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Stochastic Analysis for Vector-valued Generalized Grey Brownian Motion

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Eigene Veröffentlichungen im Zusammenhang mit dieser Arbeit

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Abstract

This dissertation presents a generalization of the generalized grey Brownian motion with componentwise independence, called a vector-valued generalized grey Brownian motion (vggBm), and builds a framework of mathematical analysis around this process with the aim of solving stochastic differential equations with respect to this process. Similar to that of the one-dimensional case, the construction of vggBm starts with selecting the appropriate nuclear triple, and construct the corresponding probability measure on the co-nuclear space. Since independence of components are essential in constructing vggBm, a natural way to achieve this is to use the nuclear triple of product spaces:

$$\mathcal{S}_d(\mathbb{R}) \subset L^2_d(\mathbb{R}) \subset \mathcal{S}'_d(\mathbb{R}),$$

where $L^2_d(\mathbb{R})$ is the real separable Hilbert space of $\mathbb{R}^d$-valued square integrable functions on $\mathbb{R}$ with respect to the Lebesgue measure, $\mathcal{S}_d(\mathbb{R})$ is the external direct sum of d copies of the nuclear space $\mathcal{S}(\mathbb{R})$ of Schwartz test functions, and $\mathcal{S}'_d(\mathbb{R})$ is the dual space of $\mathcal{S}_d(\mathbb{R})$. The probability measure used is the the d-fold product measure of the Mittag-Leffler measure, denoted by $\mu_\beta^{\otimes d}$, whose characteristic function is given by

$$\int_{\mathcal{S}'_d(\mathbb{R})} e^{i\langle \omega,\varphi\rangle} \, \mathrm{d}\mu_\beta^{\otimes d}(\omega) = \prod_{k=1}^{d} E_\beta\left(-\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right), \qquad \varphi \in \mathcal{S}_d(\mathbb{R}),$$

where $\beta \in (0,1]$, and E_β is the Mittag-Leffler function. Vector-valued generalized grey Brownian motion, denoted by $B_d^{\beta,\alpha} := (B_{d,t}^{\beta,\alpha})_{t\geq 0}$, is then defined as a process taking values in $L^2(\mu_\beta^{\otimes d};\mathbb{R}^d)$ given by

$$B_{d,t}^{\beta,\alpha}(\omega) := (\langle \omega_1, M_-^{\alpha/2}\mathbb{1}_{[0,t)}\rangle, \ldots, \langle \omega_d, M_-^{\alpha/2}\mathbb{1}_{[0,t)}\rangle), \quad \omega \in \mathcal{S}'_d(\mathbb{R}),$$

where $M^{\alpha/2}$ is an appropriate fractional operator indexed by $\alpha \in (0,2)$ and $\mathbb{1}_{[0,t)}$ is the indicator function on the interval $[0,t)$. This process is, in general, not the aforementioned d-dimensional analogues of ggBm for $d \geq 2$, since componentwise independence of the latter process holds only in the Gaussian case.

The study of analysis around vggBm starts with accessibility to Appell systems, so that characterizations and tools for the analysis of the corresponding distribution spaces are established. Then, explicit examples of the use of these characterizations and tools are given: the construction of Donsker's delta function, the existence of local times and self-intersection local times of vggBm, the existence of the derivative of vggBm in the sense of distributions, and the existence of solutions to linear stochastic differential equations with respect to vggBm.

Zusammenfassung

In dieser Dissertation wird eine Verallgemeinerung der verallgemeinerten grauen Brownschen Bewegung mit komponentenweiser Unabhängigkeit vorgestellt, die als vektorwertige verallgemeinerte graue Brownsche Bewegung (vggBm) bezeichnet wird, und es wird ein Rahmen für die mathematische Analysis dieses Prozesses geschaffen, um stochastische Differentialgleichungen in Bezug auf diesen Prozess zu lösen. Ähnlich wie im eindimensionalen Fall beginnt die Konstruktion von vggBm mit der Auswahl des geeigneten nuklearen Tripels und der Konstruktion des entsprechenden Wahrscheinlichkeitsmaßes auf dem conuklearen Raum. Da die Unabhängigkeit der Komponenten bei der Konstruktion von vggBm von wesentlicher Bedeutung ist, besteht ein natürlicher Weg, dies zu erreichen, in der Verwendung des Gel'fand-Tripels von Produkträumen:

$$\mathcal{S}_d(\mathbb{R}) \subset L^2_d(\mathbb{R}) \subset \mathcal{S}'_d(\mathbb{R}),$$

wobei $L^2_d(\mathbb{R})$ der reelle separable Hilbertraum der $\mathbb{R}^d$-wertigen quadratintegrablen Funktionen auf $\mathbb{R}$ bezüglich des Lebesgue Maßes und $\mathcal{S}_d(\mathbb{R})$ die externe direkte Summe von d Kopien des nuklearen Raumes $\mathcal{S}(\mathbb{R})$ der Schwartz Testfunktionen und $\mathcal{S}'_d(\mathbb{R})$ der Dualraum von $\mathcal{S}_d(\mathbb{R})$ ist. Als Wahrscheinlichkeitsmaß wird das d-fache Produktmaß des Mittag-Leffler-Maßes $\mu_\beta^{\otimes d}$. genutzt, Die charakteristische Funktionen ist gegeben durch:

$$\int_{\mathcal{S}'_d(\mathbb{R})} e^{i\langle \omega, \varphi \rangle}\, \mathrm{d}\mu_\beta^{\otimes d}(\omega) = \prod_{k=1}^{d} E_\beta\left(-\frac{1}{2}\langle \varphi_k, \varphi_k \rangle\right), \qquad \varphi \in \mathcal{S}_d(\mathbb{R}),$$

mit $\beta \in (0,1]$, und der Mittag-Leffler-Funktion E_β. Die vektorwertige verallgemeinerte graue Brownsche Bewegung, beschrieben durch $B_d^{\beta,\alpha} := (B_{d,t}^{\beta,\alpha})_{t\geq 0}$, ist definiert als $L^2(\mu_\beta^{\otimes d}; \mathbb{R}^d)$-wertiger Prozess durch

$$B_{d,t}^{\beta,\alpha}(\omega) := (\langle \omega_1, M_-^{\alpha/2} \mathbb{1}_{[0,t)} \rangle, \ldots, \langle \omega_d, M_-^{\alpha/2} \mathbb{1}_{[0,t)} \rangle), \quad \omega \in \mathcal{S}'_d(\mathbb{R}),$$

wobei $M^{\alpha/2}$ eingeeigneter fraktionaler Integral-oder Differentialoperator ist mit $\alpha \in (0,2)$ und $\mathbb{1}_{[0,t)}$ die Indikatorfuktion auf $[0,t)$ beschreibt. Dieser Prozess ist im Allgemeinen nicht das vorher beschriebene d-dimensionale Analog zu ggBm für $d \geq 2$, da dessen komponentenweise Unabhängigkeit lediglich im Gaußschen Fall gilt.

Die Analysis von vggBm beginnt mit der Zugänglichkeit zu Appellsystemen, so dass Charakterisierungen und Werkzeuge für die Analysis der entsprechenden Distributionenräume aufgestellt werden. Dann werden explizite Beispiele für die Verwendung dieser Charakterisierungen und Werkzeuge gegeben: die Konstruktion der Donsker-Deltafunktion, die Existenz von lokalen Zeiten und lokalen Selbsüberschneidungszeiten der vggBm, die Existenz der Ableitung der vggBm im Sinne der Distributionen, und die Existenz von Lösungen zu linearen stochastischen Differentialgleichungen bezüglich der vggBm.

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Contents

Introduction

As a particular case of Gaussian analysis, white noise theory is evolved to a tool to solve various problems in fields such as statistical mechanics, quantum field theory, quantum mechanics and polymer physics as well as in applied mathematics, stochastic analysis, Dirichlet forms, stochastic (partial) differential equations and finance. For a more detailed explanation of white noise analysis and Gaussian analysis, we refer to the monographs [HKPS93, BK95, Kuo96, Oba94, HS17] and the article [KLP+96].

In recent years, fractional Brownian motion and processes related to fractional dynamics become more and more an object of intensive study. The reason for this interest from the mathematical and applied science point of view is twofold: on one hand the processes in general lack both, the Markov and the (semi-)martingale property, which makes them mathematically challenging and not accessible by classic methods from stochastic analysis. On the other hand, due to these properties, it is possible to model processes with long-range and memory effects. The fractional Brownian motion can be represented in a very natural way in white noise theory, see e.g. [Mis08].

There is a well-known connection between PDEs and stochastic processes, provided by the Feynman-Kac formula. By investigating a heat equation, where the time derivative is a Caputo derivative of fractional order, Schneider introduced grey Brownian motion (gBm) in [Sch92]. He showed that a solution to the time-fractional heat equation is given in terms of grey Brownian motion like in the Feynman-Kac case. The link between grey Brownian motion and fractional differential equations is also studied by Mura and Mainardi in the framework of fractional diffusion equations in [MM09]. Grey Brownian motion is constructed on a probability space with a non-Gaussian measure, called Mittag-Leffler measure, whose characteristic function is given by the Mittag-Leffler function. A calculus in this setting is established in, e.g., [GJRdS15] and [GJ16]. For recent results in this framework, see also [BdS17, BGO21]. The mathematical framework, as white noise calculus, generalizes many finite dimensional methods and concepts known such as differential operators and Fourier transform.

Many applications call for processes with long-range dependence and complex correlation structures. As a generalization of Brownian motion, fractional

Brownian motion (fBm) is used to model such dynamics, based on its correlated increments, which imply short and/or long-range dependence [Mis08, BHØZ08, Nou12]. Fractional Brownian motion is neither a semi-martingale nor a Markov process, except for the Brownian motion case. Hence, it is not accessible by standard stochastic calculus, and thus challenging from the mathematical point of view. There are various ways to cast fBm into the classical Brownian motion framework, starting with the famous definition by Mandelbrot and van Ness [MvN68]. This idea is also the starting point for a characterization of fBm using an infinite superposition of Ornstein-Uhlenbeck processes with respect to the standard Wiener process; compare the works of Carmona, Coutin, Montseny, and Muravlev [CCM00, CCM03, Mur11] or also the monograph of [Mis08]. Recently, further applications of this representation have for instance been investigated in [HS19] with a focus on finance and in [BD20] in the context of optimal portfolios.

The Mandelbrot-van Ness representation can be used to represent fBm in the framework of white noise analysis [Mis08, BHØZ08, Nou12, Ben03]. White noise analysis has evolved into an infinite dimensional distribution theory, with rapid developments in mathematical structure and applications in various domains; see, e.g. the monographs [HKPS93, Oba94, Kuo96, HS17]. Various characterization theorems [PS91, KLP⁺96, GKS97, HS17, GMN21] are proven to build up a strong analytical foundation. Almost at the same time, first attempts were made to introduce a non-Gaussian infinite dimensional analysis, by transferring properties of the Gaussian measure with the help of bi-orthogonal generalized Appell systems [Dal91, ADKS96, KSWY98]. In particular, this approach is used to establish the so-called Mittag-Leffler analysis, introduced in [GJRdS15] and [GJ16]. The class of measures used in this analysis generalize from that of white noise analysis in the sense that the characteristic function of the Gaussian measure, the exponential function, is replaced by a Mittag-Leffler function. The corresponding process in this analysis, called generalized grey Brownian motion (ggBm), is in general neither a martingale nor a Markov process. Moreover, it is not possible, as in the Gaussian case, to find a proper orthonormal system of polynomials to describe the test and generalized functions. Hence, it is necessary to make use of the aforementioned Appell system of bi-orthogonal polynomials. The grey noise measure [Sch92, MM09] is included as a special case in this class of measures, which offers the possibility to apply Mittag-Leffler analysis to fractional differential equations, in particular to fractional diffusion equations [Sch90, Sch92], which carry numerous applications in science, like relaxation-type differential equations or viscoelasticity. Fractional processes were motivated by phenomena in heterogeneous media modeled by fractional partial differential equations; see [KE04, Mai10, MS04]. Corresponding stochastic processes governed by these equations have applications in science, engineering and finance [RD03, Mag09, MK00, MK04, Sca06]. Fractional time derivatives are used to model sticking of particles in porous media [SBMB03]. In statistical physics, fractional time derivatives reflects random waiting times between particle jumps [MS04]. Detailed discussion of such processes is also found in [BB03]. For a detailed study of the special class of heavy tailed processes, see [LPSŠ17, LPSŠ20, HL05]. An approach using subordination can be found in [BB03].

With the help of Mittag-Leffler analysis, a relation between the fractional heat equation and the associated process, ggBm, was proven in [GJ16]. In [BdS17], Wick-type stochastic differential equations and Ornstein-Uhlenbeck processes were solved within the framework of Mittag-Leffler analysis. In [BDdS20], the results of [Mur11] and [HS19] for fBm were extended to the non-Gaussian case of ggBm by representing it via generalized grey Ornstein-Uhlenbeck processes. Using that, ggBm can be written as a product of a nonnegative and time-independent random variable and a fBm [MMP10]. A similar representation can be found in [DVS+18].

Recently, a multidimensional analogue of ggBm was defined in [BDdS20] using an extension of the definition of (one-dimensional) ggBm given in [MP08, MM09], and gave also a representation via a generalized grey Ornstein-Uhlenbeck process. On the other hand, another multidimensional analogues of representations of ggBm were also considered in [BB03] in connection with stochastic solutions of some fractional evolution equations. While some of the results of the one-dimensional case can be carried over to the multidimensional case, the processes in both definitions have components which are, in general, not independent. Properties such as independent components in a multidimensional process are desirable in many applications.

In Chapter 1, we give a brief overview into the preliminary concepts needed to follow the theorems and proofs of the thesis. In Chapter 2, we introduce the Mittag-Leffler space and study finite products of Mittag-Leffler spaces and the corresponding distributions. We give particular characterization theorems and show the accessibility to Appell systems. In Chapter 3, the generalization of the generalized grey Brownian motion with componentwise independence, which we call a vector-valued generalized grey Brownian motion (vggBm) is defined. This process is, in general, not the aforementioned d-dimensional analogues of ggBm for $d \geq 2$, since we show that the componentwise independence of the latter process holds only in the Gaussian case. For this process, we characterize Donsker's delta function, local times, self-intersection local times as suitable Mittag-Leffler distributions.

In the last chapter of this dissertation, we focus on stochastic differential equations driven by vggBm. As first examples, we study several stochastic differential equations in higher dimensions. Indeed, we study the abstract Cauchy problem with a vggBm inhomogeneity.

Preliminary Notations

We denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, and $\mathbb{C}$ the set of positive integers, nonnegative integers, real numbers and complex numbers, respectively. For a non-empty set $\mathbb{A}$ and $n \in \mathbb{N}$, we denote by $\mathbb{A}^n$ the set of n-tuples of elements in $\mathbb{A}$. If $x \in \mathbb{A}^n$ and $j = 1, \ldots, n$, the j^{th} component of x is denoted by x_j, so that x can be written as $(x_j)_{j=1}^n := (x_1, \ldots, x_n)$.

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $n \in \mathbb{N}$ be given. The canonical basis of $\mathbb{K}^n$ is the set $\{\mathbf{e}_k := (\delta_{j,k})_{j=1}^n : k = 1, \ldots, n\}$, where $\delta_{j,k}$ is the Kronecker delta. For $x, y \in \mathbb{K}^n$, the Euclidean norm of x and the Euclidean scalar product of x and y, denoted by $|x|$ and (x, y), respectively, are defined as follows:

$$|x| := \left(\sum_{j=1}^n |x_j|^2\right)^{1/2}, \qquad (x, y) := \sum_{j=1}^n x_j \overline{y_j}.$$

In particular, for $x \in \mathbb{K}^1 = \mathbb{K}$, $|x|$ is the modulus of x, which is also the absolute value of x if $x \in \mathbb{R}$. The space $\mathbb{K}^n$ will be equipped with the Euclidean topology, and its corresponding Borel σ-algebra is denoted by $\mathcal{B}(\mathbb{K}^n)$. Any subset of $\mathbb{K}^n$ will be equipped with the subspace topology or trace topology induced by $\mathbb{K}^n$.

For a topological vector space $\mathcal{X}$ over $\mathbb{K}$, we denote by $\mathcal{X}'$ its dual space, that is, the set of all continuous linear maps from X to $\mathbb{K}$. In contrast to the usual literature on functional analysis, we adopt the convention used in, e.g., [KSWY98, HKPS93, Oba94] that the canonical dual pairing between $F \in \mathcal{X}'$ and $x \in \mathcal{X}$ is denoted by $\langle F, x\rangle := F(x)$, unless otherwise specified. If $\mathcal{X}$ is a topological vector space over $\mathbb{R}$, the complexification of $\mathcal{X}$, denoted by $\mathcal{X}_{\mathbb{C}}$, is defined by $\mathcal{X}_{\mathbb{C}} := \{[f_1, f_2] : f_1, f_2 \in \mathcal{X}\}$. The space $\mathcal{X}_{\mathbb{C}}$ is a topological vector space over $\mathbb{C}$ under the following operations:

$$[f_1, f_2] + [g_1, g_2] := [f_1 + g_1, f_2 + g_2] \quad (a + ib)[f_1, f_2] := [af_1 - bf_2, bf_1 + af_2],$$

for $[f_1, f_2], [g_1, g_2] \in \mathcal{X}_{\mathbb{C}}$ and $a, b \in \mathbb{R}$. In view of its similar structure to that of $\mathbb{C}$, the element $[f_1, f_2] \in \mathcal{X}_{\mathbb{C}}$ is denoted by $f_1 + if_2$. Any linear map L between two real linear spaces extends to a linear map between their corresponding complexifications, denoted by the same symbol, in a natural way:

$$L(f_1 + if_2) := L(f_1) + iL(f_2).$$

In the case of the complexification of the dual space $\mathcal{X}'$, denoted by $\mathcal{X}'_{\mathbb{C}}$, we preserve the same symbol $\langle \cdot, \cdot \rangle$ for the canonical dual pairing, and this pairing is defined as an extension of its real counterpart in a bilinear way:

$$\langle F_1 + iF_2, f_1 + if_2\rangle := \langle F_1, f_1\rangle - \langle F_2, f_2\rangle + i(\langle F_1, f_2\rangle + \langle F_2, f_1\rangle).$$

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. Unless otherwise specified, we denote by $(\cdot,\cdot)_{\mathcal{H}}$ and $|\cdot|_{\mathcal{H}}$ its scalar product and induced norm, respectively. If $\mathcal{H}$ is a Hilbert space over $\mathbb{C}$, the scalar product is always assumed to be linear in the first component and anti-linear in the second component. If $\mathcal{H}$ is a Hilbert space over $\mathbb{R}$, then the complexification $\mathcal{H}_{\mathbb{C}}$ is a Hilbert space over $\mathbb{C}$ under the following scalar product and induced norm:

$$(f_1 + if_2, g_1 + ig_2)_{\mathcal{H}_{\mathbb{C}}} := (f_1, g_1)_{\mathcal{H}} + (f_2, g_2)_{\mathcal{H}} + i\big((f_2, g_1)_{\mathcal{H}} - (f_1, g_2)_{\mathcal{H}}\big),$$
$$|f_1 + if_2|_{\mathcal{H}_{\mathbb{C}}} := (f_1 + if_2, f_1 + if_2)^{1/2}_{\mathcal{H}_{\mathbb{C}}}.$$

The above-mentioned Hilbert spaces will be equipped with the topology generated by its induced metric.

Chapter 1

Preliminary Concepts

This chapter presents the necessary background of this study. It consists of some facts about nuclear triples, holomorphic functions on locally convex spaces, some known integrals on vector spaces, and distributions in Gaussian and non-Gaussian analysis.

1.1 Nuclear triples

Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{R}$ and $\mathcal{N}$ be a nuclear Frechét space topologically and densely embedded in $\mathcal{H}$, that is, there is a continuous embedding $\mathcal{N} \hookrightarrow \mathcal{H}$ whose range is dense in $\mathcal{H}$. By a nuclear Frechét space, instead of the usual abstract definition (see, e.g., [Trè67, Sch99]), we use the following characterization (cf. [KSWY98, Theorem 2.1]) as a projective limit of a countable number of Hilbert spaces.

Theorem 1.1. *The nuclear Frechét space $\mathcal{N}$ can be represented as*

$$\mathcal{N} = \operatorname*{prlim}_{p\in\mathbb{N}} \mathcal{H}_p \tag{1.1}$$

where $(\mathcal{H}_p)_{p\in\mathbb{N}}$ is a family of Hilbert spaces such that for all $p_1, p_2 \in \mathbb{N}$, there exists $p \in \mathbb{N}$ such that the embeddings $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_1}$ and $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_2}$ are of Hilbert-Schmidt class. The expression (1.1) *means that as a set,*

$$\mathcal{N} = \bigcap_{p\in\mathbb{N}} \mathcal{H}_p$$

and the topology of $\mathcal{N}$ is the projective limit topology, that is, the coarsest topology such that the canonical embeddings $\mathcal{N} \hookrightarrow \mathcal{H}_p$ are continuous for all $p \in \mathbb{N}$.

Together with its (topological) dual space $\mathcal{N}'$, we obtain the following inclusions, called a *nuclear (Gel'fand) triple*:

$$\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}',$$

where we identify each $f \in \mathcal{H}$ as an element of $\mathcal{N}'$ via the Riesz isomorphism

$$\mathcal{H} \ni f \leftrightarrow (f, \cdot)_{\mathcal{H}} \in \mathcal{H}'.$$

The dual pairing between $\mathcal{N}'$ and $\mathcal{N}$ is then an extension of the scalar product on $\mathcal{H}$ given by

$$\langle f, \varphi \rangle = (f, \varphi)_{\mathcal{H}}, \quad f \in \mathcal{H}, \ \varphi \in \mathcal{N}.$$

Denote by $(\cdot, \cdot)_p$ and $|\cdot|_p$ the scalar product and induced norm of $\mathcal{H}_p$, respectively, and set $\mathcal{H}_0 := \mathcal{H}$ so that we can use the notations $(\cdot, \cdot)_0$ and $|\cdot|_0$ for the corresponding scalar product and induced norm of $\mathcal{H}$. Without loss of generality, we assume that the system of norms $(|\cdot|_p)_{p \in \mathbb{N}}$ is *ordered*, that is, whenever $p, q \in \mathbb{N}_0$ with $p < q$, we have $|\cdot|_p \le |\cdot|_q$ on $\mathcal{H}_q$. Let $\mathcal{H}_{-p} := \mathcal{H}_p'$ be the dual space of $\mathcal{H}_p$ with corresponding norm denoted by $|\cdot|_{-p}$. By general duality theory (see, e.g., [GV64]), as a set,

$$\mathcal{N}' = \bigcup_{p \in \mathbb{N}} \mathcal{H}_{-p}.$$

We equip the dual space $\mathcal{N}'$ with the *inductive limit topology* generated by the spaces $\mathcal{H}_{-p}$, $p \in \mathbb{N}$, written as

$$\mathcal{N}' = \operatorname*{indlim}_{p \in \mathbb{N}} \mathcal{H}_{-p}.$$

That is, $\mathcal{N}'$ is equipped with the finest locally convex topology such that $\mathcal{H}_{-p}$ is continuously embedded in $\mathcal{N}'$ for all $p \in \mathbb{N}$. Thus, for $p, q \in \mathbb{N}_0$ with $p < q$, we have the following chain of continuous and dense embeddings:

$$\mathcal{N} \subset \mathcal{H}_q \subset \mathcal{H}_p \subset \mathcal{H} \subset \mathcal{H}_{-p} \subset \mathcal{H}_{-q} \subset \mathcal{N}'.$$

We would also like to have a notion of tensor powers of a nuclear space. To do this, we start with the usual Hilbert tensor powers of $\mathcal{H}_p$. For $p, n \in \mathbb{N}_0$, we define the n^{th} tensor product $\mathcal{H}_p^{\otimes n}$ as follows: for $n = 0$, set $\mathcal{H}_p^{\otimes 0} := \mathbb{R}$, and for $n \in \mathbb{N}$, $\mathcal{H}_p^{\otimes n}$ is the completion of the linear span of expressions of the form $f_1 \otimes \cdots \otimes f_n$, $f_j \in \mathcal{H}_p$, under the scalar product $(\cdot, \cdot)_p$ defined by

$$(f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_n)_p := \prod_{j=1}^{n} (f_j, g_j)_p, \quad f_j, g_j \in \mathcal{H}_p.$$

For convenience, we write the n-fold tensor product $f \otimes \cdots \otimes f$, $n \in \mathbb{N}$, by $f^{\otimes n}$, and $f^{\otimes 0} := 1$. We also consider the subspace of symmetric elements of $\mathcal{H}_p^{\otimes n}$, denoted by $\mathcal{H}_p^{\widehat{\otimes} n}$, defined as follows. If $n = 0$, set $\mathcal{H}_p^{\widehat{\otimes} 0} := \mathbb{R}$. If $n \in \mathbb{N}$, then the map sym_n defined by

$$\mathrm{sym}_n(f_1 \otimes \cdots \otimes f_n) := \frac{1}{n!} \sum_{\sigma \in S_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}, \quad f_j \in \mathcal{H}_p,$$

where S_n is the set of permutations on $\{1, \ldots, n\}$, extends to a continuous linear map $\widetilde{\mathrm{sym}_n} : \mathcal{H}_p^{\otimes n} \to \mathcal{H}_p^{\otimes n}$. We define $\mathcal{H}_p^{\widehat{\otimes} n} := \widetilde{\mathrm{sym}_n}(\mathcal{H}_p^{\otimes n})$. Similar notations are used for $\mathcal{H}_{-p}$. In this case, if $n \in \mathbb{N}$, $F_1, \ldots, F_n \in \mathcal{H}_{-p}$ and $f_1, \ldots, f_n \in \mathcal{H}_p$, then

$$\langle F_1 \otimes \cdots \otimes F_n, f_1 \otimes \cdots \otimes f_n \rangle = \langle F_1, f_1 \rangle \cdots \langle F_n, f_n \rangle.$$

The norms on $\mathcal{H}_p^{\otimes n}$ and $\mathcal{H}_{-p}^{\otimes n}$ are also denoted by $|\cdot|_p$ and $|\cdot|_{-p}$, respectively. Then $\mathcal{H}_{-p}^{\otimes n}$ is the dual space of $\mathcal{H}_p^{\otimes n}$ with respect to $\mathcal{H}^{\otimes n}$. The tensor powers $\mathcal{N}^{\otimes n}$ of $\mathcal{N}$ are defined as the projective limit of the spaces $(\mathcal{H}_p^{\otimes n})_{p \in \mathbb{N}}$, so that $(\mathcal{N}^{\otimes n})'$ is the inductive limit of $(\mathcal{H}_{-p}^{\otimes n})_{p \in \mathbb{N}}$. The symmetric tensor powers $\mathcal{N}^{\widehat{\otimes} n}$ are also defined similarly. In this case, we also obtain the nuclear triples

$$\mathcal{N}^{\otimes n} \subset \mathcal{H}^{\otimes n} \subset (\mathcal{N}^{\otimes n})' \qquad \mathcal{N}^{\widehat{\otimes} n} \subset \mathcal{H}^{\widehat{\otimes} n} \subset (\mathcal{N}^{\widehat{\otimes} n})'.$$

Example 1.2. An example of a nuclear triple used in white noise analysis is the following: let $L^2(\mathbb{R}) := L^2(\mathbb{R}, \mathrm{d}x)$ be the real Hilbert space of square-integrable functions on $\mathbb{R}$ with respect to the Lebesgue measure, and $\mathcal{S}(\mathbb{R})$ be the Schwartz space of test functions on $\mathbb{R}$, that is, the space of all infinitely-differentiable functions $\phi : \mathbb{R} \to \mathbb{R}$ such that for all $k, m \in \mathbb{N}_0$,

$$\sup_{x \in \mathbb{R}} \left| x^k \frac{\mathrm{d}^m}{\mathrm{d}x^m} \phi(x) \right| < \infty.$$

It is known that $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. Define the linear operator A and the norm $|\cdot|_p$, $p \in \mathbb{N}_0$, on $\mathcal{S}(\mathbb{R})$ as follows:

$$(A\phi)(x) := \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 + 1 \right) \phi(x), \quad |\phi|_p := |A^p \phi|_{L^2(\mathbb{R})}, \quad \phi \in \mathcal{S}(\mathbb{R}),\ x \in \mathbb{R}.$$

Let $\mathcal{H}_p$ be the completion of $\mathcal{S}(\mathbb{R})$ (or more precisely, the equivalence class of $\mathcal{S}(\mathbb{R})$ in $L^2(\mathbb{R})$) with respect to $|\cdot|_p$. Then $\mathcal{S}(\mathbb{R})$ is the projective limit of $(\mathcal{H}_p)_{p \in \mathbb{N}}$ satisfying the conditions of Theorem 1.1, so that $\mathcal{S}(\mathbb{R})$ is a nuclear Frechét space. Hence, we obtain the following nuclear triple, called the *standard nuclear triple*:

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}),$$

where we write $\mathcal{S}'(\mathbb{R})$ for the dual space of $\mathcal{S}(\mathbb{R})$. For more details on the proof of this nuclear triple, see, e.g., [RS72] or [HKPS93, Appendix A.5].

Remark 1.3. We list several properties of a nuclear Frechét space $\mathcal{N}$ that are used in this dissertation.

(i) The projective limit topology on $\mathcal{N}$ coincides with the topology on $\mathcal{N}$ generated by the seminorms $(|\cdot|_p)_{p\in\mathbb{N}}$. Hence, an open neighborhood base of $\mathcal{N}$ at 0 is given by

$$U_{p,\varepsilon} := \{\varphi \in \mathcal{N} : |\phi|_p < \varepsilon\}, \quad p \in \mathbb{N},\ \varepsilon > 0.$$

In this case, a subset S of $\mathcal{N}$ is bounded if and only if $\sup_{\varphi\in S} |\varphi|_p < \infty$ for all $p \in \mathbb{N}$. Moreover, a linear map $f : \mathcal{N} \to \mathbb{R}$ is continuous if and only if there exist $p \in \mathbb{N}$ and $C \in (0,\infty)$ such that $|f(\varphi)| \leq C|\varphi|_p$ for all $\varphi \in \mathcal{N}$.

(ii) In the usual literature, the dual space $\mathcal{N}'$ is equipped with either one of the following topologies: the *strong topology*, that is, the topology generated by the seminorms

$$|\Phi|_S := \sup_{\varphi\in S} |\langle \Phi, \varphi\rangle|, \quad \Phi \in \mathcal{N}' \tag{1.2}$$

where S runs over all bounded subsets of $\mathcal{N}$; and the *weak topology*, that is, the topology generated by the seminorms (1.2) where S runs over all finite subsets of $\mathcal{N}$. It turns out that $\mathcal{N}$ is reflexive, that is, if we equip the dual space $\mathcal{N}'' := (\mathcal{N}')'$ of $\mathcal{N}'$ with the strong topology, then $\mathcal{N}''$ is isomorphic to $\mathcal{N}$. As a consequence, the strong topology coincides with the inductive limit topology on $\mathcal{N}'$ (see, e.g, [HKPS93, Appendix A.5]).

(iii) Another property of $\mathcal{N}$ is that it is a *perfect space*, that is, every closed, bounded subset of $\mathcal{N}$ is compact. As a consequence, strong and weak convergence of sequences in $\mathcal{N}'$ (and $\mathcal{N}$) coincide (see [GV64, page 73]).

(iv) One of the important results for nuclear Frechét spaces is the kernel theorem (see, e.g., [GV64]): if $F : \mathcal{N}^n \to \mathbb{R}$ is n-linear and $\mathcal{H}_p$-continuous for some $p \in \mathbb{N}$, that is, there is a constant $C \in (0,\infty)$ such that

$$|F(\varphi_1, \dots, \varphi_n)| \leq C \prod_{j=1}^{n} |\varphi_j|_p \quad \text{for all } \varphi \in \mathcal{N}^n,$$

then there exists a unique $\Phi^{(n)} \in (\mathcal{N}^{\otimes n})'$ such that

$$F(\varphi_1, \dots, \varphi_n) = \langle \Phi^{(n)}, \varphi_1 \otimes \cdots \otimes \varphi_n\rangle \quad \text{for all } \varphi \in \mathcal{N}^n.$$

(v) While Theorem 1.1 is stated for nuclear Frechét spaces over $\mathbb{R}$, the result is also applicable for nuclear Frechét spaces over $\mathbb{C}$. In particular, if $\mathcal{N}$ is a nuclear Frechét space over $\mathbb{R}$, then $\mathcal{N}_{\mathbb{C}}$ is a nuclear Frechét space over $\mathbb{C}$. Thus, all properties discussed earlier for $\mathcal{N}$ can be carried over to $\mathcal{N}_{\mathbb{C}}$, with few adjustments to the notations (see, e.g., [BK95]).

1.1.1 Nuclear triples on finite direct sums of spaces

Let $d \in \mathbb{N}$. Suppose that for each $k = 1, \ldots, d$, we have a nuclear triple $\mathcal{N}_k \subset \mathcal{H}_k \subset \mathcal{N}_k'$, and we denote by ξ_k the continuous dense embedding $\mathcal{N}_k \hookrightarrow \mathcal{H}_k$. Consider the (external) Hilbert direct sum $\bigoplus_{k=1}^d \mathcal{H}_k$, and equip the direct sum $\bigoplus_{k=1}^d \mathcal{N}_k$ with the locally convex direct sum topology, that is, the finest locally convex topology such that for each $j = 1, \ldots, d$, the canonical injection

$$\mathcal{N}_j \ni \phi \mapsto \phi \mathbf{e}_j := (0, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}}\text{position}}}{\phi}, \ldots, 0) \in \bigoplus_{k=1}^d \mathcal{N}_k$$

is continuous. Then $\bigoplus_{k=1}^d \mathcal{N}_k$ is a nuclear space (see, e.g., [Trè67, Prop. 50.1]). Moreover, the map

$$\bigoplus_{k=1}^d \mathcal{N}_k \ni \varphi := (\varphi_1, \ldots, \varphi_d) \mapsto (\xi_1(\varphi_1), \ldots, \xi_d(\varphi_d)) \in \bigoplus_{k=1}^d \mathcal{H}_k$$

is a topological embedding whose range is dense in $\bigoplus_{k=1}^d \mathcal{H}_k$. Now, suppose that $\mathcal{N}_k$ is the projective limit of Hilbert spaces $(\mathcal{H}_{k,p})_{p\in\mathbb{N}}$ satisfying Theorem 1.1, and we use the induced norm $|\cdot|_p$ on the Hilbert direct sum $\bigoplus_{k=1}^d \mathcal{H}_{k,p}$ given by

$$|\varphi|_p^2 := \sum_{k=1}^d |\varphi_k|_{k,p}^2, \quad \varphi := (\varphi_1, \ldots, \varphi_d) \in \bigoplus_{k=1}^d \mathcal{H}_{k,p}, \tag{1.3}$$

where the norm $|\cdot|_{k,p}$ on the right-hand side of (1.3) is the norm on $\mathcal{H}_{k,p}$. For notational convenience, we identify the norm $|\cdot|_{k,0}$ with the norm on $\mathcal{H}_k$, so that $|\cdot|_0$ is the induced norm on $\bigoplus_{k=1}^d \mathcal{H}_k$. Let $p_1, p_2 \in \mathbb{N}$. Since the composition of a Hilbert-Schmidt operator followed by a bounded linear operator is Hilbert-Schmidt, we can assume without loss of generality that there is $q \in \mathbb{N}$ and an orthonormal basis $(e_k^{(n)})_{n=1}^\infty$ in $\mathcal{H}_{k,q}$ such that for all $j = 1, 2$ and $k = 1, \ldots, p$,

$$\sum_{n=1}^\infty \left|i_{p_j,q}^{(k)}(e_k^{(n)})\right|_{k,p_j}^2 < \infty,$$

where $i_{p_j,q}^{(k)}$ is the embedding map $\mathcal{H}_{k,q} \hookrightarrow \mathcal{H}_{k,p_j}$. Then there is an embedding

$$\bigoplus_{k=1}^d \mathcal{H}_{k,q} \ni f \mapsto i_{p_j,q}(f) := \left(i_{p_j,q}^{(k)}(f_k)\right)_{k=1}^d \in \bigoplus_{k=1}^d \mathcal{H}_{k,p_j}.$$

Moreover, the set $\{f_{k,n} := e_k^{(n)} \mathbf{e}_k : k = 1, \ldots, d, n \in \mathbb{N}\}$ is an orthonormal basis for $\bigoplus_{k=1}^d \mathcal{H}_{k,q}$ with

$$\sum_{n=1}^\infty \sum_{k=1}^d \left|i_{p_j,q}(f_{k,n})\right|_{p_j}^2 = \sum_{k=1}^d \sum_{n=1}^\infty \left|i_{p_j,q}^{(k)}(e_k^{(n)})\right|_{k,p_j}^2 < \infty.$$

In this case, $\bigoplus_{k=1}^d \mathcal{N}_k$ is the projective limit of the spaces $\bigoplus_{k=1}^d \mathcal{H}_{k,p}$, $p \in \mathbb{N}$, satisfying (N1) and (N2), and Theorem 1.1 ensures that $\bigoplus_{k=1}^d \mathcal{N}_k$ is a nuclear Frechét space. Together with the dual space $\left(\bigoplus_{k=1}^d \mathcal{N}_k\right)'$, we obtain this nuclear triple of direct sums:

$$\bigoplus_{k=1}^d \mathcal{N}_k \subset \bigoplus_{k=1}^d \mathcal{H}_k \subset \left(\bigoplus_{k=1}^d \mathcal{N}_k\right)'. \tag{1.4}$$

Recalling that inductive limit topology on $\left(\bigoplus_{k=1}^d \mathcal{N}_k\right)'$ coincides with the strong topology, this dual space is topologically isomorphic to the product space $\prod_{k=1}^d \mathcal{N}_k'$ via the canonical identification

$$\left(\bigoplus_{k=1}^d \mathcal{N}_k\right)' \ni \omega \leftrightarrow (\omega_1, \dots, \omega_d) \in \prod_{k=1}^d \mathcal{N}_k', \quad \omega_k := \omega \circ \iota_k.$$

(see, e.g., [Bou87, Proposition 14 (IV, p. 12)], or [KN76, 18.10]). Hence, if $\omega \in \left(\bigoplus_{k=1}^d \mathcal{N}_k\right)'$ and $\varphi \in \bigoplus_{k=1}^d \mathcal{N}_k$, then (see, e.g., [KN76, 14.7])

$$\langle \omega, \varphi \rangle = \sum_{k=1}^d \langle \omega_k, \varphi_k \rangle,$$

where the dual pairing on the right-hand side is that of between $\mathcal{N}_k'$ and $\mathcal{N}_k$.

Example 1.4. Consider the standard nuclear triple $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ from Example 1.2. Then the following "d-fold sum" is also a nuclear triple:

$$\bigoplus_{k=1}^d \mathcal{S}(\mathbb{R}) \subset \bigoplus_{k=1}^d L^2(\mathbb{R}) \subset \left(\bigoplus_{k=1}^d \mathcal{S}'(\mathbb{R})\right)'.$$

For convenience, we will denote this triple by $\mathcal{S}_d(\mathbb{R}) \subset L^2_d(\mathbb{R}) \subset \mathcal{S}_d'(\mathbb{R})$.

Remark 1.5. There is an equivalent way to describe the nuclear triple (1.4) if the "summands" are all equal to a fixed nuclear triple $\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'$. Define $\mathcal{N} \otimes \mathbb{R}^d$ to be the projective limit of the Hilbert tensor products $\mathcal{H}_p \otimes \mathbb{R}^d$, $p \in \mathbb{N}$. In a similar argument as that of the tensor powers of $\mathcal{N}$, we can construct a nuclear triple

$$\mathcal{N}_p \otimes \mathbb{R}^d \subset \mathcal{H}_p \otimes \mathbb{R}^d \subset (\mathcal{N}_p \otimes \mathbb{R}^d)'.$$

Note that for each $p \in \mathbb{N}$, the map $f \otimes x \mapsto (x_k f)_{k=1}^d$ for $f \in \mathcal{H}_p$ and $x \in \mathbb{R}^d$ extends to an isomorphism between $\mathcal{H}_p \otimes \mathbb{R}^d$ and $\bigoplus_{k=1}^d \mathcal{H}_p$. Therefore, $\mathcal{N} \otimes \mathbb{R}^d$ and $\bigoplus_{k=1}^d \mathcal{N}$ are topologically isomorphic.

1.2 Holomorphy on locally convex spaces

We present some concepts from the theory of holomorphy on locally convex spaces over $\mathbb{C}$ as a prerequisite for the discussion of distribution spaces in non-Gaussian analysis in Subsection 2.1. For a more detailed discussion of the concepts in this section, we refer to [Din81].

Let $\mathcal{E}$ be a locally convex space over $\mathbb{C}$. For $n \in \mathbb{N}$, denote by $\mathcal{L}(\mathcal{E}^n)$ the space of n-linear maps from $\mathcal{E}^n$ into $\mathbb{C}$, and $\mathcal{L}_s(\mathcal{E}^n)$ the subspace of $\mathcal{L}(\mathcal{E}^n)$ consisting of symmetric n-linear maps, that is, n-linear maps $L : \mathcal{E}^n \to \mathbb{C}$ such that

$$L(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = L(x), \qquad \text{for all } x \in \mathcal{E}^n, \sigma \in S_n.$$

A map $P : \mathcal{E} \to \mathbb{C}$ is said to be a *n-homogeneous polynomial* on $\mathcal{E}$ if $P = L \circ \Delta_n$, where $L \in \mathcal{L}_s(\mathcal{E}^n)$ and Δ_n is the map from $\mathcal{E}$ into $\mathcal{E}^n$ defined by

$$\Delta_n(v) := \underbrace{(v, \dots, v)}_{n \text{ times}} \quad v \in \mathcal{E}.$$

Let $P^n(\mathcal{E})$ be the space of n-homogeneous polynomials on $\mathcal{E}$. The polarization formula

$$L(x) = \frac{1}{2^n n!} \sum_{\epsilon \in \{-1,1\}^n} \epsilon_1 \cdots \epsilon_n (L \circ \Delta_n)(\epsilon_1 x_1 + \cdots + \epsilon_n x_n), \quad L \in \mathcal{L}_s(\mathcal{E}^n),\ x \in \mathcal{E}^n, \tag{1.5}$$

yields a bijection

$$\mathcal{L}_s(\mathcal{E}^n) \ni L \leftrightarrow \widehat{L} := L \circ \Delta_n \in P^n(\mathcal{E}).$$

For $n = 0$, we set $\mathcal{L}(\mathcal{E}^0)$ to be the space of constant maps from $\mathcal{E}$ into $\mathbb{C}$, which can be identified with $\mathbb{C}$ itself in a natural way. We also set $\mathcal{L}_s(\mathcal{E}^0) := P^0(\mathcal{E}) := \mathcal{L}(\mathcal{E}^0)$.

Definition 1.6. Let $\mathcal{U} \subset \mathcal{E}$ be open. A function $F : \mathcal{U} \to \mathbb{C}$ is said to be *G-holomorphic* (*Gâteaux-holomorphic*) if for all $\eta_0 \in \mathcal{U}$ and $\eta \in \mathcal{E}$, the map $\mathbb{C} \ni \lambda \mapsto F(\eta_0 + \lambda\eta) \in \mathbb{C}$ is holomorphic in some neighborhood of zero in $\mathbb{C}$.

In a similar way as that of holomorphic functions in complex analysis, a G-holomorphic function $F : \mathcal{U} \to \mathbb{C}$ can be expressed in terms of its "Taylor series" expansion: for every $\eta \in \mathcal{U}$, there exists a unique sequence $\left(\frac{1}{n!}\widehat{\mathrm{d}^n F}(\eta)\right)_{n=0}^{\infty}$, $\widehat{\mathrm{d}^n F}(\eta) \in P^n(\mathcal{E})$, such that

$$F(\eta + \theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{\mathrm{d}^n F}(\eta)(\theta),$$

for all θ belonging to some open set $\mathcal{V} \subset \mathcal{U}$ (see [Din81, Proposition 2.4]).

Definition 1.7. Let $\mathcal{U} \subset \mathcal{E}$ be open. A function $F : \mathcal{U} \to \mathbb{C}$ is said to be *holomorphic* (on $\mathcal{U}$) if it is G-holomorphic and for all $\eta \in \mathcal{U}$, there exists an open set $\mathcal{V} \subset \mathcal{U}$ such that the map

$$\mathcal{V} \ni \theta \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{\mathrm{d}^n F}(\eta)(\theta)$$

converges and defines a continuous function on $\mathcal{V}$. The function F is said to be *holomorphic at* $\eta_0 \in \mathcal{U}$ if there is an open neighborhood $\mathcal{U}_0$ of η_0 contained in $\mathcal{U}$ such that F is holomorphic on $\mathcal{U}_0$.

A necessary and sufficient condition for a G-holomorphic function to be holomorphic is given by [Din81, Lemma 2.8] as follows.

Proposition 1.8. *Let $\mathcal{U} \subset \mathcal{E}$ be open and $F : \mathcal{U} \to \mathbb{C}$ be G-holomorphic. Then F is holomorphic if and only if F is locally bounded, that is, for every $\eta \in \mathcal{U}$, F is bounded on some neighborhood of η contained in $\mathcal{U}$.*

Remark 1.9. In the discussion of distribution spaces in Section 1.4, we are only interested in functions $F : \mathcal{N}_{\mathbb{C}} \to \mathbb{C}$, $\mathcal{N}$ a nuclear Frechét space, that are holomorphic at $0 \in \mathcal{N}_{\mathbb{C}}$. In this case, by Proposition 1.8 and the topological properties of $\mathcal{N}_{\mathbb{C}}$ (see Remark 1.3), F is holomorphic at 0 if and only if there exists $p \in \mathbb{N}$ and $\varepsilon > 0$ such that

(i) *(local boundedness)* there exists $C \in (0, \infty)$ such that $|F(\varphi)| \leq C$ for all $\varphi \in \mathcal{N}_{\mathbb{C}}$ with $|\varphi|_p \leq \varepsilon$;

(ii) *(G-holomorphy)* for all $\varphi^0, \varphi \in \mathcal{N}_{\mathbb{C}}$ such that $|\varphi^0|_p \leq \varepsilon$, the map $\mathbb{C} \ni \lambda \mapsto F(\varphi^0 + \lambda\varphi) \in \mathbb{C}$ is holomorphic at $0 \in \mathbb{C}$.

1.3 Bochner Integral

This section provides an overview of the integral for Banach space-valued functions that are used in this dissertation, the Bochner integral. For more details on these integrals, we refer to [Kuo96].

Let $(T, \mathcal{B}, \nu)$ be a measure space and X be a Banach space over $\mathbb{C}$. Equip X with the Borel σ-algebra generated by its induced metric.

Definition 1.10. A function $f : T \to X$ is said to be

(i) *countably-valued* if there exist a sequence $(x_k)_{k\in\mathbb{N}}$ in X and a sequence $(E_k)_{k\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{B}$ such that on T,

$$f = \sum_{k=1}^{\infty} \mathbb{1}_{E_k}(\cdot)x_k \tag{1.6}$$

(ii) *weakly measurable* if the map $x \mapsto \langle x', f(x)\rangle$ is measurable for each $x' \in X'$;

(iii) *almost separately-valued* if there exists $E_0 \in \mathcal{B}$ such that $\nu(E_0) = 0$ and $f(X \setminus E_0)$ is separable.

Definition 1.11. A countably-valued function $f : T \to X$ of the form (1.6) is said to be *Bochner integrable (with respect to the measure ν)* if

$$\sum_{k=1}^{\infty} \nu(E_k)|x_k|_X < \infty.$$

In this case, for any $E \in \mathcal{B}$, we define the *Bochner integral* of a countably-valued function f on E by

$$\int_E f(t)\, \mathrm{d}\nu(t) := \sum_{k=1}^{\infty} \nu(E \cap E_k) x_k.$$

In the general case, a function $F : T \to X$ is *Bochner integrable* if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of countably-valued Bochner integrable functions such that $f_n \to F$ as $n \to \infty$ ν-almost everywhere and

$$\lim_{n\to\infty} \int_T |f_n(t) - F(t)|_X \,\mathrm{d}\nu(t) = 0.$$

In this case, the *Bochner integral* of F on E is given by

$$\int_E F(t)\,\mathrm{d}\nu(t) := \lim_{n\to\infty} \int_E f_n(t)\,\mathrm{d}\nu(t).$$

Proposition 1.12. *A function $F : T \to X$ is Bochner integrable if and only if F is weakly measurable, almost separately-valued, and $\int_T |F(t)|_X \,\mathrm{d}\nu(t) < \infty$. If X is separable, then F is Bochner integrable if and only if it is weakly measurable and $|F(\cdot)|_X \in L^1(T, \nu)$.*

1.4 Distributions in infinite-dimensional analysis

Given a nuclear triple $\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'$, we equip the space $\mathcal{N}'$ with the so-called *cylinder σ-algebra $C_\sigma(\mathcal{N}')$*, defined as the σ-algebra generated by the sets

$$\{\omega \in \mathcal{N}' : (\langle \omega, \varphi_1\rangle, \dots, \langle \omega, \varphi_n\rangle) \in A\},$$

over all $n \in \mathbb{N}$, $\varphi_1, \dots, \varphi_n \in \mathcal{N}$ and $A \in \mathcal{B}(\mathbb{R}^n)$. An equivalent way of describing the cylinder σ-algebra is that it is the smallest σ-algebra $\mathcal{M}$ on $\mathcal{N}'$ such that all *cylinder maps*, that is, maps of the form

$$\mathcal{N}' \ni \omega \mapsto C_{\varphi_1,\dots,\varphi_n}(\omega) := (\langle \omega, \varphi_1 \rangle, \dots, \langle \omega, \varphi_n \rangle) \in \mathbb{R}^n, \quad n \in \mathbb{N},\ \varphi_1, \dots, \varphi_n \in \mathcal{N},$$

are $\mathcal{M}$-$\mathcal{B}(\mathbb{R}^n)$ measurable. Since $\mathcal{N}$ is a projective limit of a countable number of Hilbert spaces, $C_\sigma(\mathcal{N}')$ coincides with the Borel σ-algebra generated by the strong topology on $\mathcal{N}'$. We refer to, e.g., [BK95], for the proof of this property of the cylinder σ-algebra.

In a similar manner as that of finite-dimensional spaces, we can define a probability measure on the space $(\mathcal{N}', C_\sigma(\mathcal{N}'))$ in terms of its so-called characteristic function, described by the Bochner-Minlos theorem (see, e.g., [Oba94, Theorem 1.5.2]).

Theorem 1.13. *If μ is a probability measure on $(\mathcal{N}', C_\sigma(\mathcal{N}'))$, then its Fourier transform, defined by*

$$\mathcal{N} \ni \varphi \mapsto C_\mu(\varphi) := \int_{\mathcal{N}'} e^{i\langle \omega, \varphi \rangle} \, \mathrm{d}\mu(\omega) \in \mathbb{C},$$

is a characteristic function on $\mathcal{N}$, that is, C_μ is a continuous function from $\mathcal{N}$ into $\mathbb{C}$ such that $C_\mu(0) = 1$ and C_μ is positive definite: for all $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $\varphi_1, \dots, \varphi_n \in \mathcal{N}$, we have

$$\sum_{j,k=1}^{n} \lambda_j \overline{\lambda_k} C_\mu(\varphi_j - \varphi_k) \geq 0.$$

Conversely, if C is a characteristic function on $\mathcal{N}$, then there exists a unique probability measure μ on $(\mathcal{N}', C_\sigma(\mathcal{N}'))$ such that $C_\mu = C$.

In view of Bochner-Minlos theorem, we will also call the Fourier transform C_μ as the *characteristic function* of μ.

Throughout this dissertation, whenever we define a measure μ on $\mathcal{N}'$, we implicitly assume that $\mathcal{N}'$ is equipped with $C_\sigma(\mathcal{N}')$ as its σ-algebra, unless otherwise indicated. In this case, we denote by $\mathbb{E}_\mu$ the expectation operator, and for $p \geq 1$, the L^p spaces of complex-valued functions on $\mathcal{N}'$ with respect to μ are denoted by $L^p(\mu) := L^p(\mathcal{N}', \mu; \mathbb{C})$ with corresponding norm $\| \cdot \|_{L^p(\mu)}$. For $p = 2$, the corresponding scalar product is denoted by $(\!(\cdot, \cdot)\!)_{L^2(\mu)}$, and is given by:

$$(\!(F, G)\!)_{L^2(\mu)} := \int_{\mathcal{N}'} F(\omega) \overline{G(\omega)} \, \mathrm{d}\mu(\omega), \quad F, G \in L^2(\mu).$$

1.4.1 Distributions in Gaussian analysis

It was shown in, e.g., [Oba94, Lemma 2.1.1] that the map

$$\mathcal{N} \ni \varphi \mapsto \exp\left(-\frac{1}{2}\langle \varphi, \varphi \rangle\right) \in \mathbb{C}$$

is a characteristic function on $\mathcal{N}$. Hence, by Bochner-Minlos theorem, we obtain the following measure on $\mathcal{N}'$.

Definition 1.14. The *Gaussian measure* on $\mathcal{N}'$ is a probability measure μ_1 on $\mathcal{N}'$ whose characteristic function is given by

$$\int_{\mathcal{N}'} e^{i\langle \omega, \varphi \rangle} \, \mathrm{d}\mu_1(\omega) = \exp\left(-\frac{1}{2}\langle \varphi, \varphi \rangle\right), \quad \varphi \in \mathcal{N}. \tag{1.7}$$

Remark 1.15. The unusual notation μ_1 for the Gaussian measure is used in anticipation for its generalization, the Mittag-Leffler measure, in Chapter 2. This measure is denoted by μ_β, $\beta \in (0, 1]$, and the case $\beta = 1$ produces the Gaussian measure.

We state several properties of the probability space $(\mathcal{N}', \mu_1)$ from [Oba94] as follows. If $\varphi_1, \ldots, \varphi_n \in \mathcal{N}$ form an orthonormal set with respect to the scalar product in $\mathcal{H}$, then the image measure of μ_1 under the cylinder map $C_{\varphi_1,\ldots,\varphi_n}$ is the standard Gaussian measure on $\mathbb{R}^n$, the probability measure on $\mathbb{R}^n$ with density ρ (with respect to the Lebesgue measure on $\mathbb{R}^n$) given by

$$\rho(x) := \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}|x|^2\right), \quad x \in \mathbb{R}^n.$$

From this image measure of μ_1, we obtain the following formula:

$$\int_{\mathcal{N}'} |\langle \omega, \varphi \rangle|^2 \, \mathrm{d}\mu_1(\omega) = \langle \varphi, \varphi \rangle^2, \quad \varphi \in \mathcal{N}_{\mathbb{C}}. \tag{1.8}$$

Equation (1.8) allows us to extend the definition of the dual pairing $\langle \cdot, \cdot \rangle$ to $\mathcal{N}' \times \mathcal{H}_{\mathbb{C}}$. To see this, let $\eta \in \mathcal{H}_{\mathbb{C}}$ and $(\varphi_n)_{n=1}^\infty$ be a sequence in $\mathcal{N}_{\mathbb{C}}$ converging to η in $\mathcal{H}_{\mathbb{C}}$. Then Equation (1.8) implies that for all $m, n \in \mathbb{N}$,

$$\|\langle \cdot, \varphi_m \rangle - \langle \cdot, \varphi_n \rangle\|_{L^2(\mu_1)} = |\varphi_m - \varphi_n|_{\mathcal{H}_{\mathbb{C}}}.$$

As the sequence $(\varphi_n)_{n=1}^\infty$ is Cauchy in $\mathcal{H}_{\mathbb{C}}$, the above equality implies that the sequence $(\langle \cdot, \varphi_n \rangle)_{n=1}^\infty$ is Cauchy in $L^2(\mu_1)$. Hence, we can define

$$\langle \cdot, \eta \rangle := \lim_{n \to \infty} \langle \cdot, \varphi_n \rangle, \tag{1.9}$$

where the limit is taken with respect to $L^2(\mu_1)$. This limit is independent of the sequence in $\mathcal{N}_\mathbb{C}$ approximating η. Indeed, if another sequence $(\widetilde{\varphi}_n)_{n=1}^\infty$ in $\mathcal{N}_\mathbb{C}$ converge to η in $\mathcal{H}_\mathbb{C}$ and F is the limit of the sequence $(\langle \cdot, \widetilde{\varphi}_n\rangle)_{n=1}^\infty$ in $L^2(\mu_1)$, then

$$\begin{aligned}\|\langle \cdot,\eta\rangle - F\|_{L^2(\mu_1)} &\leq \|\langle \cdot,\eta\rangle - \langle \cdot,\varphi_n\rangle\|_{L^2(\mu_1)} + \|\langle \cdot,\varphi_n\rangle - \langle \cdot,\widetilde{\varphi}_n\rangle\|_{L^2(\mu_1)} + \|\langle \cdot,\widetilde{\varphi}_n\rangle - F\|_{L^2(\mu_1)}\\ &= \|\langle \cdot,\eta\rangle - \langle \cdot,\varphi_n\rangle\|_{L^2(\mu_1)} + |\varphi_n - \widetilde{\varphi}_n|_{\mathcal{H}_\mathbb{C}} + \|\langle \cdot,\widetilde{\varphi}_n\rangle - F\|_{L^2(\mu_1)}.\end{aligned}$$

As all the terms on the right-hand side converge to 0 as $n \to \infty$, we infer that $\langle \cdot, \eta\rangle = F$ in $L^2(\mu_1)$. From this definition of $\langle \cdot, \eta\rangle$, the following formulas hold for $\eta, \xi \in \mathcal{H}_\mathbb{C}$ and $n \in \mathbb{N}_0$:

$$\begin{aligned}\int_{\mathcal{N}'} e^{i\langle \omega,\eta\rangle}\, d\mu_1(\omega) &= \exp\left(-\frac{1}{2}\langle \eta,\eta\rangle\right),\\ \int_{\mathcal{N}'} \langle \omega,\eta\rangle^{2n+1}\, d\mu_1(\omega) &= 0,\\ \int_{\mathcal{N}'} \langle \omega,\eta\rangle^{2n}\, d\mu_1(\omega) &= \frac{(2n)!}{2^n n!}\langle \eta,\eta\rangle^n,\\ \int_{\mathcal{N}'} \langle \omega,\eta\rangle\langle \omega,\xi\rangle\, d\mu_1(\omega) &= \langle \eta,\xi\rangle.\end{aligned} \tag{1.10}$$

Example 1.16. Consider the standard nuclear triple $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ from Example 1.2. For $t \in (0,\infty)$, the indicator function $\mathbb{1}_{[0,t)}$ belongs to $L^2(\mathbb{R})$, and so $B_t := \langle \cdot, \mathbb{1}_{[0,t)}\rangle$ is a well-defined element of $L^2(\mu_1)$. Set $B_0 := 0$. Now, using (1.10), we have, for $t, s \in (0,\infty)$,

$$\int_{\mathcal{S}'(\mathbb{R})} \langle \omega, \mathbb{1}_{[0,t)}\rangle\, d\mu_1(\omega) = 0,$$

$$\int_{\mathcal{S}'(\mathbb{R})} \langle \omega, \mathbb{1}_{[0,t)}\rangle\langle \omega, \mathbb{1}_{[0,s)}\rangle\, d\mu_1(\omega) = (\mathbb{1}_{[0,t)}, \mathbb{1}_{[0,s)})_{L^2(\mathbb{R})} = \min\{t,s\},$$

and by an approximation procedure, for $t \in (0,\infty)$ and $p \in \mathbb{R}$,

$$\int_{\mathcal{S}'(\mathbb{R})} e^{ipB_t(\omega)}\, d\mu_1(\omega) = \exp\left(-\frac{1}{2}p^2 t\right) = \frac{1}{\sqrt{2\pi t}}\int_\mathbb{R} e^{ipx} e^{-\frac{1}{2t}x^2}\, dx.$$

Hence, the stochastic process $(B_t)_{t\geq 0}$ is a Gaussian centered process with covariance given by $\mathrm{Cov}(B_t, B_s) = \min\{t,s\}$. In fact, $(B_t)_{t\geq 0}$ has a continuous modification which is a Brownian motion. For more details, see, e.g., [HKPS93].

The construction of test functions and distributions on the space $(\mathcal{N}', \mu_1)$ begins with the following subspace of $L^2(\mu_1)$. Let $\mathcal{P}(\mathcal{N}')$ be the complex space of *smooth polynomials* on $\mathcal{N}'$, defined by the space of all maps of the form

$$\mathcal{N}' \ni \omega \mapsto p(\langle \omega,\varphi_1\rangle, \ldots, \langle \omega,\varphi_n\rangle) \in \mathbb{C},$$

where $n \in \mathbb{N}$, $\varphi_1, \ldots, \varphi_n \in \mathcal{N}_{\mathbb{C}}$, and p is a polynomial over $\mathbb{C}$ in n indeterminates. Since the map

$$(\mathcal{N}_{\mathbb{C}})^m \ni (\varphi_1, \ldots, \varphi_m) \mapsto \langle \omega^{\otimes m}, \varphi_1 \otimes \cdots \otimes \varphi_m \rangle = \langle \omega, \varphi_1 \rangle \cdots \langle \omega, \varphi_m \rangle \in \mathbb{C}, \qquad m \in \mathbb{N},\ \omega \in \mathcal{N}',$$

is m-linear and symmetric, the polarization formula (1.5) provides an alternative expression for $\varphi \in \mathcal{P}(\mathcal{N}')$ as the following finite sum:

$$\varphi(\omega) = \sum_{n=0}^{N} \langle \omega^{\otimes n}, \varphi^{(n)} \rangle, \quad \omega \in \mathcal{N}', \tag{1.11}$$

where $N \in \mathbb{N}_0$, and $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}$ is a (finite) linear combination of elements of the form $\xi^{\otimes n}$, $\xi \in \mathcal{N}_{\mathbb{C}}$. Note that (1.10) and (1.11) implies that $\mathcal{P}(\mathcal{N}')$ is indeed a subspace of $L^2(\mu_1)$. In fact, $\mathcal{P}(\mathcal{N}')$ is dense in $L^2(\mu_1)$ (see, e.g., [Oba94, Proposition 2.3.2]). However, we would like to rewrite smooth polynomials in such a way that a certain orthogonality relation is satisfied. Thus, we use a different set of polynomials, the so-called Wick-ordered polynomials, in which we use the following construction instead of the usual definition from, e.g., [Oba94, HKPS93]. For a fixed $n \in \mathbb{N}_0$, let $\Delta(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n})$ be the subspace of $\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}$ spanned by the elements of the form $\xi^{\otimes n}$, $\xi \in \mathcal{N}_{\mathbb{C}}$. The polarization formula (1.5) ensures that $\Delta(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n})$ is dense in each of the spaces $\mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes} n}$, $p \in \mathbb{N}_0$, and so $\Delta(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n})$ is also dense in $\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}$. The n^{th} Wick power of $\omega \in \mathcal{N}'$, denoted by $:\omega^{\otimes n}:$, is an element of $(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n})'$ defined as follows. First, the dual pairing between $:\omega^{\otimes n}:$ and an element $\xi^{\otimes n}$, $\xi \in \mathcal{N}_{\mathbb{C}}$ is given by

$$\langle :\omega^{\otimes n}:, \xi^{\otimes n} \rangle = \frac{|\xi|_{\mathcal{H}_{\mathbb{C}}}^n}{2^{n/2}} H_n\left(\frac{\langle \omega, \xi \rangle}{\sqrt{2}|\xi|_{\mathcal{H}_{\mathbb{C}}}} \right),$$

where H_n is the n^{th} *Hermite polynomial*:

$$H_n(x) := n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!(n-2k)!} (2x)^{n-2k}.$$

Then extend the dual pairing to $\Delta(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n})$ via linearity, and finally to $\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}$ via approximation of elements from $\Delta(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n})$. Then every $\varphi \in \mathcal{P}(\mathcal{N}')$ can be written uniquely as a finite sum of Wick powers:

$$\varphi(\omega) = \sum_{n=0}^{\infty} \langle :\omega^{\otimes n}:, \varphi^{(n)} \rangle, \quad \omega \in \mathcal{N}', \tag{1.12}$$

where $\varphi^{(n)} \in \Delta(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n})$, and $\varphi^{(n)} = 0$ except for a finite number of $n \in \mathbb{N}_0$ (see, e.g., [Oba94, Corollary 2.2.11]). A smooth polynomial expressed in the form (1.12) is called a *Wick-ordered polynomial* on $\mathcal{N}'$. The advantage of using Wick-ordered polynomials over the usual expression or expression (1.11) for smooth polynomials is the following orthogonality relation on the Wick powers:

$$\int_{\mathcal{N}'} \langle :\omega^{\otimes m}:, \xi^{\otimes m}\rangle \langle :\omega^{\otimes n}:, \varphi^{\otimes n}\rangle \, \mathrm{d}\mu_1(\omega) = \delta_{m,n} n! \langle \varphi, \xi\rangle^n, \quad \xi, \varphi \in \mathcal{N}_{\mathbb{C}},\ m, n \in \mathbb{N}_0. \tag{1.13}$$

Equation (1.13) implies that if $\varphi^{(n)} \in \Delta(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n})$, then

$$\| \langle :\cdot^{\otimes n}:, \varphi^{(n)}\rangle \|^2_{L^2(\mu_1)} = n! |\varphi^{(n)}|_p^2,$$

Recalling that $\Delta(\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n})$ is dense in $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$, we can define the expression $\langle :\cdot^{\otimes n}:, F^{(n)}\rangle$ for $F^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ in $L^2(\mu_1)$ in a similar manner as that of (1.9). From this definition of $\langle :\cdot^{\otimes n}:, F^{(n)}\rangle$ and the density of $\mathcal{P}(\mathcal{N}')$ in $L^2(\mu_1)$, the following result called the Wiener-Itô-Segal chaos decomposition theorem is proven.

Theorem 1.17. *For every $F \in L^2(\mu_1)$, there exists a unique sequence $(F^{(n)})_{n\in\mathbb{N}_0}$, $F^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$, such that the following equality holds in $L^2(\mu_1)$:*

$$F = \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, F^{(n)}\rangle. \tag{1.14}$$

Moreover, $\|F\|^2_{L^2(\mu_1)} = \sum_{n=0}^{\infty} n! |F^{(n)}|^2_{\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}}$.

We will call the expression on the right-hand side of (1.14) the *(Wiener-Itô-Segal)* chaos decomposition of $F \in L^2(\mu_1)$.

Example 1.18. Let $\varphi \in \mathcal{N}_{\mathbb{C}}$. The *Wick exponential*, denoted by $:e^{\langle \cdot, \varphi\rangle}:$ or $:\exp\langle \cdot, \varphi\rangle:$, is a map from $\mathcal{N}'$ to $\mathbb{C}$ defined by

$$:e^{\langle \omega, \varphi\rangle}: := \frac{e^{\langle \omega, \varphi\rangle}}{\mathbb{E}_{\mu_1}(\langle \cdot, \varphi\rangle)} = \exp\left(\langle \omega, \varphi\rangle - \frac{1}{2}\langle \varphi, \varphi\rangle \right), \quad \omega \in \mathcal{N}'.$$

The Wick exponential is a well-defined element of $L^2(\mu_1)$ whose chaos decomposition is given by

$$:e^{\langle \cdot, \varphi\rangle}: = \sum_{n=0}^{\infty} \left\langle :\cdot^{\otimes n}:, \frac{1}{n!}\varphi^{\otimes n} \right\rangle. \tag{1.15}$$

We refer to, e.g., [HKPS93, Kuo96, Oba94] for the details on the derivation of the decomposition formula (1.15).

Now, we are in a position to construct the following spaces of test functions and distributions on $(\mathcal{N}', \mu_1)$. Instead of the approach from, e.g., [KLS96, GKS97], we will use the chaos decomposition (1.14) as a starting point and follow a construction in a similar way as that of the construction of the Hida spaces of test functions and distributions (see, e.g., [HKPS93]). We will revisit the former approach when we discuss the corresponding construction for the non-Gaussian case in Subsection 1.4.2.

For $p, q \in \mathbb{N}$, define the following subspace of $L^2(\mu_1)$:

$$(\mathcal{H}_p)^1_{q,\mu_1} = \Big\{ F = \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, F^{(n)} \rangle \in L^2(\mu_1) : \ F^{(n)} \in \mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes} n}, \\ \|F\|_{p,q,\mu_1} := \left(\sum_{n=0}^{\infty} (n!)^2 2^{nq} |F^{(n)}|_p^2 \right)^{1/2} < \infty \Big\}.$$

A quick verification shows that $\|\cdot\|_{p,q,\mu_1}$ is indeed a norm on $(\mathcal{H}_p)^1_{q,\mu_1}$ induced by the scalar product $(\!(\cdot,\cdot)\!)_{p,q,\mu_1}$ on $(\mathcal{H}_p)^1_{q,\mu_1}$ defined by

$$(\!(F, G)\!)_{p,q,\mu_1} = \sum_{n=0}^{\infty} (n!)^2 2^{nq} (F^{(n)}, G^{(n)})_p, \quad F, G \in (\mathcal{H}_p)^1_{q,\mu_1}.$$

Moreover, for all $p, p', q, q' \in \mathbb{N}$ with $p \le p'$ and $q \le q'$,

$$(\mathcal{H}_{p'})^1_{q',\mu_1} \subset (\mathcal{H}_p)^1_{q,\mu_1} \subset L^2(\mu_1).$$

Next, we will show that $(\mathcal{H}_p)^1_{q,\mu_1}$ is complete. Let $(F_k)_{k\in\mathbb{N}}$, $F_k = \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, F_k^{(n)} \rangle$, be a Cauchy sequence in $(\mathcal{H}_p)^1_{q,\mu_1}$. The estimate

$$|F_k^{(n)} - F_l^{(n)}|_p^2 \le (n!)^{-2} 2^{-nq} \|F_k - F_l\|_{p,q,\mu_1}^2, \quad k, l \in \mathbb{N}, \ n \in \mathbb{N}_0,$$

shows that $(F_k^{(n)})_{k\in\mathbb{N}}$ is Cauchy in $\mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes} n}$, and thus for each $n \in \mathbb{N}_0$, there exists $F^{(n)} := \lim_{k\to\infty} F_k^{(n)}$ in $\mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes} n}$. Set $F := \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, F^{(n)} \rangle \in L^2(\mu_1)$. We infer from Fatou's lemma that

$$\|F_k - F\|_{p,q,\mu_1}^2 = \sum_{n=0}^{\infty} (n!)^2 2^{nq} \lim_{l\to\infty} |F_k^{(n)} - F_l^{(n)}|_p^2 \le \liminf_{l\to\infty} \sum_{n=0}^{\infty} (n!)^2 2^{nq} |F_k^{(n)} - F_l^{(n)}|_p^2 \\ = \liminf_{l\to\infty} \|F_k - F_l\|_{p,q,\mu_1}^2 \to 0 \quad \text{as } k \to \infty.$$

Thus, $F \in (\mathcal{H}_p)^1_{q,\mu_1}$ and that $F_k \to F$ in $(\mathcal{H}_p)^1_{q,\mu_1}$ as $k \to \infty$. This shows that $(\mathcal{H}_p)^1_{q,\mu_1}$ is a Hilbert space.

Finally, the test function space $(\mathcal{N})^1_{\mu_1}$ is defined to be the space

$$(\mathcal{N})^1_{\mu_1} := \operatorname*{prlim}_{p,q\in\mathbb{N}} (\mathcal{H}_p)^1_{q,\mu_1}.$$

It turns out that $(\mathcal{N})^1_{\mu_1}$ is a nuclear Fréchet space densely and topologically embedded in $L^2(\mu_1)$, and that the topology on $(\mathcal{N})^1_{\mu_1}$ is independent of the norms $(|\cdot|_p)_{p\in\mathbb{N}}$ topologizing $\mathcal{N}$ (cf. [KLS96, Theorem 1]). Moreover, as a projective limit of the spaces $(\mathcal{H}_p)^1_{q,\mu_1}$, every $\varphi \in (\mathcal{N})^1_{\mu_1}$ has the following unique representation in $L^2(\mu_1)$:

$$\varphi = \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, \varphi^{(n)}\rangle, \quad \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}, \tag{1.16}$$

such that

$$\|\varphi\|^2_{p,q,\mu_1} = \sum_{n=0}^{\infty} (n!)^2 2^{nq} |\varphi^{(n)}|^2_p < \infty, \quad \text{for all } p, q \in \mathbb{N}.$$

The distribution space, denoted by $(\mathcal{N})^{-1}_{\mu_1}$, is then defined to be the dual space of $(\mathcal{N})^1_{\mu_1}$. Let $(\mathcal{H}_{-p})^{-1}_{-q,\mu_1}$ be the dual space of $(\mathcal{H}_p)^1_{q,\mu_1}$ with norm denoted by $\|\cdot\|_{-p,-q,\mu_1}$. By general duality theory, we have

$$(\mathcal{N})^{-1}_{\mu_1} = \bigcup_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})^{-1}_{-q,\mu_1},$$

and so we equip $(\mathcal{N})^{-1}_{\mu_1}$ with the inductive limit topology of the spaces $(\mathcal{H}_{-p})^{-1}_{-q,\mu_1}$ over all $p, q \in \mathbb{N}$. With this, we obtain the following nuclear triple:

$$(\mathcal{N})^1_{\mu_1} \subset L^2(\mu_1) \subset (\mathcal{N})^{-1}_{\mu_1}.$$

We shall denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mu_1}$ both the dual pairing between $(\mathcal{N})^{-1}_{\mu_1}$ and $(\mathcal{N})^1_{\mu_1}$ and that of between $(\mathcal{H}_{-p})^{-1}_{-q,\mu_1}$ and $(\mathcal{H}_p)^1_{q,\mu_1}$. This dual pairing is an extension of the scalar product in $L^2(\mu_1)$ via

$$\langle\!\langle F, \varphi \rangle\!\rangle_{\mu_1} = (\!(F, \overline{\varphi})\!)_{L^2(\mu_1)}, \quad F \in L^2(\mu_1),\ \varphi \in (\mathcal{N})^1_{\mu_1}.$$

Moreover, for all $p, p', q, q' \in \mathbb{N}$ with $p \le p'$ and $q \le q'$, we obtain the following chain of continuous and dense embeddings:

$$(\mathcal{N})^1_{\mu_1} \subset (\mathcal{H}_{p'})^1_{q',\mu_1} \subset (\mathcal{H}_p)^1_{q,\mu_1} \subset L^2(\mu_1) \subset (\mathcal{H}_{-p})^{-1}_{-q,\mu_1} \subset (\mathcal{H}_{-p'})^{-1}_{-q',\mu_1} \subset (\mathcal{N})^{-1}_{\mu_1}.$$

The chaos decomposition formula also allows for a natural decomposition for distributions $\Phi \in (\mathcal{N})^{-1}_{\mu_1}$ in terms of $\Phi^{(n)} \in (\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n})'$. To see this, given $p, q \in \mathbb{N}$ and $\Phi^{(n)} \in \mathcal{H}_{-p,\mathbb{C}}^{\widehat{\otimes} n}$, define the map

$$I_n(\Phi^{(n)}) : (\mathcal{H}_p)^1_{q,\mu_1} \ni F := \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, F^{(n)} \rangle \mapsto n! \langle \Phi^{(n)}, F^{(n)} \rangle \in \mathbb{C}.$$

Note that $I_n(\Phi^{(n)})$ belongs to $(\mathcal{H}_{-p})^{-1}_{-q,\mu_1}$; indeed, for all $F := \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, F^{(n)} \rangle \in (\mathcal{H}_p)^1_{q,\mu_1}$,

$$\left| \langle\!\langle I_n(\Phi^{(n)}), F \rangle\!\rangle_{\mu_1} \right| \leq n! |\Phi^{(n)}|_{-p} |F^{(n)}|_p \leq 2^{-nq/2} |\Phi^{(n)}|_{-p} \|F\|_{p,q,\mu_1} < \infty.$$

From here, the dual space $(\mathcal{H}_{-p})^{-1}_{-q,\mu_1}$ is characterized as the following space:

$$(\mathcal{H}_{-p})^{-1}_{-q,\mu_1} = \left\{ \Phi = \sum_{n=0}^{\infty} I_n(\Phi^{(n)}) : \ \Phi^{(n)} \in \mathcal{H}_{-p,\mathbb{C}}^{\widehat{\otimes} n}, \ \|\Phi\|^2_{-p,-q,\mu_1} = \sum_{n=0}^{\infty} 2^{-nq} |\Phi^{(n)}|^2_{-p} < \infty \right\}.$$

Any $\Phi \in (\mathcal{N})^{-1}_{\mu_1}$ then has a unique decomposition

$$\Phi = \sum_{n=0}^{\infty} I_n(\Phi^{(n)}), \quad \Phi^{(n)} \in (\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n})', \tag{1.17}$$

where the sum converges in $(\mathcal{N})^{-1}_{\mu_1}$. Moreover, for all $\Phi \in (\mathcal{N})^{-1}_{\mu_1}$ and $\varphi \in (\mathcal{N})^{1}_{\mu_1}$ written in the forms (1.17) and (1.16), respectively,

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle_{\mu_1} = \sum_{n=0}^{\infty} n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle.$$

Remark 1.19. The nuclear triple $(\mathcal{N})^1_{\mu_1} \subset L^2(\mu_1) \subset (\mathcal{N})^{-1}_{\mu_1}$ is part of a family of nuclear triples

$$(\mathcal{N})^{\rho}_{\mu_1} \subset L^2(\mu_1) \subset (\mathcal{N})^{-\rho}_{\mu_1} \tag{1.18}$$

parametrized by $\rho \in [0, 1]$ (see, e.g., [KLS96]). Such triple starts from using the following norm

$$\|F\|_{p,q,\rho,\mu_1} := \left(\sum_{n=0}^{\infty} (n!)^{1+\rho} 2^{nq} |F^{(n)}|^2_p \right)^{1/2},$$

and then the triple (1.18) will be constructed in a similar manner as that of the case for $\rho = 1$. The norm on the corresponding dual space is given by

$$\|\Phi\|^2_{-p,-q,-\rho,\mu_1} = \sum_{n=0}^{\infty} (n!)^{1-\rho} 2^{-nq} |\Phi^{(n)}|^2_{-p},$$

and for $\rho, \rho' \in [0,1]$ with $\rho \le \rho'$, we obtain the following chain of inclusions:

$$(\mathcal{N})^1_{\mu_1} \subset (\mathcal{N})^{\rho'}_{\mu_1} \subset (\mathcal{N})^{\rho}_{\mu_1} \subset (\mathcal{N})_{\mu_1} \subset L^2(\mu_1) \subset (\mathcal{N})'_{\mu_1} \subset (\mathcal{N})^{-\rho}_{\mu_1} \subset (\mathcal{N})^{-\rho'}_{\mu_1} \subset (\mathcal{N})^{-1}_{\mu_1}.$$

where we use the notation $(\mathcal{N})_{\mu_1}$ and $(\mathcal{N})'_{\mu_1}$ for the spaces $(\mathcal{N})^{\rho}_{\mu_1}$ and $(\mathcal{N})^{-\rho}_{\mu_1}$, respectively, corresponding to $\rho = 0$. In this dissertation, we are only concerned with the endpoints $(\mathcal{N})^1_{\mu_1}$ and $(\mathcal{N})^{-1}_{\mu_1}$ of this chain.

1.4.2 Distributions in non-Gaussian analysis

In the previous subsection, we have constructed the spaces of test functions and distributions on $(\mathcal{N}', \mu_1)$ using the Wiener-Itô-Segal chaos decomposition theorem. The chaos decomposition (1.14) relies on the orthogonality relation of the Wick-ordered polynomials (see Equation (1.13)). However, the construction from [KLS96, GKS97] describes $(\mathcal{H}_p)^1_{q,\mu_1}$ as the completion of $\mathcal{P}(\mathcal{N}')$ with respect to the norm

$$\|\varphi\|^2_{p,q,\mu_1} := \sum_{n=0}^{\infty} (n!)^2 2^{nq} |\varphi^{(n)}|^2_p,$$

for $\varphi \in \mathcal{P}(\mathcal{N}')$ written in Wick-ordered polynomial form (1.12). It turns out that we can use this approach to construct test functions and distributions on a more general probability space $(\mathcal{N}', \mu)$. In this case, we replace the Wick-ordered polynomials by the so-called "Appell system": a pair $(\mathbb{P}^\mu, \mathbb{Q}^\mu)$ consisting of a set $\mathbb{P}^\mu$ of specialized polynomials and a set $\mathbb{Q}^\mu$ of distributions properly associated with μ. Elements of $\mathbb{P}^\mu$ generally do not satisfy an orthogonality relation similar to that of (1.13). Instead, each pair of elements in $\mathbb{P}^\mu \times \mathbb{Q}^\mu$ satisfy a certain biorthogonality relation (see Equation (1.23)). This system has been first constructed for smooth measures μ by Daletskii [Dal91], and more details and results arising from this construction were obtained in [ADKS96]. In this subsection, we use the construction of the Appell system from [GJRdS15], which is the same as that from [KSWY98] but with the exception that the former uses a stronger assumption on the measure μ than that of the latter.

As a starting point of constructing test functions and distributions using the Appell system, we assume that the measure μ on $\mathcal{N}'$ satisfy the following properties.

(A1) The measure μ has an analytic Laplace transform in a neighborhood of zero, that is, the map

$$\mathcal{N}_{\mathbb{C}} \ni \varphi \mapsto l_{\mu}(\varphi) := \int_{\mathcal{N}'} e^{\langle \omega, \varphi \rangle} \, \mathrm{d}\mu(\omega) \in \mathbb{C}$$

is holomorphic at $0 \in \mathcal{N}_{\mathbb{C}}$.

(A2) The measure μ has full topological support, that is, for any nonempty open subset $\mathcal{U} \subset \mathcal{N}'$, we have $\mu(\mathcal{U}) > 0$.

As a side note, in [KSWY98], condition **(A2)** was replaced by the weaker condition that μ is non-degenerate, that is, if $\varphi \in \mathcal{P}(\mathcal{N}')$ such that $\varphi = 0$ μ-almost everywhere, then φ is identically zero on $\mathcal{N}'$. However, it was shown in [KK99] that this assumption is not sufficient to guarantee the embedding of test functions into $L^2(\mu)$.

Remark 1.20. It was shown in [KSWY98] that the following statements are equivalent for a measure μ on $\mathcal{N}'$.

(i) Condition **(A1)** holds for μ.

(ii) There exist $p \in \mathbb{N}$ and $C \in (0, \infty)$ such that

$$\left| \int_{\mathcal{N}'} \langle \omega, \xi \rangle^n \, \mathrm{d}\mu(\omega) \right| \le n! C^n |\xi|_p^n, \quad \xi \in \mathcal{H}_{p,\mathbb{C}}, \ n \in \mathbb{N}.$$

(iii) There exist $p \in \mathbb{N}$ and $\varepsilon > 0$ such that for all $n \in \mathbb{N}$,

$$\int_{\mathcal{N}'} e^{\varepsilon |\omega|_{-p}} \, \mathrm{d}\mu(\omega) < \infty.$$

Statement (ii) implies that $\mathcal{P}(\mathcal{N}')$ is a subspace of $L^2(\mu)$. In fact, [Jah15, Theorem 2.1.4] shows that condition **(A1)** of the measure μ ensures that $\mathcal{P}(\mathcal{N}')$ is dense in $L^2(\mu)$.

Next step is to construct the Appell system itself and write decomposition formulas for elements in the space $\mathcal{P}(\mathcal{N}')$ and its dual space. We only provide an overview of the construction, and we refer to [KSWY98] for further details of this construction. First, we need to assign a topology on $\mathcal{P}(\mathcal{N}')$ to obtain the corresponding dual space. Note that every $\varphi \in \mathcal{P}(\mathcal{N}')$ can be written uniquely as

$$\varphi(\omega) = \sum_{n=0}^{\infty} \langle \omega^{\otimes n}, \varphi^{(n)} \rangle, \quad \omega \in \mathcal{N}', \tag{1.19}$$

where $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}$ such that $\varphi^{(n)} = 0$ for all but a finite number of $\varphi^{(n)}$. Expression (1.19) allows us to construct a natural topology on $\mathcal{P}(\mathcal{N}')$, namely the topology such that the bijective mapping

$$\varphi = \sum_{n=0}^{\infty} \langle \cdot^{\otimes n}, \varphi^{(n)} \rangle \longleftrightarrow \vec{\varphi} := (\varphi^{(n)})_{n \in \mathbb{N}_0}$$

becomes a topological isomorphism between $\mathcal{P}(\mathcal{N}')$ and the topological direct sum of the spaces $\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}$:

$$\mathcal{P}(\mathcal{N}') \cong \bigoplus_{n=0}^{\infty} \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}.$$

Let $\mathcal{P}'_{\mu}(\mathcal{N}')$ be the dual space of $\mathcal{P}(\mathcal{N}')$. Then by identifying $L^2(\mu)$ with its dual via the Riesz representation theorem, we obtain the following chain of inclusions:

$$\mathcal{P}(\mathcal{N}') \subset L^2(\mu) \subset \mathcal{P}'_{\mu}(\mathcal{N}').$$

Denote the dual pairing between $\mathcal{P}'_{\mu}(\mathcal{N}')$ and $\mathcal{P}(\mathcal{N}')$ by $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mu}$. This dual pairing is a bilinear extension of the scalar product on $L^2(\mu)$ given by

$$\langle\!\langle F, \varphi \rangle\!\rangle_{\mu} = (\!(F, \overline{\varphi})\!)_{L^2(\mu)}, \quad F \in L^2(\mu),\ \varphi \in \mathcal{P}(\mathcal{N}').$$

In addition, since the map $\mathbf{1} : \mathcal{N}' \longrightarrow \mathbb{C}$ defined by $\mathbf{1}(\omega) = 1$ for all $\omega \in \mathcal{N}'$ belongs to $\mathcal{P}(\mathcal{N}')$, we can extend the definition of the expectation operator from random variables to elements in $\mathcal{P}'_{\mu}(\mathcal{N}')$ by

$$\mathbb{E}_{\mu}(\Phi) := \langle\!\langle \Phi, \mathbf{1} \rangle\!\rangle_{\mu}, \quad \Phi \in \mathcal{P}'_{\mu}(\mathcal{N}').$$

and call $\mathbb{E}_{\mu}(\Phi)$ the *generalized expectation* of Φ.

To construct a decomposition formula for elements in $\mathcal{P}(\mathcal{N}')$, we introduce the μ-exponential defined as follows. Since $l_{\mu}(0) = 1$ and l_{μ} is holomorphic at $0 \in \mathcal{N}_{\mathbb{C}}$ by **(A1)**, there exists a neighborhood $\mathcal{U}_0 \subset \mathcal{N}_{\mathbb{C}}$ of zero such that $l_{\mu}(\varphi) \neq 0$ for all $\varphi \in \mathcal{U}_0$. The *$\mu$-exponential* or *normalized exponential* is then defined as follows:

$$e_{\mu}(\varphi; \omega) := \frac{e^{\langle \omega, \varphi \rangle}}{l_{\mu}(\varphi)}, \quad \varphi \in \mathcal{U}_0,\ \omega \in \mathcal{N}'_{\mathbb{C}}.$$

The holomorphy of the map $\varphi \mapsto e_{\mu}(\varphi; \omega)$ for each $\omega \in \mathcal{N}'_{\mathbb{C}}$ implies that the μ-exponential can be expressed as a power series in a similar way as that of the case for one-dimensional Appell polynomials (see, e.g., [Bou04]), that is,

$$e_{\mu}(\varphi; \omega) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n^{\mu}(\omega), \varphi^{\otimes n} \rangle, \quad \varphi \in \mathcal{U}_0,\ \omega \in \mathcal{N}'_{\mathbb{C}}, \tag{1.20}$$

for suitable mappings $P_n^\mu : \mathcal{N}'_{\mathbb{C}} \to (\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n})'$. Using this expansion, every $\varphi \in \mathcal{P}(\mathcal{N}')$ has the following unique decomposition:

$$\varphi(\omega) = \sum_{n=0}^{\infty} \langle P_n^\mu(\omega), \varphi^{(n)} \rangle, \quad \omega \in \mathcal{N}'_{\mathbb{C}}, \tag{1.21}$$

where $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}$ and $\varphi^{(n)} = 0$ except for a finite number of $n \in \mathbb{N}_0$. The set

$$\mathbb{P}^\mu := \{\langle P_n^\mu(\cdot), \varphi^{(n)} \rangle : \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}, n \in \mathbb{N}_0\}$$

is then called the $\mathbb{P}^\mu$*-system*, and its elements are called *Appell polynomials*.

For the elements in $\mathcal{P}'_\mu(\mathcal{N}')$, we define the following continuous linear operator on $\mathcal{P}(\mathcal{N}')$. Given $\Phi^{(n)} \in (\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n})'$, define the continuous linear operator $D(\Phi^{(n)})$ on $\mathcal{P}(\mathcal{N}')$ acting on the monomials $\langle (\cdot)^{\otimes m}, \varphi^{(m)} \rangle$ by

$$D(\Phi^{(n)})\langle (\cdot)^{\otimes m}, \varphi^{(m)} \rangle = \begin{cases} \dfrac{m!}{(m-n)!} \langle (\cdot)^{\otimes (m-n)} \widehat{\otimes} \Phi^{(n)}, \varphi^{(m)} \rangle, & m \geq n, \\ 0, & m < n, \end{cases}$$

Then every $\Phi \in \mathcal{P}'_\mu(\mathcal{N}')$ also has a unique decomposition:

$$\Phi = \sum_{n=0}^{\infty} Q_n^\mu(\Phi^{(n)}), \tag{1.22}$$

for suitable $\Phi^{(n)} \in (\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n})'$. In this representation, $Q_n^\mu(\Phi^{(n)}) := D(\Phi^{(n)})^* \mathbf{1}$, where $D(\Phi^{(n)})^*$ is the adjoint of $D(\Phi^{(n)})$. The set

$$\mathbb{Q}^\mu := \{Q_n^\mu(\Phi^{(n)}) : \Phi^{(n)} \in (\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n})', n \in \mathbb{N}_0\}$$

is called the $\mathbb{Q}^\mu$*-system*. The pair $(\mathbb{P}^\mu, \mathbb{Q}^\mu)$ is then called the *Appell system* generated by μ.

The main property of the Appell system is the following biorthogonality relation (see [KSWY98, Theorem 4.17]).

Theorem 1.21. *For all* $\Phi^{(n)} \in (\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n})'$ *and* $\varphi^{(m)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} m}$,

$$\langle\!\langle Q_n^\mu(\Phi^{(n)}), \langle P_m^\mu(\cdot), \varphi^{(m)} \rangle \rangle\!\rangle_\mu = \delta_{m,n} n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle. \tag{1.23}$$

From here on, we always use the representations (1.21) and (1.22) for $\varphi \in \mathcal{P}(\mathcal{N}')$ and $\Phi \in \mathcal{P}'_\mu(\mathcal{N}')$, respectively.

Finally, we use the Appell system to construct test functions and distributions on $(\mathcal{N}', \mu)$. For $p, q \in \mathbb{N}$, the space $(\mathcal{H}_p)^1_{q,\mu}$ is defined as the completion of the space $\mathcal{P}(\mathcal{N}')$ with respect to the norm

$$\|\varphi\|^2_{p,q,\mu} := \sum_{n=0}^{\infty} (n!)^2 2^{nq} |\varphi^{(n)}|^2_p, \quad \varphi \in \mathcal{P}(\mathcal{N}').$$

Note that the sum is finite since only a finite number of $\varphi^{(n)}$ is nonzero. This norm on $\mathcal{P}(\mathcal{N}')$ is induced by the scalar product $(\!(\cdot,\cdot)\!)_{p,q,\mu}$ on $\mathcal{P}(\mathcal{N}')$ defined by

$$(\!(\varphi, \psi)\!)_{p,q,\mu} := \sum_{n=0}^{\infty} (n!)^2 2^{nq} (\varphi^{(n)}, \psi^{(n)})_p, \quad \varphi, \psi \in \mathcal{P}(\mathcal{N}'),$$

and hence $(\mathcal{H}_p)^1_{q,\mu}$ is a Hilbert space. It has been shown in [KK99] that under condition **(A2)**, there exist $p', q' \in \mathbb{N}$ such that for all $p > p'$ and $q > q'$, $(\mathcal{H}_p)^1_{q,\mu}$ can be topologically embedded in $L^2(\mu)$. The test function space $(\mathcal{N})^1_\mu$ is then defined as

$$(\mathcal{N})^1_\mu := \underset{p,q\in\mathbb{N}}{\mathrm{prlim}} (\mathcal{H}_p)^1_{q,\mu}.$$

This is a nuclear space which is continuously and densely embedded in $L^2(\mu)$. Moreover, every $\varphi \in (\mathcal{N})^1_\mu$ has a representation as an entire function on $\mathcal{N}'_\mathbb{C}$ in terms of a power series:

$$\varphi(\omega) = \sum_{n=0}^{\infty} \langle P^\mu_n(\omega), \varphi^{(n)} \rangle, \quad \omega \in \mathcal{N}'_\mathbb{C}, \tag{1.24}$$

where $\varphi^{(n)} \in \mathcal{N}^{\widehat{\otimes} n}_\mathbb{C}$ such that

$$\|\varphi\|^2_{p,q,\mu} = \sum_{n=0}^{\infty} (n!)^2 2^{nq} |\varphi^{(n)}|^2_p < \infty, \quad \text{for all } p, q \in \mathbb{N}.$$

It turns out that $(\mathcal{N})^1_\mu$ is the same for all measures μ satisfying **(A1)** and **(A2)**, and so we drop the subscript μ and denote the test function space by $(\mathcal{N})^1$.

The distribution space $(\mathcal{N})^{-1}_\mu$ is defined as the dual space of $(\mathcal{N})^1$. Let $(\mathcal{H}_{-p})^{-1}_{-q,\mu}$ be the dual space of $(\mathcal{H}_p)^1_{q,\mu}$ with norm denoted by $\|\cdot\|_{-p,-q,\mu}$. The biorthogonality relation (1.23) implies that

$$(\mathcal{H}_{-p})^{-1}_{-q,\mu} = \left\{ \Phi \in \mathcal{P}'_\mu(\mathcal{N}') : \|\Phi\|^2_{-p,-q,\mu} = \sum_{n=0}^{\infty} 2^{-qn} |\Phi^{(n)}|^2_{-p} < \infty \right\}.$$

General theory of duality then implies that

$$(\mathcal{N})^{-1}_{\mu} = \bigcup_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})^{-1}_{-q,\mu},$$

and so we equip $(\mathcal{N})^{-1}_{\mu}$ with the inductive topology:

$$(\mathcal{N})^{-1}_{\mu} := \underset{p,q\in\mathbb{N}}{\operatorname{indlim}} (\mathcal{H}_{-p})^{-1}_{-q,\mu}.$$

Hence, we obtain the nuclear triple

$$(\mathcal{N})^{1} \subset L^2(\mu) \subset (\mathcal{N})^{-1}_{\mu},$$

and for all $p, p', q, q' \in \mathbb{N}$ with $p \le p'$ and $q \le q'$, we obtain the following chain of continuous embeddings:

$$\begin{aligned}\mathcal{P}'_{\mu}(\mathcal{N}') \subset (\mathcal{N})^{1} \subset (\mathcal{H}_{p'})^{1}_{q',\mu} \subset (\mathcal{H}_{p})^{1}_{q,\mu} \subset L^2(\mu)& \\ \subset (\mathcal{H}_{-p})^{-1}_{-q,\mu} \subset (\mathcal{H}_{-p'})^{-1}_{-q',\mu} \subset (\mathcal{N})^{-1}_{\mu} \subset \mathcal{P}'_{\mu}(\mathcal{N}').& \quad (1.25)\end{aligned}$$

We keep the notation $\langle\langle\cdot,\cdot\rangle\rangle_{\mu}$ for the dual pairing between $(\mathcal{N})^{-1}_{\mu}$ and $(\mathcal{N})^{1}_{\mu}$ and that of between $(\mathcal{H}_{-p})^{-1}_{-q,\mu}$ and $(\mathcal{H}_{p})^{1}_{q,\mu}$. The biorthogonality relation (1.23) implies that for any $\varphi \in (\mathcal{N})^{1}$ and $\Phi \in (\mathcal{N})^{-1}_{\mu}$, we have

$$\langle\langle \Phi, \varphi \rangle\rangle_{\mu} = \sum_{n=0}^{\infty} n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle.$$

Example 1.22. Consider the normalized exponential $e_{\mu}(\varphi;\cdot)$ for $\varphi \in \mathcal{U}_0$, a neighborhood of zero in $\mathcal{N}_{\mathbb{C}}$ such that $e_{\mu}(\varphi;\cdot)$ is well-defined as a function on $\mathcal{N}'_{\mathbb{C}}$. In view of (1.20), we have for $p, q \in \mathbb{N}$,

$$\|e_{\mu}(\varphi;\cdot)\|^2_{p,q,\mu} = \sum_{n=0}^{\infty} (n!)^2 2^{nq} \left| \frac{1}{n!} \varphi^{\otimes n} \right|^2_p = \sum_{n=0}^{\infty} 2^{nq} |\varphi|^{2n}_p.$$

Note that the last expression is finite if and only if $2^q |\varphi|^2_p < 1$. Hence, $e_{\mu}(\varphi;\cdot) \notin (\mathcal{N})^{1}$ whenever $\varphi \neq 0$, but $e_{\mu}(\varphi;\cdot) \in (\mathcal{H}_p)^{1}_{q,\mu}$ if $\varphi \in \mathcal{U}_0 \cap \mathcal{U}_{p,q}$, where

$$\mathcal{U}_{p,q} := \{\varphi \in \mathcal{N}_{\mathbb{C}} : 2^q |\varphi|^2_p < 1\}.$$

In fact, it turns out that the linear span of the set $\{e_{\mu}(\varphi;\cdot) : \varphi \in \mathcal{U}_0 \cap \mathcal{U}_{p,q}\}$ is dense in $(\mathcal{H}_p)^{1}_{q,\mu}$.

Integral transforms and characterization theorems

As in the case of Gaussian analysis, it is useful to characterize distribution spaces by certain integral transforms. In this dissertation, we only consider two of them. Note that each $\Phi \in (\mathcal{N})_\mu^{-1}$ belongs to $(\mathcal{H}_{-p})_{-q,\mu}^{-1}$ for some $p, q \in \mathbb{N}$. In view of Example 1.22 and the chain (1.25), we can choose $p, q \in \mathbb{N}$ such that the normalized exponential $e_\mu(\varphi; \cdot)$ is well-defined on $\mathcal{N}'_{\mathbb{C}}$ for all $\varphi \in \mathcal{U}_{p,q}$. With this choice of $p, q \in \mathbb{N}$, the *S_μ-transform* and the *T_μ-transform* of Φ, denoted by $S_\mu\Phi$ and $T_\mu\Phi$, respectively, are defined on $\varphi \in \mathcal{U}_{p,q}$ as the following dual product:

$$S_\mu\Phi(\varphi) := \langle\!\langle \Phi, e_\mu(\varphi; \cdot) \rangle\!\rangle_\mu, \qquad\qquad T_\mu\Phi(\varphi) := \langle\!\langle \Phi, e^{i\langle \cdot, \varphi \rangle} \rangle\!\rangle_\mu.$$

For a vector $\mathbf{\Phi}$ with components in $(\mathcal{N})_\mu^{-1}$, its S_μ-transform and T_μ-transform is a vector field defined on $\mathcal{U}_0$ whose components are the S_μ-transform and T_μ-transform, respectively, of the corresponding components of $\mathbf{\Phi}$. Of course, we have the following relationships between the two transforms:

$$T_\mu\Phi(\varphi) = l_\mu(i\varphi) \cdot S_\mu\Phi(i\varphi), \quad \varphi \in \mathcal{U}_{p,q}. \tag{1.26}$$

The characterization theorem for the space $(\mathcal{N})_\mu^{-1}$ via the S_μ-transform is done using the spaces of holomorphic functions on $\mathcal{N}_{\mathbb{C}}$. We denote by $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ the space of (equivalence class of) holomorphic functions at zero, where we identify two functions which coincides on a neighborhood of zero. See [KSWY98, Theorem 8.34] for the details and proof of the following characterization theorem.

Theorem 1.23. *The S_μ-transform is a topological isomorphism from $(\mathcal{N})_\mu^{-1}$ to* $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$.

A corollary of Theorem 1.23 is the result characterizing strongly convergent sequences in $(\mathcal{N})_\mu^{-1}$ and integrable maps with values in $(\mathcal{N})_\mu^{-1}$ in a weak sense. We refer to [Jah15, Theorem 2.2.2, Theorem 2.3.1]) for the proof of the next two results.

Theorem 1.24. *Let $(T, \mathcal{B}, \nu)$ be a measure space and $\Phi_t \in (\mathcal{N})_\mu^{-1}$ for all $t \in T$. Let $\mathcal{U}_0$ be a neighborhood of zero in $\mathcal{N}_{\mathbb{C}}$ and $C \in (0, \infty)$ such that*

(i) the map $T \ni t \mapsto S_\mu\Phi_t(\varphi) \in \mathbb{C}$ is measurable for all $\varphi \in \mathcal{U}_0$; and

(ii) $\int_T |S_\mu\Phi_t(\varphi)| \,\mathrm{d}\nu(t) \leq C$ for all $\varphi \in \mathcal{U}_0$.

Then there exists a unique $\Psi \in (\mathcal{N})_\mu^{-1}$ such that for all $\varphi \in \mathcal{U}_0$,

$$S_\mu\Psi(\varphi) = \int_T S_\mu\Phi_t(\varphi) \,\mathrm{d}\nu(t).$$

We denote Ψ by $\int_T \Phi_t \,\mathrm{d}\nu(t)$ and call it the weak integral of $(\Phi_t)_{t\in T}$.

Theorem 1.25. *A sequence* $(\Phi_n)_{n\in\mathbb{N}}$ *in* $(\mathcal{N})_\mu^{-1}$ *converges strongly in* $(\mathcal{N})_\mu^{-1}$ *if and only if there exist* $p, q \in \mathbb{N}$ *such that*

(i) $(S_\mu\Phi_n(\varphi))_{n\in\mathbb{N}}$ *is a Cauchy sequence for all* $\varphi \in \mathcal{U}_{p,q}$*;*

(ii) for each $n \in \mathbb{N}$, $S_\mu\Phi_n$ *is holomorphic on* $\mathcal{U}_{p,q}$*, and there exists a constant* $C \in (0,\infty)$ *such that* $|S_\mu\Phi_n(\varphi)| \leq C$ *for all* $\varphi \in \mathcal{U}_{p,q}$ *and* $n \in \mathbb{N}$.

A consequence of Theorem 1.25 that is used for applications to stochastic differential equations is the following sufficient condition for the derivative and the S_μ-transform to commute.

Corollary 1.26. *Let* $I \subset \mathbb{R}$ *be an interval and* $(\Phi_t)_{t\in I}$ *be a family of distributions in* $(\mathcal{N})_\mu^{-1}$*. Assume that there exist* $p, q \in \mathbb{N}$ *such that*

(i) for all $t \in I$, $S_\mu\Phi_t$ *is holomorphic on* $\mathcal{U}_{p,q}$*;*

(ii) for each $\varphi \in \mathcal{U}_{p,q}$*, the map* $I \ni t \mapsto S_\mu\Phi_t(\varphi) \in \mathbb{C}$ *is differentiable;*

(iii) there exists a constant such that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}S_\mu\Phi_t(\varphi)\right| \leq C, \qquad \text{for all } t \in I,\ \varphi \in \mathcal{U}_{p,q}.$$

Then $\Phi_\cdot$ *is differentiable in* $(\mathcal{N})_\mu^{-1}$ *at all* $t \in I$*, that is, for each* $t \in I$*,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t := \lim_{h\to 0}\frac{1}{h}\left(\Phi_{t+h} - \Phi_t\right)$$

exists as an element of $(\mathcal{N})_\mu^{-1}$*. Moreover, for each* $\varphi \in \mathcal{U}_{p,q}$ *and* $t \in I$*,*

$$S_\mu\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(\varphi) = \frac{\mathrm{d}}{\mathrm{d}t}S_\mu\Phi_t(\varphi).$$

Proof. Let $t \in I$ and $(h_n)_{n\in\mathbb{N}}$ be a nonzero sequence in $\mathbb{R}$ with $t + h_n \in I$ and $h_n \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, set

$$\Psi_n := \frac{1}{h_n}(\Phi_{t+h_n} - \Phi_t) \in (\mathcal{N})_\mu^{-1}.$$

Then for all $\varphi \in \mathcal{U}_{p,q}$, $(S_\mu\Psi_n(\varphi))_{n\in\mathbb{N}}$ is a Cauchy sequence, since

$$S_\mu\Psi_n(\varphi) = \frac{1}{h_n}(S_\mu\Phi_{t+h_n}(\varphi) - S_\mu\Phi_t(\varphi)) \to \frac{\mathrm{d}}{\mathrm{d}t}S_\mu\Phi_t(\varphi) \quad \text{as } n \to \infty.$$

Moreover, we infer by the Fundamental Theorem of Calculus that

$$|S_\mu \Psi_n(\varphi)| \leq \frac{1}{|h_n|} \left| \int_t^{t+h_n} \frac{\mathrm{d}}{\mathrm{d}s} S_\mu \Phi_s(\varphi)\, \mathrm{d}s \right| \leq C.$$

Thus, the sequence $(\Psi_n)_{n\in\mathbb{N}}$ fulfills the assumptions of Theorem 1.25, implying the existence of $\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t$ in $(\mathcal{N})_\mu^{-1}$. Moreover,

$$S_\mu \frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(\varphi) = \lim_{n\to\infty} S_\mu \Psi_n(\varphi) = \frac{\mathrm{d}}{\mathrm{d}t} S_\mu \Phi_t(\varphi). \qquad \square$$

Remark 1.27. As the space $\mathrm{Hol}_0\left(\mathcal{N}_\mathbb{C}\right)$ is an algebra, we infer from Equation (1.26) that Theorem 1.23, Theorem 1.24, Theorem 1.25, and Corollary 1.26 also hold if the S_μ-transform is replaced by the T_μ-transform.

Chapter 2

Mittag-Leffler Analysis in Product Spaces

In this chapter we will introduce the Mittag-Leffler measure. Moreover we will in addition consider finite products of the corresponding probability space. We show that for this setting, i.e. independent products of the Mittag-Leffler measure suitable distributions can be identified. In addition we give a characterization of these distributions. We show the admissibility for Appell systems for the product measure and moreover work out the well-known Donskers Delta function as example.

2.1 The Mittag-Leffler space

The definition of the Mittag-Leffler measure on $\mathcal{N}'$ relies on the following function introduced by Mittag-Leffler in a series of papers [ML03, ML04, ML05]; see also [Wim05a, Wim05b]. We also introduce two generalizations of this function first appeared in [Wim05a] and in [Pra71], respectively.

Definition 2.1. For $\beta \in (0, \infty)$, the *Mittag-Leffler function* E_β is an entire function defined by its power series

$$E_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \qquad z \in \mathbb{C}, \tag{2.1}$$

where Γ is the Gamma function. In addition, for $\rho, \gamma \in (0, \infty)$, the *two-parameter Mittag-Leffler function* $E_{\beta,\rho}$ and the *three-parameter Mittag-Leffler function* $E^\gamma_{\beta,\rho}$ are entire functions defined by the power series

$$E_{\beta,\rho}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + \rho)}, \qquad E^\gamma_{\beta,\rho}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\beta n + \rho) n!} z^n$$

for $z \in \mathbb{C}$, where $(\gamma)_n := \gamma(\gamma+1)\ldots(\gamma+n-1) = \Gamma(\gamma+n)/\Gamma(\gamma)$ is the n^{th} Pochhammer's symbol.

Note that for any $z \in \mathbb{C}$, $E^1_{\beta,\rho}(z) = E_{\beta,\rho}(z)$, $E_{\beta,1}(z) = E_\beta(z)$, and $E_1(z) = e^z$. Moreover, since E_β is entire, we can calculate its derivative by differentiating term-by-term the series in (2.1), and obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}E_\beta(z) = \frac{1}{\beta}E_{\beta,\beta}(z). \tag{2.2}$$

For $\beta \in (0,1)$, it was shown in [GJRdS15] that the following Laplace transform holds:

$$\int_0^\infty M_\beta(t)e^{-zt}\,\mathrm{d}t = E_\beta(-z), \quad z \in \mathbb{C},$$

where M_β is the so-called *M-Wright function*:

$$M_\beta(z) := \sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma(-\beta n + 1 - \beta)}, \quad z \in \mathbb{C}.$$

Now, it was shown in [Pol48] that for all $\beta \in (0,1]$, the map $x \mapsto E_\beta(-x)$ is completely monotonic on $[0,\infty)$, that is, for all $n \in \mathbb{N}$ and $x \geq 0$, we have $(-1)^n E^{(n)}_\beta(-x) \geq 0$. Using this fact, we can show in a similar manner as that of [Sch92] that the map

$$\mathcal{N} \ni \varphi \mapsto E_\beta\left(-\frac{1}{2}\langle\varphi,\varphi\rangle\right) \in \mathbb{R}$$

is a characteristic function on $\mathcal{N}$. Using the Bocher-Minlos theorem, the following definition from [GJ16] makes sense.

Definition 2.2. For $\beta \in (0,1]$, the *Mittag-Leffler measure* μ_β is defined as the unique probability measure on the space $(\mathcal{N}', C_\sigma(\mathcal{N}'))$ whose characteristic function is

$$\int_{\mathcal{N}'} e^{i\langle\omega,\varphi\rangle}\,\mathrm{d}\mu_\beta(\omega) = E_\beta\left(-\frac{1}{2}\langle\varphi,\varphi\rangle\right), \quad \varphi \in \mathcal{N}.$$

Remark 2.3. The class of Mittag-Leffler measures on $\mathcal{N}'$ includes the following.

(i) For $\beta = 1$, the Mittag-Leffler measure μ_1 on $\mathcal{N}'$ is the usual Gaussian measure on $\mathcal{N}'$ (see Subsection 1.4.1).

(ii) If $\mathcal{H} = \mathcal{N} = \mathbb{R}^n$, $n \in \mathbb{N}$, the Mittag-Leffler measure on $\mathcal{N}' = \mathbb{R}^n$ is called the *n-dimensional Mittag-Leffler measure*, and is denoted by μ^n_β. This has been studied in [Sch92].

(iii) The measure μ_β on $\mathcal{S}'(\mathbb{R})$ is also called the *grey noise (reference) measure* in [GJRdS15, GJ16].

In [GJRdS15, GJ16], the following properties of the measure μ_β are obtained.

Proposition 2.4. *For any $\varphi \in \mathcal{N}$ and $n \in \mathbb{N}_0$,*

$$\int_{\mathcal{N}'} \langle \omega, \varphi \rangle^{2n+1} \, \mathrm{d}\mu_\beta(\omega) = 0;$$
$$\int_{\mathcal{N}'} \langle \omega, \varphi \rangle^{2n} \, \mathrm{d}\mu_\beta(\omega) = \frac{(2n)!}{2^n \Gamma(\beta n + 1)} \langle \varphi, \varphi \rangle^n.$$

In particular, for all $\varphi, \psi \in \mathcal{N}$,

$$\|\langle \cdot, \varphi \rangle\|^2_{L^2(\mu_\beta)} = \frac{1}{\Gamma(\beta + 1)} |\varphi|_0^2. \tag{2.3}$$
$$\int_{\mathcal{N}'} \langle \omega, \varphi \rangle \langle \omega, \psi \rangle \, \mathrm{d}\mu_\beta(\omega) = \frac{1}{\Gamma(\beta + 1)} \langle \varphi, \psi \rangle.$$

Remark 2.5. Equation (2.3) allows us to extend the definition of the dual pairing $\langle \cdot, \cdot \rangle$ to $\mathcal{N}' \times \mathcal{H}$ in a similar manner as that of the Gaussian case (see the discussion on Subsection 1.4.1 about expression (1.9)). To find the characteristic function of $\langle \cdot, \eta \rangle$, $\eta \in \mathcal{H}$, by dropping to a subsequence, we can assume without loss of generality that $(\langle \cdot, \varphi_n \rangle)_{n=1}^\infty$ converges to $\langle \cdot, \eta \rangle$ μ_β-almost surely. Then for all $p \in \mathbb{R}$, $(e^{ip\langle \omega, \varphi_n \rangle})_{n=1}^\infty$ converges to $e^{ip\langle \omega, \eta \rangle}$. Since $|e^{ip\langle \omega, \varphi_n \rangle}| \leq 1$, we infer from Lebesgue's dominated convergence theorem that

$$\int_{\mathcal{N}'} e^{ip\langle \omega, \eta \rangle} \, \mathrm{d}\mu_\beta(\omega) = \lim_{n\to\infty} \int_{\mathcal{N}'} e^{ip\langle \omega, \varphi_n \rangle} \, \mathrm{d}\mu_\beta(\omega) = \lim_{n\to\infty} E_\beta \left(-\frac{p^2}{2} \langle \varphi_n, \varphi_n \rangle \right).$$

Since $\lim_{n\to\infty} \langle \varphi_n, \varphi_n \rangle = \langle \eta, \eta \rangle$, the continuity of E_β implies that

$$\int_{\mathcal{N}'} e^{ip\langle \omega, \eta \rangle} \, \mathrm{d}\mu_\beta(\omega) = E_\beta \left(-\frac{p^2}{2} |\eta|_0^2 \right), \quad p \in \mathbb{R}, \ \eta \in \mathcal{H}. \tag{2.4}$$

Using Equation (2.4), it was then shown in [GJRdS15, GJ16, Jah15] that Proposition 2.4 holds for elements in $\mathcal{H}$.

Proposition 2.6. *Let $\{\varphi_1, \ldots, \varphi_n\}$, $n \in \mathbb{N}$, be an orthonormal set in $\mathcal{H}$. The random variables $\langle \cdot, \varphi_1 \rangle, \ldots, \langle \cdot, \varphi_n \rangle$ on the probability space $(\mathcal{N}', \mu_\beta)$ are independent if and only if $n = 1$ or $\beta = 1$.*

Proof. Using (2.4), for all real numbers $p_1, \dots, p_n$,

$$\int_{\mathcal{N}'} \exp\left(i \sum_{r=1}^{n} p_r \langle \omega, \varphi_r \rangle\right) \mathrm{d}\mu_\beta(\omega)$$
$$= \int_{\mathcal{N}'} \exp\left(i \left\langle \omega, \sum_{r=1}^{n} p_r \varphi_r \right\rangle\right) \mathrm{d}\mu_\beta(\omega) = E_\beta\left(-\frac{1}{2} \sum_{r=1}^{n} p_r^2\right). \quad (2.5)$$

If $n \neq 1$, then independence of $\langle \cdot, \varphi_1 \rangle, \dots, \langle \cdot, \varphi_n \rangle$ holds if and and only if

$$E_\beta(-(x+y)) = E_\beta(-x) E_\beta(-y), \quad \text{for all } x, y \geq 0. \quad (2.6)$$

Indeed, if (2.6) holds, then Equations (2.5) and (2.4) imply that

$$\int_{\mathcal{N}'} \exp\left(i \sum_{r=1}^{n} p_r \langle \omega, \varphi_r \rangle\right) \mathrm{d}\mu_\beta(\omega) = \prod_{r=1}^{n} \int_{\mathcal{N}'} \exp(i p_r \langle \omega, \varphi_r \rangle) \,\mathrm{d}\mu_\beta(\omega), \quad (2.7)$$

that is, $\langle \cdot, \varphi_1 \rangle, \dots, \langle \cdot, \varphi_n \rangle$ are independent, and conversely, if (2.7) holds, then Equations (2.5) and (2.4) imply that

$$E_\beta\left(-\frac{1}{2} \sum_{r=1}^{n} p_r^2\right) = \prod_{r=1}^{n} E_\beta\left(-\frac{1}{2} p_r^2\right), \quad p_1, \dots, p_n \in \mathbb{R},$$

and in particular, (2.6) holds. Now, as E_β is entire and $E_\beta(0) = 1$, the identity theorem from complex analysis implies that (2.6) holds if and only if E_β is the exponential function, that is, $\beta = 1$. □

2.2 Finite products of Mittag-Leffler spaces

Let $d \in \mathbb{N}$, and consider the nuclear triple (1.4) in Subsection 1.1.1:

$$\widetilde{\mathcal{N}} := \bigoplus_{k=1}^{d} \mathcal{N}_k \subset \widetilde{\mathcal{H}} := \bigoplus_{k=1}^{d} \mathcal{H}_k \subset \widetilde{\mathcal{N}}' := \left(\bigoplus_{k=1}^{d} \mathcal{N}_k\right)' = \prod_{k=1}^{d} \mathcal{N}_k'.$$

In this case, the cylinder σ-algebra $C_\sigma(\widetilde{\mathcal{N}}')$ coincides with the product σ-algebra $\bigotimes_{k=1}^{d} C_\sigma(\mathcal{N}_k')$. Indeed, since $C_\sigma(\mathcal{N}_k')$ and $C_\sigma(\widetilde{\mathcal{N}}')$ coincide with the Borel σ-algebras $\mathcal{B}_\sigma(\mathcal{N}_k')$ and $\mathcal{B}_\sigma(\widetilde{\mathcal{N}}')$ generated by the strong topology on $\mathcal{N}_k'$ and $\widetilde{\mathcal{N}}'$, respectively (see Section 1.4), we use, e.g. [Els05, Kap. III, Satz 5.9], to obtain

$$C_\sigma(\widetilde{\mathcal{N}}') = \mathcal{B}_\sigma(\widetilde{\mathcal{N}}') \supset \bigotimes_{k=1}^{d} \mathcal{B}_\sigma(\mathcal{N}_k') = \bigotimes_{k=1}^{d} C_\sigma(\mathcal{N}_k'),$$

while the other inclusion follows from the fact that the cylindrical map

$$\widetilde{\mathcal{N}}' \ni \omega \mapsto (\langle \omega, \varphi^1 \rangle, \ldots, \langle \omega, \varphi^n \rangle) = \sum_{k=1}^{d} (\langle \omega_k, \varphi_k^1 \rangle, \ldots, \langle \omega_k, \varphi_k^n \rangle) \in \mathbb{R}^n \tag{2.8}$$

for $n \in \mathbb{N}$ and $\varphi^1, \ldots, \varphi^n \in \widetilde{\mathcal{N}}$ is $\bigotimes_{k=1}^{d} C_\sigma(\mathcal{N}_k')$-$\mathcal{B}(\mathbb{R}^n)$ measurable, since the k^{th} term of the sum in (2.8) is a composition of the canonical projection $\prod_{k=1}^{d} \mathcal{N}_k \to \mathcal{N}_k'$ followed by a cylinder map on $\mathcal{N}_k'$.

Now given $\varphi \in \widetilde{\mathcal{N}}$, consider the following measurable map

$$\widetilde{\mathcal{N}}' \ni \omega \mapsto (\langle \omega_1, \varphi_1 \rangle, \ldots, \langle \omega_d, \varphi_d \rangle) \in \mathbb{R}^d. \tag{2.9}$$

We would like to have a probability measure on $\widetilde{\mathcal{N}}'$ such that (2.9) is a random vector whose components are mutually independent random variables with respect to the Mittag-Leffler measure μ_β. A natural way to obtain this property is to equip the Mittag-Leffler measure μ_β on each of the spaces $\mathcal{N}_k'$, $k = 1, \ldots, d$, and use the d-fold product measure on $\widetilde{\mathcal{N}}'$, denoted by $\mu_\beta^{\otimes d}$, whose characteristic function is given by

$$\int_{\widetilde{\mathcal{N}}'} e^{i\langle \omega, \varphi \rangle} \, \mathrm{d}\mu_\beta^{\otimes d}(\omega) = \prod_{k=1}^{d} E_\beta \left(-\frac{1}{2} \langle \varphi_k, \varphi_k \rangle \right), \quad \varphi \in \widetilde{\mathcal{N}}. \tag{2.10}$$

Proposition 2.7. *For any $\varphi \in \widetilde{\mathcal{N}}$ and $n \in \mathbb{N}_0$,*

$$\int_{\widetilde{\mathcal{N}}'} \langle \omega, \varphi \rangle^{2n+1} \, \mathrm{d}\mu_\beta^{\otimes d}(\omega) = 0; \tag{2.11}$$

$$\int_{\widetilde{\mathcal{N}}'} \langle \omega, \varphi \rangle^{2n} \, \mathrm{d}\mu_\beta^{\otimes d}(\omega) = \frac{(2n)!}{2^n} \sum_{r} \frac{\langle \varphi_1, \varphi_1 \rangle^{r_1} \cdots \langle \varphi_d, \varphi_d \rangle^{r_d}}{\Gamma(\beta r_1 + 1) \cdots \Gamma(\beta r_d + 1)}, \tag{2.12}$$

where the sum in Equation (2.12) *is taken over all $r \in \mathbb{N}_0^d$ such that $r_1 + \cdots + r_d = n$, and we use the convention that $\langle \varphi_k, \varphi_k \rangle^0 = 1$, even if $\varphi_k = 0$. In particular, for all $\varphi, \psi \in \mathcal{N}$,*

$$\|\langle \cdot, \varphi \rangle\|^2_{L^2(\mu_\beta^{\otimes d})} = \frac{1}{\Gamma(\beta + 1)} |\varphi|_0^2, \tag{2.13}$$

$$\int_{\widetilde{\mathcal{N}}'} \langle \omega, \varphi \rangle \langle \omega, \psi \rangle \, \mathrm{d}\mu_\beta^{\otimes d}(\omega) = \frac{1}{\Gamma(\beta + 1)} \langle \varphi, \psi \rangle. \tag{2.14}$$

Proof. The multinomial theorem yields that for $m \in \mathbb{N}_0$,

$$\int_{\widetilde{\mathcal{N}}'} \langle \omega, \varphi \rangle^m \, \mathrm{d}\mu_\beta^{\otimes d}(\omega) = \int_{\widetilde{\mathcal{N}}'} \left(\sum_{k=1}^d \langle \omega_k, \varphi_k \rangle \right)^m \mathrm{d}\mu_\beta^{\otimes d}(\omega)$$
$$= \sum_p \frac{m!}{p_1! \cdots p_d!} \prod_{k=1}^d \int_{\mathcal{N}_k'} \langle \omega_k, \varphi_k \rangle^{p_k} \, \mathrm{d}\mu_\beta(\omega_k),$$

where the sum is taken over all $p \in \mathbb{N}_0^d$ such that $p_1 + \cdots + p_d = m$. Equations (2.11)-(2.12) then follow directly from Proposition 2.4. Set $n := 1$ to Equation (2.12) to obtain Equation (2.13), and Equation (2.14) follows immediately. □

Remark 2.8. In a similar manner as that of Remark 2.5, Equation (2.13) allows us to define $\langle \cdot, \eta \rangle$ for $\eta \in \widetilde{\mathcal{H}}$ as an $L^2(\mu_\beta^{\otimes d})$-limit of the sequence $(\langle \cdot, \varphi_n \rangle)_{n \in \mathbb{N}}$, where $(\varphi_n)_{n \in \mathbb{N}}$ is a sequence in $\widetilde{\mathcal{N}}$ that converges to η with respect to the norm in $\widetilde{\mathcal{H}}$, and this limit is independent of the approximating sequence $(\varphi_n)_{n \in \mathbb{N}}$ of η. Moreover, Lebesgue's dominated convergence theorem and the continuity of the Mittag-Leffler function E_β imply that

$$\int_{\widetilde{\mathcal{N}}'} e^{ip\langle \omega, \eta \rangle} \, \mathrm{d}\mu_\beta^{\otimes d}(\omega) = \prod_{k=1}^d E_\beta \left(-\frac{p^2}{2} |\eta_k|_{k,0}^2 \right), \quad p \in \mathbb{R}, \tag{2.15}$$

In fact, for $\mu_\beta^{\otimes d}$-almost all $\omega \in \widetilde{\mathcal{N}}'$,

$$\langle \omega, \eta \rangle = \sum_{k=1}^d \langle \omega, \eta_k \mathbf{e}_k \rangle = \sum_{k=1}^d \langle \omega_k, \eta_k \rangle. \tag{2.16}$$

Indeed, let $(\varphi^n)_{n \in \mathbb{N}}$ be a sequence in $\widetilde{\mathcal{N}}$ that converges to η in $\widetilde{\mathcal{H}}$. Then for each $k = 1, \ldots, d$, the sequences $(\varphi_k^n)_{n \in \mathbb{N}}$ and $(\varphi_k^n \mathbf{e}_k)_{n \in \mathbb{N}}$ converge to $\eta_k \in \mathcal{H}_k$ and $\eta_k \mathbf{e}_k \in \widetilde{\mathcal{H}}$, respectively. Hence, the following equality holds in $L^2(\mu_\beta^{\otimes d})$:

$$\langle \cdot, \eta \rangle = \lim_{n \to \infty} \langle \cdot, \varphi^n \rangle = \sum_{k=1}^d \lim_{n \to \infty} \langle \cdot, \varphi_k^n \mathbf{e}_k \rangle = \sum_{k=1}^d \langle \cdot, \eta_k \mathbf{e}_k \rangle,$$

and similar computations hold for the other equality. From (2.16) and Remark 2.5, we infer that Proposition 2.7 holds for elements in $\widetilde{\mathcal{H}}$. These observations can be extended to the space $\widetilde{\mathcal{H}}_\mathbb{C}$: if $\varphi := \varphi^1 + i\varphi^2 \in \widetilde{\mathcal{N}}_\mathbb{C}$,

$$\|\langle \cdot, \varphi \rangle\|_{L^2(\mu_\beta^{\otimes d})}^2 = \int_{\widetilde{\mathcal{N}}'} \left(\langle \omega, \varphi^1 \rangle^2 + \langle \omega, \varphi^2 \rangle^2 \right) \mathrm{d}\mu_\beta^{\otimes d}(\omega)$$
$$= \frac{1}{\Gamma(\beta+1)} \left(|\varphi^1|_0^2 + |\varphi^2|_0^2 \right) = \frac{1}{\Gamma(\beta+1)} |\varphi|_0^2.$$

Thus, $\langle\cdot,\eta\rangle$ for $\eta\in\widetilde{\mathcal{H}}_{\mathbb{C}}$ is defined in $L^2(\mu_\beta^{\otimes d})$ in a similar manner as that of the case for $\widetilde{\mathcal{H}}$.

Remark 2.9. A generalization of the product measure $\mu_\beta^{\otimes d}$ on $\widetilde{\mathcal{N}}'$ can be obtained as follows. Let $\beta\in(0,1]^d$. Equip each space $\mathcal{N}_k'$ with the measure μ_{β_k}, and use the product measure $\mu_{\vec{\beta}}:=\mu_{\beta_1}\times\cdots\times\mu_{\beta_d}$ on $\widetilde{\mathcal{N}}'$. In this case, the characteristic function is given by

$$\int_{\widetilde{\mathcal{N}}'} e^{i\langle\omega,\varphi\rangle}\,\mathrm{d}\mu_{\vec{\beta}}(\omega)=\prod_{k=1}^d E_{\beta_k}\left(-\frac{1}{2}\langle\varphi_k,\varphi_k\rangle\right),\quad \varphi\in\widetilde{\mathcal{N}}.$$

Moreover, by following the proof similar to that of Proposition 2.7, the following equations hold for all $\varphi\in\widetilde{\mathcal{N}}$ and $n\in\mathbb{N}_0$:

$$\int_{\widetilde{\mathcal{N}}'}\langle\omega,\varphi\rangle^{2n+1}\,\mathrm{d}\mu_{\vec{\beta}}(\omega)=0,$$

$$\int_{\widetilde{\mathcal{N}}'}\langle\omega,\varphi\rangle^{2n}\,\mathrm{d}\mu_{\vec{\beta}}(\omega)=\frac{(2n)!}{2^n}\sum_{r}\frac{\langle\varphi_1,\varphi_1\rangle^{r_1}\cdots\langle\varphi_d,\varphi_d\rangle^{r_d}}{\Gamma(\beta_1 r_1+1)\cdots\Gamma(\beta_d r_d+1)},$$

where the sum in the last equation is taken in the same manner as that of Equation (2.12). In particular, on the space $L^2(\mu_{\vec{\beta}})$, we have

$$\|\langle\cdot,\varphi\rangle\|^2_{L^2(\mu_{\vec{\beta}})}=\sum_{k=1}^d\frac{1}{\Gamma(\beta_k+1)}|\varphi_k|^2_{k,0},\quad\varphi\in\widetilde{\mathcal{N}},$$

so that we can define $\langle\cdot,\eta\rangle$, $\eta\in\widetilde{\mathcal{H}}$, on the space $L^2(\mu_{\vec{\beta}})$ in a similar way as that of Remark 2.8.

2.3 Distributions on the product Mittag-Leffler measure

As $\mu_\beta^{\otimes d}$ is generally non-Gaussian, we use the Appell system to construct test functions and distributions on $(\widetilde{\mathcal{N}}',\mu_\beta^{\otimes d})$. In the following, we show that $\mu_\beta^{\otimes d}$ satisfies **(A1)** and **(A2)**.

Lemma 2.10. *Let $\varphi\in\widetilde{\mathcal{N}}$ and $\lambda\in\mathbb{R}$. Then the exponential map $\widetilde{\mathcal{N}}'\ni\omega\mapsto e^{|\lambda\langle\omega,\varphi\rangle|}$ is integrable with respect to $\mu_\beta^{\otimes d}$ and*

$$\int_{\widetilde{\mathcal{N}}'} e^{\lambda\langle\omega,\varphi\rangle}\,\mathrm{d}\mu_\beta^{\otimes d}(\omega)=\prod_{k=1}^d E_\beta\left(\frac{\lambda^2}{2}\langle\varphi_k,\varphi_k\rangle\right).$$

Proof. By [GJRdS15, Lemma 4.1],

$$\int_{\widetilde{\mathcal{N}}'} e^{|\lambda\langle\omega,\varphi\rangle|}\,\mathrm{d}\mu_\beta^{\otimes d}(\omega) \le \int_{\widetilde{\mathcal{N}}'} \exp\left(\sum_{k=1}^{d} \left|\lambda\langle\omega_k,\varphi_k\rangle\right|\right) \mathrm{d}\mu_\beta^{\otimes d}(\omega)$$
$$= \prod_{k=1}^{d} \int_{\mathcal{N}_k'} \exp\left(\left|\lambda\langle\omega_k,\varphi_k\rangle\right|\right) \mathrm{d}\mu_\beta(\omega_k) < \infty,$$

and

$$\int_{\widetilde{\mathcal{N}}'} e^{\lambda\langle\omega,\varphi\rangle}\,\mathrm{d}\mu_\beta^{\otimes d}(\omega) = \prod_{k=1}^{d} \int_{\mathcal{N}_k'} e^{\lambda\langle\omega_k,\varphi_k\rangle}\,\mathrm{d}\mu_\beta(\omega_k) = \prod_{k=1}^{d} E_\beta\left(\frac{\lambda^2}{2}\langle\varphi_k,\varphi_k\rangle\right). \qquad \square$$

Proposition 2.11. *The map*

$$\widetilde{\mathcal{N}}_{\mathbb{C}} \ni \varphi \mapsto l_{\mu_\beta^{\otimes d}}(\varphi) := \int_{\widetilde{\mathcal{N}}'} e^{\langle\omega,\varphi\rangle}\,\mathrm{d}\mu_\beta^{\otimes d}(\omega) \in \mathbb{C}$$

is a holomorphic map from $\widetilde{\mathcal{N}}_{\mathbb{C}}$ to $\mathbb{C}$.

Proof. Note that if $\varphi := \varphi^1 + i\varphi^2 \in \widetilde{\mathcal{N}}_{\mathbb{C}}$, then Lemma 2.10 implies that

$$|l_{\mu_\beta^{\otimes d}}(\varphi)| \le \int_{\widetilde{\mathcal{N}}'} e^{\langle\omega,\varphi^1\rangle}\,\mathrm{d}\mu_\beta^{\otimes d}(\omega) = \prod_{k=1}^{d} E_\beta\left(\frac{1}{2}\langle\varphi_k^1,\varphi_k^1\rangle\right). \tag{2.17}$$

Since E_β is continuous on $\mathbb{C}$, for each $\varphi^0 \in \widetilde{\mathcal{N}}_{\mathbb{C}}$, estimate (2.17) shows that $l_{\mu_\beta^{\otimes d}}$ is bounded on the neighborhood $\{\varphi \in \widetilde{\mathcal{N}}_{\mathbb{C}} : |\varphi - \varphi^0|_0 \le 1\}$ of φ^0, and thus $l_{\mu_\beta^{\otimes d}}$ is locally bounded on $\widetilde{\mathcal{N}}_{\mathbb{C}}$. Now we show that $l_{\mu_\beta^{\otimes d}}$ is G-holomorphic, that is, the map $\mathbb{C} \ni z \mapsto f(z) := l_{\mu_\beta^{\otimes d}}(\varphi^0 + z\varphi)$, where $\varphi^0, \varphi \in \widetilde{\mathcal{N}}_{\mathbb{C}}$, is holomorphic on some neighborhood of zero in $\mathbb{C}$. Note that f is continuous: given $z \in \mathbb{C}$ and a sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ converging to z, the following estimate holds for sufficiently large n:

$$|\exp(\langle\omega,\varphi^0 + z_n\varphi\rangle)| \le \exp(|\langle\omega,\varphi^0\rangle|)\exp\left((1+|z|)|\langle\omega,\varphi\rangle|\right),$$

and thus continuity of f follows from Lemma 2.10, Cauchy-Schwarz inequality, and Lebesgue dominated convergence theorem. Moreover, if γ is a closed, bounded curve in $\mathbb{C}$, then the compactness of γ allows us to use Fubini's theorem:

$$\int_\gamma \int_{\widetilde{\mathcal{N}}'} \exp(\langle\omega,\varphi^0 + z\varphi\rangle)\,\mathrm{d}\mu_\beta^{\otimes d}(\omega)\,\mathrm{d}z = \int_{\widetilde{\mathcal{N}}'} \int_\gamma \exp(\langle\omega,\varphi^0 + z\varphi\rangle)\,\mathrm{d}z\,\mathrm{d}\mu_\beta^{\otimes d}(\omega) = 0,$$

where the last equality holds as the exponential function is holomorphic on $\mathbb{C}$. By Morera's theorem, f is holomorphic on $\mathbb{C}$, and thus, $l_{\mu_\beta^{\otimes d}}$ is G-holomorphic. By Proposition 1.8, $l_{\mu_\beta^{\otimes d}}$ is holomorphic. □

Corollary 2.12. *For $\eta \in \widetilde{\mathcal{H}}_{\mathbb{C}}$ and $z \in \mathbb{C}$, the map $\widetilde{\mathcal{N}}' \ni \omega \mapsto e^{|z\langle \omega, \eta \rangle|}$ is integrable with respect to $\mu_\beta^{\otimes d}$ and*

$$\int_{\widetilde{\mathcal{N}}'} e^{z\langle \omega, \eta \rangle} \, \mathrm{d}\mu_\beta^{\otimes d}(\omega) = \prod_{k=1}^{d} E_\beta \left(\frac{z^2}{2} \langle \eta_k, \eta_k \rangle \right).$$

Proof. In the proof of Proposition 2.11, we have shown that for all $\varphi^0, \varphi \in \widetilde{\mathcal{N}}_{\mathbb{C}}$, the map

$$\mathbb{C} \ni z \mapsto \int_{\widetilde{\mathcal{N}}'} e^{\langle \omega, \varphi^0 + z\varphi \rangle} \, \mathrm{d}\mu_\beta^{\otimes d}(\omega) \in \mathbb{C}$$

is holomorphic on $\mathbb{C}$. Since

$$\int_{\widetilde{\mathcal{N}}'} e^{\langle \omega, \xi^1 + t\xi^2 \rangle} \, \mathrm{d}\mu_\beta^{\otimes d}(\omega) = \prod_{k=1}^{d} E_\beta \left(\frac{1}{2} \langle \xi_k^1 + t\xi_k^2, \xi_k^1 + t\xi_k^2 \rangle \right), \quad t \in \mathbb{R}, \ \xi^1, \xi^2 \in \widetilde{\mathcal{N}},$$

by Lemma 2.10, we infer from the identity theorem in complex analysis that

$$\int_{\widetilde{\mathcal{N}}'} e^{\langle \omega, \xi^1 + z\xi^2 \rangle} \, \mathrm{d}\mu_\beta^{\otimes d}(\omega) = \prod_{k=1}^{d} E_\beta \left(\frac{1}{2} \langle \xi_k^1 + z\xi_k^2, \xi_k^1 + z\xi_k^2 \rangle \right), \quad z \in \mathbb{C}, \ \xi^1, \xi^2 \in \widetilde{\mathcal{N}}.$$

If $\varphi \in \widetilde{\mathcal{N}}_{\mathbb{C}}$ and $z \in \mathbb{C}$, then we set $z\varphi := \xi^1 + i\xi^2$ with $\xi_1, \xi_2 \in \widetilde{\mathcal{N}}$ to obtain

$$\begin{aligned} \int_{\widetilde{\mathcal{N}}'} e^{z\langle \omega, \varphi \rangle} \, \mathrm{d}\mu_\beta^{\otimes d}(\omega) &= \int_{\widetilde{\mathcal{N}}'} e^{\langle \omega, \xi^1 + i\xi^2 \rangle} \, \mathrm{d}\mu_\beta^{\otimes d}(\omega) \\ &= \prod_{k=1}^{d} E_\beta \left(\frac{1}{2} \langle \xi_k^1 + i\xi_k^2, \xi_k^1 + i\xi_k^2 \rangle \right) = \prod_{k=1}^{d} E_\beta \left(\frac{z^2}{2} \langle \varphi_k, \varphi_k \rangle \right). \end{aligned}$$

Now, we apply Equation (2.16) and use a proof similar to that of [GJRdS15, Lemma 4.1] to infer that Lemma 2.10 holds for $\eta \in \widetilde{\mathcal{H}}$. From here, the proof for the case $\eta \in \widetilde{\mathcal{H}}_{\mathbb{C}}$ and $z \in \mathbb{C}$ is constructed in a similar manner as that of $\widetilde{\mathcal{N}}_{\mathbb{C}}$. □

It is shown in [GJRdS15, Theorem 4.5] that the Mittag-Leffler measure satisfies **(A2)**, and so the proof that $\mu_\beta^{\otimes d}$ also satisfies **(A2)** is straightforward.

Proposition 2.13. *For any nonempty open subset $\mathcal{U} \subset \widetilde{\mathcal{N}}'$, we have $\mu_\beta^{\otimes d}(\mathcal{U}) > 0$.*

Proof. Let $\mathcal{U}$ be a nonempty open subset in $\widetilde{\mathcal{N}}'$. As $\widetilde{\mathcal{N}}'$ is a finite product space, there exist nonempty open sets $\mathcal{U}_1, \dots, \mathcal{U}_d$ in $\mathcal{N}_1', \dots, \mathcal{N}_d'$, respectively, such that $\mathcal{U}_1 \times \cdots \times \mathcal{U}_d \subset \mathcal{U}$. Since the measure μ_β on $\mathcal{N}_k'$ satisfies **(A2)** by [GJRdS15, Theorem 4.5], we have

$$\mu_\beta^{\otimes d}(\mathcal{U}) \geq \mu_\beta(\mathcal{U}_1) \cdots \mu_\beta(\mathcal{U}_d) > 0. \qquad \square$$

Since $(\widetilde{\mathcal{N}}', \mu_\beta^{\otimes d})$ satisfies **(A1)** and **(A2)**, we can construct the Appell system generated by $\mu_\beta^{\otimes d}$, and all results from Subsection 1.4.2 are applicable to $\mu_\beta^{\otimes d}$. To finish this section, we write all the formulas constructed using the Appell system generated by $\mu_\beta^{\otimes d}$ that are needed for subsequent discussions.

Now, Corollary 2.12 shows an explicit formula for the Laplace transform of $\mu_\beta^{\otimes d}$:

$$l_{\mu_\beta^{\otimes d}}(\varphi) = \int_{\widetilde{\mathcal{N}}'} e^{\langle \omega, \varphi \rangle} \, \mathrm{d}\mu_\beta^{\otimes d}(\omega) = \prod_{k=1}^{d} E_\beta\left(\frac{1}{2}\langle \varphi_k, \varphi_k \rangle\right), \quad \varphi \in \widetilde{\mathcal{N}}_{\mathbb{C}}. \tag{2.18}$$

Since E_β is entire and $E_\beta(0) = 1$, there exists $\varepsilon_\beta > 0$ such that $E_\beta(z) \neq 0$ for all $z \in \mathbb{C}$ in the open disk $\{|z| < \varepsilon_\beta\}$. Thus, $l_{\mu_\beta^{\otimes d}}(\varphi) \neq 0$ for all φ in the following neighborhood of zero in $\widetilde{\mathcal{N}}_{\mathbb{C}}$:

$$\mathcal{U}_\beta := \{\varphi \in \widetilde{\mathcal{N}}_{\mathbb{C}} : |\langle \varphi_k, \varphi_k \rangle| < \varepsilon_\beta, \text{ for all } k = 1, \dots, d\}.$$

Thus, the normalized exponential

$$e_{\mu_\beta^{\otimes d}}(\varphi; \cdot) = \frac{e^{\langle \cdot, \varphi \rangle}}{l_{\mu_\beta^{\otimes d}}(\varphi)} = e^{\langle \cdot, \varphi \rangle} \prod_{k=1}^{d} \frac{1}{E_\beta\left(\frac{1}{2}\langle \varphi_k, \varphi_k \rangle\right)}$$

is well-defined on $\widetilde{\mathcal{N}}'$ for all $\varphi \in \mathcal{U}_\beta$. Throughout this dissertation, whenever we use $e_{\mu_\beta^{\otimes d}}(\varphi; \cdot)$, we frequently refer to $\mathcal{U}_\beta$ as an explicit neighborhood of zero in $\widetilde{\mathcal{N}}_{\mathbb{C}}$ where $e_{\mu_\beta^{\otimes d}}(\varphi; \cdot)$ is defined.

Using the neighborhood $\mathcal{U}_\beta$, we can further specify the domain of the $S_{\mu_\beta^{\otimes d}}$-transform and $T_{\mu_\beta^{\otimes d}}$-transform of a distribution $\Phi \in (\widetilde{\mathcal{N}})^{-1}_{\mu_\beta^{\otimes d}}$:

$$S_{\mu_\beta^{\otimes d}}\Phi(\varphi) := \langle\!\langle \Phi, e_{\mu_\beta^{\otimes d}}(\varphi; \cdot) \rangle\!\rangle_{\mu_\beta^{\otimes d}}, \qquad T_{\mu_\beta^{\otimes d}}\Phi(\varphi) := \langle\!\langle \Phi, e^{i\langle \cdot, \varphi \rangle} \rangle\!\rangle_{\mu_\beta^{\otimes d}},$$

for $\varphi \in \mathcal{U}_{p,q}$, where $p, q \in \mathbb{N}$ are chosen such that $\Phi \in (\mathcal{H}_{-p})^{-1}_{-q,\mu_\beta^{\otimes d}}$ and $\mathcal{U}_{p,q} \subset \mathcal{U}_\beta$. In view of Equation (2.18), Equation (1.26) relating the two transforms becomes

$$T_{\mu_\beta^{\otimes d}}\Phi(\varphi) = S_{\mu_\beta^{\otimes d}}\Phi(i\varphi) \prod_{k=1}^{d} E_\beta\left(-\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right), \quad \varphi \in \mathcal{U}_{p,q}. \tag{2.19}$$

2.4 Donsker's delta of Mittag-Leffler random vectors

In this section, we want to construct a distribution in $(\widetilde{\mathcal{N}})^{-1}_{\mu_\beta^{\otimes d}}$ for $\beta \in (0, 1]$, which is a generalization of Donsker's delta of d-dimensional Brownian motion. First, for $\eta \in \widetilde{\mathcal{H}}$, define the random vector $G(\cdot, \eta)$ on $\widetilde{\mathcal{N}}'$ by

$$G(\omega, \eta) := (\langle \omega, \eta_1 \mathbf{e}_1\rangle, \ldots, \langle \omega, \eta_d \mathbf{e}_d\rangle) = (\langle \omega_1, \eta_1\rangle, \ldots, \langle \omega_d, \eta_d\rangle) \in \mathbb{R}^d, \quad \text{for } \mu_\beta^{\otimes d}\text{-a.a. } \omega \in \widetilde{\mathcal{N}}'.$$

This random vector is well defined as an element of $L^2(\mu_\beta^{\otimes d}; \mathbb{R}^d)$ by Remark 2.8. The following properties of $G(\cdot, \eta)$ follow directly from the definition of $\mu_\beta^{\otimes d}$ and Proposition 2.4.

Proposition 2.14. *Let $\eta, \zeta \in \widetilde{\mathcal{H}}$ and $p \in \mathbb{R}^d$.*

(i) The characteristic function of $G(\cdot, \eta)$ is given by

$$\mathbb{E}_{\mu_\beta^{\otimes d}}\left(e^{i(p, G(\cdot, \eta))}\right) = \prod_{k=1}^{d} E_\beta\left(-\frac{1}{2}p_k^2 |\eta_k|_{k,0}^2\right).$$

(ii) The characteristic function of $G(\cdot, \eta) - G(\cdot, \zeta)$ is given by

$$\mathbb{E}_{\mu_\beta^{\otimes d}}\left(e^{i(p, G(\cdot, \eta) - G(\cdot, \zeta))}\right) = \prod_{k=1}^{d} E_\beta\left(-\frac{1}{2}p_k^2 |\eta_k - \zeta_k|_{k,0}^2\right).$$

(iii) The expectation vector of $G(\cdot, \eta)$ is zero, and for all $i, j = 1, \ldots, d$,

$$\mathbb{E}_{\mu_\beta^{\otimes d}}\left(G(\cdot, \eta)_i \, G(\cdot, \zeta)_j\right) = \frac{1}{\Gamma(\beta + 1)} \delta_{i,j} (\eta_i, \zeta_i)_{\mathcal{H}_i}.$$

In particular,

$$\| |G(\cdot,\eta)| \|^2_{L^2(\mu_\beta^{\otimes d})} = \frac{1}{\Gamma(\beta+1)}|\eta|_0^2.$$

and the covariance matrix of $G(\cdot,\eta)$ is given by

$$\frac{1}{\Gamma(\beta+1)}\mathrm{diag}\left(|\eta_1|^2_{1,0},\dots,|\eta_d|^2_{d,0}\right).$$

(iv) *The components of $G(\cdot,\eta)$ are mutually independent.*

Proposition 2.15. *For $\eta \in \widetilde{\mathcal{H}}$, the $S_{\mu_\beta^{\otimes d}}$-transform of the random variable $\langle\cdot,\eta\rangle$ on $\widetilde{\mathcal{N}}'$ is given by*

$$S_{\mu_\beta^{\otimes d}}\langle\cdot,\eta\rangle(\varphi) = \sum_{k=1}^d \frac{E_{\beta,\beta}\left(\frac{1}{2}\langle\varphi_k,\varphi_k\rangle\right)}{\beta E_\beta\left(\frac{1}{2}\langle\varphi_k,\varphi_k\rangle\right)}\langle\varphi_k,\eta_k\rangle, \quad \varphi\in\mathcal{U}_\beta. \tag{2.20}$$

Proof. Since $\langle\cdot,\eta\rangle \in L^2(\mu_\beta^{\otimes d})$ by Remark 2.8, for $\varphi := \varphi^1 + i\varphi^2 \in \mathcal{U}_\beta$,

$$\begin{aligned} S_{\mu_\beta^{\otimes d}}\langle\cdot,\eta\rangle(\varphi) &= \frac{1}{l_{\mu_\beta^{\otimes d}}(\varphi)}\int_{\widetilde{\mathcal{N}}'}\langle\omega,\eta\rangle e^{\langle\omega,\varphi\rangle}\,\mathrm{d}\mu_\beta^{\otimes d}(\omega) \\ &= \int_{\widetilde{\mathcal{N}}'}\frac{\mathrm{d}}{\mathrm{d}t}e^{\langle\omega,\varphi\rangle+t\langle\omega,\eta\rangle}\bigg|_{t=0}\,\mathrm{d}\mu_\beta^{\otimes d}(\omega)\cdot\prod_{k=1}^d\frac{1}{E_\beta\left(\frac{1}{2}\langle\varphi_k,\varphi_k\rangle\right)}. \end{aligned} \tag{2.21}$$

Now, for $t\in[-1,1]$, the map $\widetilde{\mathcal{N}}' \ni \omega \mapsto e^{\langle\omega,\varphi\rangle+t\langle\omega,\eta\rangle}$ belongs to $L^1(\mu_\beta^{\otimes d})$ by Hölder's inequality and Corollary 2.12. Moreover, for $\mu_\beta^{\otimes d}$-a.a. $\omega\in\widetilde{\mathcal{N}}'$,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}e^{\langle\omega,\varphi\rangle+t\langle\omega,\eta\rangle}\right| = |\langle\omega,\eta\rangle|\,|e^{\langle\omega,\varphi\rangle+t\langle\omega,\eta\rangle}| \le e^{\langle\omega,\varphi^1\rangle}e^{2|\langle\omega,\eta\rangle|},$$

and the map $\widetilde{\mathcal{N}}' \ni \omega \mapsto e^{\langle\omega,\varphi^1\rangle}e^{2|\langle\omega,\eta\rangle|}$ also belongs to $L^1(\mu_\beta^{\otimes d})$ by Hölder's inequality, Lemma 2.10, and Corollary 2.12. Thus, interchanging the derivative and integral in Equation (2.21) is allowed. Since $e^{\langle\omega,\varphi\rangle+t\langle\omega,\eta\rangle} = e^{\langle\omega,\varphi+t\eta\rangle}$ for each $t\in[-1,1]$ and for $\mu_\beta^{\otimes d}$-a.a. $\omega\in\widetilde{\mathcal{N}}'$, applying Corollary 2.12 and Equation (2.2) after interchanging the derivative and integral yield

$$\begin{aligned} \int_{\widetilde{\mathcal{N}}'}\frac{\mathrm{d}}{\mathrm{d}t}e^{\langle\omega,\varphi\rangle+t\langle\omega,\eta\rangle}\bigg|_{t=0}\,\mathrm{d}\mu_\beta^{\otimes d}(\omega) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\prod_{k=1}^d E_\beta\left(\frac{1}{2}\langle\varphi_k+t\eta_k,\varphi_k+t\eta_k\rangle\right)\right)\bigg|_{t=0} \\ &= \frac{1}{\beta}\sum_{k=1}^d\langle\varphi_k,\eta_k\rangle E_{\beta,\beta}\left(\frac{1}{2}\langle\varphi_k,\varphi_k\rangle\right)\prod_{j\neq k}E_\beta\left(\frac{1}{2}\langle\varphi_j,\varphi_j\rangle\right). \end{aligned}$$

The last equation and (2.21) imply (2.20). □

Corollary 2.16. *For $\eta \in \widetilde{\mathcal{H}}$ and $\varphi \in \mathcal{U}_0$, $\mathcal{U}_0 \subset \widetilde{\mathcal{N}}_{\mathbb{C}}$ a suitable neighborhood of zero,*

$$S_{\mu_\beta^{\otimes d}} G(\cdot,\eta)(\varphi) = \sum_{k=1}^{d} \frac{E_{\beta,\beta}\left(\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)}{\beta E_\beta\left(\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)} \langle \varphi_k, \eta_k\rangle \mathbf{e}_k.$$

Proof. Apply Proposition 2.15 to the random variable $\langle \cdot, \eta_k \mathbf{e}_k\rangle$, $k = 1, \ldots, d$, and note that $G(\omega,\eta) = \sum_{k=1}^{d} \langle \omega, \eta_k \mathbf{e}_k\rangle \mathbf{e}_k$ for $\mu_\beta^{\otimes d}$-almost all $\omega \in \widetilde{\mathcal{N}}'$ by Equation (2.16). □

Theorem 2.17. *Let $\eta \in \widetilde{\mathcal{H}}$ such that $\eta_k \neq 0$ for all $k = 1, \ldots, d$. Then the d-dimensional Donsker's delta at $a \in \mathbb{R}^d$, defined via the integral representation*

$$\delta_a(G(\cdot,\eta)) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(s, G(\cdot,\eta) - a)} \, \mathrm{d}s,$$

exists in the space $(\widetilde{\mathcal{N}})^{-1}_{\mu_\beta^{\otimes d}}$ as a weak integral in the sense of Theorem 1.24.

Proof. Since $e^{i(s,G(\cdot,\eta)-a)} \in L^2(\mu_\beta^{\otimes d})$, we apply Corollary 2.12 to obtain its $T_{\mu_\beta^{\otimes d}}$-transform at $\varphi \in \widetilde{\mathcal{N}}_{\mathbb{C}}$:

$$\begin{aligned}
T_{\mu_\beta^{\otimes d}} e^{i(s,G(\cdot,\eta)-a)}(\varphi) &= \int_{\widetilde{\mathcal{N}}'} e^{i(s,G(\omega,\eta)-a)} e^{i\langle \omega,\varphi\rangle} \, \mathrm{d}\mu_\beta^{\otimes d}(\omega) \\
&= e^{-i(s,a)} \int_{\widetilde{\mathcal{N}}'} \exp\left(i \left\langle \omega, \sum_{k=1}^{d} s_k \eta_k \mathbf{e}_k + \varphi \right\rangle \right) \mathrm{d}\mu_\beta^{\otimes d}(\omega) \\
&= e^{-i(s,a)} \prod_{k=1}^{d} E_\beta\left(-\frac{1}{2} s_k^2 |\eta_k|^2_{\mathcal{H}_k} - s_k \langle \eta_k, \varphi_k\rangle - \frac{1}{2}\langle \varphi_k, \varphi_k\rangle \right) \qquad (2.22)
\end{aligned}$$

Hence, the map $\mathbb{R}^d \ni s \mapsto T_{\mu_\beta^{\otimes d}} e^{i(s,G(\cdot,\eta)-a)}(\varphi)$ is measurable for all $\varphi \in \widetilde{\mathcal{N}}_{\mathbb{C}}$. Now, for each $k = 1, \ldots, d$ and $M \in (0,\infty)$, there exists a constant $C_k \in (0,\infty)$ such that

$$\int_{\mathbb{R}} \left| E_\beta\left(-\frac{1}{2} s_k^2 |\eta_k|^2_{\mathcal{H}_k} - s_k \langle \eta_k, \varphi_k\rangle - \frac{1}{2}\langle \phi, \phi\rangle \right) \right| \mathrm{d}s_k \leq C_k.$$

for all ϕ belonging to the set $\mathcal{U}_k := \{\phi \in \mathcal{N}_{k,\mathbb{C}} : |\phi|_{\mathcal{H}_{k,\mathbb{C}}} < M\}$. Indeed, the case for $\beta \in (0,1)$ has been proven in [GJRdS15, Proposition 5.2]. The case for $\beta = 1$

is even simpler: setting $\phi := \phi_1 + i\phi_2 \in \mathcal{U}_k$, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \left| E_\beta \left(-\frac{1}{2} s_k^2 |\eta_k|^2_{\mathcal{H}_k} - s_k \langle \eta_k, \varphi_k \rangle - \frac{1}{2} \langle \phi, \phi \rangle \right) \right| \, \mathrm{d}s_k \\
&= \int_{\mathbb{R}} \exp \left(\Re \left(-\frac{1}{2} s_k^2 |\eta_k|^2_{\mathcal{H}_k} - s_k \langle \eta_k, \phi \rangle - \frac{1}{2} \langle \phi, \phi \rangle \right) \right) \mathrm{d}s_k \\
&= \int_{\mathbb{R}} \exp \left(- \left(\frac{1}{2} s_k^2 |\eta_k|^2_{\mathcal{H}_k} + s_k \langle \eta_k, \phi_1 \rangle + \frac{1}{2} \left(|\phi_1|^2_{\mathcal{H}_k} - |\phi_2|^2_{\mathcal{H}_k} \right) \right) \right) \mathrm{d}s_k \\
&= \sqrt{\frac{2\pi}{|\eta_k|^2_{\mathcal{H}_k}}} \exp \left(\frac{\langle \eta_k, \phi_1 \rangle^2}{2|\eta_k|^2_{\mathcal{H}_k}} + \frac{1}{2} |\phi_2|^2_{\mathcal{H}_k} - \frac{1}{2} |\phi_1|^2_{\mathcal{H}_k} \right).
\end{aligned}$$

Since $\phi \in \mathcal{U}_k$, we use Cauchy-Schwarz inequality to obtain the following bound:

$$\frac{\langle \eta_k, \phi_1 \rangle^2}{2|\eta_k|^2_{\mathcal{H}_k}} + \frac{1}{2} |\phi_2|^2_{\mathcal{H}_k} - \frac{1}{2} |\phi_1|^2_{\mathcal{H}_k} \leq \frac{1}{2} |\phi_2|^2_{\mathcal{H}_k} < \frac{1}{2} M^2,$$

and so

$$\int_{\mathbb{R}} \left| E_\beta \left(-\frac{1}{2} s_k^2 |\eta_k|^2_{\mathcal{H}_k} - s_k \langle \eta_k, \varphi_k \rangle - \frac{1}{2} \langle \phi, \phi \rangle \right) \right| \, \mathrm{d}s_k \leq \sqrt{\frac{2\pi}{|\eta_k|^2_{\mathcal{H}_k}}} \exp \left(\frac{1}{2} M^2 \right) < \infty.$$

Hence, for all φ belonging to $\mathcal{U}_0 := \{\varphi \in \widetilde{\mathcal{N}}_{\mathbb{C}}, |\varphi|_{\widetilde{\mathcal{H}}_{\mathbb{C}}} < M\}$, we use the last estimate and (2.22) to obtain

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| T_{\mu_\beta^{\otimes d}} e^{i(s, G(\cdot,\eta) - a)}(\varphi) \right| \mathrm{d}s \leq \frac{1}{(2\pi)^d} \prod_{k=1}^{d} C_k < \infty. \tag{2.23}$$

Therefore, $\delta_a(\langle \cdot, \eta \rangle) \in (\widetilde{\mathcal{N}})^{-1}_{\mu_\beta^{\otimes d}}$ by Theorem 1.24. □

Remark 2.18. We can use [GJRdS15, Theorem 5.3] to obtain an explicit formula for the $T_{\mu_\beta^{\otimes d}}$-transform of the Donsker's delta at $a = 0$: for all $\varphi \in \mathcal{U}_0$, $\mathcal{U}_0$ as in the proof of Theorem 2.17,

$$T_{\mu_\beta^{\otimes d}} \delta_0(G(\cdot, \eta))(\varphi) = \frac{1}{(2\pi)^{d/2}} \prod_{k=1}^{d} \langle \eta_k, \eta_k \rangle^{-1/2} H^{1\,1}_{1\,2} \left(\frac{1}{2} \langle \varphi_k, \varphi_k \rangle - \frac{\langle \eta_k, \varphi_k \rangle}{\langle \eta_k, \eta_k \rangle} \middle| \begin{matrix} (\frac{1}{2}, 1) \\ (0, 1), (\frac{1}{2}\beta, \beta) \end{matrix} \right),$$

where H is the Fox H-function (see, e.g., [GJRdS15, Appendix A] for the definition of this function). Note that this particular Fox H-function has a representation in terms of the three-parameter Mittag-Leffler function (see, e.g., [GKMR14]):

$$H^{1\,1}_{1\,2} \left(-z \middle| \begin{matrix} (1-\gamma, 1) \\ (0, 1), (1-\rho, \beta) \end{matrix} \right) = \Gamma(\gamma) \, E^{\gamma}_{\beta,\rho}(z), \quad z \in \mathbb{C},$$

so that

$$T_{\mu_\beta^{\otimes d}}\delta_0(G(\cdot,\eta))(\varphi) = \frac{1}{2^{d/2}}\prod_{k=1}^{d}\langle\eta_k,\eta_k\rangle^{-1/2}E^{1/2}_{\beta,1-\frac{1}{2}\beta}(z_k), \quad z_k := \frac{\langle\eta_k,\varphi_k\rangle}{\langle\eta_k,\eta_k\rangle} - \frac{1}{2}\langle\varphi_k,\varphi_k\rangle.$$

The expectation of the Donsker's delta at 0 is given by

$$\begin{aligned}\mathbb{E}_{\mu_\beta^{\otimes d}}(\delta_0(G(\cdot,\eta))) = \langle\!\langle \delta_0(G(\cdot,\eta)), \mathbf{1}\rangle\!\rangle_{\mu_\beta^{\otimes d}} &= T_{\mu_\beta^{\otimes d}}\delta_0(G(\cdot,\eta))(0)\\ &= \frac{1}{2^{d/2}\Gamma(1-\frac{1}{2}\beta)^d}\prod_{k=1}^{d}\langle\eta_k,\eta_k\rangle^{-1/2}.\end{aligned}$$

While Donsker's delta is defined merely as a distribution in $(\widetilde{\mathcal{N}})^{-1}_{\mu_\beta^{\otimes d}}$, if the space $L^2(\mu_\beta^{\otimes d})$ is separable, then the following result shows that it can be approximated in $(\widetilde{\mathcal{N}})^{-1}_{\mu_\beta^{\otimes d}}$ by Bochner integrable functions with values in $L^2(\mu_\beta^{\otimes d})$.

Theorem 2.19. *Suppose that $L^2(\mu_\beta^{\otimes d})$ is separable. Let $\eta \in \widetilde{\mathcal{H}}$ such that $\eta_k \neq 0$ for all $k = 1,\dots,d$. Then for $a \in \mathbb{R}^d$ and $n \in \mathbb{N}$, the function*

$$[-n,n]^d \ni s \mapsto e^{i(s,G(\cdot,\eta)-a)} \in L^2(\mu_\beta^{\otimes d}) \tag{2.24}$$

is Bochner integrable with respect to the Lebesgue measure on $[-n,n]^d$, and

$$\delta_a(G(\cdot,\eta)) = \lim_{n\to\infty}\frac{1}{(2\pi)^d}\int_{[-n,n]^d} e^{i(s,G(\cdot,\eta)-a)}\,\mathrm{d}s \qquad \text{in } (\widetilde{\mathcal{N}})^{-1}_{\mu_\beta^{\otimes d}}, \tag{2.25}$$

where the integral on the right-hand side of (2.25) *is a Bochner integral.*

Proof. Note that for all $F \in L^2(\mu_\beta^{\otimes d})$, the function

$$[-n,n]^d \ni s \mapsto (\!(e^{i(s,G(\cdot,\eta)-a)}, F)\!)_{L^2(\mu)} \in \mathbb{C}$$

is measurable. Thus, the function (2.24) is weakly measurable by Riesz representation theorem. Moreover,

$$\int_{[-n,n]^d}\left\|e^{i(s,G(\cdot,\eta)-a)}\right\|_{L^2(\mu_\beta^{\otimes d})}\mathrm{d}s = (2n)^d < \infty.$$

Since $L^2(\mu_\beta^{\otimes d})$ is separable, we conclude that (2.24) is Bochner integrable by Proposition 1.12. Now, to prove (2.25), we choose $p,q \in \mathbb{N}$ such that $\mathcal{U}_{p,q} \subset \mathcal{U}_0$, the neighborhood of zero from the proof of Theorem 2.17. For convenience, set

$$\Phi_n := \frac{1}{(2\pi)^d}\int_{[-n,n]^d} e^{i(s,G(\cdot,\eta)-a)}\,\mathrm{d}s \in L^2(\mu_\beta^{\otimes d}), \quad n \in \mathbb{N}.$$

By applying similar arguments as that of Theorem 2.17, the integral in the definition of Φ_n also exists as a weak integral in the sense of Theorem 1.24, and we use (2.22) to obtain for $\varphi \in \mathcal{U}_{p,q}$ the following:

$$T_{\mu_\beta^{\otimes d}}\Phi_n(\varphi) = \frac{1}{(2\pi)^d}\int_{[-n,n]^d} e^{-i(s,a)} \prod_{k=1}^{d} E_\beta\left(-\frac{1}{2}s_k^2|\eta_k|^2_{\mathcal{H}_k} - s_k\langle \eta_k, \varphi_k\rangle - \frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right) \mathrm{d}s.$$

As $n \longrightarrow \infty$, the function

$$\mathbb{R}^d \ni s \mapsto h_n(s) := \mathbb{1}_{[-n,n]^d}(s)e^{-i(s,a)} \prod_{k=1}^{d} E_\beta\left(-\frac{1}{2}s_k^2|\eta_k|^2_{\mathcal{H}_k} - s_k\langle \eta_k, \varphi_k\rangle - \frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)$$

converges pointwisely in $s \in \mathbb{R}^d$ to

$$\mathbb{R}^d \ni s \mapsto h(s) := e^{-i(s,a)} \prod_{k=1}^{d} E_\beta\left(-\frac{1}{2}s_k^2|\eta_k|^2_{\mathcal{H}_k} - s_k\langle \eta_k, \varphi_k\rangle - \frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)$$

for all $\varphi \in \mathcal{U}_{p,q}$. In addition, estimate (2.23) shows that h_n and h is integrable with respect to the Lebesgue measure on $\mathbb{R}^d$. Hence, by dominated convergence theorem and (2.22),

$$T_{\mu_\beta^{\otimes d}}\Phi_n(\varphi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} h_n(s)\,\mathrm{d}s \overset{n\to\infty}{\longrightarrow} \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} h(s)\,\mathrm{d}s = T_{\mu_\beta^{\otimes d}}\delta_a(G(\cdot,\eta))(\varphi),$$
$$\varphi \in \mathcal{U}_{p,q}. \quad (2.26)$$

In particular, $(T_{\mu_\beta^{\otimes d}}\Phi_n(\varphi))_{n\in\mathbb{N}}$ is a Cauchy sequence. Finally, estimate (2.23) shows that there is $C \in (0,\infty)$ such that $|T_{\mu_\beta^{\otimes d}}\Phi_n(\varphi)| \leq C$ for all $\varphi \in \mathcal{U}_{p,q}$. Therefore, the sequence $(\Phi_n)_{n\in\mathbb{N}}$ converges in $(\widetilde{\mathcal{N}})^{-1}_{\mu_\beta^{\otimes d}}$ by Theorem 1.25, and convergence (2.25) follows from (2.26). □

Chapter 3

Finite-Dimensional Grey Noise Analysis

In this chapter, we will generalize the definition of the generalized grey Brownian motion given by Schneider in [Sch92] and formalized by Grothaus and Jahnert [GJ16]. In particular we will use the results from the previous chapter to construct vector-valued generalized grey Brownian motion, which components are independent. In addition we will obtain properties of this process and use the characterization theorem to identify the local time and self-intersection local time of this process in a suitable distribution space.

3.1 Overview of generalized grey Brownian motion

Here, we discuss the definition of the generalized grey Brownian motion $B^{\beta,\alpha} := (B_t^{\beta,\alpha})_{t\geq 0}$ for $\beta \in (0,1]$, $\alpha \in (0,2)$, introduced by Schneider in [Sch92] and further studied by Mura, Mainardi and Pagnini [MP08, MM09]. The construction of this process was done originally via proper nuclear triples, indexed by α, and corresponding characteristic functions given by $E_\beta(-\|\cdot\|_\alpha^2)$, where $\|\cdot\|_\alpha$ is the norm of the Hilbert space in the nuclear triple. Here, we follow the construction by Grothaus and Jahnert [GJ16], starting with the Mittag-Leffler measure μ_β on $\mathcal{S}'(\mathbb{R})$ and construct the process in the probability space $(\mathcal{S}'(\mathbb{R}), \mu_\beta)$ for all α. Let $\alpha \in (0,2)$ be given and define the fractional operator $M_\pm^{\alpha/2}$ on $\mathcal{S}(\mathbb{R})$ as follows:

$$\mathcal{S}(\mathbb{R}) \ni \varphi \mapsto M_\pm^{\alpha/2}\varphi := \begin{cases} K_{\alpha/2} D_\pm^{(1-\alpha)/2}\varphi, & \alpha \in (0,1), \\ \varphi, & \alpha = 1, \\ K_{\alpha/2} I_\pm^{(\alpha-1)/2}\varphi, & \alpha \in (1,2), \end{cases}$$

where $K_{\alpha/2} := \sqrt{\alpha \sin(\alpha\pi/2)\Gamma(\alpha)}$ is a normalization constant, and for $r > 0$, $D^r_\pm$ is the right-sided and the left-sided Marchaud fractional derivative of order r, while $I^r_\pm$ denote the right-sided and the left-sided Riemann-Liouville fractional integral of order r. Although $M^{\alpha/2}_\pm$ is defined on $\mathcal{S}(\mathbb{R})$, its actual domain is larger. In particular, the domain includes the indicator function $\mathbb{1}_{[0,t)}$, $t \geq 0$, and that $M^{\alpha/2}_\pm \mathbb{1}_{[0,t)} \in L^2(\mathbb{R})$ (see [GJ16, Remark 3.2]). Moreover, by [GJ16, Corollary 3.5], the following scalar product holds.

Proposition 3.1. *For all $t, s \geq 0$, $\alpha \in (0,2)$ and $m, n \in \mathbb{N}$,*

$$(M^{\alpha/2}_- \mathbb{1}_{[0,t)}, M^{\alpha/2}_- \mathbb{1}_{[0,s)})_{L^2(\mathbb{R})} = \frac{1}{2}\left(t^\alpha + s^\alpha - |t-s|^\alpha\right). \tag{3.1}$$

Definition 3.2. For $\beta \in (0,1]$, $\alpha \in (0,2)$ and $t \geq 0$, define $B^{\beta,\alpha}_t$ as follows:

$$\mathcal{S}'(\mathbb{R}) \ni \omega \mapsto B^{\beta,\alpha}_t(\omega) := \langle \omega, M^{\alpha/2}_- \mathbb{1}_{[0,t)} \rangle,$$

The process $B^{\beta,\alpha} := (B^{\beta,\alpha}_t)_{t\geq 0}$ takes values in $L^2(\mu_\beta)$, and is called a *generalized grey Brownian motion* (briefly *ggBm*). If $\alpha = \beta$, the process $B^{\beta,\beta}$ is denoted by $B^\beta := (B^\beta_t)_{t\geq 0}$, and is called a *grey Brownian motion* (briefly *gBm*).

We state the following properties of the process $B^{\beta,\alpha}$ from [GJ16].

Proposition 3.3. *Let $\beta \in (0,1]$ and $\alpha \in (0,2)$.*

(i) $B^{\beta,\alpha}$ has mean zero and covariance

$$\mathbb{E}_{\mu_\beta}\left(B^{\beta,\alpha}_t B^{\beta,\alpha}_s\right) = \frac{1}{2\,\Gamma(\beta+1)}\left(t^\alpha + s^\alpha - |t-s|^\alpha\right) =: \frac{1}{2} G_{\beta,\alpha}(t,s), \quad t, s \geq 0.$$

(ii) For all $p \in \mathbb{N}$, there exists $K < \infty$ such that

$$\mathbb{E}_{\mu_\beta}\left(\left|B^{\beta,\alpha}_t - B^{\beta,\alpha}_s\right|^{2p}\right) \leq K|t-s|^{\alpha p}, \quad t, s \geq 0,$$

and hence $B^{\beta,\alpha}$ has a continuous modification by Kolmogorov's continuity theorem.

(iii) The density $f_{\beta,\alpha}$ of the finite-dimensonal (marginal) distributions of $B^{\beta,\alpha}$ are given as follows: for $x \in \mathbb{R}^n$ and $0 \leq t_1 < \cdots < t_n < \infty$,

$$f_{\beta,\alpha}(x; t_1, \ldots, t_n) = \frac{(2\pi)^{-n/2}}{\sqrt{\Gamma(1+\beta)^n \det G_{\beta,\alpha}}} \int_0^\infty \frac{M_\beta(\tau)}{\tau^{n/2}} \exp\left(-\frac{1}{2}\frac{x^T G^{-1}_{\beta,\alpha} x}{\tau \Gamma(1+\beta)}\right) \mathrm{d}\tau$$

if $\beta \in (0,1)$*, and if* $\beta = 1$*,*

$$f_{1,\alpha}(x;t_1,\ldots,t_n) = \frac{1}{\sqrt{(2\pi)^n \det G_{1,\alpha}}} \exp\left(-\frac{1}{2}x^T G_{1,\alpha}^{-1} x\right).$$

Here, $G_{\beta,\alpha} \in \mathbb{R}^{n\times n}$ *whose* (i,j)*-entry is* $G_{\beta,\alpha}(t_i,t_j)$*.*

Remark 3.4. In view of Proposition 3.3, the family of processes $B^{\beta,\alpha}$, $\beta \in (0,1]$, $\alpha \in (0,2)$, includes the following:

(i) The process $B^1 = B^{1,1}$ is a standard one-dimensional Brownian motion (see [HKPS93]).

(ii) The process $B^{1,\alpha}$ is a one-dimensional fractional Brownian motion with Hurst parameter $\alpha/2$ (see [Ben03]).

Remark 3.5. There is a representation of a scaled version of ggBm due to [GJ16, Remark 3.10] and [MP08, Proposition 3]:

$$(B^{\beta,\alpha}_{2^{1/\alpha}t})_{t\geq 0} \stackrel{\mathrm{d}}{=} (\sqrt{L_\beta} X^\alpha_t)_{t\geq 0}, \tag{3.2}$$

where $\stackrel{\mathrm{d}}{=}$ denotes equality in the sense of finite-dimensional distributions, $(X^\alpha_t)_{t\geq 0}$ is a fBm with Hurst parameter $\alpha/2$, and L_β is a nonnegative random variable independent of X^α whose Laplace transform is $E_\beta(-\cdot)$.

3.2 Vector-valued generalized grey Brownian motion (vggBm)

In this section, we construct a random vector whose components are independent ggBms with the same parameters α and β. For this, we start with the nuclear triple $\mathcal{S}_d(\mathbb{R}) \subset L^2_d(\mathbb{R}) \subset \mathcal{S}'_d(\mathbb{R})$ from Example 1.4 and equip $\mathcal{S}'_d(\mathbb{R})$ with the product measure $\mu_\beta^{\otimes d}$. For convenience in writing, the test function and distribution spaces $(\mathcal{S}_d(\mathbb{R}))^1_{\mu_\beta^{\otimes d}}$ and $(\mathcal{S}_d(\mathbb{R}))^{-1}_{\mu_\beta^{\otimes d}}$ on the space $(\mathcal{S}'_d(\mathbb{R}), \mu_\beta^{\otimes d})$ are denoted by $(\mathcal{S}_d)^1$ and $(\mathcal{S}_d)^{-1}$, respectively.

Definition 3.6. For $d \in \mathbb{N}$, $\beta \in (0,1]$, $\alpha \in (0,2)$, and $t \geq 0$, define $B^{\beta,\alpha}_{d,t}$ as follows:

$$B^{\beta,\alpha}_{d,t}(\omega) := G\left(\omega, \textstyle\sum_{k=1}^d M_-^{\alpha/2}\mathbb{1}_{[0,t)}\mathbf{e}_k\right) = (\langle \omega_1, M_-^{\alpha/2}\mathbb{1}_{[0,t)}\rangle, \ldots, \langle \omega_d, M_-^{\alpha/2}\mathbb{1}_{[0,t)}\rangle),$$

for $\mu_\beta^{\otimes d}$-a.a. $\omega \in \mathcal{S}'_d(\mathbb{R})$. The process $B_d^{\beta,\alpha} := (B_{d,t}^{\beta,\alpha})_{t\geq 0}$ takes values in $L^2(\mu_\beta^{\otimes d};\mathbb{R}^d)$, and is called a *vector-valued generalized grey Brownian motion* (briefly *vggBm*). If $\alpha=\beta$, the process $B_d^{\beta,\beta}$ is denoted by $B_d^{\beta} := (B_{d,t}^{\beta,\beta})_{t\geq 0}$, and is called a *vector-valued grey Brownian motion* (briefly *vgBm*).

If we define a vggBm using the standard Mittag-Leffler measure μ_β on $\mathcal{S}'_d(\mathbb{R})$ for $d \geq 2$, then in view of Proposition 2.6, it has independent components if and only if it is a fractional Brownian motion. However, since this process is defined using the product measure $\mu_\beta^{\otimes d}$, we get the following result.

Proposition 3.7. *The process $B_d^{\beta,\alpha}$ has the following properties.*

(i) For each $t \geq 0$, $B_{d,t}^{\beta,\alpha}$ has characteristic function

$$\mathbb{E}_{\mu_\beta^{\otimes d}}\left(e^{i(p,B_{d,t}^{\beta,\alpha})}\right) = \prod_{k=1}^{d} E_\beta\left(-\frac{1}{2}p_k^2 t^\alpha\right), \quad p \in \mathbb{R}^d.$$

(ii) For each $t \geq 0$, $B_{d,t}^{\beta,\alpha}$ has expectation zero, and for all $i,j = 1,\dots,d$ and $t,s \geq 0$,

$$\mathbb{E}_{\mu_\beta^{\otimes d}}\left((B_{d,t}^{\beta,\alpha})_i(B_{d,s}^{\beta,\alpha})_j\right) = \frac{1}{2\,\Gamma(\beta+1)}\delta_{i,j}\left(t^\alpha + s^\alpha - |t-s|^\alpha\right).$$

In particular, for each $t \geq 0$, the covariance matrix of $B_{d,t}^{\beta,\alpha}$ is given by $\frac{t^\alpha}{\Gamma(\beta+1)}I_d$, where I_d is the identity matrix of order d.

(iii) For each $t \geq 0$, $B_{d,t}^{\beta,\alpha}$ has independent components.

(iv) $B_d^{\beta,\alpha}$ is $\alpha/2$ self-similar: for every $a > 0$, $(B_{d,at}^{\beta,\alpha})_{t\geq 0} \overset{\mathrm{d}}{=} (a^{\alpha/2}B_{d,t}^{\beta,\alpha})_{t\geq 0}$.

(v) $B_d^{\beta,\alpha}$ has stationary increments in the strict sense: for every $h \geq 0$, $(B_{d,t+h}^{\beta,\alpha} - B_{d,h}^{\beta,\alpha})_{t\geq 0} \overset{\mathrm{d}}{=} (B_{d,t}^{\beta,\alpha})_{t\geq 0}$.

Proof. Since statements (i)-(iii) follow directly from Proposition 2.14 and Equation (3.1), we only need to prove (iv) and (v). Let $p := (p^1,\dots,p^n) \in (\mathbb{R}^d)^n$ and $0 \leq t_1 < t_2 < \ldots < t_n < \infty$. To show $\alpha/2$ self-similarity, we need to show that for all $a > 0$,

$$\mathbb{E}_{\mu_\beta^{\otimes d}}\left(\exp\left(i\sum_{r=1}^{n}(p^r, B_{d,at_r}^{\beta,\alpha})\right)\right) = \mathbb{E}_{\mu_\beta^{\otimes d}}\left(\exp\left(i\sum_{r=1}^{n}(p^r, a^{\alpha/2}B_{d,t_r}^{\beta,\alpha})\right)\right). \tag{3.3}$$

By Proposition 2.14(i), Equation (3.3) is equivalent to

$$\prod_{k=1}^{d} E_\beta\left(-\frac{1}{2}\left|\sum_{r=1}^{n} p_k^r M_-^{\alpha/2}\mathbb{1}_{[0,at_r)}\right|^2_{L^2(\mathbb{R})}\right) = \prod_{k=1}^{d} E_\beta\left(-\frac{1}{2}\left|a^{\alpha/2}\sum_{r=1}^{n} p_k^r M_-^{\alpha/2}\mathbb{1}_{[0,t_r)}\right|^2_{L^2(\mathbb{R})}\right),$$

and this equation holds. Indeed, for all $k = 1, \ldots, d$, we use Equation (3.1) to infer that

$$\begin{aligned}
\left|\sum_{r=1}^{n} p_k^r M_-^{\alpha/2}\mathbb{1}_{[0,at_r)}\right|^2_{L^2(\mathbb{R})} &= \sum_{r=1}^{n}\sum_{s=1}^{n} p_k^r p_k^s \left(M_-^{\alpha/2}\mathbb{1}_{[0,at_r)}, M_-^{\alpha/2}\mathbb{1}_{[0,at_s)}\right)_{L^2(\mathbb{R})} \\
&= \sum_{r=1}^{n}\sum_{s=1}^{n} \frac{1}{2} p_k^r p_k^s \left((at_r)^\alpha + (at_s)^\alpha - |at_r - at_s|^\alpha\right) \\
&= a^\alpha \sum_{r=1}^{n}\sum_{s=1}^{n} \frac{1}{2} p_k^r p_k^s \left(t_r^\alpha + t_s^\alpha - |t_r - t_s|^\alpha\right) \\
&= a^\alpha \left|\sum_{r=1}^{n} p_k^r M_-^{\alpha/2}\mathbb{1}_{[0,t_r)}\right|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

A similar procedure may be applied in order to prove that the increments are stationary; we have to show that for all $h \geq 0$,

$$\mathbb{E}_{\mu_\beta^{\otimes d}}\left(\exp\left(i\sum_{r=1}^{n}(p^r, B^{\beta,\alpha}_{d,t_r+h} - B^{\beta,\alpha}_{d,h})\right)\right) = \mathbb{E}_{\mu_\beta^{\otimes d}}\left(\exp\left(i\sum_{r=1}^{n}(p^r, B^{\beta,\alpha}_{d,t_r})\right)\right). \tag{3.4}$$

Indeed, for each $r, s = 1, \ldots, n$,

$$\begin{aligned}
&\left(M_-^{\alpha/2}\mathbb{1}_{[0,t_r+h)} - M_-^{\alpha/2}\mathbb{1}_{[0,h)}, M_-^{\alpha/2}\mathbb{1}_{[0,t_s+h)} - M_-^{\alpha/2}\mathbb{1}_{[0,h)}\right)_{L^2(\mathbb{R})} \\
&= \frac{1}{2}\Big(\left((t_r+h)^\alpha + (t_s+h)^\alpha - |t_r - t_s|^\alpha\right) - \left((t_r+h)^\alpha + h^\alpha - t_r^\alpha\right) - \left(h^\alpha + (t_s+h)^\alpha - t_s^\alpha\right) + 2h^\alpha\Big) \\
&= \frac{1}{2}\left(t_r^\alpha + t_r^\alpha - |t_r - t_s|^\alpha\right) = \left(M_-^{\alpha/2}\mathbb{1}_{[0,t_r)}, M_-^{\alpha/2}\mathbb{1}_{[0,t_s)}\right)_{L^2(\mathbb{R})},
\end{aligned}$$

so that for all $k = 1, \ldots, d$,

$$\left|\sum_{r=1}^{n} p_k^r (M_-^{\alpha/2}\mathbb{1}_{[0,t_r+h)} - M_-^{\alpha/2}\mathbb{1}_{[0,h)})\right|^2_{L^2(\mathbb{R})} = \left|\sum_{r=1}^{n} p_k^r M_-^{\alpha/2}\mathbb{1}_{[0,t_r)}\right|^2_{L^2(\mathbb{R})}.$$

Equation (3.4) then follows from Proposition 2.14(i). □

Proposition 3.8. *For all $r \in \mathbb{N}$ and $t, s \geq 0$,*

$$\mathbb{E}_{\mu_\beta^{\otimes d}}\left(\left|B_{d,t}^{\beta,\alpha} - B_{d,s}^{\beta,\alpha}\right|^{2r}\right) = \frac{r!}{2^r}\left(\sum_p \prod_{k=1}^d \frac{(2p_k)!}{\Gamma(\beta p_k + 1)p_k!}\right)|t-s|^{\alpha r}, \tag{3.5}$$

where the sum in (3.5) *is taken over all $p \in \mathbb{N}_0^d$ such that $p_1 + \cdots + p_d = r$.*

Proof. Recalling that Proposition 2.7 holds for elements in $L_d^2(\mathbb{R})$ by Remark 2.8, we have

$$\begin{aligned}
&\mathbb{E}_{\mu_\beta^{\otimes d}}\left(\left|B_{d,t}^{\beta,\alpha} - B_{d,s}^{\beta,\alpha}\right|^{2r}\right)\\
&= \int_{\mathcal{S}_d'(\mathbb{R})}\left(\sum_{k=1}^d \langle \omega, (M_-^{\alpha/2}\mathbb{1}_{[0,t)} - M_-^{\alpha/2}\mathbb{1}_{[0,s)})\mathbf{e}_k\rangle^2\right)^r \mathrm{d}\mu_\beta^{\otimes d}(\omega)\\
&= \sum_p \frac{r!}{p_1!\cdots p_d!}\prod_{k=1}^d \int_{\mathcal{S}'(\mathbb{R})}\langle \omega_k, M_-^{\alpha/2}\mathbb{1}_{[0,t)} - M_-^{\alpha/2}\mathbb{1}_{[0,s)}\rangle^{2p_k}\,\mathrm{d}\mu_\beta(\omega_k)\\
&= \sum_p \frac{r!}{p_1!\cdots p_d!}\prod_{k=1}^d \frac{(2p_k)!}{2^{p_k}\Gamma(\beta p_k + 1)}\left|M_-^{\alpha/2}\mathbb{1}_{[0,t)} - M_-^{\alpha/2}\mathbb{1}_{[0,s)}\right|_{L^2(\mathbb{R})}^{2p_k}\\
&= \frac{r!}{2^r}\sum_p \prod_{k=1}^d \frac{(2p_k)!}{\Gamma(\beta p_k + 1)p_k!}|t-s|^{\alpha p_k}\\
&= \frac{r!}{2^r}\left(\sum_p \prod_{k=1}^d \frac{(2p_k)!}{\Gamma(\beta p_k + 1)p_k!}\right)|t-s|^{\alpha r}.
\end{aligned}$$

□

Remark 3.9. Proposition 3.8 shows that for each $\gamma \in (0, \frac{\alpha}{2})$, $B_d^{\beta,\alpha}$ has a modification whose trajectories are locally γ-Hölder continuous. Indeed, choose $r \in \mathbb{N}$ such that $\gamma < \frac{\alpha}{2} - \frac{1}{2r}$. Proposition 3.8 shows that there is a constant $C \in (0, \infty)$ such that

$$\mathbb{E}_{\mu_\beta^{\otimes d}}\left(\left|B_{d,t}^{\beta,\alpha} - B_{d,s}^{\beta,\alpha}\right|^{2r}\right) \leq C|t-s|^{(\alpha r - 1)+1}, \quad t, s \in [0, \infty).$$

Kolmogorov's continuity theorem then implies that $B_d^{\beta,\alpha}$ has a modification whose trajectories are locally γ-Hölder continuous.

Remark 3.10. An elliptically contoured random field is a scale mixture of Gaussian random fields. Examples include Gaussian distributions of course, but also others such as Student's t-distributions [Ma13b, AM19], hyperbolic [DLMS12] and Mittag-Leffler distributions [Ma13a]. A more general example of a class of such fields is the class of isotropic random fields with infinitely divisible marginal distributions [WLM18]. It is known (see, e.g., [FKN90]) that the characteristic function of an elliptically contoured random field $X := (X_1, \dots, X_d)^T$, $d \in \mathbb{N}$, is given by

$$\mathbb{E}(e^{ip^T X}) = e^{ip^T \mu}\phi(p^T \Sigma p), \quad p \in \mathbb{R}^d, \tag{3.6}$$

for some $\mu \in \mathbb{R}^d$, a symmetric positive definite $\Sigma \in \mathbb{R}^{d\times d}$, and a map $\phi : \mathbb{R} \to \mathbb{R}$. It has been shown in [GJ16] that all finite-dimensional marginal distributions of ggBm are elliptically contoured. However, for all $t > 0$ and dimensions $d \geq 2$, $B^{\beta,\alpha}_{d,t}$ is elliptically contoured if and only if $\beta = 1$, that is, it is Gaussian. Indeed, if $B^{\beta,\alpha}_{d,t}$ has the characteristic function given by (3.6), then Proposition 3.7(ii) and [CHS81, Theorem 4] imply that Σ is a diagonal matrix. Since $B^{\beta,\alpha}_{d,t}$ has independent components, [Kel70, Lemma 5] would imply that $B^{\beta,\alpha}_{d,t}$ has a Gaussian distribution.

Remark 3.11. We can use representation (3.2) and the independence of the components of vggBm to obtain a similar realization for the scaled vggBm $\widehat{B}^{\beta,\alpha}_d := (B^{\beta,\alpha}_{d,2^{1/\alpha}t})_{t\geq 0}$:

$$\widehat{B}^{\beta,\alpha}_d \overset{\mathrm{d}}{=} \left(\sqrt{L^{(1)}_\beta} X^{\alpha,1}, \dots, \sqrt{L^{(d)}_\beta} X^{\alpha,d}\right), \tag{3.7}$$

where $X^{\alpha,k}$, $k = 1, \dots, d$, are independent one-dimensional fBm with Hurst parameter $\alpha/2$, and $L^{(k)}_\beta$, $k = 1, \dots, d$, are i.i.d. nonnegative random variables with Laplace transform $E_\beta(-\cdot)$ which are independent of $X^{\alpha,k}$, $k = 1, \dots, d$. Representation (3.7) seems to be particularly useful from a modeling point of view, since the properties of ggBm can be studied using results known in fBm without delving too much into the details of its construction in infinite-dimensional analysis. However, performing analysis for vggBm using (3.7) requires a realization of a d-dimensional fBm, a construction of a similar d-dimensional realization of the process after incorporating the random variables $(L^{(k)}_\beta)_{k=1,\dots,d}$, and then proceed with the calculations. Performing such calculations, either in the framework of classical stochastic analysis or white noise analysis, would be considerably much difficult than that of the realization of vggBm using Definition 3.6, as we shall see in further discussions.

Remark 3.12. We can use the same framework to further generalize vggBm in such a way that its components are independent (one-dimensional) ggBm with possibly different indices β and α. Let $\beta \in (0, 1]^d$ and consider the product measure $\mu_{\vec{\beta}}$ from Remark 2.9. For $\alpha \in (0, 2)^d$, define the process $\vec{B}^{\beta,\alpha} := (\vec{B}^{\beta,\alpha}_t)_{t\geq 0}$

taking values in $L^2(\mu_{\vec{\beta}};\mathbb{R}^d)$ given by the following:

$$\mathcal{S}_d'(\mathbb{R}) \ni \omega \mapsto \vec{B}^{\beta,\alpha}(\omega) := (\langle \omega_1, M_-^{\alpha_1/2}\mathbb{1}_{[0,t)}\rangle, \dots, \langle \omega_d, M_-^{\alpha_d/2}\mathbb{1}_{[0,t)}\rangle).$$

A slight modification of the proofs of Proposition 2.11 and Proposition 2.13 shows that the product measure $\mu_{\vec{\beta}}$ also satisfies (A1) and (A2), so that the Appell system can be utilized to study the process $\vec{B}^{\beta,\alpha}$.

To determine the derivative of the vggBm $B_d^{\beta,\alpha}$ in the sense of Corollary 1.26, we use Corollary 2.16 and [GJ16, Equation (21)] to infer that for all $t \geq 0$,

$$\begin{aligned} S_{\mu_\beta^{\otimes d}} B_{d,t}^{\beta,\alpha}(\varphi) &= \sum_{k=1}^{d} \frac{E_{\beta,\beta}\left(\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)}{\beta E_\beta\left(\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)} \langle \varphi_k, M_-^{\alpha/2}\mathbb{1}_{[0,t)}\rangle \mathbf{e}_k \\ &= \sum_{k=1}^{d} \frac{E_{\beta,\beta}\left(\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)}{\beta E_\beta\left(\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)} \int_0^t \left(M_+^{\alpha/2}\varphi_k\right)(x)\,\mathrm{d}x\, \mathbf{e}_k, \end{aligned}$$

on the set $\mathcal{U}_\beta$. We infer from the continuity of $M_+^{\alpha/2}\varphi_k$ on $\mathbb{R}$ (see Theorem 2.7 in [Ben03]) that for $\varphi \in \mathcal{U}_\beta$,

$$\frac{\mathrm{d}}{\mathrm{d}t} S_{\mu_\beta^{\otimes d}} B_{d,t}^{\beta,\alpha}(\varphi) = \sum_{k=1}^{d} \frac{E_{\beta,\beta}\left(\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)}{\beta E_\beta\left(\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)} \left(M_+^{\alpha/2}\varphi_k\right)(t)\, \mathbf{e}_k.$$

Now, as the Mittag-Leffler functions are holomorphic, there are $p, q \in \mathbb{N}$ and a constant $K < \infty$ such that $\mathcal{U}_{p,q} \subset \mathcal{U}_\beta$, and for all $\varphi \in \mathcal{U}_{p,q}$ and $k = 1, \dots, d$,

$$\left| \frac{E_{\beta,\beta}\left(\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)}{\beta E_\beta\left(\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)} \right| \leq K.$$

Moreover, [Ben03, Theorem 2.3] shows that there exists $p' \in \mathbb{N}$ and a constant $C < \infty$ such that for all $k = 1, \dots, d$, and $x \in \mathbb{R}$,

$$\left| \left(M_+^{\alpha/2}\varphi_k\right)(x) \right| \leq C|\varphi_k|_{p'}.$$

Thus, by choosing $p^* > \max\{p, p'\}$, the following estimate holds for all $t \geq 0$, $\varphi \in \mathcal{U}_{p^*,q}$, and $k = 1, \dots, d$,

$$\left| \frac{E_{\beta,\beta}\left(\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)}{\beta E_\beta\left(\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)} \left(M_+^{\alpha/2}\varphi_k\right)(t) \right| \leq KC|\varphi|_{p^*} \leq 2^{-q}KC. \tag{3.8}$$

Therefore, by Corollary 1.26, we establish the existence of the derivative of $B_{d,t}^{\beta,\alpha}$ in $(\mathcal{S}_d)^{-1}$, called the *vggBm noise*.

Proposition 3.13. *For each $t \geq 0$, $\frac{\mathrm{d}}{\mathrm{d}t}B^{\beta,\alpha}_{d,t}$ exists as a vector with components in $(\mathcal{S}_d)^{-1}$ in the sense of Corollary 1.26. Moreover, for all φ belonging to a suitable neighborhood of zero in $\mathcal{S}_d(\mathbb{R})_{\mathbb{C}}$,*

$$S_{\mu_\beta^{\otimes d}} \frac{\mathrm{d}}{\mathrm{d}t} B^{\beta,\alpha}_{d,t}(\varphi) = \sum_{k=1}^{d} \frac{E_{\beta,\beta}\left(\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)}{\beta E_\beta\left(\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)} \left(M_+^{\alpha/2}\varphi_k\right)(t)\, \mathbf{e}_k, \quad t \geq 0.$$

3.3 Local time and self-intersection local time for vggBm

Here, we consider the local time and the self-intersection local time for vggBm, which are given respectively by

$$L_{\beta,\alpha}(a,T) := \int_{[0,T]} \delta_a(B^{\beta,\alpha}_{d,t})\, \mathrm{d}t, \quad a \in \mathbb{R}^d,\ T > 0,$$

$$L^s_{\beta,\alpha}(T) := \int_{[0,T]} \int_{[0,T]} \delta_0(B^{\beta,\alpha}_{d,s} - B^{\beta,\alpha}_{d,u})\, \mathrm{d}u\, \mathrm{d}s, \quad T > 0.$$

The vggBm local time $L_{\beta,\alpha}(a,T)$ is used to measure the amount of time the sample path of a vggBm spends at $a \in \mathbb{R}^d$ within the time interval $[0,T]$, while the vggBm self-intersection local time $L^s_{\beta,\alpha}(T)$ is intended to measure the amount of time in which the sample path of a vggBm spends intersecting itself also within the time interval $[0,T]$. A priori, the expressions above have no mathematical meaning, since Lebesgue integration of Dirac delta distribution is not defined. In the following, we prove that under some constraints, we can make sense of these objects as weak integrals in the sense of Theorem 1.24.

Theorem 3.14. *For $d \in \mathbb{N}$, $\alpha \in (0, \frac{2}{d})$, $\beta \in (0,1)$, $T > 0$ and $a \in \mathbb{R}^d$, the vggBm local time $L_{\beta,\alpha}(a,T)$ and the vggBm self-intersection local time $L^s_{\beta,\alpha}(T)$ exist in $(\mathcal{S}_d)^{-1}$ as weak integrals in the sense of Theorem 1.24. Moreover, for all φ belonging to a suitable neighborhood $\mathcal{U}_0 \subset \mathcal{S}_d(\mathbb{R})_{\mathbb{C}}$ of zero,*

$$T_{\mu_\beta^{\otimes d}} L_{\beta,\alpha}(a,T)(\varphi) = \int_{[0,T]} T_{\mu_\beta^{\otimes d}} \delta_a(B^{\beta,\alpha}_{d,t})(\varphi)\, \mathrm{d}t.$$

$$T_{\mu_\beta^{\otimes d}} L^s_{\beta,\alpha}(T)(\varphi) = \int_{[0,T]} \int_{[0,T]} T_{\mu_\beta^{\otimes d}} \delta_0(B^{\beta,\alpha}_{d,s} - B^{\beta,\alpha}_{d,u})(\varphi)\, \mathrm{d}u\, \mathrm{d}s.$$

Proof. Let $\varphi \in \mathcal{S}_d(\mathbb{R})_{\mathbb{C}}$ with $|\varphi| < M$, for some $M < \infty$, and for convenience, set $\eta_t := M_-^{\alpha/2} 1\!\!1_{[0,t)}$ for $t \in [0,T]$. Following the same calculations from [GJRdS15, Proposition 5.2] and using Proposition 3.1, we have

$$
\begin{aligned}
&\int_{[0,T]} \left| T_{\mu_\beta^{\otimes d}} \delta_a(B_{d,t}^{\beta,\alpha})(\varphi) \right| \mathrm{d}t \\
&\quad \leq \frac{1}{(2\pi)^d} \int_{[0,T]} \prod_{k=1}^{d} \int_{\mathbb{R}} \left| E_\beta \left(-\frac{1}{2} s^2 \langle \eta_t, \eta_t \rangle - \frac{1}{2} \langle \varphi_k, \varphi_k \rangle - s \langle \eta_t, \varphi_k \rangle \right) \right| \mathrm{d}s \, \mathrm{d}t \\
&\qquad \leq \frac{1}{(2\pi)^d} \int_{[0,T]} \prod_{k=1}^{d} \left[\sqrt{\frac{2\pi}{t^\alpha}} \int_0^\infty M_\beta(r) r^{-1/2} \exp\left(\frac{1}{2} M^2 r \right) \mathrm{d}r \right] \mathrm{d}t.
\end{aligned}
$$

By [GJRdS15, Lemma A.4],

$$
K := \int_0^\infty M_\beta(r) r^{-1/2} \exp\left(\frac{1}{2} M^2 r \right) \mathrm{d}r < \infty,
$$

so that

$$
\int_{[0,T]} \left| T_{\mu_\beta^{\otimes d}} \delta_a(B_{d,t}^{\beta,\alpha})(\varphi) \right| \mathrm{d}t \leq \frac{K^d}{(2\pi)^{d/2}} \int_{[0,T]} t^{-\alpha d/2} \, \mathrm{d}t = \frac{2}{2 - \alpha d} K^d T^{1 - \alpha d/2} < \infty.
$$

Therefore, $L_{\beta,\alpha}(a,T) \in (\mathcal{S}_d)^{-1}$ by Theorem 1.24. A similar computation holds for the case of $L_{\beta,\alpha}^s(T)$: for all $\varphi \in \mathcal{S}_d(\mathbb{R})_{\mathbb{C}}$ with $|\varphi| < M$, $M < \infty$, and $s, u \in [0,T]$,

$$
\begin{aligned}
&\left| T_{\mu_\beta^{\otimes d}} \delta(B_{d,s}^{\beta,\alpha} - B_{d,u}^{\beta,\alpha})(\varphi) \right| \\
&\leq \frac{1}{(2\pi)^d} \prod_{k=1}^{d} \int_{\mathbb{R}} \left| E_\beta \left(-\frac{1}{2} \lambda_k^2 \langle \eta_s - \eta_u, \eta_s - \eta_u \rangle - \frac{1}{2} \langle \varphi_k, \varphi_k \rangle - \lambda_k \langle \eta_s - \eta_u, \varphi_k \rangle \right) \right| \mathrm{d}\lambda_k \\
&\quad \leq \frac{1}{(2\pi)^d} \prod_{k=1}^{d} \left(\sqrt{\frac{2\pi}{\langle \eta_s - \eta_u, \eta_s - \eta_u \rangle}} \int_0^\infty M_\beta(r) r^{-1/2} \exp\left(\frac{1}{2} M^2 r \right) \mathrm{d}r \right) \\
&\qquad = \frac{K^d}{(2\pi)^{d/2}} |s - u|^{-\alpha d/2},
\end{aligned}
$$

so that

$$
\begin{aligned}
\int_{[0,T]} \int_{[0,T]} \left| T_{\mu_\beta^{\otimes d}} \delta(B_{d,s}^{\beta,\alpha} - B_{d,u}^{\beta,\alpha})(\varphi) \right| \mathrm{d}u \, \mathrm{d}s &\leq \frac{2K^d}{(2\pi)^{d/2}} \int_{[0,T]} \int_{[0,s]} (s-u)^{-\alpha d/2} \, \mathrm{d}u \, \mathrm{d}s \\
&= \frac{8 K^d T^{2 - \alpha d/2}}{(2\pi)^{d/2} (2 - \alpha d)(4 - \alpha d)} < \infty.
\end{aligned}
$$

The conclusion follows from Theorem 1.24. □

Remark 3.15. Let $d \in \mathbb{N}$ and $\alpha \in (0, \frac{2}{d})$.

(i) If $\beta \in (0,1)$, then by Theorem 3.14 and Remark 2.18, the generalized expectation of $L_{\beta,\alpha}(0,T)$ is given by

$$\mathbb{E}_{\mu_\beta^{\otimes d}}(L_{\beta,\alpha}(0,T)) = \int_{[0,T]} T_{\mu_\beta^{\otimes d}}\delta_0(B_{d,t}^{\beta,\alpha})(0)\,\mathrm{d}t = \frac{T^{1-\alpha d/2}}{2^{d/2-1}\Gamma(1-\frac{1}{2}\beta)^d(2-\alpha d)}, \tag{3.9}$$

while the generalized expectation of $L_{\beta,\alpha}^s(T)$ is given by

$$\mathbb{E}_{\mu_\beta^{\otimes d}}(L_{\beta,\alpha}^s(T)) = \int_{[0,T]}\int_{[0,T]} T_{\mu_\beta^{\otimes d}}\delta_0(B_{d,s}^{\beta,\alpha} - B_{d,u}^{\beta,\alpha})(0)\,\mathrm{d}u\,\mathrm{d}s = \frac{T^{2-\alpha d/2}}{2^{d/2-2}\Gamma(1-\frac{1}{2}\beta)^d(2-\alpha d)(4-\alpha d)}. \tag{3.10}$$

(ii) Consider the case $\beta = 1$, in which the process $B_d^{\beta,\alpha}$ is a d-dimensional fBm with Hurst parameter $H = \alpha/2$. In this case, the assumption that $\alpha d < 2$ reduces to $Hd < 1$, and Corollary 4.10(a) in [HØ02] shows that the right-hand side of (3.9) corresponds to the generalized expectation of the fBm local time at 0. Moreover, a simple application of Lebesgue's dominated convergence theorem to Equation (14) in [HN05] shows that the right-hand side of (3.10) corresponds to the expectation of the L^2-limit of the approximated self-intersection local time I_ε of fBm defined by Equation (2) in [HN05].

Chapter 4

Stochastic Differential Equations Perturbed By Vector-valued Grey Noise

Stochastic differential equations (SDEs) are one core element in classical stochastic analysis. First, SDEs in the case of Mittag-Leffler analysis have been studied by Bock and Silva [BdS17]. In particular, they studied Ornstein-Uhlenbeck processes. In this chapter, we generalize these results to vggBm and consider in particular linear SDEs driven by vggBm.

4.1 Linear stochastic differential equations driven by vggBm noise

In this section, we study linear stochastic differential systems of the form

$$\begin{cases} \mathrm{d}X_t = A(t)X_t\,\mathrm{d}t + \sigma\,\mathrm{d}B^{\beta,\alpha}_{d,t}, & t \in [0,T] \\ X_0 = x_0 \in \mathbb{R}^d, \end{cases} \tag{4.1}$$

where we assume that for $T > 0$, $A : [0,T] \to \mathbb{R}^{d\times d}$ is continuous, and $\sigma \in \mathbb{R}$. As in the case for white noise analysis, we rewrite (4.1) as a system of equations in $(\mathcal{S}_d)^{-1}$:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}X_t = A(t)X_t + \sigma\,\dfrac{\mathrm{d}}{\mathrm{d}t}B^{\beta,\alpha}_{d,t}, & t \in [0,T] \\ X_0 = x_0 \in \mathbb{R}^d, \end{cases} \tag{4.2}$$

and seek a vector-valued process X_t with components taking values in $(\mathcal{S}_d)^{-1}$ that solves the system (4.2) for all $t \in [0,T]$.

First, assume that there exists such a process X_t. If we apply the $S_{\mu_\beta^{\otimes d}}$-transform to both sides of $(4.2)_1$, then by Corollary 1.26, for some neighborhood $\mathcal{U}_0$ of zero in $\mathcal{S}_d(\mathbb{R})_\mathbb{C}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}S_{\mu_\beta^{\otimes d}}X_t(\varphi) = A(t)S_{\mu_\beta^{\otimes d}}X_t(\varphi) + \sigma S_{\mu_\beta^{\otimes d}}\frac{\mathrm{d}}{\mathrm{d}t}B^{\beta,\alpha}_{d,t}(\varphi), \quad t \in [0,T],\ \varphi \in \mathcal{U}_0. \tag{4.3}$$

Note that in (4.3), the matrix $A(t)$ and $S_{\mu_\beta^{\otimes d}}$-transform commute since $A(t)$ is independent of $\omega \in \mathcal{S}'_d(\mathbb{R})$. Set $Y_t(\varphi) := S_{\mu_\beta^{\otimes d}}X_t(\varphi)$ and use Proposition 3.13 to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}Y_t(\varphi) = A(t)Y_t(\varphi) + \sigma C_{\beta,\alpha}(\varphi,t), \quad t \in [0,T],\ \varphi \in \mathcal{U}_0, \tag{4.4}$$

where, for convenience, we set

$$C_{\beta,\alpha}(\varphi,t) := \sum_{k=1}^{d} \frac{E_{\beta,\beta}\left(\frac{1}{2}\langle\varphi_k,\varphi_k\rangle\right)}{\beta E_\beta\left(\frac{1}{2}\langle\varphi_k,\varphi_k\rangle\right)}\left(M_+^{\alpha/2}\varphi_k\right)(t)\,\mathbf{e}_k.$$

Equation (4.4) is a linear nonhomogeneous ordinary differential system, with initial condition

$$Y_0(\varphi) = S_{\mu_\beta^{\otimes d}}X_0(\varphi) = x_0. \tag{4.5}$$

It has a unique solution on $[0,T]$ for each $\varphi \in \mathcal{U}_0$, since both A and $M_+^{\alpha/2}\varphi_k$, $k = 1,\dots,d$, are continuous on $[0,T]$. The solution of (4.4) with initial condition (4.5) is computed using the method of variation of constants:

$$Y_t(\varphi) = V(t)V(0)^{-1}x_0 + \sigma V(t)\int_0^t V(s)^{-1}C_{\beta,\alpha}(\varphi,s)\,\mathrm{d}s, \quad t \in [0,T],\ \varphi \in \mathcal{U}_0, \tag{4.6}$$

where $V : [0,T] \to \mathbb{R}^{d\times d}$ is the fundamental matrix to the homogeneous system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}(t) = A(t)\mathbf{y}(t), \quad t \in [0,T]. \tag{4.7}$$

Now, let $u_{j,k} : \mathbb{R} \to \mathbb{R}$, $j,k = 1,\dots,d$, be the (j,k)-entry of V^{-1}, extended to zero outside $[0,T]$. Since $u_{j,k}$ is continuously differentiable on the compact interval $[0,T]$, for each $t \in [0,T]$, the product $\mathbb{1}_{[0,t)}u_{j,k}$ belongs to $L^q(\mathbb{R})$ for every $1 \le q \le \infty$. Moreover, $M_-^{\alpha/2}\left(\mathbb{1}_{[0,t)}u_{j,k}\right) \in L^2(\mathbb{R})$. Indeed, this is clear for $\alpha = 1$. If $1 < \alpha < 2$, then the statement follows from Theorem 5.3 in [SKM93], since $\mathbb{1}_{[0,t)}u_{j,k} \in L^{2/\alpha}(\mathbb{R})$. For $0 < \alpha < 1$, the function $\mathbb{1}_{[0,t)}u_{j,k}$ is piecewise Lipschitz with a finite number of discontinuities and $\mathbb{1}_{[0,t)}(x)u_{j,k}(x) \to 0$ as

$|x| \to \infty$, so that $M_-^{\alpha/2}\left(\mathbb{1}_{[0,t)}u_{j,k}\right) \in L^2(\mathbb{R})$ by Theorem 11.7 and Theorem 6.1 in [SKM93]. Furthermore, by following a proof similar to that of Lemma 2.5 in [Ben03], we have the following duality relation for all $\phi \in \mathcal{S}(\mathbb{R})_{\mathbb{C}}$:

$$\langle \phi, M_-^{\alpha/2}\left(\mathbb{1}_{[0,t)}u_{j,k}\right)\rangle = \langle M_+^{\alpha/2}\phi, \mathbb{1}_{[0,t)}u_{j,k}\rangle = \int_0^t (M_+^{\alpha/2}\phi)(s)\, u_{j,k}(s)\,\mathrm{d}s. \tag{4.8}$$

For each $t \in [0,T]$ and $j = 1, \ldots, d$, set

$$\eta_{t,j} := \sum_{k=1}^{d} M_-^{\alpha/2}\left(\mathbb{1}_{[0,t)}u_{j,k}\right)\mathbf{e}_k.$$

Then for all $t \in [0,T]$ and $j = 1, \ldots, d$, $\eta_{t,j}$ belongs to $L^2_d(\mathbb{R})$. Moreover, by Proposition 2.15 and Equation (4.8),

$$\begin{aligned} S_{\mu_\beta^{\otimes d}}\langle \cdot, \eta_{t,j}\rangle(\varphi) &= \sum_{k=1}^{d} \frac{E_{\beta,\beta}\left(\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)}{\beta E_\beta\left(\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)}\langle \varphi_k, M_-^{\alpha/2}\left(\mathbb{1}_{[0,t)}u_{j,k}\right)\rangle \\ &= \sum_{k=1}^{d} \frac{E_{\beta,\beta}\left(\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)}{\beta E_\beta\left(\frac{1}{2}\langle \varphi_k, \varphi_k\rangle\right)} \int_0^t (M_+^{\alpha/2}\varphi_k)(s)\, u_{j,k}(s)\,\mathrm{d}s, \quad \varphi \in \mathcal{U}_0, \end{aligned}$$

so that by Theorem 1.23, each component of the right-hand side of (4.6) is holomorphic at zero in $\mathcal{S}_d(\mathbb{R})_{\mathbb{C}}$ for all $t \in [0,T]$, and that

$$X_t = S^{-1}_{\mu_\beta^{\otimes d}} Y_t = V(t)V(0)^{-1}x_0 + \sigma V(t)\sum_{j=1}^{d}\langle \cdot, \eta_{t,j}\rangle \mathbf{e}_j \ \in L^2(\mu_\beta^{\otimes d}; \mathbb{R}^d).$$

Finally, we show that the components of $(X_t)_{t\in[0,T]}$ satisfy the assumptions of Corollary 1.26, that is, for some $p, q \in \mathbb{N}$, each component of the right-hand side of (4.4) is uniformly bounded in $t \in [0,T]$ and $\varphi \in \mathcal{U}_{p,q}$. Now, using Estimate (3.8), we can choose $p, q \in \mathbb{N}$ and a constant $K < \infty$ such that

$$|C_{\beta,\alpha}(\varphi, t)|_{\mathrm{euc}} \leq K, \quad t \geq 0,\ \varphi \in \mathcal{U}_{p,q}.$$

Then the continuity of A, V, V^{-1} on the compact interval $[0,T]$ yield a uniform bound of $|Y_t(\varphi)|_{\mathrm{euc}}$, and hence, of $\left|\frac{\mathrm{d}}{\mathrm{d}t}Y_t(\varphi)\right|_{\mathrm{euc}}$, in $t \in [0,T]$ and $\varphi \in \mathcal{U}_{p,q}$. Therefore, by Corollary 1.26, we obtain the following result.

Theorem 4.1. *The process*

$$X_t = V(t)V(0)^{-1}x_0 + \sigma V(t)\sum_{j=1}^{d}\langle \cdot, \eta_{t,j}\rangle \mathbf{e}_j, \quad t \in [0,T],$$

with values in $L^2(\mu_\beta^{\otimes d};\mathbb{R}^d)$ *solves (4.2) as a system of equations in* $(\mathcal{S}_d)^{-1}$*, where* $V:[0,T]\longrightarrow\mathbb{R}^{d\times d}$ *is a fundamental matrix to the homogeneous system (4.7),*

$$\eta_{t,j} := \sum_{k=1}^{d} M_-^{\alpha/2}\left(\mathbb{1}_{[0,t)}u_{j,k}\right)\mathbf{e}_k,$$

and for $j,k=1,\dots,d$*,* $u_{j,k}:\mathbb{R}\longrightarrow\mathbb{R}$ *is the* (j,k)*-entry of* V^{-1}*, extended to zero outside* $[0,T]$*. Its* $S_{\mu_\beta^{\otimes d}}$*-transform is given by*

$$S_{\mu_\beta^{\otimes d}}X_t(\varphi) = V(t)V(0)^{-1}x_0+\sigma V(t)\int_0^t V(s)^{-1}\sum_{k=1}^{d}\frac{E_{\beta,\beta}\left(\frac{1}{2}\langle\varphi_k,\varphi_k\rangle\right)}{\beta E_\beta\left(\frac{1}{2}\langle\varphi_k,\varphi_k\rangle\right)}\left(M_+^{\alpha/2}\varphi_k\right)(s)\,\mathbf{e}_k\,\mathrm{d}s,$$

for $t\in[0,T]$ *and* $\varphi\in\mathcal{U}_0$*, where* $\mathcal{U}_0$ *is a suitable neighborhood of zero in* $\mathcal{S}_d(\mathbb{R})_{\mathbb{C}}$*.*

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