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On a Phenomenological Generalized Boltzmann Equation

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Abstract
In this paper we prove existence and uniqueness of the solution for a generalized Boltzmann equation and we discuss the positivity of this solution. We will give two series representations of the solution of our equation.

Key words: Generalized Boltzmann Equation, Differential Equations in Banach Spaces

1) Introduction

Due to several space research projects, e.g. the european space project HERMES, there is a growing interest in the study of so called real gas effects. To describe such effects on a kinetic level, one needs a generalized Boltzmann equation. The present paper is concerned with such an equation. We will study the initial value problem for the spatially homogeneous case.

The plan of our paper is as follows: In section 2 we will describe the Boltzmann equation and introduce some notation. The subsequent section contains the definition of the function spaces for the scattering cross sections and we will discuss there some basic physical estimates. In section 4 we will prove the existence theorems and we will show in section 5, that the solution of our kinetic equation is positive, if the initial condition is nonnegative and the inelastic part of the scattering cross section is positive. Section 6 contains two series representations of the solution of the Boltzmann equation and we will prove there an existence theorem for the case of initial conditions which are negative on sets with positive, but sufficiently small Lebesgue measure.

2) The Boltzmann equation

The evolution of the distribution function of a spatially homogeneous gas consisting of molecules with internal energy is given by:

$$\frac{\partial}{\partial t} f(t,v,\varepsilon_1) = J(\sigma,f,f)(t,v,\varepsilon_1)$$

$$\text{with } J(\sigma,f,g) = \frac{1}{2}\int_{\Pi'} \sqrt{1 - e_1' - e_2'}\; \sigma(E,e_1,e_2,e_1',e_2'.\eta.\eta')\left[f'g'_* + f'_*g' - fg_* - f_*g\right] d\Omega(\eta')\, de_1' de_2'\, d\varepsilon_2 dw. \tag{2.1}$$

In (2.1) we have used the following notations:

$$\Pi' = \mathbb{R}^3 x \mathbb{R}_+ x \Delta_1 x S_2 \quad \text{with } \Delta_1 = \{(e_1',e_2') : 0 \le e_1',\ 0 \le e_2' \text{ and } e_1' + e_2' \le 1\}, \tag{2.2}$$

$$E = \frac{1}{2}|v-w|^2 + \varepsilon_1 + \varepsilon_2 \;,\; c' = \sqrt{2E(1 - e_1' - e_2')} \quad \text{and } e_i = \varepsilon_i/E,\ i=1,2. \tag{2.3}$$

$$v' = \frac{1}{2}(v + w + \eta'c'), \qquad \varepsilon_1' = e_1'E,$$
$$w' = \frac{1}{2}(v + w - \eta'c'), \qquad \varepsilon_2' = e_2'E, \tag{2.4}$$

$$f' = f(t,v',\varepsilon_1'),\quad f'_* = f(t,w',\varepsilon_2'),\quad f_* = f(t,w,\varepsilon_2) \tag{2.5}$$

and the scattering cross section has the form

$$\sigma(E,e_1,e_2,e_1',e_2',x) = \sigma_1(E,e_1,e_2,e_1',e_2',x) + \frac{\sigma_2(E,e_1,e_2,x)}{\sqrt{1 - e_1' - e_2'}}\,\delta(e_1-e_1')\,\delta(e_2-e_2'). \tag{2.6}$$

This scattering cross section shows that one has to distinguish between two types of collisions: on the one hand there are inelastic ones described by σ_1, one the other hand we have elastic collisions: the relative velocity of the two colliding particles can change, the internal energies remain unchanged.

3) The space of scattering cross sections and basic physical estimates

In this section we introduce the function spaces for the scattering cross sections for both inelastic and elastic collisions and show some properties of the collision operator. As usual we denote by $C(X \to Y)$ the space of continous functions from a metric space X into a metric space Y.

Definition 3.1: The set S of the inelastic scattering cross sections is the set of all measurable real valued functions k defined on $\mathbb{R}_+ x \Delta_1 x \Delta_1 x S_2$ which have the properties:

(i) $k \in C(\mathbb{R}_+ x \Delta_1 \to L_1(\Delta_1 x[-1,1]))$

(ii) $k(E,e,e',x) = k(E,e',e,x)$ a.e (3.1)

(iii) $e_1 + e_2 = 1 \Rightarrow k(E,e,e',x) = 0$ a.e.

The set of all nonnegative functions in S will be denoted by S_+.

$$S_0 = \{ k \in S : \|k\|_{S_0} = \sup_{(E,e)} \int_{\Delta_1 x S_2} |k(E,e,e',x)| \sqrt{1 - e_1' - e_2'}\, de'dx < \infty \},$$

$$S_1 = \{ k \in S : \|k\|_{S_1} = \sup_{(E,e)} \frac{1}{1+E} \int_{\Delta_1 x S_2} |k(E,e,e',x)| \sqrt{1 - e_1' - e_2'}\, de'dx < \infty \}.$$

As usual we denote : $$k_\tau(E,e) = 2\pi \int_{\Delta_1 x S_2} |k(E,e,e',x)| \sqrt{1 - e_1' - e_2'}\, de'dx .$$

Remark: Condition (3.1) is the so called detailed balance condition[2]. Condition (iii) ensures that particles which have relative velocity zero can not collide. We remark that $(S_0, \| \,.\, \|_{S_0})$ and $(S_1, \| \,.\, \|_{S_1})$ are Banach spaces.

Analogously to definition 3.1 we introduce the function space of the elastic scattering cross sections

Definition 3.2: I is the set of all measurable real valued functions σ defined on $\mathbb{R}_+ x \Delta_1 x S_2$ which have the properties:

(i) $\sigma \in C(\mathbb{R}_+ x \Delta_1 \to L_1([-1,1]))$

(ii) $e_1 + e_2 = 1 \Rightarrow \sigma(E,e,x) = 0$ a.e.

I_+ denotes the set of all nonnegative $\sigma \in I$.

$$I_o = \{ k \in I : \|k\|_{I_o} = \sup_{(E,e)} \int_{[-1,1]} |k(E,e,x)| \, dx < \infty \},$$

$$I_1 = \{ k \in I : \|k\|_{I_1} = \sup_{(E,e)} \frac{1}{1+E} \int_{[-1,1]} |k(E,e,x)| dx < \infty \}$$

and we write: $\sigma_\tau(E,e) = 2\pi \int_{[-1,1]} |\sigma(E,e,x)| dx.$

The Boltzmann equation (2.1) and equation (2.6) indicate that one is interested in solutions, which depend on pairs of scattering cross sections. Therefore we introduce

Definition 3.3: W_o is the cartesian product of S_o and I_o equipped with the norm:

$$W_o \ni k=(k_1,k_2) \quad \to \quad \|k\|_o = \|k_1\|_{S_o} + \|k_2\|_{I_o}$$

W_1 is the cartesian product of S_1 and I_1 equipped with the norm

$$W_1 \ni k=(k_1,k_2) \quad \to \quad \|k\|_1 = \|k_1\|_{S_1} + \|k_2\|_{I_1}.$$

For any element of W_o or W_1 we write: $k_\tau(E,e) = k_{1\tau}(E,e) + k_{2\tau}(E,e)$.

Notation: For any nonnegative integer k we introduce

$$\|f\|_k = \int_{\mathbb{R}^3 x \mathbb{R}_+} (1 + |v|^2 + \varepsilon_1)^k \, |f(v,\varepsilon_1)| \, d\varepsilon_1 dv$$

with corresponding function spaces $L_{1,k}$. As usual we denote $L_{1,o}$ by L_1 if there is no confusion possible.

The detailed balance condition in Defintion 2.1 ensures that the collision operator in (2.1) has the property[1)]:

$$\int \varphi(v,\varepsilon_1) J(k,f,g)\, dv d\varepsilon_1 =$$

$$\frac{1}{8} \int_{\Pi'} \sqrt{1 - e_1' - e_2'}\; \sigma(E,e_1,e_2,e_1',e_2',\eta\cdot\eta')\,[f'g'_* + f'_*g' - f g_* - f_*g] \quad (3.2)$$

$$\cdot[\varphi + \varphi_* - \varphi' - \varphi'_*]\; d\Omega(\eta')\, de_1' de_2'\, d\varepsilon_2 dw$$

for measurable functions φ,f and g and $\sigma \in W$ for which the integral on left hand side of (3.2) converges. By inspection of (2.1) the collision operator can be split into a gain and a loss term

$$J(\sigma,f,g) = G(\sigma,f,g) - V(\sigma,f,g) \quad (3.4)$$

where the functions G and V are nonnegative if the scattering cross section and the functions f and g are nonnegative. We have the following proposition; for the proof we refer to ref. 1):

<u>Proposition 3.1:</u> Let σ be in W_o . Then both $G(\sigma,\cdot,\cdot)$ and $V(\sigma,\cdot,\cdot)$ are mappings from $L_1 x L_1$ into L_1 and there hold the estimates:

$$\|V(\sigma,f,g)\|_o \le 2\pi \|\sigma\|_o \|f\|_o \|g\|_o \quad \text{and} \quad \|G(\sigma,f,g)\|_o \le 2\pi \|\sigma\|_o \|f\|_o \|g\|_o . \quad (3.5)$$

Moreover $G(\sigma,\cdot,\cdot)$ and $V(\sigma,\cdot,\cdot)$ are mappings from $L_{1,k} x L_{1,k}$ into $L_{1,k}$, $k \ge 1$, and we have

$$\|V(\sigma,f,g)\|_k \le \pi \|\sigma\|_o (\|f\|_o \|g\|_k + \|g\|_o \|f\|_k)$$
$$\|G(\sigma,f,g)\|_k \le \pi \|\sigma\|_o (\|f\|_o \|g\|_k + \|g\|_o \|f\|_k) . \quad (3.6)$$

If we define for $\sigma \in W_{o,+}$ the operator

$$Q_h(\sigma,f,g) = J(\sigma,f,g) + \frac{h}{2} \Big\{ f \int_{\mathbb{R}^3 x \mathbb{R}_+} g(w,\varepsilon_2)\, d\varepsilon_2 dw + g \int_{\mathbb{R}^3 x \mathbb{R}_+} f(w,\varepsilon_2)\, d\varepsilon_2 dw \Big\}, \quad (3.7)$$

then we have for $h \ge 2\pi \|\sigma\|_o$ the following monotonicity properties:

$$(i)\; 0 \le f,\; 0 \le g \;\Rightarrow\; Q_h(\sigma,f,g) \ge 0$$
$$(ii)\; 0 \le g \le f \;\Rightarrow\; Q_h(\sigma,g,g) \le Q_h(\sigma,f,f) . \quad (3.8)$$

Now suppose we have found a solution $f(\cdot)$ of (2.1) in $C([0,t_o] \to L_{1,o})$. Then, because of (3.2), we have :

$$\int_{\mathbb{R}^3 x \mathbb{R}_+} f(t,v,\varepsilon_1)\, d\varepsilon_1 dv = \int_{\mathbb{R}^3 x \mathbb{R}_+} f_o(v,\varepsilon_1)\, d\varepsilon_1 dv + \int_0^t \int_{\mathbb{R}^3 x \mathbb{R}_+} J(\sigma,f(s),f(s))(v,\varepsilon_1)\, d\varepsilon_1 dv\, ds$$
$$= \int_{\mathbb{R}^3 x \mathbb{R}_+} f_o(v,\varepsilon_1)\, d\varepsilon_1 dv \quad (3.9)$$

From this equation it follows that $f(\cdot)$ solves

$$\frac{\partial}{\partial t} f(t,v,\varepsilon_1) + h f(t,v,\varepsilon_1) \int_{\mathbb{R}^3 \times \mathbb{R}_+} f_o(v,\varepsilon_1)\, d\varepsilon_1 dv \quad = \quad Q_h(\sigma,f,f)(t,v,\varepsilon_1) \tag{3.10}$$

with initial value f_o for any $h > 0$, which is equivalent to

$$f(t) \quad = \quad f_o e^{-ht} + \int_0^t e^{-h(t-s)} Q_h(\sigma,f(s),f(s))\, ds\ . \tag{3.11}$$

Otherwise, if we have a solution of (3.10) in $C([0,t_o]\to L_{1,o})$ then the function

$$z(t) \quad = \quad \int_{\mathbb{R}^3 \times \mathbb{R}_+} f(t,v,\varepsilon_1)\, d\varepsilon_1 dv$$

solves $d_t z(t) + h z_o z(t) = h z^2(t)$ with initial condition $z(0) = z_o$, which implies $z(t) = z_o$, so that $f(\cdot)$ solves (2.1). In the following we call (3.10) *Arkeryd's equation*[5)].

At the end of this section we note a scaling property of the solution of (2.1). Suppose $f(\cdot) \in C([0,t_o]\to L_{1,k})$ solves (2.1) with initial condition f_o. For any $\lambda \in \mathbb{R}_+$ we introduce $g_o = \lambda f_o$. If we define $g(t) = \lambda f(\lambda t)$ then we have $g(\cdot) \in C([0,t_o/\lambda]\to L_{1,k})$ and

$$\begin{aligned} g(t) \quad = \quad \lambda f(\lambda t) \quad &= \quad \lambda f_o + \lambda \int_0^{\lambda t} J(\sigma,f,f)(s)\, ds \quad = \quad \lambda f_o + \int_0^t J(\sigma,\lambda f(\lambda s),\lambda f(\lambda s))\, ds \\ &= \quad g_o + \int_0^t J(\sigma,g(s),g(s))\, ds \end{aligned}$$

which means, that $g(\cdot)$ solves (2.1) with initial condition λf_o.

4) The existence theorems

Theorem 4.1: Let f_o be a nonnegative function with $\|f_o\|_o = 1$ and let σ be a nonnegative element of W_o. For any $t_o > 0$ there exists a unique function $f(\cdot) \in C([0,t_o]\to L_{1,o})$ which solves (2.1). In addition we have the properties:

$$(\,i\,)\ \forall\, t \geq 0 :\ f(t) \geq 0 \quad \text{and} \quad \|f(t)\|_o = 1\,. \tag{4.1}$$

(ii) If we have $\|f_o\|_1 = C < \infty$, then there holds:

$$\begin{aligned} \forall\, t \geq 0 :\ \int_{\mathbb{R}^3 \times \mathbb{R}_+} v\, f(t,v,\varepsilon_1)\, d\varepsilon_1 dv \quad &= \quad \int_{\mathbb{R}^3 \times \mathbb{R}_+} v\, f_o(v,\varepsilon_1)\, d\varepsilon_1 dv\ , \\ \|f(t)\|_1 \quad &= \quad \|f_o\|_1. \end{aligned} \tag{4.2}$$

(iii) Suppose we have $\sigma_\tau(E,e) \leq C(1+E)$ and $\|f_o\|_k < \infty$ for a $k \geq 2$. Then there exists a constant C' depending only on $\|f_o\|_1$, t_1, C such that

$$\forall\ t \in [0,t_1] : \|f(t)\|_k \leq C' \|f_o\|_k \tag{4.3}$$

For k = 2 we have $\|f(t)\|_2 \leq \|f_o\|_2 \exp[4C\ \beta_2\ \|f_o\|_1\ t]$ with some $\beta_2 > 0$.

Remark: (4.3) is the equivalent of Povzner's inequality[3)] for the present case.

Proof: Let t_o be a positive fixed time. Because of (3.9) and (3.10) we seek a solution of Arkeryd's equation in $C([0,t_o]\to L_{1,o})$. To this end we define the following sequence $\{f_n(\cdot)\}$:

$$\begin{aligned} f_1(t) &= f_o e^{-ht}, \\ f_{n+1}(t) &= f_o e^{-ht} + \int_0^t e^{-h(t-s)}\ Q_h(\sigma,f_n(s),f_n(s))\ ds,\quad n \geq 1, \end{aligned} \tag{4.4}$$

where we have $h \geq 2\pi \|\sigma\|_o$. We note that each f_n is in $C([0,t_o]\to L_{1,o})$ because f_1 has this property. Moreover, due to (3.8), we can see easily by induction:

$$\forall\ n \in \mathbb{N} : f_{n+1}(t) \geq f_n(t) \geq 0. \tag{4.5}$$

Moreover we have an upper bound for the L_1 norms of $f_n(t)$ for any positive time:

$$\|f_{n+1}(t)\|_o = \int f_{n+1}(t,v,\varepsilon_1)\ d\varepsilon_1 dv = \|f_o\|_o e^{-ht} + \int_0^t e^{-h(t-s)} \|f_n(s)\|_o ds\ , \tag{4.6}$$

which yields by induction: $\|f_{n+1}(t)\|_o \leq 1$. As a consequence of (4.5) and (4.6) and of the Levi proerty of $L_{1,o}$[5)], the sequence $\{f_n(t)\}$ converges pointwise in $L_{1,o}$ towards a function f(t). Moreover, because $L_1([0,t_o]x\mathbb{R}^3x\mathbb{R}_+)$ has the Levi property too, we get in addition:

$$f(\cdot) \in L_1([0,t_o]x\mathbb{R}^3x\mathbb{R}_+) \quad \text{and} \quad \forall\, n \in \mathbb{N}, t \in [0,t_o]:\ f(t) \geq f_n(t) \geq 0\ . \tag{4.7}$$

We get from the monotonicity (3.8) of Q_h that $f(\cdot)$ solves (3.11) in $L_1([0,t_o]x\mathbb{R}^3x\mathbb{R}_+)$. To see that $f(\cdot)$ is in $C([0,t_o]\to L_{1,o})$ we simply calculate:

$$\begin{aligned} \|f(t) - f_n(t)\|_o &= \int_{\mathbb{R}^3x\mathbb{R}_+} [\ f(t,v,\varepsilon_1) - f_n(t,v,\varepsilon_1)]\ d\varepsilon_1 dv \\ &= \int_0^t \int_{\mathbb{R}^3x\mathbb{R}_+} e^{-h(t-s)}\ [Q_h(\sigma,f(s),f(s)) - Q_h(\sigma,f_n(s),f_n(s))]\ ds \end{aligned} \tag{4.8}$$

which means that $\|f(t) - f_n(t)\|_o$ is increasing in time. Therefore pointwise convergence at t_o implies uniform convergence on $[0,t_o]$. This yields, that $f(\cdot)$ solves Arkeryd's equation, and (4.1) is proved.

To prove (ii) we simply notice that the functions $f_n(\cdot)$ are in $C([0,t_o]\to L_{1,1})$, if f_o is in $L_{1,1}$. In addition we have convergence of $\{f_n(\cdot)\}$ towards the function $f(\cdot)$ of part (i) of the proof, so that $f(\cdot)$ solves Arkeryd's equation in $C([0,t_o]\to L_{1,1})$. Now (4.2) is an easy consequence of (3.2) and the non-negativity of $f(\cdot)$.

To prove (iii) we first notice two simple estimates:

- for any (v,ε_1), $(w,\varepsilon_2) \in \mathbb{R}^3 x \mathbb{R}_+$ we have

$$1 + \frac{1}{2}|v-w|^2 + \varepsilon_1 + \varepsilon_2 \le (1 + |v|^2 + \varepsilon_1) + (1 + |w|^2 + \varepsilon_2) \tag{4.9}$$

- for real numbers $a,b \ge 0$ and $s \ge 1$ and $0 \le \delta s \le 1$ we have[3]:

$$(a^s + b^s) \le (a + b)^s \le a^s + b^s + \beta_s(a^{\delta s}b^{(1-\delta)s} + a^{(1-\delta)s}b^{\delta s}) \tag{4.10}$$

Because of the monotonicity and the nonnegativity of the functions $f_n(\cdot)$ we have:

$$\begin{aligned} f_{n+1}(t) &= f_o + \int_0^t J(\sigma,f_n(s),f_n(s))\, ds \\ &\quad + \int_0^t [\, h\, f_n(s) \int_{\mathbb{R}^3 x \mathbb{R}_+} f_n(s,w,\varepsilon_2) d\varepsilon_2 dw - h\, f_{n+1}(s)\,]\, ds \\ &\le f_o + \int_0^t J(\sigma,f_n(s),f_n(s))\, ds \,. \end{aligned} \tag{4.11}$$

For arbitrary functions $f,g \in L_{1,k}$ we have from proposition 3.1 $\|J(\sigma,f,g)\|_k < \infty$ and there holds:

$$\begin{aligned} &\int_{\mathbb{R}^3 x \mathbb{R}_+} (1 + |v|^2 + \varepsilon_1)^k\, J(\sigma,f,g)\, d\varepsilon_1 dv = \\ &\frac{1}{4} \int_{(\mathbb{R}^3 x \mathbb{R}_+)^2 x \Delta_1 x S_2} \sigma_1(E,e,e',\eta\cdot\eta') \sqrt{1 - e_1' - e_2'} \; [(1 + |v'|^2 + \varepsilon_1')^k + (1 + |w'|^2 + \varepsilon_2')^k \\ &\qquad - (1 + |v|^2 + \varepsilon_1)^k - (1 + |w|^2 + \varepsilon_2)^k] \\ &\qquad \cdot [f(v,\varepsilon_1)g(w,\varepsilon_2) + f(w,\varepsilon_2)g(v,\varepsilon_1)] d\Omega(\eta') de' d\varepsilon_2 dw d\varepsilon_1 dv \\ &+ \frac{1}{4} \int_{(\mathbb{R}^3 x \mathbb{R}_+)^2 x S_2} \sigma_2(E,e,\eta\cdot\eta') \; [(1 + |\tilde{v}|^2 + \varepsilon_1)^k + (1 + |\tilde{w}|^2 + \varepsilon_2)^k \\ &\qquad - (1 + |v|^2 + \varepsilon_1)^k - (1 + |w|^2 + \varepsilon_2)^k] \\ &\qquad \cdot [f(v,\varepsilon_1)g(w,\varepsilon_2) + f(w,\varepsilon_2)g(v,\varepsilon_1)] d\Omega(\eta') d\varepsilon_2 dw d\varepsilon_1 dv \end{aligned} \tag{4.12}$$

In (4.12) we have used the notations (2.3) and (2.4) and in addition

$$\tilde{v} = \frac{1}{2}(v + w + \eta'|v-w|)$$

Due to (4.10) we have:

$$(1 + |v'|^2 + \varepsilon_1)^k + (1 + |w'|^2 + \varepsilon_2)^k \le (1 + 1 + |v'|^2 + |w'|^2 + \varepsilon_1' + \varepsilon_2')^k$$

$$= (1 + 1 + |v|^2 + |w|^2 + \varepsilon_1 + \varepsilon_2)^k$$

$$\le (1 + |v|^2 + \varepsilon_1)^k + (1 + |w|^2 + \varepsilon_2)^k$$

$$+ \beta_k[(1 + |v|^2 + \varepsilon_1)^{\delta k}(1 + |w|^2 + \varepsilon_2)^{(1-\delta)k}$$

$$+ (1 + |v|^2 + \varepsilon_1)^{(1-\delta)k}(1 + |w|^2 + \varepsilon_2)^{\delta k}]$$

and there is an analogous estimate for $(1 + |\tilde{v}|^2 + \varepsilon_1)^k + (1 + |\tilde{w}|^2 + \varepsilon_2)^k$. Using this inequality we get with the help of (4.9) and (4.12):

$$\|f_{n+1}(t)\|_k = \int_{\mathbb{R}^3 x \mathbb{R}_+} (1 + |v|^2 + \varepsilon_1)^k f_{n+1}(t,v,\varepsilon_1)\, d\varepsilon_1 dv$$

$$\le \|f_0\|_k + \int_0^t \int_{\mathbb{R}^3 x \mathbb{R}_+} 2C\, [(1 + |v|^2 + \varepsilon_1) + (1 + |w|^2 + \varepsilon_2)]$$

$$\cdot \beta_k[(1 + |v|^2 + \varepsilon_1)^{\delta k}(1 + |w|^2 + \varepsilon_2)^{(1-\delta)k}$$

$$+ (1 + |v|^2 + \varepsilon_1)^{(1-\delta)k}(1 + |w|^2 + \varepsilon_2)^{\delta k}]$$

$$\cdot \frac{1}{2}\, [f_n(s,v,\varepsilon_1) f_n(s,w,\varepsilon_2)]\, d\varepsilon_2 dw d\varepsilon_1 dv\, ds\,.$$

This yields:

$$\|f_{n+1}(t)\|_k \le \|f_0\|_k + \int_0^t 2C\beta_k\, [\|f_n(s)\|_{\delta k+1} \|f_n(s)\|_{(1-\delta)k} + \|f_n(s)\|_{1+(1-\delta)k} \|f_n(s)\|_{\delta k}]\, ds\,. \tag{4.13}$$

We choose now δ such that we have $\delta k = 1$ and we get from (4.13):

$$\|f_{n+1}(t)\|_k \le \|f_0\|_k + \int_0^t 2C\beta_k\, [\|f_n(s)\|_2 \cdot \|f_n(s)\|_{k-1} + \|f_n(s)\|_1 \|f_n(s)\|_k] ds \tag{4.14}$$

Let us consider the special case $k = 2$ first. From (4.14) and part (ii) of the proof we get:

$$\|f_{n+1}(t)\|_2 \le \|f_0\|_2 + \int_0^t 4C\beta_2\, \|f_0\|_1\, \|f_n(s)\|_2\, ds,$$

which yields by induction:

$$\forall n \in \mathbb{N}: \quad \|f_{n+1}(t)\|_2 \le \|f_0\|_2 \exp[4C\beta_2\, \|f_0\|_1 t] \tag{4.15}$$

Because of the convergence theorems of Lebesgue this upper bound holds for the limit function $f(\cdot)$ too. So (4.3) is proved for the special case $k = 2$. To prove (4.3) for any $k > 2$ we notice: If we have for a sequence $\{x_n(\cdot)\}$ of functions the estimate:

$$x_{n+1}(t) \le x_o + \lambda_1 t + \int_0^t \lambda_2 x_n(s)\, ds \quad \text{with } \lambda_2 > 0 \text{ and } x_o > 0 ,$$

then there holds: $x_n(t) \le x_o \exp[\lambda_2 t] + \frac{\lambda_1}{\lambda_2}(\exp[\lambda_2 t] - 1)$

Now (4.3) follows from (4.14) and (4.15) by induction on k.

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To prove existence and uniqueness of the solution of (2.1) for the case $\sigma \in W_1$ we approximate such a σ by a sequence of bounded scattering cross sections and show convergence of of the corresponding solutions. We remind the reader on the splitting (3.4) of the collision operator into a gain and a loss part.

Theorem 4.2: Let $\sigma \in W_1$ and $f_o \in L_{1,k}$, $k \ge 2$, be nonnegative functions. For any $t_o > 0$ there exists a unique function $f \in C([0,t_o] \to L_{1,1})$ with the properties:

(i) $\forall\, t \ge 0 : f(t) \ge 0$.

(ii) $f(0) = f_o$ and $d_t f(t) = J(\sigma, f(t), f(t))$

(iii) $\|f(t)\|_o = \|f_o\|_o$ and $\|f(t)\|_1 = \|f_o\|_1$.

Remark: As usual we assume: $\|f_o\|_o = 1$.

Proof: Let t_o be a positive fixed time. We have

$$\sigma_\tau(E,e) \le C(1+E) \tag{4.16}$$

We first notice a simple estimate: For real numbers $C \ge 0$, $x,y \ge 1$ we have

$$2Cxy \ge C(x+y) \tag{4.17}$$

To perform a truncation of σ we introduce for $m \in \mathbb{N}$ the function

$$\Theta_m(x) = \begin{cases} 1 & , x \in [0,m] \\ 1-(x-m) & , x \in]m,m+1] \\ 0 & , x > m \end{cases}$$

and denote:

$$\sigma_m(E,e,e',x) = \Theta_m(E)\sigma(E,e,e',x) \tag{4.18}$$

We introduce for functions $f,g \in L_{1,1}$ the operators

$$Q_h'(\sigma_m,f,g) \;=\; J(\sigma_m f,g) + \frac{1}{2}\, h\Big[\, \psi\, f \int_{\mathbb{R}^3\times\mathbb{R}_+} \psi(w,\varepsilon_2)\; g(w,\varepsilon_2)\,dw\,d\varepsilon_2 \qquad\qquad \psi\, g \int_{\mathbb{R}^3\times\mathbb{R}_+} \psi(w,\varepsilon_2)\; f(w,\varepsilon_2)\,dw\,d\varepsilon_2 \Big] \tag{4.19}$$

and

$$Q_h''(\sigma_m,f,g) \;=\; G(\sigma_m,f,g) - V(\sigma,f,g) + \frac{1}{2}\, h\Big[\, \psi\, f \int_{\mathbb{R}^3\times\mathbb{R}_+} \psi(w,\varepsilon_2)\; g(w,\varepsilon_2)\,dw\,d\varepsilon_2 \qquad\qquad \psi\, g \int_{\mathbb{R}^3\times\mathbb{R}_+} \psi(w,\varepsilon_2)\; f(w,\varepsilon_2)\,dw\,d\varepsilon_2 \Big] \tag{4.20}$$

where we have: $\psi(v,\varepsilon_1) = (1 + |v|^2 + \varepsilon_1)$. We notice the following monotonicity properties, which are easy consequences of (4.16) and (4.17): For $h \geq 4\pi C$ we have

$$\begin{aligned} &- \quad 0 \leq f,g \;\Longrightarrow\; 0 \leq Q_h''(\sigma_m,f,g) \leq Q_h'(\sigma_m,f,g) \quad \text{for any } m \in \mathbb{N} \\ &- \quad j \leq m \text{ and } 0 \leq f,g \;\Longrightarrow\; 0 \leq Q_h''(\sigma_j,f,g) \leq Q_h''(\sigma_m f,g) \end{aligned} \tag{4.21}$$

To construct our solution of (2.1) in $C([0,t_o]\to L_{1,1})$ we proceed now in an analogous way to the proof of Arkeryd[5] for the case of the Boltzmann equation for monoatomic gases. We first note that, because of theorem 4.1, there exists for any $m \in \mathbb{N}$ a unique function $f_m'(\cdot) \in C([0,t_o]\to L_{1,1})$ which solves (2.1) and that, because of (4.2) this function is a solution of

$$d_t\, f_m'(t) + h\psi\, \|f_o\|_1 f_m'(t) \;=\; Q_h'(\sigma_m, f_m'(t), f_m'(t))\,. \tag{4.22}$$

Moreover, using (4.21) and an analogous iteration procedure as (4.4) we see that there is a unique solution $f_m''(\cdot) \in C([0,t_o]\to L_{1,1})$ of

$$d_t\, f_m''(t) + h\psi\|f_o\|_1\, f_m''(t) \;=\; Q_h''(\sigma_m, f_m''(t), f_m''(t)) \tag{4.23}$$

which has in addition the properties:

$$\begin{aligned} &- \quad 0 \leq f_m''(t) \leq f_m'(t) \quad \text{which implies : } \|f_m''(t)\|_1 \leq \|f_m'(t)\|_1 = \|f_o\|_1 \\ &- \quad \text{for } j \leq m \text{ we have } f_j''(t) \leq f_m''(t)\,. \end{aligned} \tag{4.24}$$

Now due to the Levi property of $L_{1,1}$ we obtain, that $\{f_m(.)\}$ converges in $L_1([0,t_o]\times\mathbb{R}^3\times\mathbb{R}_+)$ towards a function $f(\cdot)$ and because of (4.21) this function solves in $L_1([0,t_o]\times\mathbb{R}^3\times\mathbb{R}_+)$

$$f(t) \;=\; f_o e^{-h\psi t} + \int_0^t e^{-h\psi(t-s)} Q_h''(\sigma, f(s), f(s))\; ds \tag{4.25}$$

and we have $\|f(t)\|_1 \le \|f_o\|_1$. In addition, due to (4.21) and (4.24), $\|f(t)-f_m''(t)\|_1$ is monotonically increasing in time so that we get that $f(\cdot)$ solves (4.25) in $C([0,t_o]\rightarrow L_{1,1})$. To see that $f(\cdot)$ solves (2.1) for the given σ we have to show $\|f(t)\|_1 = \|f_o\|_1$. To this end we simply notice that due to (4.3) $\|f_m'(t)\|_2$ is uniformly bounded in m and this implies[5)]:

$$\lim_{m\rightarrow\infty} \|f_m'(t) - f_m''(t)\|_1 = 0.$$

on sufficiently small time intervalls. Now an iteration procedure yields the desired result.

It remains to be proved that $f(\cdot)$ is the unique solution of (2.1) which conserves energy. To see this we assume that there is another nonnegative function $g \in C([0,t_o]\rightarrow L_{1,1})$ which solves (2.1) with $\|g(t)\|_1 = \|f_o\|_1$. As a consequence of this, $g(\cdot)$ solves (4.25) which implies $g(\cdot) \ge f(\cdot)$. Because $g(\cdot)$ is assumed to be different from $f(\cdot)$, there are some time t, for which we have $g(t) > f(t)$ on a set with positive Lebesgue measure. But this implies: $\|g(t)\|_1 > \|f(t)\|_1 = \|f_o\|_1$ and we obtain the desired contradiction.

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5) Positivity of the solution

What has been shown so far is, that there is a nonnegative solution of (2.1), if we have a nonnegative initial condition and a nonnegative scattering cross section in W_o or W_1. In this section we will strengthen our requirements on $\sigma = (\sigma_1,\sigma_2)$ a little bit. We assume the inelastic scattering cross section σ_1 to be an almost everywhere positive function. If this happens we will show that the solution of (2.1) is almost everywhere positive for any positive time. The key fact to prove this claim is, that we have from the definition of the approximating sequence (4.4) and the monotonicity property (4.21) of $Q_h''(\sigma,\cdot,\cdot)$ and the implicit formula (4.25) for the solution of (2.1) the following estimate:

$$f(t) \ge \int_0^t e^{h(t-s)} G(\sigma_1,f(s),f(s))\, ds \ge \int_0^t e^{-h\psi(t-s)} G(\sigma_1\, f_o,f_o)\; ds\,, \tag{5.1}$$

where we have, as in (4.20), $\psi(v,\varepsilon_1) = (1+|v|^2+\varepsilon_1)$. $G(\sigma_1,\cdot,\cdot)$ is the gain part of J for a purely inelastic scattering cross section $(\sigma_1,0)$.

The proof of our claim will be done in two steps. In the first step we will show that the positivity of $f(t)$, $t \in\,]0,t_o]$, on an intervall implies the positivity on $\mathbb{R}^3 x \mathbb{R}_+$. By an intervall we denote a set of the form

$$B(\delta,v_1)xI(\delta,\varepsilon_1) \text{ with}$$
$$B(\delta,v_1) = \{v \in \mathbb{R}^3\colon |v-v_1| \le \delta\} \quad \text{and} \quad I(\delta,\varepsilon_1) = \{\varepsilon \in \mathbb{R}_+\colon |\varepsilon-\varepsilon_1| \le \delta\}. \tag{5.2}$$

In a second step we show, that there is an intervall, on which $G(\sigma_1,f_o,f_o)$ is positive, provided that f_o is positive on a set of positive Lebesgue measure.

Lemma 5.1: Let σ_1 be a positive function in W_1. Suppose the assumptions of theorem 4.2 hold and in addition, that there are a $\delta > 0$, a $v_1 \in \mathbb{R}^3$ and an $\varepsilon_1 \in \mathbb{R}_+$, such that the solution $f(\cdot)$ of (4.25) has the property:

$$f(t) > 0 \text{ on } B(\delta,v_1)xI(\delta,\varepsilon_1), \text{ if } t > 0. \tag{5.3}$$

Then there exists a $z > 0$, independent of δ, v_1 and ε_1, such that

$$f(t) > 0 \text{ on } B(\delta(1+z),v_1)xI(\delta(1+z),\varepsilon_1). \tag{5.4}$$

Proof: By (4.21) and (5.1) we see, that (5.4) is proved if we can show, that (5.3) implies

$$\exists\, z > 0 \text{ , independent of } \delta,\ v_1 \text{ and } \varepsilon_1:$$

$$G(\sigma_1,f(s),f(\sigma)) > 0 \text{ on } B(\delta(1+z),v_1)xI(\delta(1+z),\varepsilon_1)\setminus B(\delta,v_1)xI(\delta,\varepsilon_1), \text{ if } s > 0. \tag{5.5}$$

To this end we make for fixed $v \in \mathbb{R}^3$ the transformation of integration variables $w \rightarrow c = v-w$ and obtain

$$G(\sigma_1,f(s),f(s))(v,\varepsilon) =$$

$$\int_{\mathbb{R}^3x\mathbb{R}_+x\Delta_1xS_2} \sqrt{1 - e_1' - e_2'}\ \sigma_1(E,e,e',\eta\cdot\eta')\ f(s,v',\varepsilon_1')\ f(s,w'\varepsilon_2')\ d\Omega(\eta')de'd\varepsilon_2dc\ , \tag{5.6}$$

where we have

$$E = \frac{|c|^2}{2} + \varepsilon + \varepsilon_2\ ,$$

$$v' = v + \frac{c}{2} + \frac{\eta'}{2}\sqrt{2E(1 - e_1' - e_2')}\ , \qquad \varepsilon_1' = e_1'E\ ,$$

$$w' = v + \frac{c}{2} - \frac{\eta'}{2}\sqrt{2E(1 - e_1' - e_2')}\ , \qquad \varepsilon_2' = e_2'E\ .$$

Notice, that the collision energy in (5.6) is independent of v. We have to distinguish between two cases:

- $I(\delta,\varepsilon_1) = [0,b]$, $b \leq 2\delta$
- $I(\delta,\varepsilon_1) = [a,b]$, $a,b > 0$.

Because the techniques of the proof are exactly the same in both cases we consider here only the first one. Obviously, a set of the form (5.5) can be written as a disjoint union of three sets:

$$B(\delta(1+z),v_1)xI(\delta(1+z),\varepsilon_1)\setminus B(\delta,v_1)xI(\delta,\varepsilon_1) =$$

$$\left[\bigcup_{v\in\partial B(\delta,v_1)} A_1(v,z)\right] \cup B(\delta,v_1)x]b,b(1+z)] \cup \left[\bigcup_{v\in\partial B(\delta,v_1)} A_2(v,z)\right] \tag{5.7}$$

where we have $A_1(v,z) = \{(1+\xi_1)(v-v_1)\}x[0,b],\ \xi_1 \in [0,z]\}$

and $A_2(v,z) = \{((1+\xi_1)(v-v_1),(1+\xi_2)b),\ \xi_1,\xi_2 \in [0,z]\}$.

Let $v_o \in \partial B(\delta,v_1)$ be a given fixed vector. We set $\tilde{v} = v_o - v_1$ and $\tilde{\varepsilon} = b$. We choose $\eta \perp \tilde{v}$ and $c_o = -\lambda\,\tilde{v}$. Then we have for $\varepsilon_2 \in \mathbb{R}_+$:

$$E^o = \frac{1}{2}|c_o|^2 + \varepsilon_2 + b = \frac{1}{2}\lambda_o^2\,\delta_1^2 + \varepsilon_2 + b, \text{ which implies}$$

$$|v' - v_1|^2 = |w' - v_1|^2 = \delta_1^2(1 - \frac{\lambda_o^2}{2}) + \frac{E^o}{2}(1 - e_1' - e_2').$$

If we choose now $\varepsilon_2^o = \frac{b}{2}$ and $e_1'^o = e_2'^o = \dfrac{\frac{3}{4}b}{\frac{1}{2}\lambda_o^2\,\delta_1^2 + \frac{3}{2}b}$,

$$\varepsilon_1'^o = e_1'^o E^o,\quad \varepsilon_2'^o = e_2'^o E^o \text{ and } \lambda_o = 0.1,$$

then we have $|v' - v_1| \le \delta$ and, due to the continuity of the functions v', w', ε_1' and ε_2', we obtain the following statement:

There exists a $z_3 > 0$ such that there are for all $(v,\varepsilon) \in A(v_o,z_3)$ sets $U_1(\xi_1) \subset \mathbb{R}^3$, $U_2(\xi_2) \subset \mathbb{R}_+$, $U_3(\xi_1,\xi_2) \subset \Delta_1$ and $U_4(\xi_1,\xi_2) \subset S_2$ with the property:

$$\forall\ (c,\varepsilon,e',\eta') \in U_1(\xi_1)xU_2(\xi_2)xU_3(\xi_1,\xi_2)xU_4(\xi_1,\xi_2):$$

$$v' \in B(\delta,v_1),\ w' \in B(\delta,v_1),\quad \varepsilon_1' \in [0,b],\quad \varepsilon_2' \in [0,b].$$

But now, due to (5.6), we see that $G(\sigma_1,f(s),f(s))(v,\varepsilon)$ is positive. In addition, the above construction shows that z_3 is independent of the particular choice of v_o which implies

$$G(\sigma_1,f(s),f(s)) > 0 \text{ on } \left[\bigcup_{v\in\partial B(\delta,v_1)} A_2(v,z_3)\right]$$

An anologous discussion of the two other sets yields the existence of two numbers z_1 and z_2 such that we have

$$G(\sigma_1,f(s),f(s)) > 0 \text{ on } \left[\bigcup_{v\in\partial B(\delta,v_1)} A_1(v,z_1)\right] \text{ and}$$

$$G(\sigma_1,f(s),f(s)) > 0 \text{ on } B(\delta,v_1)x]b,b(1+z_2)]$$

We take now $z = \min(z_1,z_2,z_3)$ and we get (5.5) from the decomposition (5.7).

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Corollary: Suppose the assumptions of theorem 4.2 hold and σ_1 is a positive function. Then the positivity of f(t), t $\in$ $]0,t_o]$, on an intervall $B(\delta,v_1)xI(\delta,\varepsilon_1)$ implies the positivity of f(t) on $\mathbb{R}^3 x \mathbb{R}_+$.

Lemma 5.2: Suppose the assumptions on theorem 4.2 hold, and let σ_1 be a positive function. Then there exist a $\delta > 0$, $v_1 \in \mathbb{R}^3$ and an $\varepsilon_1 \in \mathbb{R}_+$ such that the solution f($\cdot$) of (4.25) has the property:

$$f(t) > 0 \text{ on } B(\delta,v_1)xI(\delta,\varepsilon_1), \text{ if } t > 0. \tag{5.8}$$

Proof: Due to (5.1), the Lemma is proved, if we can show, that $G(\sigma_1,f_o,f_o)$ is positive on an intervall. We set $\Omega = \{(v,\varepsilon_1) : f_o(v,\varepsilon_1) > 0\}$. Ω has a positive Lebesque measure, because we have $\|f_o\|_o = 1$.
We choose a Vitali covering[6)] of Ω with axis parallel cubes, such that there are $\delta > 0$, $\tilde{v}_1 \in \mathbb{R}^3$, $\tilde{\varepsilon}_1 \in \mathbb{R}_+$ with:

$$\text{(i) } 0 < \delta < \frac{1}{10} \tag{5.9}$$

$$\text{(ii) } I(4\delta, \tilde{\varepsilon}_1) = [-4\delta + \tilde{\varepsilon}_1, \tilde{\varepsilon}_1 + 4\delta] \text{ , which implies } \tilde{\varepsilon}_1 \geq 4\delta \tag{5.10}$$

$$\text{(iii) } \forall (v,\varepsilon_1) \in \mathbb{R}^3 x \mathbb{R}_+ \text{ with } B(\delta,v)xI(\delta,\varepsilon_1) \subset B(4\delta,\tilde{v}_1)x\, I(4\delta,\tilde{\varepsilon}_1) \text{ holds:} \tag{5.11}$$

$$\mu(\Omega \cap B(\delta,v)xI(\delta,\varepsilon_1)) \geq \frac{8}{10}\, \mu(B(\delta,v)xI(\delta,\varepsilon_1)) \text{ ,}$$

Here μ is the 4 dimensional Lebesque measure. To discuss $G(\sigma_1,f_o,f_o)$ we start with (5.6) and we perform the following additional transformations of integration variables

$$(e_1',e_2') \to (z = e_1'+e_2', y = e_1'-e_2') \quad , \quad \Delta_1 \to \{(z,y) : z \in [0,1] \text{ , } -z\leq y\leq z\}$$

$$y \to y' = \frac{y}{z} \text{ , } \quad z \to z' = 1 - z \text{ , } \quad z' \to r = \sqrt{\frac{Ez'}{2}}$$

$$y' \to y'' = \frac{1}{2}(E-2r^2)(1+y')$$

and we obtain:

$$G(\sigma_1,f_o,f_o)(v,\varepsilon_1) \geq$$
$$\int_{\mathbb{R}^3 x \mathbb{R}_+} \int_{B(\sqrt{E/2},0)} \int_0^{E-2|x|^2} 4\sqrt{2}\, \frac{1}{E^2\sqrt{E}}\, \sigma_1(E,e,e',\eta\cdot\eta') \cdot [f_o(v + \frac{c}{2} + x,y'')\, f_o(v + \frac{c}{2} - x,E - 2|x|^2 - y'')] \cdot dy''d^3x\, d\varepsilon_2 d^3c \tag{5.12}$$

$$\text{with : } E = \frac{c^2}{2} + \varepsilon_1 + \varepsilon_2 \text{ , } e' = (e_1',e_2') : \quad e_1' = \frac{y''}{E} \text{ and } e_2' = \frac{E-2|x|^2-2y''}{2E} \tag{5.13}$$

and $\eta' = \frac{x}{|x|}$.

Consider now a fixed $v \in B(\delta, \tilde{v}_1)$ and $\varepsilon_1 \in [\tilde{\varepsilon}_1 - \delta, \tilde{\varepsilon}_1 + \delta]$. Obviously we have from (5.12):

$$G(\sigma_1, f_0, f_0)(v, \varepsilon_1) \geq$$

$$\int_{B(\delta,0)} \int_{\tilde{\varepsilon}_1}^{\tilde{\varepsilon}_1 + 2\delta} \int_{B(\sqrt{E/2},0)} \int_0^{E-2|x|} 4\sqrt{2} \, \frac{1}{E^2\sqrt{E}} \, \sigma_1(E,e,e',\eta\cdot\eta') \qquad (5.14)$$

$$\cdot [f_0(v + \frac{c}{2} + x, y'') \, f_0(v + \frac{c}{2} - x, E - 2|x|^2 - y'')]$$

$$\cdot \, dy'' d^3x \, d\varepsilon_2 d^3c$$

Due to (5.10) and (5.13) we have in (5.14): $E \geq 6\delta$. Moreover, for $x \in B(\delta,0)$ we have

$$\frac{E}{2} - |x|^2 - \delta \;\geq\; \frac{E}{2} - \delta^2 - \delta \geq 0 \quad \text{and} \quad \frac{E}{2} - |x|^2 + \delta \leq E - 2|x|^2 \; .$$

These two estimates yield together with (5.9):

$$G(\sigma_1, f_0, f_0)(v, \varepsilon_1) \geq$$

$$\int_{B(\delta,0)} \int_{\tilde{\varepsilon}_1}^{\tilde{\varepsilon}_1 + 2\delta} \int_{\tilde{A}} 4\sqrt{2} \, \frac{1}{E^2\sqrt{E}} \, \sigma_1(E,e,e',\eta\cdot\eta') \qquad (5.15)$$

$$\cdot [f_0(v + \frac{c}{2} + x, y'') \, f_0(v + \frac{c}{2} - x, E - 2|x|^2 - y'')]$$

$$\cdot \, dy'' \otimes d^3x \, d\varepsilon_2 d^3c$$

with $\tilde{A} = B(\delta,0) \times [\frac{E}{2} - |x|^2 - \delta, \frac{E}{2} - |x|^2 + \delta]$. We notice, that the mapping

$$B(\delta,0) \times [\frac{E}{2} - |x|^2 - \delta, \frac{E}{2} - |x|^2 + \delta] \;\rightarrow\; B(\delta,0) \times [\frac{E}{2} - |x|^2 - \delta, \frac{E}{2} - |x|^2 + \delta] \; , \qquad (5.16)$$

$$(x, y'') \rightarrow (-x, E - 2|x|^2 - y'')$$

is a measure preserving bijection. Now we want to show, that the right hand side of (5.15) is positive. To this end, due to (5.16), we have to show

$$\mu\Big(\Omega \cap B(\delta, v + \frac{c}{2}) \times [\frac{E}{2} - |x'|^2 - \delta, \frac{E}{2} - |x'|^2 + \delta]\Big) > \frac{1}{2} \, \mu\Big(B(\delta, v + \frac{c}{2}) \times [\frac{E}{2} - |x'|^2 - \delta, \frac{E}{2} - |x'|^2 + \delta]\Big), \quad (5.17)$$

where we have $x = x' + v + \frac{c}{2}$, $x' \in B(\delta,0)$. To prove (5.17) we define

$$A = B(\delta, v + \frac{c}{2}) \times [\frac{E}{2} - |x'|^2 - \delta, \frac{E}{2} - |x'|^2 + \delta]$$

$$A^- = B(\delta, v + \frac{c}{2}) \times [\frac{E}{2} - |x'|^2 - \delta, \frac{E}{2} - \delta], \qquad A^+ = B(\delta, v + \frac{c}{2}) \times [\frac{E}{2} - |x'|^2 + \delta, \frac{E}{2} + \delta]$$

$$A_1 = B(\delta, v + \frac{c}{2}) \times [\frac{E}{2} - \delta, \frac{E}{2} - |x'|^2 + \delta] \, .$$

We have: $A = A_1 \cup A^-$ and $A_1 \cup A^+ = B(\delta, v+\frac{c}{2}) x [\frac{E}{2} - \delta, \frac{E}{2} + \delta]$.

Due to (5.11) we have: $\mu(\Omega \cap [A_1 \cup A^+]) \geq 0.8\, \mu(A_1 \cup A^+) = 0.8\, \mu(A) = \frac{4}{5}\, \frac{8}{3} \pi \delta^4$.
Now we can calculate:

$\mu(\Omega \cap A) = \mu(\Omega \cap [A_1 \cup A^-]) \geq \mu(\Omega \cap A_1)$ and

$\mu(\Omega \cap [A_1 \cup A^+]) = \mu(\Omega \cap A_1) + \mu(\Omega \cap A^+)$, which yields:

$$\mu(\Omega \cap A_1) = \mu(\Omega \cap [A_1 \cup A^+]) - \mu(\Omega \cap A^+) \geq \mu(\Omega \cap [A_1 \cup A^-]) - \mu(A^-)$$

$$\geq \left[\frac{4}{5} - \frac{3}{10} \delta \right] \frac{8}{3} \pi \delta^4 = \frac{77}{100} \mu(A)$$

which proves (5.17) and so the Lemma is proved.

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Lemma 5.1 and Lemma 5.2 together yield the following theorem.

Theorem 5.1: Suppose the assumptions of theorem 4.2 hold and let σ_1 be a positive function. Then the solution $f(\cdot)$ of (4.25) has the property:

$\forall\, t > 0 : f(t) > 0$ a.e on $\mathbb{R}^3 x \mathbb{R}_+$.

6) Series representation of the solution of the Boltzmann equation

In this section we will introduce two series representations of the solution of the Boltzmann equation (2.1) for bounded scattering cross sections. Both are of the form

$$f(t,\sigma) = \sum_{i=0}^{\infty} \Psi_i(t)\, l_i(\sigma) , \tag{6.1}$$

with real valued functions Ψ_i. The functions $l_i(\sigma)$ take their values in $L_{1,k}$ if f_0 is in $L_{1,k}$. They can be calculated recursively from f_0. Because of the special form (6.1) of the solution of (2.1), such series are well suited for the study of the dependence of the solution $f(\cdot,\sigma)$ of (2.1) on the scattering cross section σ which will be the topic of a subsequent paper.[7] It should be noticed here, that series representations of solutions of kinetic equations have been used so far mainly in the case of model equations[8] or to get explicit solutions of the Boltzmann equation for monoatomic gases.[9,10] The first of our series comes from the proof of local existence and uniqueness of (2.1) by means of classical Banach space techniques. It is not required, that σ is nonnegative. We note a proposition which follows from proposition 3.1; for the proof we refer to ref. 1).

Proposition 6.1: Let σ be in W_o. Suppose the series (6.1) converges absolutely for some t_o in $L_{1,k}$, $k \geq 0$. Then we have

$$J(\sigma,f(t_o,\sigma),f(t_o,\sigma)) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Psi_{n-k}(t_o)\, \Psi_k(t_o)\, J(\sigma,l_{n-k}(\sigma),l_k(\sigma)) \text{ in } L_{1,k} \qquad (6.2)$$

Motivated by this proposition we introduce the following sequence of functions

Definition 6.1: Let σ be in W_o and f_o in L_1. Then we define the following sequence $\{G_n(\sigma)\}$ of functions:

$$G_o(\sigma) = f_o$$
$$G_n(\sigma) = \frac{1}{n} \sum_{\mu=0}^{n-1} J(\sigma,G_{n-1-\mu}(\sigma),G_\mu(\sigma)), \; n \geq 1. \qquad (6.3)$$

Corollary of proposition 6.1: Using (3.5) and proposition 6.1 we can see easily from (6.3):

$$\|G_n(\sigma)\|_o \leq \frac{1}{n} \sum_{\mu=0}^{n-1} 4\pi \,\|\sigma\|\, \|G_{n-1-\mu}(\sigma)\|_o \,\|G_\mu(\sigma)\|_o ,$$

which yields: $\|G_n(\sigma)\|_o \leq \left[4\pi \,\|\sigma\|\right]^n.$ (6.4)

Theorem 6.1: Let be $\sigma \in W_o$ and $f_o \in L_1$ with $\|f_o\|_o = 1$. Let $\{G_n(\sigma)\}$ be the sequence defined in (6.3). Then there holds:

For any t_o with $|t_o| < \frac{1}{4\pi \,\|\sigma\|}$ converges $f(t,\sigma) = \sum_{n=0}^{\infty} t^n\, G_n(\sigma)$ uniformly on $[-t_o,t_o]$. Moreover $f(\cdot,\sigma) \in C([-t_o,t_o]\rightarrow L_{1,o})$ and $f(\cdot,\sigma)$ solves (2.1) in $C([-t_o,t_o]\rightarrow L_{1,o})$.

Remark: With the help of the sequence $\{G_n(\sigma)\}$ we have a representation of the solution of (2.1) also for sufficiently small negative times.

Proof: The uniform convergence of $\sum_{n=0}^{\infty} t^n\, G_n(\sigma)$ follows from (6.4), if the absolute value of t is smaller then $[4\pi\|\sigma\|]^{-1}$. Moreover the mapping

$$[-t_o,t_o] \ni t \rightarrow \sum_{n=0}^{\infty} t^n \, G_n(\sigma)$$

is in $C([-t_o,t_o] \rightarrow L_{1,o})$, if $|t_o| < \frac{1}{4\pi \, \|\sigma\|}$. (6.5)

Due to proposition 6.1 we have for arbitrary $t \in [-t_o,t_o]$:

$$J(\sigma,f(t,\sigma),f(t,\sigma)) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} t^n \; J(\sigma,G_{n-k}(\sigma),G_k(\sigma))$$

$$= \sum_{n=0}^{\infty} t^n \, (n+1) \, G_{n+1}(\sigma) \;,$$

which yields for $t \in [-t_o,t_o]$ because of the uniform convergence of the series:

$$\int_0^t J(\sigma,f(s,\sigma),f(s,\sigma)) \, ds = \int_0^t \sum_{n=0}^{\infty} s^n \, (n+1) \, G_{n+1}(\sigma) \; ds$$

$$= \sum_{n=0}^{\infty} \int_0^t s^n \, (n+1) \, G_{n+1}(\sigma) \; ds$$

$$= \sum_{n=0}^{\infty} G_{n+1}(\sigma) = f(t,\sigma) - f_o.$$

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Corollary of theorem 6.1: From the theorem of the extension of the solution of differential equations in Banach spaces[4)] we can deduce easily, that the solution of (2.1) is analytic on $[0,\infty[$, if both the scattering cross section and the initial condition are nonnegative.

So far we have constructed a series representation (6.5) of the solution of (2.1) which converges for sufficiently small positive and negative times. Starting from (6.5) we will construct a series representation of the solution of (2.1) which converges for any nonnegative time, if the initial condition f_o and the scattering cross section σ are nonnegative.

Definition 6.2: We introduce for nonnegative $\sigma \in W_o$ and $f_o \in L_{1,o}$ and $h \geq 2\pi\|\sigma\|$ the following sequence $\{H_n(\sigma)\}$ of functions:

$$H_o(\sigma) = f_o$$

$$H_n(\sigma) = \frac{1}{n\,h} \sum_{\mu=0}^{n-1} Q_h(\sigma,H_{n-1-\mu}(\sigma),H_\mu(\sigma)) \quad , \text{ if } n \geq 1 \,, \tag{6.6}$$

where $Q_h(\sigma,\cdot,\cdot)$ is given by (3.7).

Proposition 6.2: The functions $\{H_n(\cdot)\}$ of (6.6) have the properties

(i) $\forall\ n \geq 0 : H_n(\sigma) \geq 0$ (6.7)

(ii) $\forall\ n \geq 0 : \quad \|H_n(\sigma)\|_o = \int_{\mathbb{R}^3 x \mathbb{R}_+} H_n(v,\varepsilon_1)\, d\varepsilon_1 dv = 1$, if $\|f_o\|_o = 1$. (6.8)

(iii) From $\|f_o\|_o = 1$ and $\|f_o\|_1 < \infty$ we have: $\|H_n(\sigma)\|_1 = \|f_o\|_1$. (6.9)

Proof: Part (i) is an immediate consequence of the definition of H_n and (3.8). To prove (ii) and (iii) we remark, that we have for functions f and $g \in L_{1,k}$, k=0,1, the property

$$\int_{\mathbb{R}^3 x \mathbb{R}_+} (1 + |v|^2 + \varepsilon_1)^m\, J(\sigma,f,g)\, d\varepsilon_1 dv = 0 \tag{6.10}$$

for m = 0, if k = 0 and m = 0,1, if k = 1. Because of (6.7) we get

$$\begin{aligned}\|H_n(\sigma)\|_o &= \int_{\mathbb{R}^3 x \mathbb{R}} H_n(v,\varepsilon_1)\, d\varepsilon_1 dv \\ &= \frac{1}{nh}\sum_{\mu=0}^{n-1} 0 + h\Big[\int_{\mathbb{R}^3 x \mathbb{R}_+} H_{n-1-\mu}(\sigma)\, d\varepsilon_1 dv\Big]\Big[\int_{\mathbb{R}^3 x \mathbb{R}_+} H_\mu(\sigma)\, d\varepsilon_1 dv\Big].\end{aligned}$$

and (6.8) follows by induction. To prove (iii) we use (6.10) to get:

$$\begin{aligned}\|H_n(\sigma)\|_1 &= \int_{\mathbb{R}^3 x \mathbb{R}_+} (1 + |v|^2 + \varepsilon_1) H_n(v,\varepsilon_1)\, d\varepsilon_1 dv \\ &= \frac{1}{nh}\sum_{\mu=0}^{n-1} 0 + \frac{h}{2}\int_{\mathbb{R}^3 x \mathbb{R}_+} (1 + |v|^2 + \varepsilon_1) H_{n-1-\mu}(\sigma)\, d\varepsilon_1 dv \int_{\mathbb{R}^3 x \mathbb{R}_+} H_\mu(\sigma)\, d\varepsilon_1 dv \\ &\qquad + \frac{h}{2}\int_{\mathbb{R}^3 x \mathbb{R}_+} (1 + |v|^2 + \varepsilon_1) H_\mu(\sigma)\, d\varepsilon_1 dv \int_{\mathbb{R}^3 x \mathbb{R}_+} H_{n-1-\mu}(\sigma)\, d\varepsilon_1 dv\end{aligned}$$

and (6.9) follows from (6.8) by induction.

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Lemma 6.1: Let $\sigma \in W_o$ and $f_o \in L_{1,o}$ be nonnegative functions and suppose $h \geq 2\pi\, \|\sigma\|$. Then we have for the equation

$$\begin{aligned} d_t\, \Phi(t,\sigma) &= Q_h(\sigma,\Phi(t,\sigma),\Phi(t,\sigma)) \\ \Phi(0,\sigma) &= f_o \end{aligned} \tag{6.11}$$

a solution only for positive times $t < h^{-1}$ and this solution is unique. Moreover it is given by:

$$\Phi(t,\sigma) = \sum_{n=0}^{\infty} t^n\, h^n\, H_n(\sigma)\ . \tag{6.12}$$

Proof: Local existence and uniqueness can be obtained by standard techniques of the theory of differential equations in Banach spaces. [4] In addition we get from those techniques the nonnegativity of the solution Φ. Therefore we have:

$$d_t \int_{\mathbb{R}^3 \times \mathbb{R}_+} \Phi(t,\sigma)\, d\varepsilon_1 dv = d_t \, \|\Phi(t,\sigma)\|_o$$

$$= \int_{\mathbb{R}^3 \times \mathbb{R}_+} Q_h(\sigma,\Phi(t,\sigma),\Phi(t,\sigma))\, d\varepsilon_1 dv$$

$$= h \, \|\Phi(t,\sigma)\|_o^2 \quad ,$$

which yields: $\|\Phi(t,\sigma)\|_o = \dfrac{1}{1 - ht}$.

So we have, that h^{-1} is an upper bound for existence time for the solution of (6.11). To prove, that there is a solution of (6.11) for $t < h^{-1}$, we consider the series (6.12). Due to proposition 6.2, the partial sums

$$\sum_{n=0}^{N} t^n \, h^n \, H_n(\sigma)$$

of this series converge absolutely and uniformly in $L_{1,o}$ for $0 \le t < h^{-1}$. This implies, that the function

$$\Psi(t,\sigma) = \sum_{n=0}^{\infty} t^n \, h^n \, H_n(\sigma)$$

is in $C([0,t_o] \to L_{1,o})$ if $t_o < h^{-1}$. In addition we have[1]:

$$Q_h(\sigma,\Psi(t,\sigma),\Psi(t,\sigma) = \sum_{n=0}^{\infty} t^n \, h^n \sum_{\mu=0}^{n} Q_h(\sigma,H_{n-\mu}(\sigma),H_\mu(\sigma)) \, ,$$

$$= \sum_{n=0}^{\infty} t^n \, h^{n+1} \, H_{n+1}(\sigma),$$

which yields:

$$\Psi(t,\sigma) - f_o = \int_o^t Q_h(\sigma,\Psi(s,\sigma),\Psi(s,\sigma))\, ds \quad \text{for } 0 < t < h^{-1}.$$

///

<u>Theorem</u> <u>6.2:</u> Let $\sigma \in W_o$, $f_o \in L_{1,o}$ be nonnegative functions with $\|f_o\|_o = 1$. Suppose we have $h \ge 2\pi \, \|\sigma\|$. Let $\Psi(\cdot,\sigma)$ be the solution of (6.11). Then the function

$$\mathbb{R}_+ \ni t \to f(t,\sigma) = e^{-ht}\Psi(\tau(t),\sigma) \quad \text{with } \tau(t) = h^{-1}(1 - e^{-ht}) \tag{6.13}$$

solves the Boltzmann equation (2.1).

Proof: We first note: $\tau(\cdot) \in C^{\infty}(\mathbb{R}_+ \to [0,h^{-1}[)$. Due to this property of $\tau(\cdot)$ and due to Lemma 6.1, we have

$$f(\cdot,\sigma) \quad \in C^1([0,\infty[\to L_{10}) \tag{6.14}$$

Due to the fact, that $\Psi(\cdot,\sigma)$ solves (6.11) we have

$$\begin{aligned} d_t\, f(t,\sigma) &= -hf(t,\sigma) + e^{-ht}\, Q_h\big(\sigma,\Psi(\tau(t),\sigma),\Psi(\tau(t),\sigma)\big)\, e^{-ht} \\ &= -hf(t,\sigma) + Q_h\big(\sigma,e^{-ht}\Psi(\tau(t),\sigma),e^{-ht}\Psi(\tau(t),\sigma)\big) \\ &= -hf(t,\sigma) + Q_h(\sigma,f(t,\sigma),f(t,\sigma)), \end{aligned}$$

which implies that the function $f(\cdot,\sigma)$ of (6.13) solves Arkeryd's equation.

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<u>Remark:</u> We have shown, that the solution (2.1) for nonnegative scattering cross sections and initial conditions can be represented in the form:

$$f(t,\sigma) = \sum_{n=0}^{\infty} e^{-ht}\,(1 - e^{-ht})^n\; H_n(\sigma) \quad . \tag{6.15}$$

<u>Theorem 6.3:</u> Let $\sigma \in W_o$ be a nonnegative function. Suppose we have an initial condition $f_o \in L_{1,o}$ with:

$$1 > a_o = \int_{\mathbb{R}^3 \times \mathbb{R}_+} f_o(v,\varepsilon_1)\, d\varepsilon_1 dv > 0 \quad \text{and} \quad \|f_o\|_o = 1.$$

Then there exists a unique solution of (2.1) for $0 \le t < \dfrac{-\ln(1-a_o)}{2\pi\, \|\sigma\|\, a_o}$

Proof: As in the case of nonnegative initial conditions a function $f(\cdot,\sigma)$ solves (2.1) iff it solves

$$\begin{aligned} d_t\, f(t,\sigma) + h\, f(t,\sigma) \int_{\mathbb{R}^3 \times \mathbb{R}_+} f_o(v,\varepsilon_1) d\varepsilon_1 dv &= Q_h(\sigma,f(t,\sigma),f(t,\sigma)) \\ f(0,\sigma) &= f_o. \end{aligned} \tag{6.16}$$

We define the following sequence $\{H_n'(\sigma)\}$

$$H_o'(\sigma) = f_o$$

$$H_n'(\sigma) = \frac{1}{nh} \sum_{\mu=0}^{n-1} Q_h(\sigma,H_{n-1-\mu}'(\sigma),H_\mu'(\sigma)),\ n \ge 1,$$

where we have $h \ge 2\pi\|\sigma\|$. We consider the following sequence of functions:

$$Y_m(t,\sigma) = \sum_{n=0}^{m} \exp[-a_o ht] \, (1 - \exp[-a_o ht])^n \, H_n'(\sigma) \, [a_o h]^{-n}$$

and note: If $\{Y_m(\cdot,\sigma)\}$ converges absolutely and uniformly on $[0,t_o]$ in $L_{1,o}$ for some $t_o > 0$, then $Y(\cdot,\sigma)$ solves (6.16) in $C([0,t_o] \to L_{1,o})$ and so it solves (2.1).

For functions f, g $\in L_{1,o}$ and nonnegative $\sigma \in W_o$ and $h \geq 2\pi \|\sigma\|$ we have the following estimate

$$\begin{aligned}
|Q_h(\sigma,f,g)|(v,\varepsilon_1) \leq\ & |G(\sigma,f,g)|(v,\varepsilon_1) + \\
& \frac{1}{2}\Big| \int_{\mathbb{R}^3 \times \mathbb{R}_+} [h - 2\pi\, \sigma^t(E,e)] \\
& \qquad \cdot [f(v,\varepsilon_1)g(w,\varepsilon_2) + f(w,\varepsilon_2)g(v,\varepsilon_1)]\, d\varepsilon_2 dw \Big| \\
\leq\ & G(\sigma,|f|,|g|) + \\
& \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}_+} [h - 2\pi\, \sigma^t(E,e)] \\
& \qquad \cdot [|f|(v,\varepsilon_1)|g|(w,\varepsilon_2) + |f|(w,\varepsilon_2)|g|(v,\varepsilon_1)]\, d\varepsilon_2 dw \\
=\ & Q_h(\sigma,|f|,|g|),
\end{aligned}$$

which yields by induction $\|H_n'(\sigma)\|_o \leq h^n$. Therefore we get

$$\|Y_m(t,\sigma)\|_o \leq \sum_{n=0}^{m} \exp[-a_o ht] \, (1 - \exp[-a_o ht])^n \, a_o^{-n}.$$

which shows, that the sequence $\{Y_m(\cdot,\sigma)\}$ converges absolutely and uniformly on intervalls $[0,t]$, if t satisfies the inequality

$$1 > \frac{1 - \exp[-a_o ht]}{a_o} \iff t < \frac{\ln\,(1-a_o)}{a_o h}.$$

If we choose now $h = 2\pi \|\sigma\|$, we get the desired result.

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