Mathematical Models for Vehicular Traffic

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Summary. This survey contains a description of different types of mathematical models used for the simulation of vehicular traffic. It includes models based on ordinary differential equations, fluid dynamic equations and on equations of kinetic type. Connections between the different types of models are mentioned. Particular emphasis is put on kinetic models and on simulation methods for these models.

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1 Introduction

Modeling and computer simulation play an increasing role in the optimization of traffic flow. Traditionally there have been three types of approach to the problem.

The first and most basic models are microscopic or car following models, modeling the actual response of individual vehicles to their predecessor by ordinary differential equations based on Newtons law. They have been investigated by many authors, see, e.g., [1, 2, 3, 4].

Macroscopic models based on fluid dynamic equations have been proposed by a large number of authors, see, e.g., [5, 6, 7, 8, 9, 10, 11]. However, some of these models have been subject to considerable controversy, concerning their validity and applicability to traffic flow [12].

Kinetic models present an intermediate step between the above two types of model. They are based on Boltzmann type kinetic equations. On the one hand they can be more fundamentally justified than the standard macroscopic models, leading to a better justification of the macroscopic models and potentially to more accurate results. On the other side, compared to microscopic models, computation time is strongly decreased. This may make the kinetic models applicable to the description of real life situations and traffic control problems. Kinetic models in traffic flow started originally with the work of Prigogine et al. [13, 14, 15, 16], who introduced a kinetic term to account for the slowing down interactions. However, they and most of their successors (see [17, 18, 19, 20]) treated the acceleration of the vehicles by means of a heuristic relaxation term. Only recently, Nelson [21] succeded in developing reasonable kinetic equations by using a kinetic description for the acceleration.

In this survey we will review these approaches from a kinetic point of view. The article is organized in the following way: In Section 2 the basic quantities are introduced and the concept of homogeneous traffic flow is explained. Section 3 contains a description of the most common microscopic models like car following and cellular automata models, the latter being already similar to very simple kinetic models. In Section 4 macroscopic models based on fluid dynamic descriptions are discussed. Section 5 contains the description of kinetic models. Finally, in Section 6 we show some results of numerical simulations for particular traffic flow situations.

2 Basic Concepts

2.1 Levels of Description

As stated in the introduction we distinguish different levels of description. On the first and most detailed level each vehicle is described on the "microscopic scale" by its space and speed coordinates for time $t \in \mathbb{R}_0^+$:

$$x_j(t) \in \mathbb{R}, \quad v_j(t) \in [0, v_m], \quad j = 1, \ldots, N$$

where $v_m \in [0, \infty)$ is the maximum speed. Here N is the total number of vehicles under consideration. The evolution is decribed in the simplest cases by second order ODE's with a time lag, see Section 3.1.

In contrast to the discrete structure of microscopic models, a homogenized description is used on the kinetic (mesoscopic) level. Here a distribution function $f(x, v, t)$ describing the number of vehicles with a certain location and speed at time t is considered.

$$N_M = \int_M f(x, v, t) dx dv$$

denotes the number of vehicles with space coordinate x and speed coordinate v, $(x, v) \in M \subset \mathbb{R} \times [0, v_m]$. In particular,

$$N = \int_{\mathbb{R} \times [0, v_m]} f(x, v, t) dx dv$$

The time evolution of the distribution function is usually given by an integro-differential equation of Boltzmann type (the kinetic equation). Since the number of vehicles with coordinates (x_j, v_j) in a set M is also determined on the microscopic level by

$$N_M = \sum_{j=1}^{N} \chi_M(x_j, v_j)$$

with

$$\chi_M(x, v) = \begin{cases} 1 & : \quad (x, v) \in M \\ 0 & : \quad (x, v) \notin M \end{cases}$$

the kinetic equation should be derivable from the microscopic evolution of vehicles.

The last and coarsest level is the macroscopic description. In most applications one is neither interested in the exact evolution of the single vehicles nor in the distribution function f. The main quantities are the density ρ, the mean speed V and the speed variance θ of the vehicles. From the kinetic point of view, these quantities are given by

$$\rho(x,t) = \int_0^{v_m} f(x,v,t)dv$$

$$V(x,t) = \frac{1}{\rho(x,t)} \int_0^{v_m} vf(x,v,t)dv$$

$$\theta(x,t) = \frac{1}{\rho(x,t)} \int_0^{v_m} (v - V(x,t))^2 f(x,v,t)dv$$

Sometimes it is more convenient to deal with the traffic flow q (or "volume") and the traffic pressure $\mathcal{P}$ instead of the mean speed V and the speed variance θ

$$q(x,t) = \rho(x,t)V(x,t) = \int_0^{v_m} vf(x,v,t)dv$$

$$\mathcal{P}(x,t) = \rho(x,t)\theta(x,t) = \int_0^{v_m} (v - V(x,t))^2 f(x,v,t)dv$$

Macroscopic theories are based on generalized balance equations for the basic quantities ρ, V, θ. In principle, such equations should be derivable from a kinetic traffic flow equation. We discuss this point in Section 5.2.3.

2.2 Homogeneous Traffic Flow

The simplest situation in traffic flow theories is the so-called homogeneous equilibrium traffic flow, i.e. a uniform flow of vehicles not depending on space and time variables. In this situation, measurements show that the state of the traffic flow is only dependent on the density of vehicles.

On the kinetic level this state is described by a global equilibrium distribution function $f_e(\rho, v)$, which must be the stationary solution of the homogeneous (space independent) kinetic equation. The homogeneous kinetic equation describes the time evolution of the distribution function f towards this equilibrium state f_e (see Section 5.2.1). Numerical calculations of equilibrium distribution functions are shown in Section 6.3.1.

On the macroscopic level the basic quantities for homogeneous traffic flow are determined by ρ as well. The connection to the kinetic theory is again given by the moments of the distribution function. That is,

$$V_e(\rho) = \frac{1}{\rho} \int_0^{v_m} vf_e(\rho, v)dv \qquad (1)$$

and analogous expressions for θ_e, q_e and $\mathcal{P}_e$.

Historically, the functional dependence of the mean speed on the density, $V_e(\rho)$ (the so-called fundamental diagram), is most important [22]. There is a large amount of available data, and a variety of different models. A very simple, but qualitatively reasonable, model is given by a linear dependence

$$V_e(\rho) = v_m(1 - \frac{\rho}{\rho_m})$$

where ρ_m is the maximal density of vehicles in a jam (bumper to bumper density). More sophisticated models are used for example by Kühne et al. [23], and Kerner and Kohnhäuser [24]. The expression Kühne uses is

$$V_e(\rho) = v_m(1 - (\frac{\rho}{\rho_m})^{n_1})^{n_2} \qquad (2)$$

where n_1, n_2 are fitted to special traffic flow situations. Kerner and Kohnhäuser take

$$V_e(\rho) = v_m(\frac{1}{\eta(\rho)} - \frac{1}{\eta(\rho_m)}) \quad \text{with} \quad \eta(\rho) = 1 + \exp(n_1(\frac{\rho}{\rho_m} - n_2))$$

Figure 1 shows a plot of $q_e(\rho) = \rho V_e(\rho)$ for the model (2).

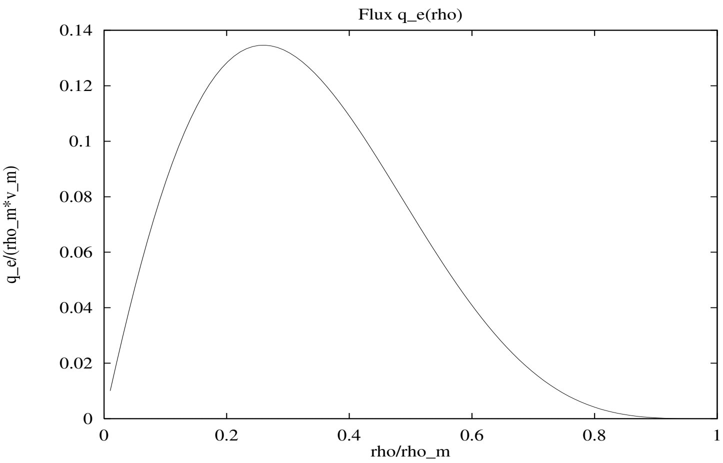

Figure 1: Fundamental Diagram

One of the main tasks of homogeneous kinetic theory (see Section 5.2.1) is to give expressions for these fundamental diagrams using (1). In Section 6.3.1 fundamental diagrams derived from a special kinetic model are shown.

3 Microscopic Models

3.1 Car following Models

From 1950 on, many authors considered car following or follow the leader models. A survey can be found in Gazis et al. [2]. The models consisted mainly of second order ODE's of the form

$$\ddot{x}_i(t+T) = a(v_i(t))^m \frac{(v_{i-1}(t) - v_i(t))}{(x_{i-1}(t) - x_i(t))^l}$$

with parameters T, a, m, l. The basic idea is that the acceleration at time $t+T$ depends on the speed of the vehicle at time t, the relative speeds of the vehicle and its leading vehicle at time t and the distance between the vehicles. T is a typical reaction time of the driver. a, m, l are fitted to special situations (see [2]).

3.2 Psycho-Physiological Car Following Models

Todosiev [3] and Wiedemann [4] introduced psycho-physiological considerations into the car following models. Wiedemann considers so-called reaction thresholds to distinguish different regions of driver behaviour. In the following we present an extensions of the Wiedemann model given by Fritsche [25] (for a similar model see [26]).

The considerations take place in the phase space spanned by the speed of the regarded vehicle v_1, the speed of the leading vehicle v_2, and the distance $h = x_2 - x_1$. From vision perception modeling "perception thresholds" are derived, i.e. distances, where the driver realizes positive and negative differences of speed between himself and its leading vehicle:

- H_1: $v_2 - v_1 = -k_-(h-h_0)^2 - v_0$ perception threshold (negative)
- H_2: $v_2 - v_1 = \quad k_+(h-h_0)^2 + v_0$ perception threshold (positive)

These thresholds separate the regimes of reaction and no reaction of the driver. Here, $h_0 = 1/\rho_m$ is the distance at speed 0. Typical values for the physiological parameters v_0 and $k_\pm$ are (see [26])

$$v_0 = 0.3m/s \qquad k_- = 4 \cdot 10^{-4} m^{-1}s^{-1} \qquad k_+ = 8 \cdot 10^{-4} m^{-1}s^{-1}$$

Besides these perception thresholds, some other thresholds for slowing down and acceleration procedures are introduced:

- H_3: $h = h_0 + T_r v_2$ risky distance
- H_4: $h = h_0 + T_s v_1$ safe distance
- H_5: $h = h_0 + T_d v_1$ desired distance

Typical values are (see[26])

$$T_d = 1.5s \qquad T_s = 0.9s \qquad T_r = 0.45s$$

In Figure 2 the different thresholds are shown for a fixed speed v_2.

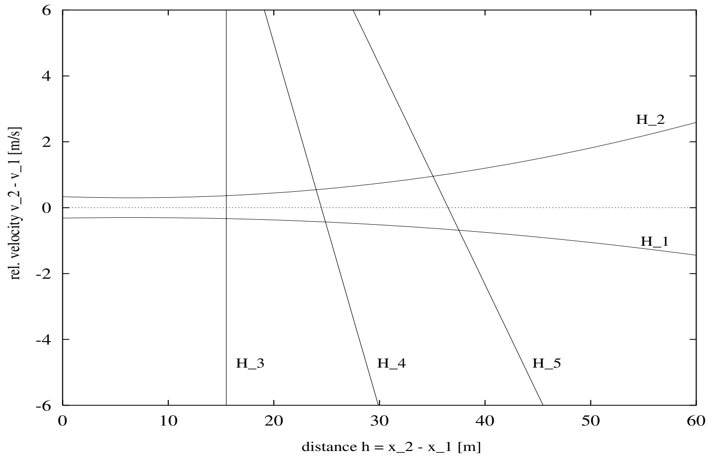

Figure 2: Thresholds

The phase space is subdivided by these five thresholds $H_1, \ldots, H_5$ in five regions (see Figure 3): danger, closing in, following I, following II, and free driving.

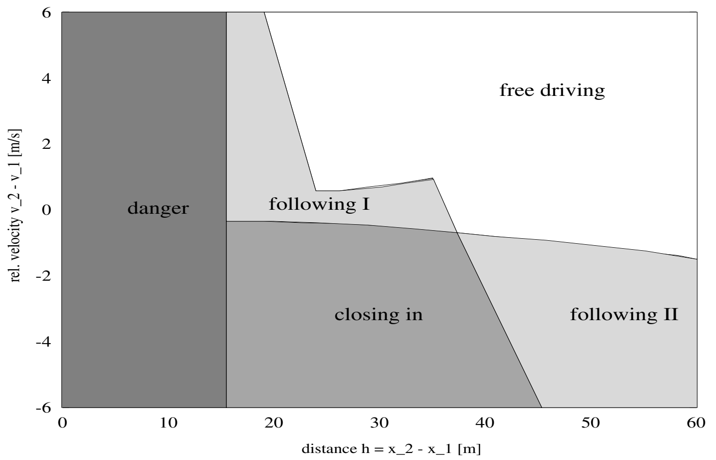

Figure 3: Regions

In the model of Fritsche the driver behaviour is determined by expected and unexpected events.

Expected events: Taking into account his present state (space coordinate, speed and acceleration) and the present state of the leading vehicle, each driver changes its acceleration only, if he passes certain thresholds. Thereby, the acceleration is always adjusted immediately, if the driver crosses a threshold to the areas “danger”, “free driving” and “closing in”. When moving into the areas “following I” and “following II”, no change of acceleration takes place. If the driver crosses the threshold to the danger area, deceleration is necessary. If the free driving area is entered, the driver accelerates. Entering the closing in area, the driver will decelerate slightly.

Unexpected events: A following vehicle reacts to a change of acceleration of the leading vehicle after a certain reaction time.

For the details of choosing the acceleration in response to these events and for the generalization to a multi-lane model we refer to [25].

3.3 Cellular Automata Models

Cellular automata models for traffic flow are derived in analogy to lattice gas automata in gas dynamics, see [27, 28, 29]. This type of models simplifies the behaviour of the drivers extremely. The vehicles are described by discrete lattice points and discrete speeds. Actually, these models could be already viewed as a type of simple kinetic models rather than a microscopic model. We concentrate in the following on the model of Nagel and Schreckenberger [27, 30]. Consider a one lane traffic flow and let *gap* denote the number of empty sites in front of the vehicle. The behaviour of the vehicles is given by the following rules:

- *Acceleration:*

  If the speed of a vehicle is lower than v_m, and if there is enough room ahead ($v \leq gap - 1$), then the speed is increased by one ($v = \max[v_m, v + 1]$).

- *Slowing down (due to other vehicles):*

  If the next vehicle ahead is too close($v \geq gap + 1$), the speed is reduced to *gap*, i.e. ($v = gap$).

- *Randomization:*

  With probability p, the speed of each vehicle is decreased by one ($v = \max[v - 1, 0]$).

- *Vehicle motion:*

  Each vehicle is advanced v sites.

The randomization includes effects due to fluctuations at maximum speed, retarded acceleration and over reaction at braking.

These automata seem to be a reasonable, simple approach. They are able to describe important features of traffic flow and may, due to their simplicity, also be used in network simulations.

4 Macroscopic Models

4.1 Basic Models

The basic approach to macroscopic models was given by Lighthill and Whitham, see [5]. They used the fundamental property of conservation of the number of vehicles, which is in a local form expressed by the continuity equation

$$\rho_t(x,t) + q_x(x,t) = 0$$

Moreover, to obtain a closed equation for ρ, one needs a functional dependence of q on ρ. The basic idea of Lighthill and Whitham was to use the "steady state assumption"

$$q(x,t) = q_e(\rho(x,t)) = \rho(x,t)V_e(\rho(x,t)) \tag{3}$$

where q_e and V_e are derived from the case of homogeneous traffic flow. Expressions for V_e were given in Section 2.2. Of course, this approach is based on the assumption of a local equilibrium flow and this is certainly not valid in more complicated situations. The resulting equation

$$\rho_t + q_e'(\rho)\rho_x = 0$$

with

$$q_e'(\rho) = \frac{d}{d\rho} q_e(\rho) = V_e(\rho) + \rho V_e'(\rho)$$

nevertheless describes the most important features of traffic flow. In particular, backwards propagating shocks may appear if $q_e'(\rho)$ has negative parts. For example, the linear model

$$V_e(\rho) = v_m(1 - \frac{\rho}{\rho_m})$$

leads to

$$\rho_t + v_m(1 - 2\frac{\rho}{\rho_m})\rho_x = 0$$

which may have backwards propagating disturbances depending on the values of ρ.

Moreover, to smear out the shock structure, Whitham has introduced a diffusion term into the equation, which leads to

$$\rho_t + q_e'(\rho)\rho_x = \nu\rho_{xx}$$

4.2 Models with an Acceleration Equation (Payne, Kühne)

To overcome the steady state assumption (3), which is certainly not valid in some of the most interesting traffic flow situations, Payne [8] introduced an additional equation for the mean speed V including a relaxation of V within a certain time τ towards its equilibrium value $V_e(\rho)$. This leads to the following system of equations

$$\rho_t + (\rho V)_x = 0$$
$$V_t + VV_x = -\frac{1}{\rho}(\mathcal{P}_e(\rho))_x + \frac{1}{\tau}(V_e(\rho) - V) \tag{4}$$

The term

$$-\frac{1}{\rho}(\mathcal{P}_e(\rho))_x = -\frac{\mathcal{P}_e'(\rho)}{\rho}\rho_x$$

is an anticipation term taking into account the awareness of the drivers for the traffic condition ahead. Beside the fundamental diagram $V_e(\rho)$, the equations make use of an additional equilibrium function $\mathcal{P}_e(\rho)$, which may be viewed as the equilibrium traffic pressure. Payne used an anticipation term determined by the fundamental diagram:

$$\mathcal{P}_e'(\rho) = \frac{1}{2\tau}|V_e'(\rho)|$$

In general we have

$$\mathcal{P}_e(\rho) = \rho\theta_e(\rho)$$

Analogous to the situation for $V_e(\rho)$, there have been different suggestions for the function $\theta_e(\rho)$. E.g., Kühne [31], and Kerner and Kohnhäuser [24, 32] suggested to take a constant value

$$\theta_e(\rho) = c_0^2$$

whereas Phillips [20] suggested to take a linear relation

$$\theta_e(\rho) = \theta_m(1 - \frac{\rho}{\rho_m})$$

As mentioned in Section 2.2, the dependence of speed V_e and variance θ_e on the density in the equilibrium case can be derived by means of kinetic theory from the equilibrium distribution function.

However the above model shows unrealistic results for strong changes of the densities (e.g. for shock waves), see Hauer et al. [33].

To overcome this difficulty, a viscosity term is introduced in the above equations smoothing out the solutions (see Kühne [10]). The acceleration equation then becomes

$$V_t + VV_x = -\frac{1}{\rho}(\mathcal{P}_e(\rho))_x + \frac{1}{\tau}(V_e(\rho) - V) + \frac{\mu}{\rho}V_{xx}$$

An important feature of inhomogeneous traffic flow is the development of instabilities, i.e., for example, stop-start patterns and the spontaneous occurence of traffic jams.

Instabilities can be detected in the equations by an analysis as in Kühne et al. [34]. We describe it shortly in the following:

Consider the model (4) with $\theta_e(\rho) = c_0^2$. One introduces the linearized version of the above equation governing the development of the perturbations $(\delta\rho, \delta V)$ of the quantities $(\rho,\, V)$ from a certain equilibrium value (ρ_0, V_0):

$$\rho = \rho_0 + \delta\rho, \quad V = V_0 + \delta V.$$

Introducing a special form of the perturbations

$$\delta\rho = (\delta\rho)_0 \exp(\omega\frac{t}{\tau} + i\frac{k}{\tau c_0}x), \quad \delta V = (\delta V)_0 \exp(\omega\frac{t}{\tau} + i\frac{k}{\tau c_0}x), \quad \omega \in \mathbb{C}, k \in \mathbb{R}$$

in the linearized equations, one obtains an algebraic equation for ω depending on the wavenumber k (dispersion relation). This equation can be derived from the vanishing coefficient determinant of the equation

$$(\omega - ik\frac{V_0}{c_0})\frac{\delta\rho}{\rho_0} - ik\frac{\delta V}{c_0} = 0$$

$$(\omega - ik\frac{V_0}{c_0})\frac{\delta V}{c_0} = (-a - 1 - ik)\frac{\delta\rho}{\rho_0} + (-1 - \nu k^2)\frac{\delta V}{c_o}$$

with $a = -1 - (\rho_0/c_0)V_e'(\rho_0)$ and $\nu = \mu/(\rho_0 c_0^2 \tau)$.

In Figure 4 a plot of $\mathrm{Re}(\omega)$ is shown for $a, \nu > 0$.

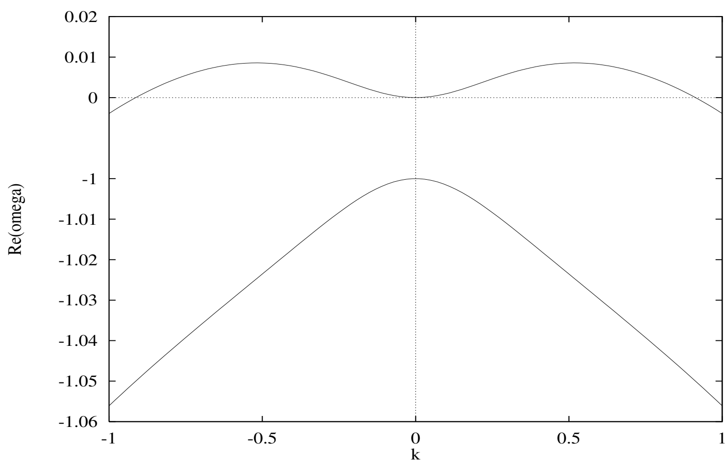

Figure 4: Dispersion relation $\mathrm{Re}(\omega)(k)$

The region of instability is determined by the values

$$\mathrm{Re}(\omega) > 0$$

which corresponds to the regions in k-space with

$$a > \nu k^2$$

Looking at this condition one observes, that instability is only possible, if the traffic parameter a is positive.

Instable traffic flow patterns like stop-start waves and the spontaneous occurence of traffic jams can also be analyzed by solving the complete nonlinear equations numerically, see Kerner and Kohnhäuser [24, 32] and the simulations in Section 6.2.

4.3 Euler and Navier Stokes like Models (Helbing)

Up to this level the speed variance was handled as an equilibrium quantity. However, for non equilibrium situations, the variance may be better treated as a dynamic variable with a further equation describing its evolution. In particular to predict traffic jams, an increase of the variance appears to be a very important indicator. A detailed treatment of this quantity seems to be necessary. Helbing [11] proposed, on the basis of kinetic theory (see Section 5.2.3), the following set of equations

$$\rho_t + (\rho V)_x = 0$$

$$V_t + VV_x = -\frac{1}{\rho}(\rho\theta)_x + \frac{1}{\tau}(V_e(\rho) - V) + \frac{\mu}{\rho}V_{xx}$$

$$\theta_t + V\theta_x = -2\theta V_x + 2\frac{\mu}{\rho}(V_x)^2 + \frac{\mu}{\rho}V_{xx} + \frac{\kappa}{\rho}\theta_{xx} + \frac{2}{\tau}(\theta_e(\rho) - \theta)$$

with θ_e depending on ρ in the same way as V_e in the model of Kerner and Kohnhäuser (see 2.2).

We note that as in gas dynamics, the above model is called an Euler type model if μ and κ are equal to zero, otherwise it is a Navier Stokes type model. Moreover, Helbing introduced a modified model taking into account the finite space requirement of the vehicles. A numerical simulation and a stability analysis of the equations can be found in Helbing [11].

5 Kinetic Models

Historically, microscopic (see section 3) and macroscopic models (see section 4) were independent starting points for the description of vehicular traffic. Analogous to gas dynamics, the missing link between these models is given by mesoscopic (kinetic) models. Kinetic models should be derivable from microscopic models and, moreover, macroscopic models should be derivable from kinetic models.

5.1 Presentation and Foundations of Kinetic Models

5.1.1 The Prigogine Model

Kinetic models started with the work of Prigogine [14], see the book of Prigogine and Herman [16]. As mentioned in Section 2, the basic quantities are the distribution functions $f(x,v,t)$. Prigogine modeled the flow by a combination of free flow, a slowing down term and a relaxation term:

$$f_t + v f_x = J_\rho(f,f) = S_\rho(f,f) + R_\rho(f)$$

The slowing down term

$$S_\rho(f,f) = (1 - P(\rho)) f \int_0^{v_m} (v' - v) f(v') dv'$$

describes the breaking down interactions of a vehicle and depends on the probability of overtaking P. This term is derived in a similar fashion to the usual derivation of the Boltzmann equation in rarefied gas dynamics. A detailed derivation for a more general model, including this term of the Prigogine model, is given in Section 5.1.3. Usually P was assumed to depend on ρ in a linear way, see [16]

$$P(\rho) = 1 - \frac{\rho}{\rho_m}$$

The relaxation term

$$R_\rho(f) = -\frac{f - f_0}{T(\rho)}$$

takes into account the acceleration of the vehicles to their desired speed. The desired speed distribution f_0 was assumed to be of the form

$$f_0(x,v,t) = \rho(x,t) F_0(v)$$

where F_0 is a given function not depending on t. The relaxation time T was also assumed to depend on ρ

$$T(\rho) = \frac{\rho}{\rho_m - \rho}$$

Obviously this relation cannot be valid for ρ close to 0. In this case the expression should be changed, in order to obtain nonzero relaxation times.

There are, however, several weak points of the model. In particular, the desired speed distribution F_0 is fixed and does not depend on the evolution. It should be a Lagrangian quantity following the motion of the vehicles instead of being assigned a priori (see [17]). This motivated Paveri-Fontana to develop his improved kinetic traffic model. We will describe it in the next section. We mention here also the work of Lampis ([18]). There, queueing vehicles have been included in the Prigogine equation by introducing a speed distribution for queues.

5.1.2 The Paveri-Fontana Model

To avoid the above mentioned problem, Paveri-Fontana [17] extended the state space for his model, introducing a generalized distribution function

$$g(x,v,t;w)$$

It describes the number of vehicles with speed v and desired speed w. The usual distribution function f is recovered by

$$f(x,v,t) = \int_0^{v_m} g(x,v,t;w)dw$$

and the desired speed distribution f_0 by

$$f_0(x,v,t) = \int_0^{v_m} g(x,v,t;w)dv$$

The kinetic equation reads

$$g_t + vg_x = S_\rho(f,g) + R_\rho(g)$$

with a slowing down term

$$S_\rho(f,g) = (1-P)f\int_v^{v_m}(v'-v)g(x,v',t;w)dv' - g\int_0^v (v-v')f(x,v',t)dv'$$

This term is derived in exactly the same way as for the Prigogine model. Integrating it over w gives the Prigogine braking term. The relaxation term, $R_\rho(g)$, is

$$R_\rho(g) = -\partial_v(\frac{w-v}{T}g)$$

Here, the basic idea is to describe an exponential acceleration of the vehicle to its desired speed w with a relaxation time T.

To compare the Paveri-Fontana equation with the Prigogine equation, one might integrate it with respect to v and w. This yields

$$f_t + vf_x + \partial_v(\frac{\int_0^{v_m} wg(x,v,t;w)dw - vf}{T}) = (1-P)f\int_0^{v_m}(v'-v)f(x,v',t)dv'$$

and

$$\partial_t f_0 + \partial_x(\int vg(x,v,t;w)dv) = 0$$

The equations for f and f_0 obviously depend on the time development of g. In particular, the equation for f_0 shows that the desired speed distribution is dependent on the time evolution of the system. Therefore, this improved model overcomes the above mentioned criticism of the Prigogine model. However, the dimensionality of the model has, compared to the Prigogine model, increased.

5.1.3 New Developments

Comparing the classical models we see that, on the one hand the Prigogine model is not completely justified microscopically. On the other hand, the Paveri-Fontana model has one more speed dimension, increasing the complexity of the problem considerably.

These points motivated Nelson [21] to derive a model for the usual distribution function $f(x, v, t)$ strictly from microscopical considerations. He treated the acceleration term in a way similar to the one Prigogine used for the braking term. However, as he himself states, his model is a caricature of traffic flow and should be seen only as a first step in obtaining a kinetic equation that is also suitable for real applications.

Wegener and Klar [35] used Nelsons [21] ideas to construct a general model which is suitable for applications. We will now describe the derivation of this general model in more detail, since the derivation of the Prigogine and Paveri-Fontana braking down terms is very similar. The basic ideas of the Nelson model will also be shown during the derivation procedure.

In [35], as in [21], one starts with a microscopic model similar to the one explained in Section 3.2. The model in [35] is the following: Consider a vehicle 1 at place x_1 with the speed v_1 and its leading vehicle 2 at x_2 with speed v_2. v_1 and v_2 are assumed to be in $[0, v_m]$, where v_m denotes again the maximal speed. Let N be the number of thresholds under consideration. If vehicle 1 is crossing a threshold or, in other words, if the headway h of vehicle 1 (the distance $x_2 - x_1$) is becoming larger or smaller than a certain threshold $h = H_i(v_1, v_2), i \in \{1, \ldots N\}$, then vehicle 1 is changing his speed into the new speed v. v_1 and v_2 are, for each threshold, out of a certain range Ω_i of values associated to the threshold H_i. The new speed v is obtained instantaneously due to a distribution function

$$\sigma_i(v; v_1, v_2) \quad \text{with} \quad (v_1, v_2) \in \Omega_i$$

σ_i may depend also on x and t. Since σ_i is a density function, it has to fulfill

$$\int_0^{v_m} \sigma_i(v; v_1, v_2) dv = 1$$

A slowing down line could be given, for example, by a function $\epsilon(v_2)$ with $H_i(v_1, v_2) = \epsilon(v_2)$, where ϵ represents the minimal acceptable distance to a leading vehicle with speed v_2. Since a vehicle is slowing down only if its speed is larger than that of his leading vehicle, Ω_i might be given in this case by $\{(v_1, v_2); v_1 > v_2\}$. It has to be noted that, in contrast to Nelson [21], a vehicle changes its speed in this model only if a threshold is crossed.

Now one derives from the microscopic model directly a kinetic equation. Let $f(x, v, t)$ denote the distribution function for the number of vehicles at place x and time t with speed v. Writing down the change of the total number of vehicles leads, as usual, to a kinetic equation:

$$f_t + v f_x = J_\rho(f, f) = \left(\frac{\delta f}{\delta t}\right)_g - \left(\frac{\delta f}{\delta t}\right)_l$$

where $t \in [0, \infty)$, $v \in [0, v_m]$, and $\left(\frac{\delta f}{\delta t}\right)_g$ and $\left(\frac{\delta f}{\delta t}\right)_l$, are the gain and loss terms due to interactions, respectively. They can be written as

$$\left(\frac{\delta f}{\delta t}\right)_g = \sum_{i=1}^{N} \left(\frac{\delta f}{\delta t}\right)_g^i, \quad \left(\frac{\delta f}{\delta t}\right)_l = \sum_{i=1}^{N} \left(\frac{\delta f}{\delta t}\right)_l^i$$

where $\left(\frac{\delta f}{\delta t}\right)_g^i$ and $\left(\frac{\delta f}{\delta t}\right)_l^i$ are describing the gain and loss terms due to the i-th threshold.

By $Q_i(v_2, \tau; x, v_1, t)$ one denotes the probability that a vehicle (with speed v_1 at the place x and time t) has a leading vehicle (with speed v_2) undergoing an interaction with him in the time intervall $[t, t+\tau]$ due to the crossing of the threshold H_i. Then $\frac{dQ_i}{d\tau}(v_2, 0; x, v_1, t)$ is the rate of change of this probability. The rate of change of the probability for vehicle 1 of having a leading vehicle with arbitrary speed and an interaction with it is therefore equal to

$$\int_{(v_1,v_2)\in\Omega_i} \frac{dQ_i}{d\tau}(v_2, 0; x, v_1, t) dv_2$$

Since, in the case of an interaction, the resulting speed of vehicle 1 is no longer its initial speed, one gets that the net rate of loss for the total number of vehicles at x and v due to the crossing of the i-th threshold is

$$\left(\frac{\delta f}{\delta t}\right)_l^i = f(x, v, t) \int_{(v,v_2)\in\Omega_i} \frac{dQ_i}{d\tau}(v_2, 0; x, v, t) dv_2$$

The corresponding net rate of gain is

$$\left(\frac{\delta f}{\delta t}\right)_g^i = \int_{(v_1,v_2)\in\Omega_i} \frac{dQ_i}{d\tau}(v_2, 0; x, v_1, t) f(x, v_1, t) \sigma_i(v; v_1, v_2) dv_1 dv_2$$

since $\sigma_i(v; v_1, v_2)$ is the distribution function for the new speed of vehicle 1 with original speed v_1.

Let $l(h, v_2; x, v_1, t)$ denote the distribution of the leading vehicles with headway h and speed v_2 for a vehicle with speed v_1 and space x at time t. Then

$$Q_i(v_2, \tau; x, v_1, t) = \chi_{\Omega_i}(v_1, v_2) \int_{M_i} l(h, v_2; x, v_1, t) dh$$

where χ_{Ω_i} denotes the characteristic function of the set Ω_i and

$$M_i = \{h \in [0, \infty);\ h + (v_2 - v_1)\tau' = H_i(v_1, v_2), \tau' \in [0, \tau]\}$$

Since the vehicles are moving with a relative speed $v_1 - v_2$, M_i contains all headways that lead during the time intervall $[0, \tau]$ to a crossing of the threshold.

A simple calculation leads to the gain and loss terms:

$$\begin{aligned}
\left(\frac{\delta f}{\delta t}\right)_g &= \sum_{i=1}^N \int_{\Omega_i} |v_1 - v_2| \sigma_i(v; v_1, v_2) f(x, v_1, t) l(H_i(v_1, v_2), v_2; x, v_1, t) dv_1 dv_2 \\
\left(\frac{\delta f}{\delta t}\right)_l &= f(x, v, t) \sum_{i=1}^N \int_{(v,v_2)\in\Omega_i} |v - v_2| l(H_i(v, v_2), v_2; x, v, t) dv_2 \qquad (5)
\end{aligned}$$

To obtain from these equations a true equation for f, one has to express l by f. This can be done along the lines suggested in Nelson [21], who introduced a correlation model using a basic assumption, which he termed 'generalized vehicular chaos assumption'. It leads to the following form of l:

$$l(h, v_2; x, v_1, t) = f(x, v_2, t) k(h, \rho(x, t)) \tag{6}$$

Nelson used an explicit form for k

$$k(h, \rho) = \exp(-\rho(h - h_0))$$

where $h_0 = 1/\rho_m$ is the headway at speed 0. Substituting (6) into (5) one is thus lead to an equation for f.

The next aim is to obtain, by specifying H_i, Ω_i and σ_i, a model that is as simple as possible but reproduces already the essential features of traffic flow. Following [35] just one threshold for slowing down and acceleration interactions is assumed.

Slowing Down Interactions: The slowing down threshold is given by a headway

$$h = H_1(v_1, v_2) = \epsilon$$

where ϵ is a positive constant. Ω_1 is, for slowing down, given by

$$\Omega_1 = \{(v_1, v_2) \in [0, v_m]^2; v_1 > v_2\}$$

It is not only allowed to slow down to the actual speed of the leading vehicle, but to a range of speeds smaller than this one. One allows slowing down of vehicle 1 to a speed in $[\beta v_2, v_2]$ where β is some positive constant smaller than 1. In this range, an uniform distribution of speeds is assumed, due to the lack of more precise knowledge. If the following vehicle is faster than the leading one, one has (in addition to slowing down) to take into account the possibility of passing (with probability P). The speed of the passing vehicle is assumed to remain the same as before. This gives

$$\sigma_1(v; v_1, v_2) = P\delta(v_1 - v) + (1 - P)\frac{1}{v_2(1 - \beta)}\chi_{[\beta v_2, v_2]}(v)$$

where $\delta(v)$ is the delta function at v. The probability of passing is a function of ρ,

$$P = 1 - \frac{\rho}{\rho_m}$$

where ρ_m stands for the maximal vehicular concentration or density as before. This passing probability has been used by Prigogine and Herman, see Section 5.1.1. We mention here that, in particular, one can recover the Prigogine slowing down term by using $H_i = 0$, σ_i equal to a delta distribution $\delta(v - v_2)$, and $\Omega_i = \{(v_1, v_2); v_1 > v_2\}$.

Acceleration Interactions: The acceleration threshold is assumed to be given by the same constant

$$h = H_2(v_1, v_2) = \epsilon$$

Ω_2 is, for acceleration, given by

$$\Omega_2 = \{(v_1, v_2) \in [0, v_m]^2; v_1 < v_2\}$$

One assumes that vehicle 1 may accelerate from its actual speed v_1 to the range of speeds between v_1 and $v_1 + \alpha(w - v_1)$. Moreover, one assumes that the speed obtained by acceleration is uniformly distributed in the range $[v_1, v_1 + \alpha(v_m - v_1)]$. This leads to the acceleration term

$$\sigma_2(v; v_1, v_2) = \frac{1}{\alpha(v_m - v_1)} \chi_{[v_1, v_1 + \alpha(v_m - v_1)]}(v)$$

α is assumed to depend on the density:

$$\alpha = \alpha_0 (1 - \frac{\rho}{\rho_m})$$

where $\alpha_0 \leq 1$ is given by some characteristic time scale.

We mention here that, in contrast to the Prigogine theory, there is no need to introduce a distribution function of desired speeds. In particular the number of parameters, compared with their equation, is greatly reduced. The introduction of the desired speed distribution in the Prigogine model seems to be justified for a very low concentration of vehicles. In this case the distribution of the speeds does not depend on the interactions between the vehicles, but essentially on the drivers wishes. However, for higher densities the drivers wishes are dominated by the drivers response to the interaction with his leading vehicle.

In the above model a constant desired speed v_m is used. Obviously, it would be reasonable, and possible, to include a distribution of desired speeds similar to that in Paveri-Fontana (see Section 5.1.2).

5.2 Applications and Restrictions of Kinetic Models

5.2.1 Homogeneous Traffic Flow

Usually, kinetic models are used to calculate equilibrium situations and homogeneous traffic flow. The homogeneous equation is

$$f_t = J_\rho(f, f) \tag{7}$$

for $f = f(v, t)$. It has to be solved with some initial condition $f(v, 0) = f_{init}(v)$.

First of all, we remark that an integration over v of the interaction term $J_\rho(f, f)$ gives 0 in all models, since the number of vehicles is not changed during an interaction. This means, in particular, that the inital density is conserved in a homogeneous situation:

$$\rho = \int_0^{v_m} f(v, t) dv = \int_0^{v_m} f_{init}(v) dv$$

The homogeneous equation should have a one parameter family of stationary distributions $f_e(\rho, v)$ depending only on the density. For ρ fixed

$$f(v,t) \to f_e(\rho, v) \quad \text{for} \quad t \to \infty$$

must be valid. As explained in Section 2.2, the significant quantities (the fundamental diagram $V_e(\rho)$, $P_e(\rho)$ and $\theta_e(\rho)$) are determined by this family of stationary distributions. For example, in the Prigogine model, the equilibrium distribution function is given implicitely by the equation

$$f_e(\rho, v)(1 + \frac{1-P}{T}\rho(V_e - v)) = f_0(v)$$

We note that V_e depends again on f_e. In particular, $f_0 \neq f_e$. Numerous examples of stationary distribution function for different f_0 can be found in [16]. In the model described in Section 5.1.3, the f_e are found numerically (see Section 6.3.1).

5.2.2 Inhomogeneous Traffic Flow

Usually kinetic equation are not used to compute inhomogeneous, real life traffic flow situations. In particular, kinetic equations with a local interaction term, as in the above models, may lead to densities exceeding the maximal value in some situations (see [36]). This might be avoided by introducing non local collision kernels. These would take into account the fact that the interaction between vehicles is nonlocal, since drivers react to events in front of their vehicle.

A modified equation, on the basis of the kinetic equation derived in 5.1.3, may be set up (see [36]) in analogy to the Enskog equation in gas dynamics by changing the collision term $J_\rho(f,f)$ in 5.1.3 in the following way

$$J_\rho(f,f) \longrightarrow J_{\bar{\rho}}(f,\bar{f})$$

with

$$\bar{f}(x,v,t) = \frac{1}{\epsilon}\int_x^{x+\epsilon} f(x',v,t)dx', \qquad \bar{\rho} = \int_0^{v_m} \bar{f}(x,v',t)dv'$$

Here ϵ is the typical threshold distance (compare 5.1.3). This takes into account that the interaction partner of the driver is at a certain distance in front of the vehicle. Simulations using this model are shown in Section 6.3.2.

5.2.3 Derivation of Macroscopic Models

As in gas dynamics, macroscopic equations may be derived by taking the moments of the kinetic equation. Consider

$$\partial_t f + v\partial_x f = J_\rho(f,f)$$

Multipying this equation with $v^k, k = 0, 1, \ldots$, and integrating it with respect to v leads to

$$\partial_t m_k + \partial_x m_{k+1} = \int_0^{v_m} v^k J_\rho(f, f) dv$$

with

$$m_k = \int_0^{v_m} v^k f dv$$

For example, with the notation ρ for m_0 and ρV for m_1, we get the usual continuity equation

$$\partial_t \rho + \partial_x(\rho V) = 0$$

The above system of equations is not closed, it forms an infinite hierachy. This hierachy must be closed, expressing m_{k+1} and the integrals over the interaction term by $m_j, j \leq k$. Setting, for example, $V = V_e(\rho)$ we obtain the Lighthill-Whitham equation. Gas dynamics provides us with many possibilities to achieve this closure. For example, one may use the so-called Hilbert or Chapman Enskog procedures. Helbing [37] derived from the Paveri-Fontana equation the first three moment equations, closing the hierachy by the above mentioned strategies.

6 Simulations and Numerical Results

6.1 Simulation Methods for Microscopic Models

Simulations of microscopic follow the leader or cellular automata models have been performed by numerous authors (see for example Wiedemann [4] and Nagel [30], respectively). They obtain a variety of different features of traffic flow, such as the description of jams at bottlenecks, phantom traffic jams or network description.

6.2 Simulation Methods for Macroscopic Models

Simulations have been performed, for example, by Payne [8], Kühne et al. [23], Kerner and Kohnhäuser [24, 32], Papagorgiou [38], Nelson [21] and Helbing [11]. They used fluid dynamic schemes like upwind and Godunov schemes for hyperbolic equations and implicit methods for parabolic equations.

For the following example an implicit integration procedure with centered differences and correct treatment of the nonlinearities by a Newtonian iteration procedure is used, see [23], [24, 32] and [39].

The following results from Sailer [39] show a 3 km long stretch of a 3 lane highway with a bottleneck (a reduction of the lanes from three to two) at the space marks km 1.8 and km 2.2. In the bottleneck a speed limit is introduced. Within the bottleneck a high density regime is formed. The minimum of the speed lies at the first third of the bottleneck. At the outlet of the bottleneck speed increases again. Later on, a traffic jam is developing and moving backwards from the bottleneck. The Figures 5,6 and Figures 7,8 respectively show the development of the speed and density course for different time ranges and different parameter conditions in the bottleneck. In Figures 7,8 additional small undulations occur within the bottleneck - the annoying stop and start waves. These undulations run backwards with the traffic jam.

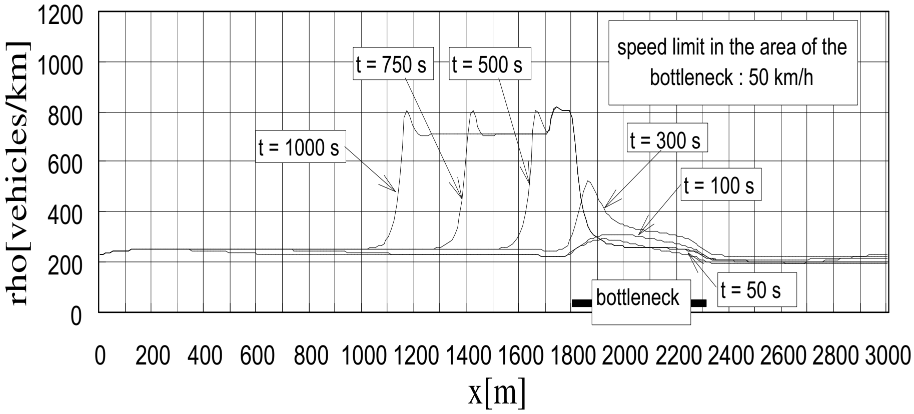

Figure 5: Density

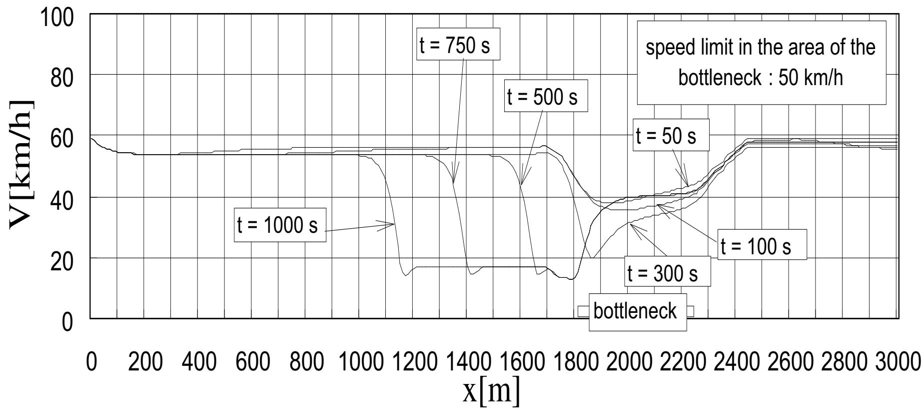

Figure 6: Mean Speed

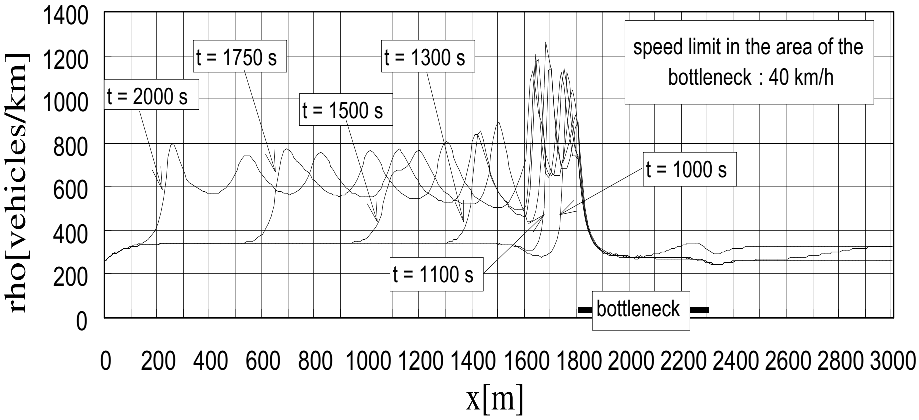

Figure 7: Density, Stop-Start Patterns

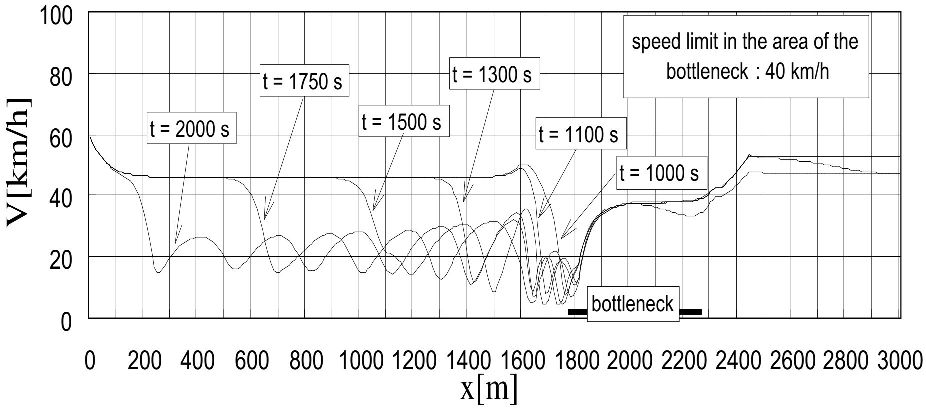

Figure 8: Mean Speed Stop-Start Patterns

6.3 Simulation Methods for Kinetic Models

In 6.3.1 we describe simulations of the homogeneous kinetic equation

$$f_t = J_\rho(f, f)$$

using a model for the interactions as described at the end of 5.1.3.

In 6.3.2 the inhomogeneous equation is simulated on the basis of the improved model from Section 5.2.2.

$$f_t + v f_x = J_{\bar{\rho}}(f, \bar{f})$$

The model for the interactions is the same as before. We refer to [35, 36] for more details.

For the following calculations a discretization scheme is used, that is roughly described as follows:

A simple standard discretization of the equation in speed space needs a large number of discretization points in order to describe correctly the influence of the singularities appearing at $v = 0$ and $v = v_m$. Therefore, the speed space is divided into a certain number of cells and one calculates the transition rates between the cells given by the kinetic equation. This is a discrete speed approximation of the kinetic equation similar to the discrete velocity models in gas dynamics. The scheme needs a much lower number of cells than a standard discretization scheme. Therefore considerably less computing time is necessary to obtain the same results. Moreover, and this is most important, the number of vehicles in the scheme is automatically preserved.

6.3.1 Simulation of Homogeneous Traffic Flow

The stationary distribution f_e of the homogeneous kinetic equation turns out to depend only on the density of the initial distribution.

In Figure 9 a normalized version of the stationary distribution $F_e(\rho, v) = f_e(\rho, v)/\rho$ for different values of ρ is shown. Here, α_0 and β have been chosen equal to 0.3.

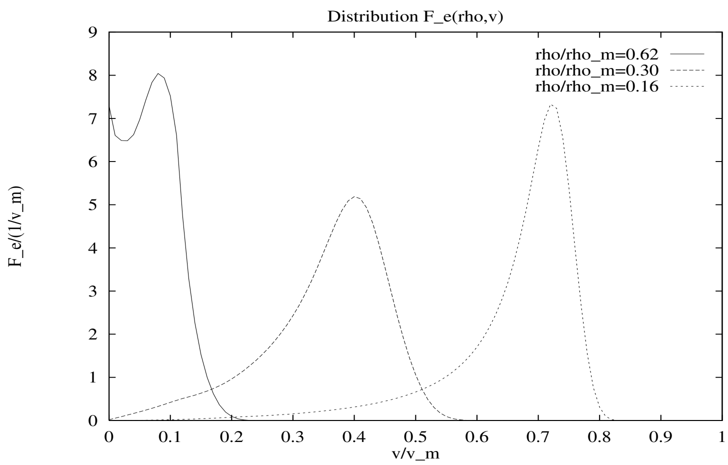

Figure 9: Stationary distributions for $\rho = 0.62,\ 0.3,\ 0.16$

In Figure 10 the corresponding fundamental diagram is shown. We plotted the flux

$$q_e(\rho) = \rho V_e(\rho) = \rho \int_0^{v_m} v F_e(v) dv$$

versus the density for the whole range of densities.

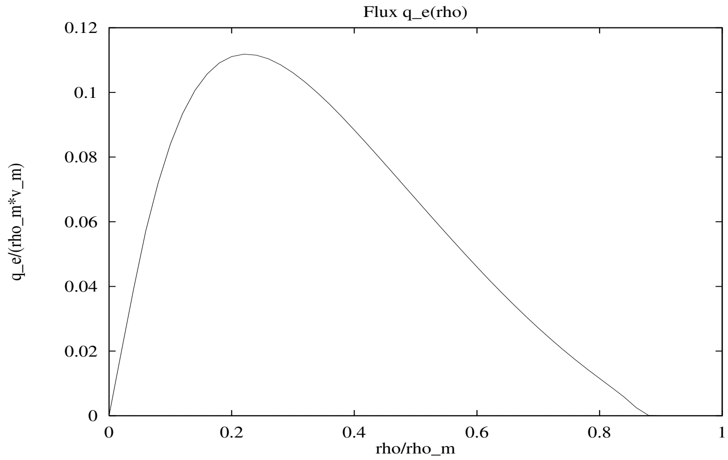

Figure 10: Fundamental diagram (Wegener and Klar)

By adjusting the parameters α_0 and β a whole range of fundamental diagrams may be obtained. The parameters can be chosen such that the fundamental diagrams fit very well to different traffic flow situtations.

6.3.2 Simulation of Inhomogeneous Traffic Flow

Here an example is calculated simulating as in the macroscopic case a highway with three lanes but no speed restriction. In Figures 11 to 14 a highway of 3 km length with a bottleneck between km 1.8 and km 2.2, where the number of lanes is reduced from 3 to 2, is seen. We start with an "empty" highway and incoming vehicles at km 0. The space-speed distribution of the vehicles is shown for different times in Figures 11 to 13. Figure 14 shows the time development of the density of vehicles. When the stretch finally is filled with vehicles, the speed distribution drops at the bottleneck reflecting the formation of a traffic jam. The jam is running backwards as in the macroscopic calculation. In the bottleneck itself the speed distribution recreates showing normal traffic flow with lower density. Due to the parameter configuration used in this example stop and start waves are avoided in these calculations. This corresponds to the macroscopic example shown in Figures 5, 6.

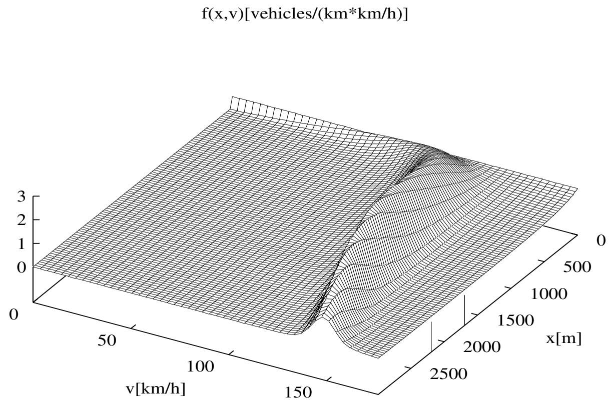

Figure 11: Space-velocity distribution function at $t = 80s$

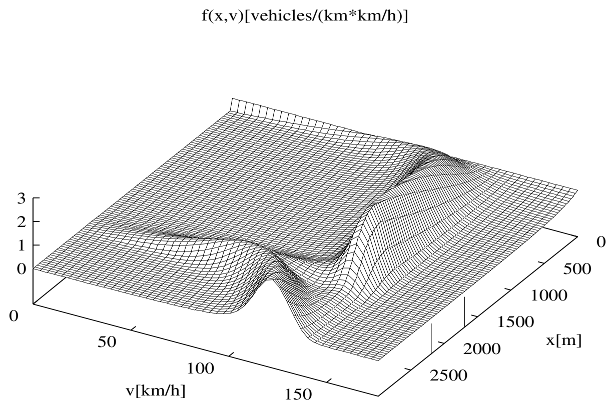

Figure 12: Space-velocity distribution function at $t = 160s$

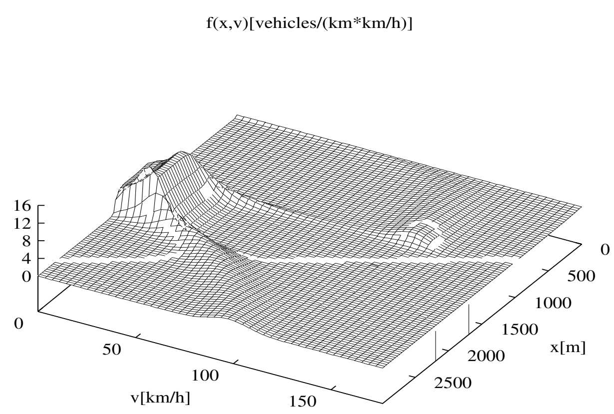

Figure 13: Space-velocity distribution function at $t = 360s$

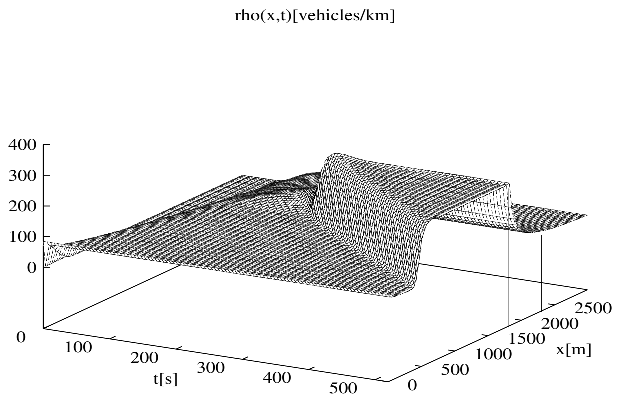

Figure 14: Time development of the density

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