finite, undirected, bipartite graph with $V = P\dot{\cup}T$, $T = \mathcal{T}_1\dot{\cup}\mathcal{T}_2\dot{\cup}...\dot{\cup}\mathcal{T}_q$, where $q \le |T|$. Then there exists for G a group-wise balanced edge coloring with $\tilde{k}$ colors w.r.t. T. $\lhd$

Proof. As described in [11, 12, 21], the edge coloring problem associated with (WP) can be formulated as node coloring problem on a hypergraph (see [5] for a definition). Moreover, it can be shown [28] that finding a group-wise balanced edge coloring corresponds to finding an equitable node coloring in a unimodular hypergraph.

Since any unimodular hypergraph H has for every $\tilde{k} \geq 2$ an equitable node coloring with $\tilde{k}$ colors (de Werra, 1971, according to [21]), the result follows. $\square$

Network flow techniques can be used to find the group-wise balanced edge coloring of Theorem 3.8 [21]. A pseudocode can be found in [28].

One can extend the (IP) formulation of the (WP) problem formulation by including the condition of group-wise balancedness to obtain (WP$_{gb}$) as follows.

$$\sum_{k=1}^{|DpW|} x_{ijk} = p_{ij}, \text{ for all } i \in \mathcal{N}, j \in \mathcal{M} \tag{12}$$

$$\left\lfloor \sum_{i=1}^{n} \frac{p_{ij}}{|DpW|} \right\rfloor \leq \sum_{i=1}^{n} x_{ijk} \leq \left\lceil \sum_{i=1}^{n} \frac{p_{ij}}{|DpW|} \right\rceil, \quad \text{for all } j \in \mathcal{M}, k = 1, ..., |DpW|, \tag{13}$$

$$\left\lfloor \sum_{j=1}^{m} \frac{p_{ij}}{|DpW|} \right\rfloor \leq \sum_{j=1}^{m} x_{ijk} \leq \left\lceil \sum_{j=1}^{m} \frac{p_{ij}}{|DpW|} \right\rceil, \quad \text{for all } i \in \mathcal{N}, k = 1, ..., |DpW|, \tag{14}$$

$$\left\lfloor \frac{p_{ij}}{|DpW|} \right\rfloor \leq x_{ijk} \leq \left\lceil \frac{p_{ij}}{|DpW|} \right\rceil, \quad \text{for all } i \in \mathcal{N}, j \in \mathcal{M}, k = 1, ..., |DpW|, \tag{15}$$

$$\left\lfloor \sum_{\{j|T_j \in \mathcal{T}_s\}} \frac{p_{ij}}{|DpW|} \right\rfloor \leq \sum_{\{j|T_j \in \mathcal{T}_s\}} x_{ijk} \leq \left\lceil \sum_{\{j|T_j \in \mathcal{T}_s\}} \frac{p_{ij}}{|DpW|} \right\rceil, \quad \text{for all } i \in \mathcal{N}, k = 1, ..., |DpW|, s = 1, ..., q \tag{16}$$

$$x_{ijk} \in \mathbb{Z}_0^+, \quad \text{for all } i \in \mathcal{N}, j \in \mathcal{M}, k = 1, ..., |DpW|. \tag{17}$$

Since it is not desirable to have two therapy sessions of the same type directly succeeding each other, it is necessary to modify the daily plans (DP^k). This is done by first splitting the total number of periods per day in half, i.e., by considering morning and afternoon periods where we assume that $|PpD| = |\mathcal{P}_1| + |\mathcal{P}_2|$. In this section, we assume that $|\mathcal{P}_i| = \frac{|PpD|}{2}$, and discuss the more general case in the subsequent section.

Thus, we solve (WP$_{gb}$), succeeded by another problem of the same type called *phase of day* (PDP$^k_{gb}$), in which we only allow two colors.

Find x^k_{ijl}, for $l = 1, 2$, such that

$$x^k_{ij1} + x^k_{ij2} = b^k_{ij}, \quad \text{for all } i \in \mathcal{N}, j \in \mathcal{M}, k = 1, ..., |DpW|, \tag{18}$$

$$\left\lfloor \sum_{i=1}^{n} \frac{b^k_{ij}}{2} \right\rfloor \leq \sum_{i=1}^{n} x^k_{ijl} \leq \left\lceil \sum_{i=1}^{n} \frac{b^k_{ij}}{2} \right\rceil, \quad \text{for all } j \in \mathcal{M}, k = 1, ..., |DpW|, \tag{19}$$

$$\left\lfloor \sum_{j=1}^{m} \frac{b^k_{ij}}{2} \right\rfloor \leq \sum_{j=1}^{m} x^k_{ijl} \leq \left\lceil \sum_{j=1}^{m} \frac{b^k_{ij}}{2} \right\rceil, \quad \text{for all } i \in \mathcal{N}, k = 1, ..., |DpW|, \tag{20}$$

$$\left\lfloor \frac{b^k_{ij}}{2} \right\rfloor \leq x^k_{ijl} \leq \left\lceil \frac{b^k_{ij}}{2} \right\rceil, \quad \text{for all } i \in \mathcal{N}, j \in \mathcal{M}, k = 1, ..., |DpW|, \tag{21}$$

$$\left\lfloor \sum_{\{j|T_j \in \mathcal{T}_s\}} \frac{b^k_{ij}}{2} \right\rfloor \leq \sum_{\{j|T_j \in \mathcal{T}_s\}} x^k_{ijl} \leq \left\lceil \sum_{\{j|T_j \in \mathcal{T}_s\}} \frac{b^k_{ij}}{2} \right\rceil, \quad \text{for all } i \in \mathcal{N}, k = 1, ..., |DpW|, s = 1, ..., q \tag{22}$$

$$x^k_{ijl} \in \mathbb{Z}_0^+, \quad \text{for all } i \in \mathcal{N}, j \in \mathcal{M}, k = 1, ..., |DpW|, \tag{23}$$

with $b^k_{ij} := x_{ijk}$ obtained by (WP$_{gb}$) and

$$x^k_{ijl} := \begin{cases} 1, & \text{patient } P_i \text{ meets therapist } T_j \\ & \text{on day } k \text{ in phase } l; \\ 0, & \text{else.} \end{cases}$$

We then pass the solution of this problem to (DP_l^k), $l=1,2$. Find $x_{ijs}^{k,l}$, such that $\triangleleft$

$$\sum_{s=1}^{|\mathcal{P}_l|} x_{ijs}^{k,l} = a_{ij}^{k,l}, \quad \text{for all } i \in \mathcal{N}, j \in \mathcal{M}, k=1,...,|DpW|, \tag{24}$$

$$\sum_{i=1}^{n} x_{ijs}^{k,l} \leq 1, \quad \text{for all } j \in \mathcal{M}, s=1,...,|\mathcal{P}_l|, k=1,...,|DpW|, \tag{25}$$

$$\sum_{j=1}^{m} x_{ijs}^{k,l} \leq 1, \quad \text{for all } i \in \mathcal{N}, s=1,...,|\mathcal{P}_l|, k=1,...,|DpW|, \tag{26}$$

$$x_{ijs}^{k,l} \in \mathbb{B}, \quad \text{for all } i \in \mathcal{N}, j \in \mathcal{M}, s=1,...,|\mathcal{P}_l|, k=1,...,|DpW|, \tag{27}$$

with $a_{ij}^{k,l} := x_{ijl}^k$ obtained by (PDP_{gb}^k) and

$$x_{ijs}^{k,l} := \begin{cases} 1, & \text{if patient } P_i \text{ meets therapist } T_j \\ & \text{on day } k \text{ in phase } l \text{ in period } s; \\ 0, & \text{else.} \end{cases}$$

Note that (PDP_{gb}^k) can be solved similar to (WP_{gb}), whereas (DP_l^k) is again an ordinary edge coloring problem. We denote the resulting problem by (PTTP_{gb}). Furthermore, we say that in (PTTP_{gb}) a solution of (WP_{gb}) is *aligned* with (PDP_{gb}^k), if this solution is used to build the corresponding (PDP_{gb}^k) and then a feasible solution of (PDP_{gb}^k) is considered further. Thus, we call this process in the following *alignment*.

Theorem 3.9
Let $|PpD| = |\mathcal{P}_1| + |\mathcal{P}_2|$, with $|\mathcal{P}_1| = |\mathcal{P}_2|$. Each solution of (WP_{gb}) aligned with (PDP_{gb}^k), $k=1,...,|DpW|$, implies a solvable system of (DP_l^k), for $l=1,2$, $k=1,...,|DpW|$, if and only if

$$\sum_{i=1}^{n} p_{ij} \leq |DpW| \cdot |PpD|, \text{ for all } j \in \mathcal{M}, \text{ and}$$

$$\sum_{j=1}^{m} p_{ij} \leq |DpW| \cdot |PpD|, \text{ for all } i \in \mathcal{N}.$$

Proof. "⇐": Let a solution of (WP_{gb}) be given together with a, from this solution obtained, feasible solution of (PDP_{gb}^k), $k=1,...,|DpW|$. This is possible by Theorem 3.8. By Proposition 3.4 we know, that (DP_l^k), is feasible for $l=1,2$, if and only if

$$\sum_{i=1}^{n} a_{ij}^{k,l} \leq \frac{|PpD|}{2}, \text{ for all } j \in \mathcal{M}, \; k=1,...,|DpW| \text{ and}$$

$$\sum_{j=1}^{m} a_{ij}^{k,l} \leq \frac{|PpD|}{2}, \text{ for all } i \in \mathcal{N}, \; k=1,...,|DpW|,$$

and we have, for k,l,j arbitrary, that

$$\begin{aligned}
\sum_{i=1}^{n} a_{ij}^{k,l} &= \sum_{i=1}^{n} x_{ijl}^k && (28)\\
&\leq \left\lceil \sum_{i=1}^{n} \frac{b_{ij}^k}{2} \right\rceil && (29)\\
&= \left\lceil \sum_{i=1}^{n} \frac{x_{ijk}}{2} \right\rceil = \left\lceil \frac{1}{2} \cdot \sum_{i=1}^{n} x_{ijk} \right\rceil && (30)\\
&\leq \left\lceil \frac{1}{2} \cdot \left\lceil \sum_{i=1}^{n} \frac{p_{ij}}{|DpW|} \right\rceil \right\rceil && (31)\\
&\leq \left\lceil \frac{1}{2} \cdot \left\lceil \frac{|DpW| \cdot |PpD|}{|DpW|} \right\rceil \right\rceil && (32)\\
&= \left\lceil \frac{1}{2} \cdot \lceil |PpD| \rceil \right\rceil = \left\lceil \frac{1}{2} \cdot |PpD| \right\rceil && (33)\\
&= \frac{1}{2} \cdot |PpD| && (34)
\end{aligned}$$

Here (28) holds by definition of $a_{ij}^{k,l}$, the first inequality (29) is correct by (PDP_{gb}^k), (30) by definition of b_{ij}^k, the second inequality (31) by (WP_{gb}) and (32) by assumption. The last equality (34) holds true, since $|PpD|$ is an even number. $\sum_{j=1}^{m} a_{ij}^{k,l} \leq \frac{|PpD|}{2}$, k,l,i arbitrary, can be shown analogously.

"⇒": Now we assume, that each solution of (WP_{gb}) with an arbitrary from it originated solution of (PDP_{gb}^k) implies a solvable system of (DP_l^k), $l=1,2$, $k=1,...,|DpW|$. Let such a solution be given. Its existence is ensured by Theorem 3.8. Thus, by (12)-(13), (18), for $j \in \mathcal{M}$ arbitrary,

$$\begin{aligned}\sum_{i=1}^{n} p_{ij} &= \sum_{i=1}^{n} \sum_{k=1}^{|DpW|} x_{ijk} = \sum_{k=1}^{|DpW|} \sum_{i=1}^{n} x_{ijk} \\ &= \sum_{k=1}^{|DpW|} \sum_{i=1}^{n} b_{ij}^{k} = \sum_{k=1}^{|DpW|} \sum_{i=1}^{n} (x_{ij1}^{k} + x_{ij2}^{k}) \\ &= \sum_{k=1}^{|DpW|} \sum_{i=1}^{n} (a_{ij}^{k,1} + a_{ij}^{k,2}) \\ &= \sum_{k=1}^{|DpW|} (\sum_{i=1}^{n} a_{ij}^{k,1} + \sum_{i=1}^{n} a_{ij}^{k,2}) \\ &\leq \sum_{k=1}^{|DpW|} (\frac{|PpD|}{2} + \frac{|PpD|}{2}) \\ &= |DpW| \cdot |PpD|.\end{aligned}$$

Analogously for $\sum_{j=1}^{m} p_{ij}$, $i \in \mathcal{N}$. $\square$

4 Hierarchical Edge Colorings on Bipartite Graphs

The problem sequence (WP_{gb}), (PDP_{gb}^{k}), (DP_{l}^{k}) of a patient-therapist-timetable problem (PTTP_{gb}) considered in the previous section can be interpreted as special case of *hierarchical edge colorings*, by considering three levels L_1, L_2, L_3 and three required (maximum) numbers of colors F_1, F_2, F_3 for each of these levels (see Figure 1), where F_3 depends on the previous level for $|\mathcal{P}_1| \neq |\mathcal{P}_2|$.

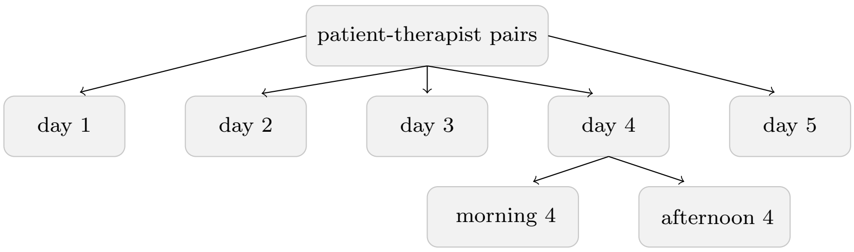


Figure 1: Levels of (PTTP_{gb}).

In the general problem, each level L_i, $i = 1, ..., z$, corresponds to a partition of the edge set E, where $L_1 = E$ and each L_{i+1} is a partition of E into a family of subsets, each of which has been obtained by solving a coloring problem on the previous level L_i, $i = 1, ..., z-1$. Thus, in Figure 1 $L_1 = E$ corresponds to the node *patient-therapist pairs*, L_2 corresponds to the nodes *day i*, $i = 1, ..., 5$, and L_3 to the nodes *morning i, afternoon i*, $i = 1, ..., 5$. $L_3 = L_z$ is the last level where an (ordinary) edge coloring problem has to be solved.

Example 4.1
The problem sequence (WP_{gb}), (PDP^k_{gb}), (DP^k_l) from the previous section, the *patient-therapist-timetable* problem with group-wise balanced constraints (PTTP_{gb}), is a hierarchical edge coloring problem, (*hierarchical* $PTTP_{gb}$), where:

$$\begin{aligned} &L_1 : \text{group-wise balanced edge coloring,} \\ &\quad F_1 = |DpW|; \\ &L_2 : \text{group-wise balanced edge coloring,} \\ &\quad F_2 = 2; \\ &L_3 : \text{ordinary edge coloring,} \\ &\quad F_3(j) = |\mathcal{P}_j|, \text{ for } j = 1, 2. \end{aligned}$$

Note that F_3 depends on color j chosen on level 2. ◁

Note further, that only hierarchical edge coloring problems with at least one level requesting an ordinary coloring are interesting in terms of feasibility considerations, since we know by the proof of Theorem 3.8 that feasibility is always ensured in the other cases.

Using the concept of hierarchical colorings, we can drop in the (PTTP_{gb}) of Section 3 the assumption $|\mathcal{P}_1| = |\mathcal{P}_2|$. It can be seen [28], that Theorem 3.9 is no longer correct without this assumption. However, the following more general result [28] can be shown.

Theorem 4.2
Let (PTTP_{gb}) be defined by $|DpW|$, $|PpD| = |\mathcal{P}_1| + |\mathcal{P}_2|$, $|\mathcal{P}_1|, |\mathcal{P}_2|$ arbitrary, and $\tilde{P} = (p_{ij})_{\mathcal{N}\times\mathcal{M}}$. Then

1) If

$$\sum_{i=1}^{n} p_{ij} \leq 2 \cdot \min_{l=1,2} |\mathcal{P}_l| \cdot |DpW|, \forall j \in \mathcal{M}, \text{ and}$$
$$\sum_{j=1}^{m} p_{ij} \leq 2 \cdot \min_{l=1,2} |\mathcal{P}_l| \cdot |DpW|, \forall i \in \mathcal{N},$$

then each solution of (WP_{gb}) aligned with (PDP^k_{gb}), $k = 1, ..., |DpW|$, gives a feasible system of (DP^k_l), $l = 1, 2$, $k = 1, ..., |DpW|$. This bound is tight.

2) If each solution of (WP_{gb}) aligned with (PDP^k_{gb}), $k = 1, ..., |DpW|$ gives a feasible system of (DP^k_l), $l = 1, 2$, $k = 1, ..., |DpW|$, then

$$\sum_{i=1}^{n} p_{ij} \leq |DpW| \cdot |PpD|, \forall j \in \mathcal{M},$$
$$\sum_{j=1}^{m} p_{ij} \leq |DpW| \cdot |PpD|, \forall i \in \mathcal{N}.$$

◁

Proof. 1) Let an arbitrary solution for (WP_{gb}) be given, together with an from it originated arbitrary feasible solution of (PDP^k_{gb}), by Theorem 3.8. By Proposition 3.4 we have that (DP^k_l) is feasible, for $l = 1, 2$, if and only if

$$\sum_{i=1}^{n} a^{k,l}_{ij} \leq |\mathcal{P}_l|,$$
$$\text{for all } j \in \mathcal{M},\ k = 1, ..., |DpW|,\ l = 1, 2,$$
$$\sum_{j=1}^{n} a^{k,l}_{ij} \leq |\mathcal{P}_l|,$$
$$\text{for all } i \in \mathcal{N},\ k = 1, ..., |DpW|,\ l = 1, 2,$$

and for arbitrary j, k, l,

$$\begin{aligned} \sum_{i=1}^{n} a^{k,l}_{ij} = \sum_{i=1}^{n} x^k_{ijl} &\leq \left\lceil \sum_{i=1}^{n} \frac{b^k_{ij}}{2} \right\rceil \\ &= \left\lceil \sum_{i=1}^{n} \frac{x_{ijk}}{2} \right\rceil \\ &= \left\lceil \frac{1}{2} \sum_{i=1}^{n} x_{ijk} \right\rceil \\ &\leq \left\lceil \frac{1}{2} \left\lceil \sum_{i=1}^{n} \frac{p_{ij}}{|DpW|} \right\rceil \right\rceil \\ &\leq \left\lceil \frac{1}{2} \left\lceil \frac{2}{|DpW|} \cdot \min_{l=1,2} |\mathcal{P}_l| \cdot |DpW| \right\rceil \right\rceil \\ &= \left\lceil \min_{l=1,2} |\mathcal{P}_l| \right\rceil \\ &= \min_{l=1,2} |\mathcal{P}_l| \leq |\mathcal{P}_l|. \end{aligned}$$

Analogously it holds that $\sum_{j=1}^{m} a_{ij}^{k,l} \leq |\mathcal{P}_l|$, for all $i \in \mathcal{N}$, $k = 1, ..., |DpW|$, $l = 1, 2$.

That this bound is tight, is shown in an Example in [28], where we have $\sum_{j=1}^{3} p_{ij} = 3$, for all $i \in \mathcal{N}$, and $2 \cdot \min_{l=1,2} |\mathcal{P}_l| \cdot |DpW| = 2$.

2) Assume that each solution of (WP$_{gb}$) with a from it originated arbitrary feasible solution of (PDP$^k_{gb}$) gives a feasible system of (DP^k_l), $l = 1, 2$, $k = 1, ..., |DpW|$. By Theorem 3.8 such a solution exists. Thus, the corresponding (DP^k_l), $l = 1, 2$, $k = 1, ..., |DpW|$, are feasible, i.e., by Proposition 3.4 we have, for $l = 1, 2$,

$$\sum_{i=1}^{n} a_{ij}^{k,l} \leq |\mathcal{P}_l|,$$

for all $j \in \mathcal{M}$, $k = 1, ..., |DpW|$, $l = 1, 2$,

$$\sum_{j=1}^{n} a_{ij}^{k,l} \leq |\mathcal{P}_l|,$$

for all $i \in \mathcal{N}$, $k = 1, ..., |DpW|$, $l = 1, 2$.

Now consider, for $j \in \mathcal{M}$,

$$\begin{aligned}\sum_{i=1}^{n} p_{ij} &= \sum_{k=1}^{|DpW|} \left(\sum_{i=1}^{n} a_{ij}^{k,1} + \sum_{i=1}^{n} a_{ij}^{k,2}\right)\\ &\leq \sum_{k=1}^{|DpW|} (|\mathcal{P}_1| + |\mathcal{P}_2|)\\ &= |DpW| \cdot (|\mathcal{P}_1| + |\mathcal{P}_2|)\\ &= |DpW| \cdot |PpD|.\end{aligned}$$

Analogously for $\sum_{j=1}^{m} p_{ij}$, $i \in \mathcal{N}$. □

For (PTTP$_{wgb}$), the patient-therapist-timetable problem without group-wise balanced constraints, the following theorem can be shown [28] (the proof is omitted, since it is quite technical and lengthy).

Theorem 4.3
Let (PTTP$_{wgb}$) be defined by $|DpW|$, let $|PpD| = |\mathcal{P}_1| + |\mathcal{P}_2|$, such that $||\mathcal{P}_1| - |\mathcal{P}_2|| = 1$, and let $\tilde{P} = (p_{ij})_{\mathcal{N}\times\mathcal{M}}$ be satisfying

$$\sum_{i=1}^{n} p_{ij} \leq |DpW| \cdot |PpD|, \forall j \in \mathcal{M},$$

$$\sum_{j=1}^{m} p_{ij} \leq |DpW| \cdot |PpD|, \forall i \in \mathcal{N}.$$

Furthermore, consider a particular solution for L_1, i.e., a feasible edge coloring of L_1.

If there are two different solutions for L_2, where one is feasible for L_3 and the other is not, then the one that is infeasible can be transfered into the feasible one by redying a finite sequence of alternating (in terms of color), elementary paths and cycles. ◁

Unfortunately, the condition in Theorem 4.3 is not sufficient to guarantee the feasibility of the complete problem. One can find examples [28] showing that no such sequence exists starting from a fixed solution for L_1, while a modification of this solution can yield a feasible solution. Hence, only the following corollary can be established [28].

Corollary 4.4
Let (PTTP$_{wgb}$) be defined by $|DpW|$, let $|PpD| = |\mathcal{P}_1| + |\mathcal{P}_2|$, such that $||\mathcal{P}_1| - |\mathcal{P}_2|| = 1$, and let $\tilde{P} = (p_{ij})_{\mathcal{N}\times\mathcal{M}}$ be satisfying

$$\sum_{i=1}^{n} p_{ij} \leq |DpW| \cdot |PpD|, \forall j \in \mathcal{M},$$

$$\sum_{j=1}^{m} p_{ij} \leq |DpW| \cdot |PpD|, \forall i \in \mathcal{N},$$

If an edge coloring for L_1 and L_2 is given, where the latter is infeasible for L_3, it holds that either a finite sequence of alternating elementary paths can be redyed in L_2, such that the edge coloring is still balanced and becomes feasible for L_3 or there is no feasible solution with that edge coloring for L_1 of (PTTP_{wgb}). $\triangleleft$

Algorithm 1 shows a pseudocode using the result of Corollary 4.4. Its proof of correctness can be found in [28].

Note that the *feasibility constraint* (FC_3) in Algorithm 1 corresponds to

$$\sum_{i=1}^{n} a_{ij}^{k,l} \leq |\mathcal{P}_l|, \text{ for all } j \in \mathcal{M},$$
$$k = 1, ..., |DpW|, \; l = 1, 2,$$
$$\sum_{j=1}^{n} a_{ij}^{k,l} \leq |\mathcal{P}_l|, \text{ for all } i \in \mathcal{N},$$
$$k = 1, ..., |DpW|, \; l = 1, 2,$$

where $a_{ij}^{k,l} = x_{ijl}^{k}$ is the solution of (PDP_{gb}^{k}).

Furthermore, *cand along* W in line 10 of Algorithm 1 means, that in particular $u \in \mathcal{V} \cup \bar{\mathcal{V}}$ has to hold, if $(., u) \in E(W)$ has color $j_2 = 1$ and $u \in \bar{\bar{\mathcal{V}}}$, if $(., u) \in E(W)$ has color $j_2 = 2$. Thus, *cand along* W is the set of *candidate nodes* along an elementary alternating path W. A candidate being a node which can be the last node on such a path W, such that W can be redyed without destroying any of the desired properties.

5 Conclusion

In this paper, we gave an overview on rehabilitation clinics in Germany. We pointed out the differences between the facilities in Germany and other countries and focused on the fact that inpatient rehabilitation clinics differ very much from regular hospitals with acute care, both in their structure and in their focus.

We presented a survey on patient scheduling related to rehabilitation clinics and gave further reference to similar areas of research.

Then we showed how a simple class-teacher model can be modified to meet a first set of constraints which are typical in a rehabilitation environment. With the concept of group-wise coloring, we introduced a kind of edge coloring, which has to the best of out knowledge so far not been considered in the literature. We noticed, that for bipartite graphs it can be seen as searching for an equitable node coloring on a hypergraph using an argument of [21]. To obtain an even better balancing of therapy sessions for the considered patients, we introduced a coloring in hierarchies and made some first attempts to approach a special case of this problem algorithmically.

Current research deals with the development of lifting procedures if line 19 of Algorithm 1 comes to pass. We want to be able to modify the coloring of the first level based on information obtained on the second level without needing to compute all possible edge colorings of the first level. The latter is, in general, impossible, since it can be shown [28], that there can be exponentially many balanced edge colorings for an undirected bipartite graph G, even if $|DpW|$ is assumed to be constant. Another immediate research topic is the determination of the complexity status for hierarchical edge coloring problems on bi-

Algorithm 1 Pseudocode (Alternating Paths)

Input: $|DpW|$, $|PpD| = |\mathcal{P}_1| + |\mathcal{P}_2|$, with $|\mathcal{P}_2| = |\mathcal{P}_1| + 1$, $\sum\limits_{i=1}^{n} p_{ij} \leq |DpW| \cdot |PpD|, \forall j \in \mathcal{M}$, $\sum\limits_{j=1}^{m} p_{ij} \leq |DpW| \cdot |PpD|, \forall i \in \mathcal{N}$, edge coloring for L_1 and aligned one of L_2

Output: $\mathcal{W} = (\mathcal{W}_1, ..., \mathcal{W}_{|DpW|})$ as in Corollary 4.4 demanded or the result, that for L_1 non-existent

for all $j_1 = 1, ..., |DpW|$ **do**
  $\mathcal{W}_{j_1} \leftarrow \emptyset$
  $\mathcal{V} \leftarrow \{v \in V | \deg_{H_{(j_1,1)}}(v) > |\mathcal{P}_1|\}$
  $\bar{\mathcal{V}} \leftarrow \{v \in V \setminus \mathcal{V} | f(1)(v)$ can be reduced by 1, without destroying equitability of the edge coloring of L_2 in $v\} = \{v \in V \setminus \mathcal{V} | f(2)(v)$ can be increased by 1, without destroying equitability of the edge coloring of L_2 in $v\}$
  $\bar{\bar{\mathcal{V}}} \leftarrow \{v \in V | f(1)(v)$ can be increased by 1, without destroying the equitability of the edge coloring of L_2 in v or $(FC_3)\}$
  $\tilde{H}_{(j_1)} \leftarrow (V, E(\tilde{H}_{(j_1)}))$
  $E(\tilde{H}_{(j_1)}) \leftarrow \{e \in E(H_{(j_1)}) \setminus \{$parallel edges of even number with half of the edges in $E(H_{(j_1,1)})$ and half in $E(H_{(j_1,2)})\}\}$
  while $\mathcal{V} \neq \emptyset$ **do**
    choose $v \in \mathcal{V}$
    if $\exists$ alt. el. v-u-path W in $\tilde{H}_{(j_1)}$: $u \in$ ($cand$ along W), starting with $j_2 = 1$ **then**
      if $u \in \mathcal{V}$ **then**
        $\mathcal{V} \leftarrow \mathcal{V} \setminus \{u\}$
      end if
      $\tilde{H}_{(j_1)} \leftarrow \tilde{H}_{(j_1)}$, where W is redyed
      $\mathcal{W}_{j_1} \leftarrow \mathcal{W}_{j_1} \cup W$
      $\mathcal{V} \leftarrow \mathcal{V} \setminus \{v\}$
      update $\bar{\mathcal{V}}$, $\bar{\bar{\mathcal{V}}}$
    else
      break there is no solution with this edge coloring of L_1 existent
    end if
  end while
  reconstruct a new $H_{(j_1)}$ based on $\tilde{H}_{(j_1)}$
  $j_1 \leftarrow j_1 + 1$
end for

partite graphs.

We further aim to approach the topic of therapy sessions of different lengths and to model the problem that patients could need sometimes more breaks in between therapy sessions if their health status demands it. We also plan to investigate the underlying polyhedra of our problems and perhaps also introduce LP-relaxations. The final goal would be to work with a rehabilitation clinic to be able to obtain computational results on real data.

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