Equivariant tropical cycles and moduli spaces of discrete admissible covers

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Preface

Zusammenfassung:

Der Ursprung dieses Projekts liegt in der Divisorentheorie der diskreten Graphen und abstrakten tropischen Kurven, genauer gesagt in ihrer *geometrischen* Gonalität. Der Begriff der geometrischen Gonalität ist definiert über den Grad harmonischer Morphismen von tropischen Modifikationen der Quellkurve nach einem Baum. Wir beschreiben ein tropisches schnitttheoretisches Framework, das die Modulräume der tropischen Kurven mit positivem Geschlecht enthält, motiviert durch einen (zur Anfangszeit des Projekts) neuen kombinatorischen Beweis dieses folgenden Ergebnisses: Jeder Graph vom Geschlecht g hat eine tropische Modifikation, die die Quellkurve einer tropischen Überlagerung vom Grad $\lceil \frac{g}{2} \rceil + 1$ eines Baumes ist. Die Gestaltung unseres Frameworks liefert nicht nur einen anderen Beweis des bisherigen Ergebnisses, sondern bietet auch genügend Spielraum für ein systematisches Argument, das die bisherigen Methoden und Ergebnisse verallgemeinert. Nachdem wir den Grad d der Überlagerung, das Geschlecht h und die Anzahl der Markierungen m der Zielkurve sowie die Verzweigungsprofile über den markierten Enden festgelegt haben, können wir insbesondere die geometrischen Orte der Kurven studieren, die die Quellkurve einer tropischen Überlagerung mit diesen Bedingungen sind. Zu diesen geometrischen Orten kommen wir in zwei Schritten: Zunächst bilden wir einen Zykel im Modulraum der diskreten zulässigen Überlagerungen (discrete admissible covers), dann konstruieren wir den Pushforward dieses Zykels durch die entsprechenden Quell- und vergesslichen Morphismen. Außerdem ist das erstgenannte Ergebnis nur eine Folge der Irreduzibilität dieser Modulräume und des Spezialfalls $d = \lceil \frac{g}{2} \rceil + 1$, $h = 0$ und $m = 3g$. Unser Framework erklärt nicht nur die Struktur dieser geometrischen Orte und ihr allgemeines Verhalten, sondern ermöglicht auch unmittelbar einige enumerative Berechnungen mit diesen Zykeln, die neue Beweise und Verallgemeinerungen bekannter Ergebnisse liefern.

Abstract:

This project originates in the divisor theory of discrete graphs and (abstract) tropical curves, and more precisely in their *geometric* gonality. The notion of geometric gonality concerns the degree of harmonic morphisms from tropical modifications of the source graph to a tree. We describe a tropical intersection theoretic framework that includes moduli spaces of tropical curves of positive genus (and possibly marked) motivated by a recent (at the time of the project) combinatorial proof of the following result: Every genus-g graph has a tropical modification that is the source of a degree-$(\lceil \frac{g}{2} \rceil + 1)$ tropical cover of a tree. The design of our framework not only furnishes a different proof of the previous result but also provides enough leeway for a systematic argument that generalizes the previous methods. In particular, after fixing the degree d of the cover, the marking m and genus h of the target, and the ramification profiles above the marked ends we are able to study the loci of marked tropical curves that are the source of a tropical cover with these conditions. We arrive at these loci in two steps: first, we produce a tropical cycle in the corresponding moduli space of discrete admissible covers, then we pushforward this cycle through the corresponding source and forgetting-the-marking morphisms. Moreover, the first-mentioned result is just a consequence of the irreducibility of these moduli spaces (in a precise sense) and the special case of $d = \lceil \frac{g}{2} \rceil + 1$, $h = 0$, and $m = 3g$. Our framework does not only shed new light into the underlying structure of these loci, and their general behavior, but also provides immediate access to some enumerative computations concerning these cycles, giving new proofs and generalizations of known results.

Results of this thesis

We present a tropical intersection theoretic framework for spaces concocted similarly to the moduli spaces of marked tropical curves with positive genus. Our approach is performed through the introduction of poic-fibrations, and we speak of cycles equivariant with respect to the fibration. Special interest is given to the case of tropical curves and discrete admissible covers. For the latter, we introduce the corresponding moduli spaces and show that the standard weight association defines a Minkowski weight therein. We subsequently pushforward this Minkowski weight to produce a tropical cycle in the desired moduli space of tropical curves. The cycles so obtained correspond to the loci of curves (with a given marking and genus) having a tropical modification that is the source of a discrete admissible cover of a fixed degree (with prescribed ramification profile above marked legs) of a curve of a fixed genus and marking. Special cases of these tropical cycles yield the following result: *A generic genus-g r-marked tropical curve Γ (with $g + r$ even) has $C_{\frac{g+r}{2}}$ many r-marked discrete ad-*

missible covers of degree $\frac{g+r}{2}+1$ *of a tree (counted with multiplicity) that have a tropical modification of* Γ *as a source, where* $C_{\frac{g+r}{2}}$ *is the* $(\frac{g+r}{2})$*th Catalan number.*

Introduction and motivation

Perhaps the most fitting label to describe the contents of this project is that of *equivariant combinatorial tropical intersections*. It may very well be the case that a more precise label would include the term *fan* or *fan cycles*. But it is most definitely the case that, instead of this terse self-flagellation in pursuit of the most adequate label, we can concretely describe the origin of this project and convey the general spirit that carried it out.

An origin of this project can be traced back to the divisor theory of graphs and (abstract) tropical curves [BN07] and more precisely in their *geometric* gonality. A divisor on a graph has two important quantities associated to it: its (Baker-Norine) rank and its degree. Motivated by classical algebraic geometry, recall that for nonnegative integers g, r, and d, the Brill-Noether number is defined to be $\rho(g,r,d) = g-(r+1)\cdot(g-d+r)$. Baker's conjecture for graphs states that: If $\rho(g,r,d) \geq 0$, then every genus-g graph has a divisor D of rank r and degree $\leq d$. This is still an open problem in the area, and its metric analog (namely, the case of tropical curves) has been established for some time (see [Bak08]).

The notion of gonality refers to the minimal degree of a rank-1 divisor. In this situation, we observe that for a fixed g the smallest possible d with $\rho(g,1,d) \geq 0$ is $\lceil \frac{g}{2} \rceil + 1$, and hence, the idea would be to show that every graph has a rank 1 divisor of degree $\leq \lceil \frac{g}{2} \rceil + 1$. This is also still open for discrete graphs. If we were in the context of algebraic geometry and wanted to produce a divisor of rank 1, then we would look at morphisms from our space to $\mathbb{P}^1$ and pull-back the divisor defined by a single point (i. e. the line bundle $\mathscr{O}_{\mathbb{P}^1}(1)$). In the context of tropical curves, our $\mathbb{P}^1$ is changed to a genus 0 tropical curve (i. e. a metric tree), and the corresponding notion of morphism is that of *harmonic morphism* (this analogy is pushed further, and eventually we just look at harmonic morphisms with nonnegative Riemann-Hurwitz numbers). This is what we refer to as *geometric gonality*: namely, the smallest degree of a morphism from a (modification) of the graph to a tree. It is known [GSv20] that this geometric gonality is different than the usual gonality, but we take this algebro-geometric motivation to the heart and utilize it to study the loci of genus-g graphs that have a tropical modification as the source of a discrete admissible cover (a special kind of harmonic morphisms). These tropical modifications refer to grafting trees to a graph, which for divisorial purposes are rather insignificant. Baker's conjecture on gonality of tropical curves (metric graphs) was originally established by means of algebraic geometry [Bak08], and subsequently [DV21] provided a purely combinatorial proof (for

geometric gonality) following a detailed analysis and classification of (certain) tropical covers of degree $\lceil \frac{g}{2} \rceil + 1$ from genus-g connected tropical curves to trees. This result and the enumerative result of [VDL22] gave rise to the primum movens of this project: to describe these loci as tropical cycles in this space.

Some words about the construction of this space are in order. We consider connected discrete graphs (possibly with legs) with all vertices at least 3-valent, and following our original motivation on divisors on graphs, we do not consider genus functions on the vertices. If A is a finite set, an A-marked graph is just a discrete graph with a bijection from A to its set of legs. A metric on a discrete graph is a function on the set of edges taking values in the positive real numbers, and an abstract tropical curve is then simply a discrete graph with a given metric. Fixing the underlying discrete graph and varying the metric gives rise to an open cone where the faces of its closure correspond to contractions of edges of the graph. We associate to each discrete graph the partially open polyhedral cone, containing this open cone, and with the additional faces that correspond to genus preserving edge contractions. For a fixed integer $g \geq 0$ and a fixed finite set A with $2g - 2 + \#A > 0$, the moduli space $\mathcal{M}^{\mathrm{trop}}_{g,A}$ of genus-g A-marked tropical curves is obtained by taking all the partially open integral cones associated to (isomorphism classes of) genus-g A-marked discrete graphs and identifying points representing isometric tropical curves. The presence of possible non-trivial automorphisms at the associated cone of a discrete graph of positive genus forces this presentation to not be a partially open fan (or partially open polyhedral complex). This is in contrast to the case of trivial genus where the space is naturally a simplicial fan. The gist of our approach is to shift our attention to cycles equivariant (or compatible) with respect to a nice map from somewhere where we can define them, instead of describing cycles *in* these type of spaces.

In the previous paragraph the first, and perhaps best, example that we found in nature was already described: moduli spaces of tropical curves with positive genus. We seek to understand cycles of $\mathcal{M}^{\mathrm{trop}}_{g,A}$ by means of rational curves through "cutting" and "glueing" edges. More precisely, let $\mathbb{g} = \{1, \dots, g, 1^*, \dots, g^*\}$ (where i^* is just a symbol for $1 \leq i \leq g$) and consider the map

$$\mathrm{st}_{g,A} \colon \mathcal{M}^{\mathrm{trop}}_{0,A\sqcup\mathbb{g}} \times \mathbb{R}^g_{>0} \to \mathcal{M}^{\mathrm{trop}}_{g,A},$$

where $\mathrm{st}_{g,A}(\Gamma, \delta_1, \dots, \delta_g)$ is the graph given by connecting the i- and i^*-legs by an interval of length $\delta_i > 0$ (and forgetting the i- and i^*-marked legs). Combinatorially, we are regarding the input tree as a spanning tree of the target graph. This map is an example of what we call a poic-fibration, which is a class of maps that we use to describe tropical cycles of the target as particular subcycles of the source. More precisely, we speak of equivariant cycles regarding the poic-fibration, where the leading idea is that

of invariance with respect to identifications (or isomorphisms) of the base space. Thus, the decision to place the label *equivariant* in this context.

The previous construction is our motivating example and, as it shows, prompts us directly to partially open integral cones, partially open fans (as in [GO17]), and more generally to introduce poic-complexes. Loosely speaking, these are to partially open fans, as cone complexes [Gro18] are to fans. Our perspective on poic-complexes is rather categorical and, admittedly, rather avoidable if these were our only objects. However, we endorse this point of view and assert its usefulness through our introduction of poic-fibrations and the role of automorphisms in the whole. In other words, adopting this perspective is what led us to the description of poic-fibrations, which are the ultimate tool that make our enterprise fully operational.

The plan-of-attack for our primum movens is then to introduce, similarly to $\mathcal{M}^{\mathrm{trop}}_{g,A}$, a moduli space of discrete admissible covers with a (sufficiently proper) *source* map, describe a poic-fibration of this space, obtain an equivariant cycle of this fibration to successfully pushforward through the *source* and *forgetting the marking morphisms* (also sufficiently proper!), and finally arrive at our desired loci of tropical curves. It turns out (as expected) that $\mathrm{st}_{g,A}$ is irreducible (meaning that there is only one top dimensional $\mathrm{st}_{g,A}$-equivariant cycle up to scaling), and in some specific cases of interest we produce a non-trivial top-dimensional cycle, which we can then mindlessly affirm is supported in the whole space. Thus showing that any genus-g graph has a tropical modification (this comes from forgetting the marking) that is the source of a cover of degree $\lceil \frac{g}{2} \rceil + 1$. Furthermore, the irreducibility aspect allows us to also answer the natural enumerative question: How many covers have (a tropical modification of) a given tropical curve as a source? It turns out that a small adaptation of the very special case of caterpillar of loops of [VDL22] produces, through our framework, several cases where we provide new answers. We must mention that the space of admissible covers that we consider depends uniquely on the degree of the covers, the genus of the target, the markings of the target, and the ramification profiles above the legs of the target. We show that these data always give rise to a cycle (whose dimension naturally depends on these parameters), but we do not address the existence problem: Does there exist an admissible cover with the given parameters? This means that unless it can be previously determined that the cycle will be non-trivial (this is the case on our special situations) it may very well happen that we end up with a trivial cycle. We close this introduction with a depiction in Figure 1 of the deformation process involved in the balancing of these cycles. In this case we have discrete admissible covers of degree 3 of a genus-1 8-marked graph, where the ramification above the marked legs is always simple. The weights of the edges of the source are specified whenever they are strictly bigger than 1, and the coloring of the marked legs specifies the discrete admissible covers.

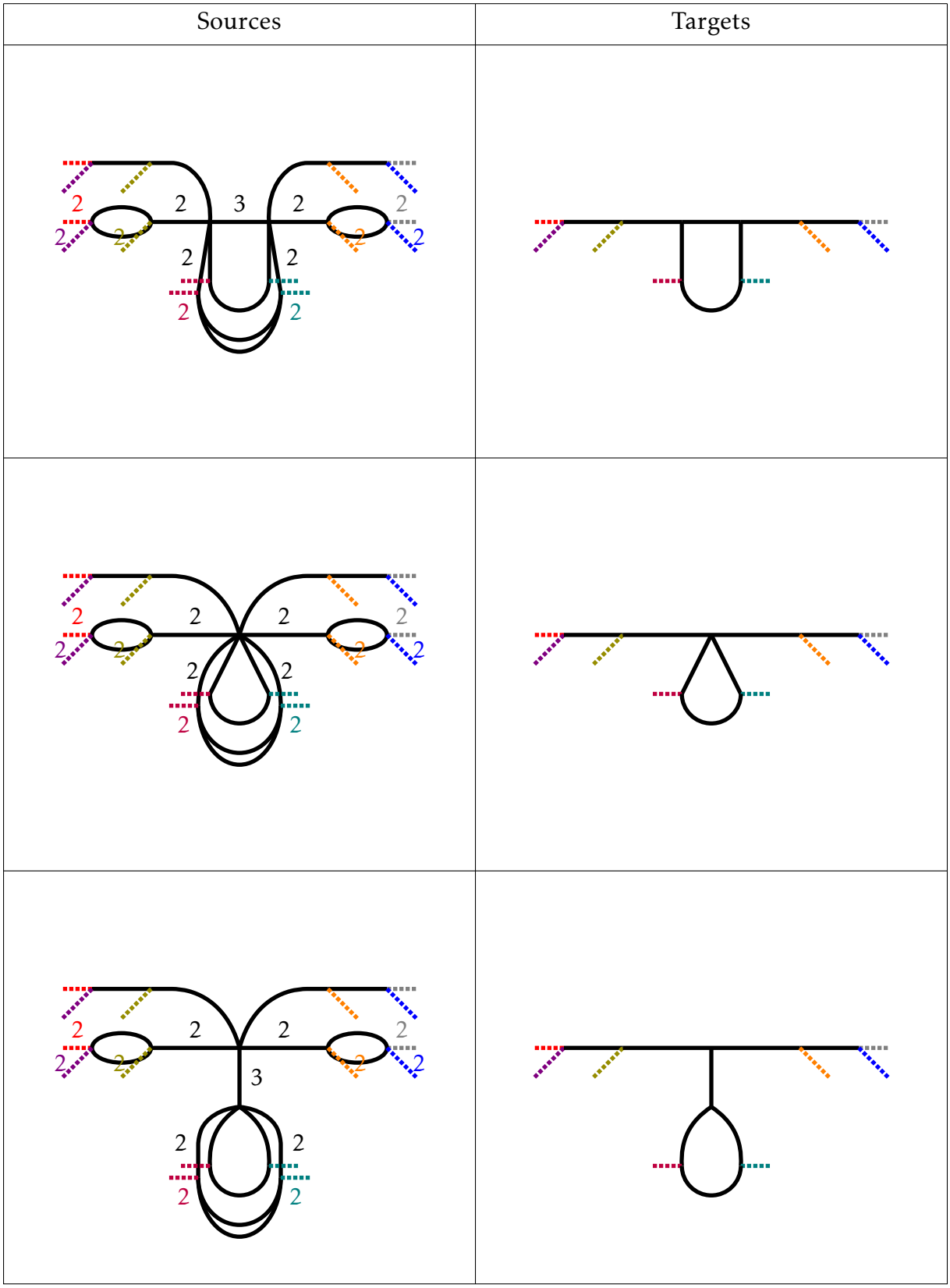

Figure 1: Depiction of the deformation of discrete admissible covers involved in the corresponding balancing.

Tropical intersection theory

Tropical intersection theory is a branch in the realm of tropical geometry that was initially developed under the motivation of several enumerative computations ([Mik05], [GM08], [KM09]). Before its christening under this name it dealt originally with cycles in tropical fans and tropical varieties [GKM09], then it was established (in particular named) and generalized for more general (abstract) polyhedral spaces [AR10], also generalized to weakly embedded cone complexes [Gro18], and has recently been extended to more general tropical spaces [CGM22]. It is also worth mentioning that some extensions to include partially open polyhedra have been developed ([GMO17] and [GO17]). Nevertheless, these extensions have not abandoned their enumerative motivations and applications, and certainly the extension described in this work neither does so. We first handle the case of (linear) poic-complexes, then we use these together with the notion of poic-fibrations to describe cycles of poic-spaces. In earnest, the cycles we described are best thought of as (equivariant) cycles of (or relative to) the poic-fibration, instead of the base space. Our enumerative motivations follow the results of [VDL22] (which in turn follow those of [EH87]), and apply to the cases of moduli spaces of tropical curves.

Moduli spaces of tropical curves

Moduli spaces of tropical curves have been thoroughly studied throughout this millennium ([ACP15], [AK06], [CGM22], [CGP21], [GKM09], [SS04], [CHMR16], [Mik07], [Koz09]), and perhaps the best understood case (in a tropical sense) is that of trivial genus ([GKM09], [Mik07], [Rau16]). Our presentation of these spaces (for general genus) is similar in spirit to that of [CGP21], but a bit different in execution. One small difference involves the fixing of objects representing the isomorphism classes, which in our context unfortunately led to some rather terse bookkeeping. Another difference concerns non-trivial genus functions on the vertices, which we do not consider following our divisorial motivation and due to subsequent constructions involving discrete admissible covers. Our approach to study the tropical intersection theory of these spaces is purely combinatorial, and certainly we are not the first to study such aspects of these spaces. In [CGM22] a different approach to tropical intersection theory for them (and more generally for *tropical spaces*) has been proposed, with a stacky perspective in relation to the site of tropical spaces. Unfortunately, there are some obstructions to the introduction of tropical cycles on these (moduli) spaces, but there is an open substack where it is possible. We relate our combinatorial framework to the case of this open substack.

Graphs, their divisors and tropical covers

The theory of divisors on graphs is tied to chip-firing games thereon and resembles in multiple facets its algebro-geometric counterpart of divisors on curves (see [BN07]). This resemblance is further enforced by the results of [Bak08] and the notion of harmonic morphisms [BN09], which strongly parallels the theory of holomorphic maps of Riemann surfaces. In the realm of tropical curves (metric graphs) this notion of harmonic morphisms is slightly modified to that of tropical covers ([CJM10b], [BBM11]). This parallelism is not only stronger, but has also been proven useful in multiple instances, such as the study of several properties of Hurwitz numbers ([BCM13], [CJM10a], [CJM11], [CMR16]). Moduli spaces of tropical covers to a fixed tropical curve (or more generally of stable maps to a fixed tropical variety) have readily been studied ([BM15], [BBBM17], [GMO17], [GO17]), but these did not allow the deformation of the target curve. Our divisorial motivations align with the existence of such spaces, and prompt us to their description. Naturally, some further restrictions have to be imposed, but largely there is a suitable description of these spaces whose nice structure favors our program.

Structure of this thesis

This thesis is organized in 6 chapters, which we now describe:

1. The first chapter deals with introducing and fixing notation of several essential objects. More precisely, we first delineate several notions concerning partially open polyhedral cones and recall some basics about discrete graphs. Subsequently, we discuss the categories of graphs that will be of our interest, explain in detail their morphisms, and construct the moduli spaces of tropical curves. Afterwards, we handle discrete admissible covers of graphs in our terms, their categories, and their moduli spaces. Essentially, this chapter just contains adaptations of several well known results into our context. We make an effort to provide and illustrate several examples of interest.

2. The second chapter deals with the introduction of (linear) poic-complexes, their tropical cycles, and the pushforwards of these through (weakly-)proper morphisms. As we have previously mentioned, poic-complexes are a simple generalization of cone complexes that allow the inclusion of partially open polyhedral cones. We take a rather categorical and combinatorial perspective on these objects, and decide to introduce them as separate spaces that are not necessarily embedded in a given space (read as either topological or vector space). We take the usual approach to describe cycles as Minkowski weights of subdivisions.

However, to even discuss Minkowski weights we are forced to require the further specification of a morphism (of poic-complexes) to a vector space with a given lattice (just as in the case of weakly-embedded cone complexes [Gro18]). These further specification permits the introduction of Minkowski weights and, by means of subdivisions, of tropical cycles. We close the chapter with the description of pushforwards for (weakly-)proper morphisms.

3. In this chapter the notions of poic-spaces, poic-fibrations and equivariant cycles of the latter are introduced. The notion of poic-spaces is quite simple and mimicks the presentation of the moduli spaces of tropical curves. In other words, a poic-space is a space glued out of quotients of partially open integral cones by finitely many automorphisms. A (linear) poic-fibration is a special kind of morphism from a (linear) poic-complex to a poic-space with certain lifting properties. This is an essential notion for all of our ventures, and the subsequent constructions of the chapter are built upon it. Following the standard roadmap, we first discuss equivariant Minkowski weights with respect to a poic-fibration, then we discuss subdivisions of the source (linear) poic-complex that are *compatible* with the poic-fibration, and finally we introduce the equivariant tropical cycles of a poic-fibration. We then present our favourite family of poic-fibrations: the spanning tree fibrations of the moduli spaces of tropical curves. These naturally come with a "forgetting the marking" morphism, which we also describe and will use in the next chapter. We close the chapter by describing the extended spanning tree fibration, namely a poic-fibration over the moduli space of admissible covers (of a fixed degree, genus of the target, marking and ramification profile above the marked legs) that essentially comes from our previous family of poic-fibrations.

4. In this section we prove the central result of our project: Theorem 4.1.3. Namely, we use the standard weight assignment of tropical covers (as in [CMR16]) and show that this produces an equivariant Minkowski weight of the extended spanning tree fibration. The proof of this theorem basically takes the whole chapter and is performed by first analyzing a local balancing, and then globalizing this local balancing. We set out to prove this local balancing inspired by the methods of [GMO17], namely we show that our tropical (combinatorial) multiplicities can be interpreted as the degree of the pullback of a particular divisor. The algebro-geometric *spaces* of interest are those involved in a very special instance of J. Li's degeneration formula (see [Li02]). We close the chapter with a generalization of the existing enumerative results on Catalan many morphisms of [VDL22], and the relation with [DV21] and [VDL22].

5. This chapter deals with the connections of our framework to the existing ones, as

well as some further wonderful constructions involving our spanning tree fibrations. To be more precise, we explain how to interpret the classical tropical intersection theory of [AR10], as well as the case of weakly embedded cone complexes of [Gro18], in our terms. We also explain a relationship between our constructions and methods, and the tropical cycles (in their sense) of the moduli space of *Mumford curves* of [CGM22].

6. This last chapter deals with several technical constructions that tie the remaining loose ends of the project. In the spirit of clarity, readability, and harmony, several proofs of foundational (but technical) results from previous chapters have been postponed hereto. These are meticulously dealt with, trying to convey the general natural idea behind the proof and carrying it out, while avoiding suffocation by an ever-so-engulfing generality.

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Chapter 1

Preamble

This chapter is dedicated to fix notation and to state precise definitions for our basic objects of interest. We first describe partially open integral (polyhedral) cones, together with their corresponding category and realization functor. Both are fundamental pieces for our subsequent constructions and endeavors. We then focus on discrete graphs, and describe a particular category of them, as well as a functor from this category to that of partially open integral cones. These constructions are then used to give a presentation of the moduli space of tropical curves of a fixed genus and marking. Subsequently we discuss discrete admissible covers, recollect some well-known results about them, and proceed in a similar manner as with discrete graphs. Namely, we describe a corresponding category of discrete admissible covers, a functor from this category to that of partially open integral cones, and use this to obtain a presentation of the moduli space of discrete admissible covers of curves with a fixed genus, marking, and specified ramification above the legs.

1.1 Partially open integral cones

Suppose N is a free abelian group of finite rank and let $N_{\mathbb{R}}$ denote the vector space $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. The dual of N is the free abelian group (of finite rank) given by $N^\vee = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. There is a natural (injective) morphism $N^\vee \to N_{\mathbb{R}}^\vee$ that gives rise to an isomorphism $N^\vee \otimes_{\mathbb{Z}} \mathbb{R} \cong N_{\mathbb{R}}^\vee$. Thereby, an element $f \in N^\vee$ correspondingly gives a closed half-space and an open half-space of the vector space $N_{\mathbb{R}}$:

$$H_f := \{x \in N_{\mathbb{R}} \colon f(x) \geq 0\}, \qquad H_f^o := \{x \in N_{\mathbb{R}} \colon f(x) > 0\}.$$

Definition 1.1.1. A *partially open integral cone of* $N_{\mathbb{R}}$ is a non-empty subset $\sigma \subset N_{\mathbb{R}}$

given as a finite intersection of open and closed half-spaces, i. e.

$$\sigma = \bigcap_{i=0}^{n} H_{f_i} \cap \bigcap_{j=0}^{m} H^o_{g_j} \tag{1.1}$$

where $f_i \in N^\vee$ and $g_j \in N^\vee$ for $0 \leq i \leq n$ and $0 \leq j \leq m$.

Notation 1.1.2. If σ is a partially open integral cone of $N_\mathbb{R}$ given by (1.1), then the relative interior σ^0 of σ is also a partially open integral cone of $N_\mathbb{R}$. In addition, we denote denote by $\mathrm{Lin}(\sigma)$ the linear subspace generated by σ, and by $\dim(\sigma)$ the dimension of σ (that is $\dim(\sigma) = \dim \mathrm{Lin}(\sigma)$).

Definition 1.1.3. Suppose σ is a partially open integral cone of $N_\mathbb{R}$. A *face* τ of σ is a partially open integral cone given by $\tau = \sigma \cap H_f$, where $f \in N^\vee$ is such that $\sigma \subset H_{-f}$. This situation is denoted by $\tau \leq \sigma$. Additionally, if τ is a face of σ with $\tau \neq \sigma$, then τ is called a *proper face*. To emphasize this distinction, we denote this situation by $\tau \lneq \sigma$.

Naturally, if σ is a partially open integral cone of $N_\mathbb{R}$ and $\tau \leq \sigma$, then $\mathrm{Lin}(\tau)$ is a subspace of $\mathrm{Lin}(\sigma)$ and $\dim(\tau) \leq \dim(\sigma)$.

Definition 1.1.4. A *partially open integral cone* (hereafter just poic) is a pair (σ, N) where N is a free abelian group of finite rank and σ is a full-dimensional partially open polyhedral cone of $N_\mathbb{R}$. A *face* of a poic (σ, N) is a poic $(\tau, \mathrm{Lin}_N(\tau))$ where τ is a face of σ in $N_\mathbb{R}$ and $\mathrm{Lin}_N(\tau) = N \cap \mathrm{Lin}(\tau)$. A *subcone* of (σ, N) is a poic (ξ, M) such that $\xi \subset \sigma$ and $M = N \cap \mathrm{Lin}(\xi)$.

Remark. Naturally, every face is a subcone. However, not every subcone is a face. We have already introduced an simple example of a subcone: If (σ, N) is a poic, then (σ^0, N) is also a poic and is a subcone of (σ, N).

We remark that Definition 1.1.1 is different from Definition 1.1.4 in a very subtle manner. The latter consists of a pair, whereas the former merely refers to a subset of a specified vector space. This difference is due to subsequent sections, where we want to consider multiple cones that do not inhabit naturally a common vector space. We apologize for its slyness, and compensate by drawing attention to it.

Notation 1.1.5. When there is no risk for confusion, if the pair (σ, N) is poic, we will just denote the lattice N by N^σ, and the whole poic by σ.

Definition 1.1.6. Suppose σ and ξ are two poics. A *morphism* $f : \sigma \to \xi$ consists of a linear morphism $f : N^\sigma \to N^\xi$, such that the induced linear map $f_\mathbb{R} : N^\sigma_\mathbb{R} \to N^\xi_\mathbb{R}$ sends σ to ξ. We say that the morphism $f : \sigma \to \xi$ is a *face-embedding*, if it identifies σ with a face of ξ. Namely, if $f_\mathbb{R}$ is injective and $f_\mathbb{R}(\sigma)$ is a face of ξ.

Example 1.1.7. If τ is a subcone of σ then the inclusion $\tau \to \sigma$ is a morphism, where the morphism between the lattices is the natural inclusion. If τ were a face of σ, then this morphism would undoubtedly be a face-embedding.

An important construction that we will continuously use is that of products.

Construction 1.1.8. Suppose (σ, N) and (ξ, M) are two poics. There is a natural isomorphism $N_{\mathbb{R}} \oplus M_{\mathbb{R}} \cong (N \oplus M)_{\mathbb{R}}$. Separately, the product $\sigma \times \xi$ is a partially open integral cone of $N_{\mathbb{R}} \oplus M_{\mathbb{R}}$. Abusing the aforementioned natural identification, let us denote by $\sigma \times \xi$ the corresponding cone of $(N \oplus M)_{\mathbb{R}}$, so that the pair $(\sigma \times \xi, N \oplus M)$ is also a poic.

Naturally, if σ and ξ are two poics, then $\sigma \times \xi$ is also a poic and both projections $\sigma \times \xi \to \sigma$ and $\sigma \times \xi \to \xi$ are morphisms of poics.

The data of poics and morphisms thereof, together with natural composition of maps, defines a category, which we denote by POIC. A poic σ naturally has an underlying topological space $|\sigma|$ given by the cone itself as a subspace of the real vector space. In other words, if (σ, N^σ) is a poic, then $|\sigma|$ is the topological space given by $\sigma \subset N_{\mathbb{R}}$ with the Euclidean topology. If ξ is also a poic and $f : \sigma \to \xi$ is a morphism, then f induces a linear map $f : N^\sigma_{\mathbb{R}} \to N^\xi_{\mathbb{R}}$ which restricts to a continuous (also linear!) map $|f| : |\sigma| \to |\xi|$. This previous association actually gives rise to a functor

$$|\bullet| : \mathrm{POIC} \to \mathrm{Top}, \sigma \mapsto |\sigma|, \tag{1.2}$$

which we call the *realization functor*. Furthermore, if σ is a poic then we call $|\sigma|$ the *realization of* σ (we follow analogous conventions with morphisms).

1.2 Discrete graphs

There are several definitions for discrete graphs in the literature, and a choice of one seems more of a matter of personal taste or convenience. In the present case we decided to employ the definition of discrete graph with legs from [LUZ24], and for convenience and sanity purposes we decided to be as explicit as possible. There is no discernible advantage from one definition or the other. We adopted this one purely out of its amenity with the description of our constructions of interest.

Definition 1.2.1. A *discrete graph (possibly with legs)* G consists of the data of a triple $(F(G), r_G, \iota_G)$ where:

- $F(G)$ is a finite set.
- $r_G : F(G) \to F(G)$ is a map of sets, called the root map.
- $\iota_G : F(G) \to F(G)$ is an involution subject to $\iota_G \circ r_G = r_G$.

The *set of vertices* of G is the subset of $F(G)$ given by $\mathrm{Im}(r_G)$. It will be denoted by $V(G)$ and its complement by $H(G)$.

Suppose G is a discrete graph. The involution map ι_G satisfies $\iota_G \circ r_G = r_G$, hence it restricts to an involution of $H(G)$. Therefore, ι_G partitions $H(G)$ into orbits of size 2 or 1. The following definitions will be used:

- A *leg* of G is a size-1 orbit of $H(G)$. The set of legs will be denoted by $L(G)$.
- For $\ell \in L(G)$ given by $\ell = \{L\} \subset H(G)$, the *boundary* is $\partial\ell := \{r_G(L)\}$.
- An *edge* of G is a size-2 orbit of $H(G)$. The set of edges will be denoted by $E(G)$.
- For $e \in E(G)$ with $e = \{E, E'\} \subset H(G)$, the *boundary* is $\partial e := \{r_G(E), r_G(E')\}$.
- A vertex is *incident* to an edge or a leg if it is in its boundary.
- The graph G is *connected*, if for any two different vertices $V, W \in V(G)$ there exists a sequence $e_1, \ldots, e_n \in E(G)$ such that:
  1. $V \in \partial e_1$ and $W \in \partial e_n$.
  2. For any $1 \leq i \leq n-1$, $\partial e_i \cap \partial e_{i+1} \neq \varnothing$.
- The *genus* of a connected graph G is the number $g(G) := \#E(G) - \#V(G) + 1$.
- A *tree* is a connected graph of genus 0.
- A *subgraph* K of G is a graph $(F(K), r_K, \iota_K)$ such that $F(K) \subset F(G)$ with $r_G|_{F(K)} = r_K$ and $\iota_G|_{F(K)} = \iota_K$.
- The *valency* of $V \in V(G)$ is the number $\mathrm{val}(v) := \#r_G^{-1}(V) - 1$.

After these many definitions, it is perhaps a good moment to do an explicit example.

Example 1.2.2. The theta graph (see Figure 1.1) is given by two vertices A and B, and three edges e_1, e_2 and e_3 such that $\partial e_i = \{A, B\}$ (for $1 \leq i \leq 3$). To present this as a discrete graph we consider the set (the elements are just names)

$$F(G) := \{AB_1, AB_2, AB_3, BA_1, BA_2, BA_3, A, B\},$$

together with the root map

$$r_G \colon F(G) \to F(G), \quad \begin{array}{ll} r_G(AB_i) = A, & 1 \leq i \leq 3, \\ r_G(BA_i) = B, & 1 \leq i \leq 3, \\ r_G(A) = A, & \\ r_G(B) = B, & \end{array}$$

and involution

$$\iota_G\colon F(G)\to F(G),\quad \begin{aligned}\iota_G(AB_i)&=BA_i,\quad 1\le i\le 3,\\ \iota_G(BA_i)&=AB_i,\quad 1\le i\le 3,\\ \iota_G(A)&=A,\\ \iota_G(B)&=B.\end{aligned}$$

It is immediate to check that the previous data gives rise to a graph. In addition, $V(G)=\{A,B\}$, $L(G)=\varnothing$, and $E(G)=\{e_1,e_2,e_3\}$ where for $1\le i\le 3$

$$e_i:=\{AB_i,BA_i\}.$$

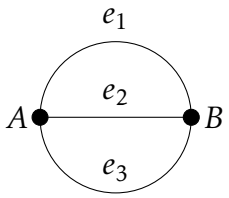


Figure 1.1: The theta graph

Notation 1.2.3. When depicting a discrete graph, we will usually do so as in Figure 1.1: by labelling both the vertices and the edges.

It is worth mentioning that we are not mainly interested in general maps between graphs, but rather in some special types of maps between them. However, it is neither burdensome nor fruitless to state a general definition and later describe our special types of interest.

Definition 1.2.4. Let G_1 and G_2 be discrete graphs. A *map of graphs* $f\colon G_1\to G_2$ is a map of sets $f\colon F(G_1)\to F(G_2)$ such that:

- It commutes with the root maps: $r_{G_2}\circ f=f\circ r_{G_1}$.
- It commutes with the involution maps: $\iota_{G_2}\circ f=f\circ\iota_{G_1}$.

If the map $f\colon F(G_1)\to F(G_2)$ is a bijection, then the map of graphs is additionally called *bijective*. In this case, it follows that f induces a set bijection $E(f)\colon E(G_1)\to E(G_2)$.

Now, we can describe our special type of maps of interest: edge contractions. It will be apparent afterwards, but these types of maps lie at the core of our program. For this reason, we separate this construction and address notation meticulously.

Construction 1.2.5. Let G be a discrete graph, and suppose $e \in E(G)$ is an edge of G. Let $F(G/e)$ denote the set obtained from $F(G)$ by identifying the subset $e \cup \partial e \subset F(G)$ into a single element V_e. In other words,

$$F(G/e) = F(G) \backslash (e \cup \partial e) \cup \{V_e\}.$$

This identification gives rise to a natural surjective map $p_e \colon F(G) \to F(G/e)$. Furthermore, the following maps are well-defined:

$$r_{G/e} \colon F(G/e) \to F(G/e), \qquad H \mapsto \begin{cases} p_e(r_G(H)), & \text{if } H \neq V_e, \\ V_e, & \text{if } H = V_e, \end{cases}$$

$$\iota_{G/e} \colon F(G/e) \to F(G/e), \qquad H \mapsto \begin{cases} p_e(\iota_G(H)), & \text{if } H \neq V_e, \\ V_e, & \text{if } H = V_e. \end{cases}$$

Lemma 1.2.6. *Following the notation of Construction 1.2.5, the maps $r_{G/e}$ and $\iota_{G/e}$ satisfy $\iota_{G/e} \circ r_{G/e} = r_{G/e}$.*

Proof. If $H \in F(G/e)$ is such that $r_{G/e}(H) \neq V_e$, then the equality $\iota_{G/e} \circ r_{G/e}(H) = r_{G/e}(H)$ readily follows. It is also clear that the equality holds when $H = V_e$, so it suffices to check the case where $H \in F(G)$ with $p_e(r_G(H)) = V_e$. Since $\iota_{G/e}(V_e) = V_e$, the desired equality follows. □

Definition 1.2.7. For a discrete graph G with $e \in E(G)$, the *contraction of e in G* is the graph G/e given by the triple $(F(G/e), r_{G/e}, \iota_{G/e})$.

Lemma 1.2.8. *Let G be a discrete graph, with $e_1, e_2 \in E(G)$ two different edges with a common incident vertex. Then $(G/e_1)/e_2$ is naturally bijective to $(G/e_2)/e_1$.*

Proof. By definition of the contraction, the identity map $F(G) \to F(G)$ induces a bijective map $F((G/e_1)/e_2) \to F((G/e_2)/e_1)$. The commutativity of this map with the respective root and involution maps is immediate. □

Observe that Lemma 1.2.8 shows that we can contract finitely many edges in any order and produce naturally bijective graphs. More precisely, suppose K is a legless subgraph of G. Then contracting the edges of K in G in different orders will produce naturally bijective distinct graphs, where the distinction between them only comes from the newly introduced vertices in the last step of the successive contractions. We want to dispose these natural identifications and replace these contractions by a single object.

Definition 1.2.9. Let K be a legless subgraph of G, with $K = K_1 \sqcup \cdots \sqcup K_m$ its connected components. The *contraction of K in G* is the graph G/K obtained by the successive contractions of the edges of $K_1, \ldots, K_m$, where the new vertex obtained by the contraction of all the edges of the component K_i is denoted by V_{K_i}.

If K is a legless subgraph of G, then the contraction of K in G is a graph G/K whose set of vertices consists of the set $V(G)\backslash V(K)$ together with one additional vertex per component of K. Naturally, if G is connected, then so is G/K (for arbitrary legless K), and when K is connected

$$g(G/K) = g(G) - g(K).$$

Example 1.2.10. In Figure 1.2 the graph on the right is obtained as the contraction of the subgraph given by the edges a and b of the left graph. This situation is made explicit as follows. Consider the graph G given by

$$F(G) = \{AB, AC, AD, BA, BC, BD, CA, CB, CD, DA, DB, DC, A, B, C, D\},$$

together with root map

$$\begin{array}{lllll} r_G(AB) = A, & r_G(BA) = B, & r_G(CA) = C, & r_G(DA) = D, & r_G(A) = A, \\ r_G(AC) = A, & r_G(BC) = B, & r_G(CB) = C, & r_G(DB) = D, & r_G(B) = B, \\ r_G(AD) = A, & r_G(BD) = B, & r_G(CD) = C, & r_G(DC) = D, & r_G(C) = C, \\ & & & & r_G(D) = D, \end{array}$$

and involution map

$$\begin{array}{lllll} \iota_G(AB) = BA, & \iota_G(BA) = AB, & \iota_G(CA) = AC, & \iota_G(DA) = AD, & \iota_G(A) = A, \\ \iota_G(AC) = CA, & \iota_G(BC) = CB, & \iota_G(CB) = BC, & \iota_G(DB) = BD, & \iota_G(B) = B, \\ \iota_G(AD) = DA, & \iota_G(BD) = DB, & \iota_G(CD) = DC, & \iota_G(DC) = CD, & \iota_G(C) = C, \\ & & & & \iota_G(D) = D. \end{array}$$

This graph corresponds to the leftmost graph of Figure 1.2 by the identification of edges

$$\begin{array}{llll} e_1 = \{AC, CA\}, & e_2 = \{AB, BA\}, & e_3 = \{CD, DC\}, \\ e_4 = \{BD, DB\}, & a = \{BC, CB\}, & b = \{AD, DA\}. \end{array}$$

Consider the subgraph K of G given by a and b. Namely, $F(K)$ is the following subset of $F(G)$:

$$F(K) = \{AD, BC, CB, DA, A, B, C, D\}.$$

Its connected components are the subgraphs K_a and K_b, given by $F(K_a) = \{BC, CB, B, C\}$ and $F(K_b) = \{AD, DA, A, D\}$. By definition, the contraction of K in G gives rise to the

graph G/K given by

$$F(G/K) = \{AB, AC, BA, BD, CA, CD, DB, DC\} \sqcup \{V_{K_a}, V_{K_b}\},$$

with root map

$$\begin{aligned} r_G(AB) = V_b, \quad r_G(BA) = V_a, \quad r_G(CA) = V_a, \quad r_G(DB) = V_b, \quad r_G(V_a) = V_a, \\ r_G(AC) = V_b, \quad r_G(BD) = V_a, \quad r_G(CD) = V_a, \quad r_G(DC) = V_b, \quad r_G(V_b) = V_b, \end{aligned}$$

and the evident involution map. An identification of edges analogous to the one before shows that this graph describes the rightmost graph of Figure 1.2.

Figure 1.2: An example of a contraction

Lemma 1.2.11. *Suppose $f : G_1 \to G_2$ is a map of graphs such that:*

- *For every $V \in V(G_2)$, the inverse image $f^{-1}(V) \subset F(G_1)$ is a legless tree.*
- *For every $e \in E(G_2) \cup L(G_2)$, its inverse image under f consists of a single edge or leg of G_1.*

Then there is a natural bijective map of graphs $G_1/f^{-1}(V(G_2)) \to G_2$.

Proof. We show the lemma in the case where there is a vertex $V \in V(G_2)$ such that for all $W \in V(G_2)\backslash\{V\}$ the inverse image $f^{-1}(W)$ is just a vertex (a trivial legless tree). This is sufficient, since every map of graphs as in the statement can be decomposed as a composition of these. In this case, if $f^{-1}(V)$ is just a vertex, then f is a bijective map between G_1 and G_2. Therefore,suppose $f^{-1}(V)$ is a non-trivial legless tree. The second condition implies that f restricts to a bijection between $F(G_1)\backslash f^{-1}(V)$ and $F(G_2)\backslash\{V\}$, and hence induces a bijection $F(G_1/f^{-1}(V)) \to F(G_2)$. The commutation with respect to root and involution maps follows directly from that of f. □

The following definition is motivated by the previous lemma, and describes precisely the types of maps of graphs that primarily concern us.

Definition 1.2.12. A map of graphs $f : G_1 \to G_2$ is a *contraction* if for every $V \in V(G_2)$ the subgraph defined by the inverse image $f^{-1}(V)$ is a legless tree of G_1.

1.3 Some categories of graphs and their structures

Suppose A is a finite set. A *discrete graph with A-marked legs* is a discrete graph G with a bijection

$$\ell_\bullet(G)\colon A \to L(G), a \mapsto \ell_a(G),$$

which we will call the *marking*. For an element $a \in A$, the *a-leg of G* is simply $\ell_a(G) \in L(G)$. A *map of A-marked graphs* is simply a map of graphs that commutes with the respective markings.

It is clear that a composition of contractions is a contraction. Suppose $2g+\#A-2>0$ and let $\mathbb{G}_{g,A}$ denote the category specified by:

- The objects of $\mathbb{G}_{g,A}$ consist of the connected genus-g discrete graphs with A-marked legs whose vertices are at least 3-valent.
- For two objects G_1 and G_2 of $\mathbb{G}_{g,A}$, the set of morphisms $\mathrm{Hom}_{\mathbb{G}_{g,A}}(G_1, G_2)$ consists of the contractions $f\colon G_1 \to G_2$.
- Composition of morphisms is just composition of maps.

Remark. The reader may find that an accurate name for this category would be that of *weightless stable genus-g A-marked graphs*. Unfortunately this name is suggestive of the existence of a non-trivial weight function, which is absent in our considerations. Hence, we simply refrain from this name.

Notation 1.3.1. In case $A = \{1,\dots,n\}$ for $n \in \mathbb{N}$, then the category $\mathbb{G}_{g,A}$ will simply be denoted by $\mathbb{G}_{g,n}$. In this case, an A-marked graph G will simply be called an n-marked graph, and for $1 \le i \le n$ the *i-th leg* will be $\ell_i(G)$.

Remark. Naturally, the maps of graphs in Definition 1.2.4 and the morphisms of $\mathbb{G}_{g,A}$ are quite distinct.

Example 1.3.2. For $g = 2$ and $n = 1$, Figure 1.3 depicts a skeleton of $\mathbb{G}_{2,1}$ that excludes the automorphisms of the objects and where the arrows denote the contracted edge.

Lemma 1.3.3. *There are finitely many isomorphism classes in $\mathbb{G}_{g,A}$.*

Proof. Note first that any object of $\mathbb{G}_{g,A}$ can be obtained up to isomorphism by a sequence of edge contractions from a 3-valent graph. Therefore,it suffices to show that there are finitely many connected 3-valent graphs of genus-g up to isomorphism. Observe that if it is shown that the number of edges and vertices is bounded, then we can fix a set Z so that up to isomorphism any connected 3-valent discrete graph G is isomorphic to a graph given by a triple (X, r_X, ι_X) where $X \subset Z$. Let G be a 3-valent graph of $\mathbb{G}_{g,A}$, then $3\#V(G) = 2\cdot\#E(G) + \#A$ and $\#E(G) - \#V(G) + 1 = g$. These two equations fix the sizes of $V(G)$ and $E(G)$ in this case, and hence the lemma. □

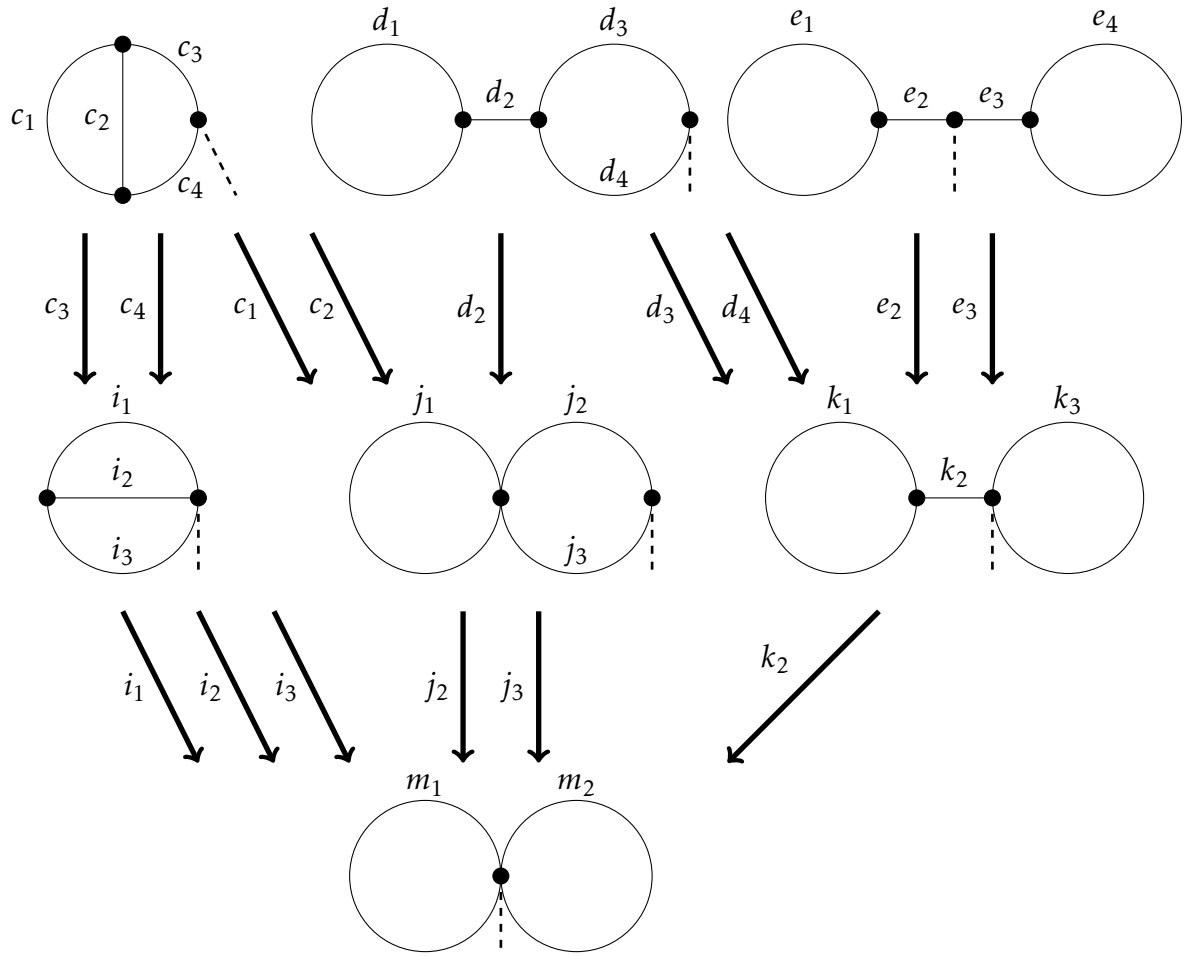


Figure 1.3: Skeleton of $\mathbb{G}_{2,1}$ without automorphisms of objects

Proposition 1.3.4. *Any object of $\mathbb{G}_{0,A}$ has no non-trivial automorphisms, and any skeleton of this category is a finite poset.*

Proof. This is a well-known fact: sufficiently marked trees have no non-trivial automorphisms. Nonetheless, we provide a proof to convey familiarity with the notation and for the sake of completeness. Let T be an object of $\mathbb{G}_{0,A}$ and $f : T \to T$ be an automorphism. We intend to show that f is the identity. For this, let $V_0(T) \subset V(T)$ denote the subset of vertices incident to legs and set $F_0(T) = \left(\bigcup_{\ell \in L(T)} \ell\right) \cup V_0(T)$. Since f is a map of A-graphs, it must preserve the legs and also every vertex incident to a leg. In particular, it restricts to the identity on $F_0(T)$. We set for $i \geq 0$

$$F_{i+1}(T) = r_T^{-1}\left(\bigcup_{j \leq i} V_j(T)\right), \qquad V_{i+1}(T) = r_T\left(\iota_T\left(F_{i+1}(T)\right)\right).$$

We observe that $F_0(T) \subset F_1(T) \subset F_2(T) \subset \cdots$, and moreover there is an $N > 0$ such that $F_N(T) = F(T)$. Hence, to show that f the identity, it suffices to show the following statement: If f restricts to the identity on $F_i(T)$, then it does too on $F_{i+1}(T)$. We now

argue by means of contradiction, and suppose that f does not restrict to the identity on $F_{i+1}(T)$. Observe that f necessarily restricts to the identity on $V_i(T)$, because it does so on $F_i(T)$. Now, if f is not the identity on $F_{i+1}(T)$, then there are two elements H and H' of $F_{i+1}(T)$, with $f(H) = H'$ and that are incident to the same vertex V of $V_i(T)$. Consider the vertices $W = r_T(\iota_T(H))$ and $W' = r_T(\iota_T(H'))$. Then f must map $r_G^{-1}(W)$ to $r_G^{-1}(W')$, and the same holds between $r_G^{-1}\left(\iota_G\left(r_G^{-1}(W)\right)\right)$ and $r_G^{-1}\left(\iota_G\left(r_G^{-1}(W')\right)\right)$. Continuing in this fashion we will either arrive to legs of T, or to a path between these vertices that does not go through V. Both cases would yield a contradiction: the former because f must fix the legs, and the latter because $g(T) = 0$. Thus, it follows that f must restrict to the identity on $F_{i+1}(T)$ and therefore on $V_{i+1}(T)$. The finiteness of isomorphism classes follows from Lemma 1.3.3 □

Remark. By briefly entertaining the sin of forward referencing, we can produce a non-circular argument yielding a different proof of Lemma 1.3.3 for positive g. Namely, we can invoke Theorem 3.5 of [GKM09] to show that there are finitely many isomorphism classes of $\mathbb{G}_{0,A}$ (these correspond to combinatorial types thereof), and we can now apply Proposition 3.5.3.

1.4 Moduli spaces of tropical curves

Tropical curves are a special kind of metric spaces obtained by a discrete graph together with a metric thereon. In this section, we define the cone of metrics associated to a discrete graph and use this construction to obtain a presentation of the moduli spaces of tropical curves of a fixed genus and marking. These spaces lie at the heart of our enterprise and not only constitute our leading examples of the ensuing constructions in later chapters, but also served as the motivating natural picture for the design of our methods.

Definition 1.4.1. A *metric* on a discrete graph G is a function $\delta\colon E(G) \to \mathbb{R}_{>0}$.

A pair (G, δ), where G is a discrete graph and δ is a metric on G, gives rise to a metric space as follows: to each edge an interval of length prescribed by δ is associated, to each leg a half line $\mathbb{R}_{\geq 0}$ is associated, and all these are glued according to the incidence relations of G.

Definition 1.4.2. For a finite set A, an *A-marked tropical curve* Γ is a metric space so arising from a pair (G, δ), where G is an A-marked discrete graph and δ is a metric on G.

Naturally, many different pairs can yield isometric spaces, which means that there are non-trivial relations to take into account. We want to understand and approach the

space parametrizing connected genus-g A-marked tropical curves, namely the moduli space of tropical curves $\mathcal{M}^{\mathrm{trop}}_{g,A}$. Our previously introduced category $\mathbb{G}_{g,A}$ provides a neat construction of this space that by design includes the previously mentioned non-trivial relations. We first deal with some general notation.

Notation 1.4.3. For a finite set X, we let $\mathbb{R}^X_{\geq 0}$ denote the poic given by the non-negative orthant in the vector space $(\mathbb{Z}^X)_{\mathbb{R}}$ and the lattice $\mathbb{Z}^X$. Analogously, we let $\mathbb{R}^X_{>0}$ denote the poic given by the positive orthant in this same vector space and the same lattice. Clearly the latter is a subcone of the former. It is also clear that subsets $Y \subset X$ give rise to face-embeddings $\mathbb{R}^Y_{\geq 0} \to \mathbb{R}^X_{\geq 0}$. In addition, any face of $\mathbb{R}^X_{\geq 0}$ is given by a subset of X.

Construction 1.4.4. Suppose A is a fixed finite set, and g is a non-negative number with $2\#A + g - 2 > 0$. Let G be an object of $\mathbb{G}_{g,A}$. Observe that subsets of $E(G)$ correspond to unmarked subgraphs of G that do not have vertices as connected components. These give, in turn, particular face-embeddings of the poic $\mathbb{R}^{E(G)}_{\geq 0}$. If K is an unmarked subgraph of G whose connected components are non-trivial trees, then:

- G/K is an object of $\mathbb{G}_{g,A}$,
- $E(G/K) = E(G)\backslash E(K)$,
- the inclusion $E(G/K) \subset E(G)$ gives rise to a face-embedding which we denote by $f_K \colon \mathbb{R}^{E(G/K)}_{\geq 0} \to \mathbb{R}^{E(G)}_{\geq 0}$.

Definition 1.4.5. Let G be an object of $\mathbb{G}_{g,A}$, with g and A as above. The *cone of metrics* of G is the poic σ_G defined as the subcone of $\mathbb{R}^{E(G)}_{\geq 0}$, given by $\mathbb{R}^{E(G)}_{>0}$ and all the images $f_K(\mathbb{R}^{E(G/K)}_{>0})$ where K is an unmarked subgraph of G whose connected components are non-trivial trees.

Remark. A metric on G corresponds uniquely to a point of the underlying polyhedral cone of $\mathbb{R}^{E(G)}_{>0}$.

Example 1.4.6. In Figure 1.4 the cones of metrics of a full set of representatives of $\mathbb{G}_{1,2}$ are depicted. In order to keep depictions as light and clear as possible, we do not make the lattice explicit. We represent non-present facets by dashed lines, and non-present vertices by unfilled points.

We now turn our attention to the relationship between morphisms of $\mathbb{G}_{g,A}$ and the cones of metrics.

Construction 1.4.7. Let A, g and G be as before (namely, like in Construction 1.4.4). Suppose G' is an additional object of $\mathbb{G}_{g,A}$ and $f \in \mathrm{Hom}_{\mathbb{G}_{g,A}}(G', G)$ (so f is a contraction). Observe that by definition $f^{-1}(V(G))$ is an unmarked subgraph of G'. Let K denote the subgraph of G' obtained from $f^{-1}(V(G))$ by removing the isolated vertices. It follows from Lemma 1.2.11 that $G'/K \cong G$. The contraction f gives rise to:

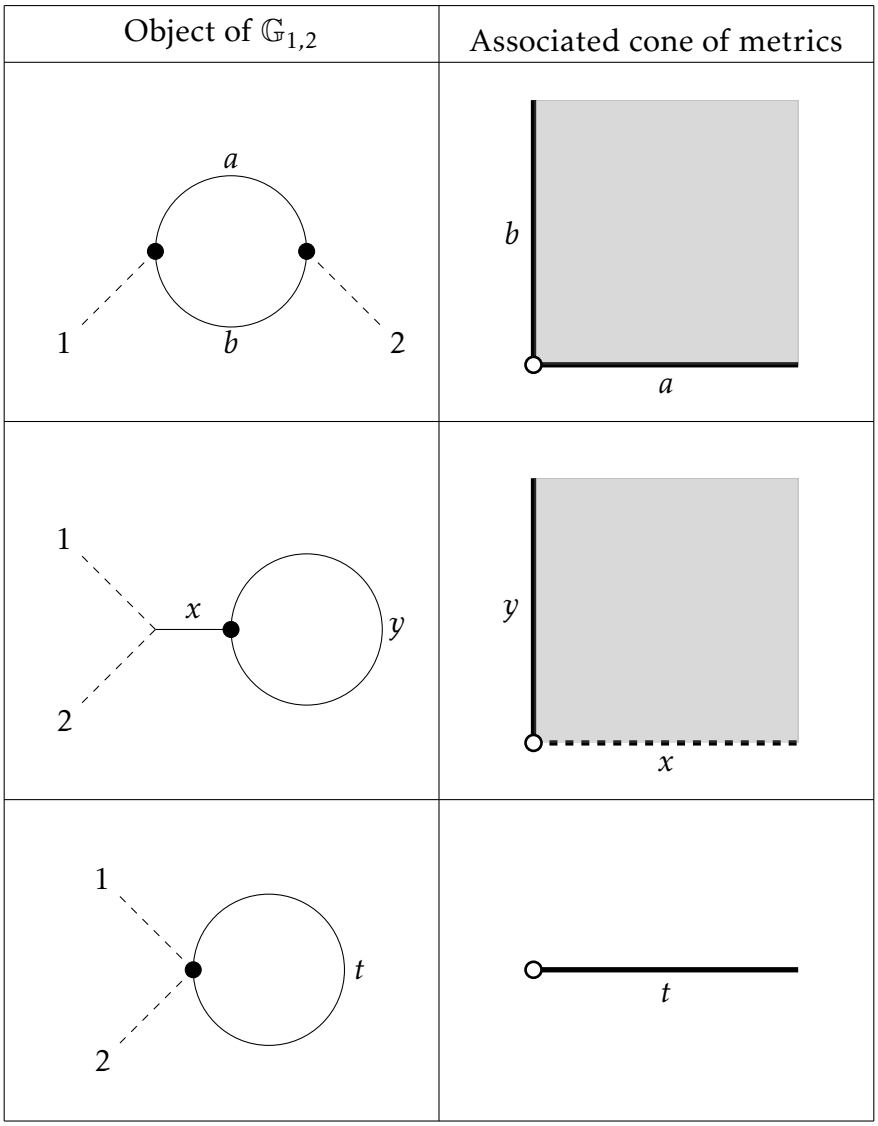


Figure 1.4: Examples of cones of metrics

- A face-embedding $\iota_K \colon \sigma_{G'/K} \to \sigma_{G'}$, given by the natural inclusion $E(G'/K) \subset E(G')$.

- A bijection $E(G) \to E(G'/K)$ which gives an isomorphism of poics $L_f \colon \sigma_G \to \sigma_{G'/K}$.

This means that the composition $\iota_K \circ L_f$ is a face-embedding. We will denote this face-embedding by $\iota_f \colon \sigma_G \to \sigma_{G'}$.

The next lemma brings Constructions 1.4.4 and 1.4.7 together into what was announced in the introduction of the section: the cones of metrics define a POIC-valued functor on our categories of graphs. The contravariant nature of this association is already implicit and witnessed in Construction 1.4.7, as faces of the cone of metrics naturally correspond to contractions of the graph.

Lemma 1.4.8. *For a non-negative integer g and a finite set A with $2g+\#A-2>0$, the cone of metrics construction gives rise to a functor*

$$\mathrm{M}^{\mathrm{trop}}_{g,A}\colon \mathbb{G}^{\mathrm{op}}_{g,A} \to \mathrm{POIC}, \tag{1.3}$$

defined as follows:

- *At an object G of $\mathbb{G}_{g,A}$, the poic $\mathrm{M}^{\mathrm{trop}}_{g,A}(G)$ is simply σ_G.*
- *At a morphism $f \in \mathrm{Hom}_{\mathbb{G}_{g,A}}(G',G)$, the poic morphism $\mathrm{M}^{\mathrm{trop}}_{g,A}(f)$ is the face-embedding $\iota_f\colon \sigma_G \to \sigma_{G'}$.*

Moreover if G is an object of $\mathbb{G}_{g,A}$ and τ is a face of $\mathrm{M}^{\mathrm{trop}}_{g,A}(G)$, then there exists up to isomorphism a unique $h \in \mathrm{Hom}_{\mathbb{G}_{g,A}}(G,H)$ such that $\mathrm{M}^{\mathrm{trop}}_{g,A}(h)\colon \sigma_H \to \sigma_G$ is isomorphic to $\tau \le \sigma_G$.

Proof. For the functoriality, we need to show that if $f\colon G' \to G$ and $g\colon G'' \to G'$ are two morphisms of $\mathbb{G}_{g,A}$, then $\iota_{f\circ g} = \iota_g \circ \iota_f$. Recall that these morphisms are contractions, and hence we take the following subgraphs into consideration:

- Let $K' \subset G$ be the maximal subgraph contained in $f^{-1}(V(G))$ whose connected components are non-trivial trees.
- Let $K'' \subset G$ denote the maximal subgraph contained in $g^{-1}(V(G'))$ whose connected components are non-trivial trees.
- Let $H'' \subset G''$ be the maximal subgraph contained in $(f\circ g)^{-1}(V(G))$ whose connected components are non-trivial trees.

Now, Lemma 1.2.11 shows that f, g, and $f\circ g$ induce respective bijective maps of graphs

$$G'/K' \to G, \qquad G''/K'' \to G', \qquad G''/H'' \to G. \tag{1.4}$$

The above produces the following containment relations

$$E(G'/K') \subset E(G'), \qquad E(G''/H'') \subset E(G''/K'') \subset E(G''). \tag{1.5}$$

By definition of the face-embeddings ι_f, ι_g, and $\iota_{f\circ g}$, the equation $\iota_{f\circ g} = \iota_g \circ \iota_f$ follows from the following commutative diagram of sets induced by (1.5), and the induced maps between edge sets of (1.4):

$$\begin{array}{ccccc} E(G''/K'') & \xrightarrow[\sim]{(1.4)} & E(G') & \xleftarrow[\sim]{(1.5)} & E(G'/K') \\ {\scriptstyle (1.5)}\uparrow & & & & \wr\downarrow{\scriptstyle (1.4)} \\ E(G''/H'') & & \xrightarrow[(1.4)]{\sim} & & E(G). \end{array}$$

It remains to show the second statement of the lemma. For this, consider an object G of $\mathbb{G}_{g,A}$ and observe that, by definition, a face $\tau \leq \mathrm{M}^{\mathrm{trop}}_{g,A}(G) = \sigma_G$ corresponds to a unmarked subgraph K of G whose components are non-trivial trees. Therefore,G/K is also an object of $\mathbb{G}_{g,A}$, and $\tau \leq \mathrm{M}^{\mathrm{trop}}_{g,A}(G)$ is given by the contraction $G \to G/K$. This identification is only up to isomorphism, because the only possible alternatives are given by a bijective map of graphs of G or G/K. □

This section is closed with our definition of the moduli space of genus-g A-marked tropical curves, and a depiction of a special case. It is worth noting that since $\mathbb{G}^{\mathrm{op}}_{g,A}$ has finitely many isomorphism classes, it is necessarily equivalent to a finite category. Recall that small colimits are representable in Top, and therefore the colimit of any functor from $\mathbb{G}^{\mathrm{op}}_{g,A}$ to Top is representable.

Definition 1.4.9. The *moduli space of A-marked genus-g tropical curves* is the topological space

$$\mathcal{M}^{\mathrm{trop}}_{g,A} := \mathrm{colim}_{\mathbb{G}^{\mathrm{op}}_{g,A}} \left|\mathrm{M}^{\mathrm{trop}}_{g,A}\right|.$$

This definition shows that after having a full set of representatives for $\mathbb{G}^{\mathrm{op}}_{g,A}$, the space $\mathcal{M}^{\mathrm{trop}}_{g,A}$ consists of taking the cones of metrics of these representatives and identifying points representing isometric curves (taking the quotient by the corresponding automorphisms and glueing according to face-embeddings). This presentation is rather useful for our purposes, because it highlights the combinatorial relations from the cones of metrics essential to our context: face relations are edge contractions, and point identifications are given by automorphisms.

Example 1.4.10. Let $g = 1$ and $n = 3$, we can convey a depiction of $\mathcal{M}^{\mathrm{trop}}_{1,3}$ as follows. Analogous to the Example 1.3.2, Figure 1.5 depicts a skeleton of $\mathbb{G}_{1,3}$ which excludes the automorphisms of the objects. In the spirit of decongestion, we have also excluded the labeling of both vertices and edges, and only label the necessary legs to distinguish the isomorphism classes. We remark that this space is three-dimensional, but we can nicely decompose it as a product. More precisely, let $\Delta_{1,3}$ denote the subspace given by the tropical curves whose total edge length sum is 1. Then the space $\mathcal{M}^{\mathrm{trop}}_{1,3}$ decomposes as the product $\Delta_{1,3} \times \mathbb{R}_{>0}$, where to a tuple $(\Gamma, t) \in \Delta_{1,3} \times \mathbb{R}_{>0}$ corresponds the tropical curve $t \cdot \Gamma$ given by dilating every edge of Γ by a factor of t. This space $\Delta_{1,3} \times \mathbb{R}_{>0}$ can actually be depicted in a way which we now describe.

For the depiction of $\Delta_{1,3}$, we first observe that to each graph of the leftmost column of Figure 1.5 corresponds to a two-dimensional cell of $\Delta_{1,3}$ obtained by taking a quotient of a subspace of the 2-simplex by an action of a finite group (the group of automorphisms of the graph). Similarly, the graphs on the middle column give rise to 1-dimensional cells similarly obtained as a quotient of a subspace of the 1-simplex. The graph on the rightmost column (a terminal object of $\mathbb{G}_{1,3}$) simply corresponds to

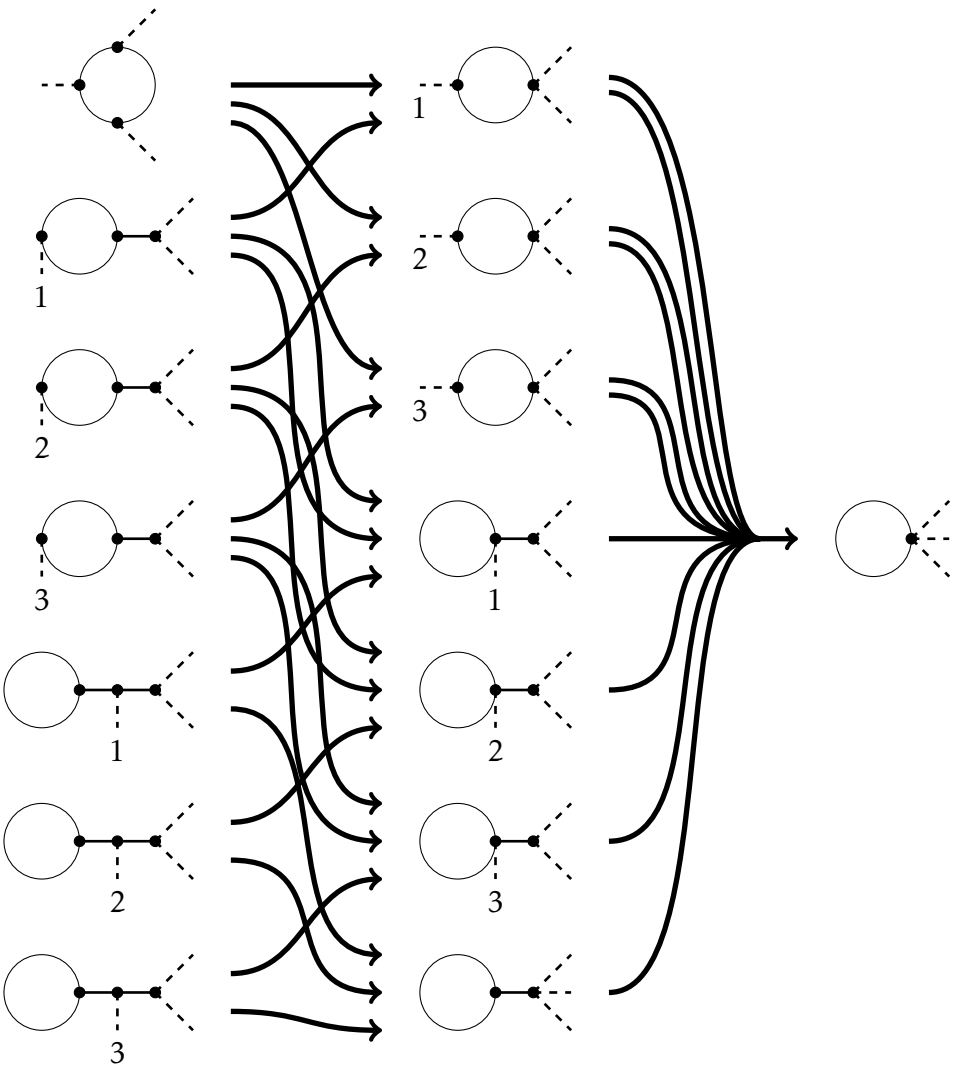


Figure 1.5: Skeleton of $\mathbb{G}_{1,3}$ without automorphisms of objects

a point $\bullet_{1,3}$ of $\Delta_{1,3}$ that also lies in each of the previous cells. We focus on the cells produced by the leftmost column, and explain how the others are glued together to them. The three main cases to consider are depicted in Figure 1.6

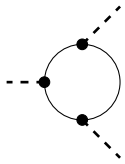
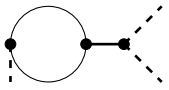
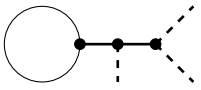
Case (a)	Case (b)	Case (c)
		

Figure 1.6: Main cases to consider for $\Delta_{1,3}$

Case (a) In this case every point of the 2-simplex represents a metric on this graph, and hence we must not take a proper subspace. So we can only focus on the re-

spective identifications. Every edge of this 2-simplex represents a different non-isomorphic graph (because of the labelling), but these edges have to be folded in half because of the new symmetry. Every vertex of this 2-simplex is identified with the single point $\bullet_{1,3}$. In conclusion, this case yields a tetrahedron with apex corresponding to $\bullet_{1,3}$ and with the other vertices corresponding to graphs obtained from a single edge contraction of this case where the two edges have the same length. Every edge that joins one of these vertices with the apex will correspond to an edge of Case (b), while the other edges (those that do not involve $\bullet_{1,3}$ are not relevant to any other of the main cases.

Case (b) In this case not every point of the 2-simplex represents a metric on this graph. Namely, we have to remove one vertex of the 2-simplex which corresponds to the contraction of the only cycle of the graph. This is the only point we have to remove, and the identifications require us to fold this 2-simplex along the half segment coming from the point we have removed to the middle of the opposite edge. This identifies the two half edges obtained from removing the vertex, and folds the closed edge in half. The remaining 2 vertices are thus identified and correspond to $\bullet_{1,3}$. There are no further identifications, and the edge we have folded in half has to be adjoined to one of the edges of Case (a) joining a vertex with the apex. From this, we obtain three cells that consist of a triangle with a removed vertex.

Case (c) In this case we must also not take the whole 2-simplex, but rather just remove one entire edge. This edge, with its two vertices, represents the contraction of the loop. Hence, the subspace we obtain has 1 vertex (corresponding to $\bullet_{1,3}$) and 2 half edges, and there are no relations to quotient by in this case. We obtain three different cells like this, and each of these is glued to only one of the cells of Case (b) along one of the half edges. The other half edges of these three cells are identified with each other (this corresponds to the metrics on the bottom graph of the middle column of Figure 1.5).

We gather these depictions in Figure 1.7, where we have additionally depicted all the discrete graphs involved color coded to describe the cell they determine. In addition, Figure 1.8 is an illustration of the whole space $\Delta_{1,3}$. The dashed edges and white vertices mean that these have been removed, the non-thick edges correspond to edges that are in these cells but do not come from one of the isomorphism classes of $\mathbb{G}_{1,3}$, and the thick black vertex corresponds to $\bullet_{1,3}$.

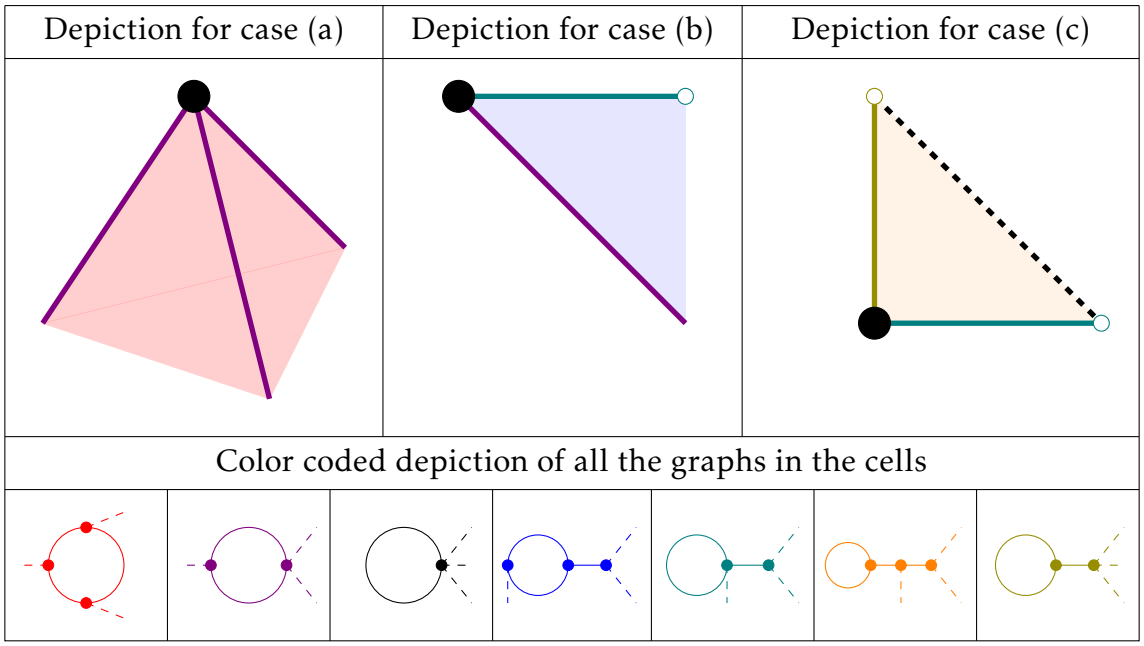


Figure 1.7: Depiction of the cells of the main cases of $\Delta_{1,3}$

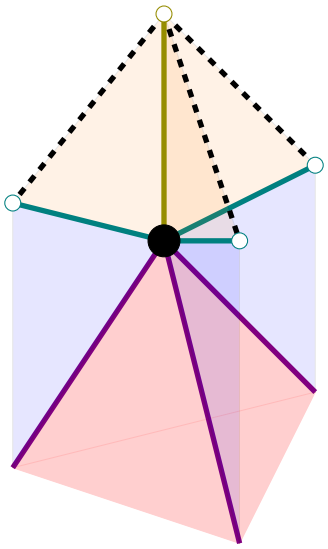


Figure 1.8: Depiction of the whole $\Delta_{1,3}$

1.5 Discrete admissible covers

In this section G_1 and G_2 will denote two discrete graphs. We will say that a map of graphs $f\colon G_1 \to G_2$ is *non-contracting* if it does not involve the contraction of edges or legs of G_1 to vertices. Put differently, we have that $f^{-1}(V(G_2)) \subset V(G_1)$.

Definition 1.5.1. A *degree assignment* on G_1 is a map $d\colon F(G_1) \to \mathbb{Z}_{\geq 0}$ such that $d \circ \iota = d$.

Notation 1.5.2. Let d be a degree assignment on G_1. If $e \in E(G_1)$, then there is an $E \in F(G_1)$ such that $e = \{E, \iota(E)\}$. By definition $d(E) = d(\iota(E))$, so we will abuse notation and refer to this common value simply as $d(e)$. Analogously, if $\ell \in L(G_1)$ and $\ell = \{L\}$,

then $d(\ell)$ will simply denote $d(L)$.

Remark. Equivalently, a degree assignment on G_1 is the same as an integral valued function on $V(G_1) \cup E(G_1) \cup L(G_1)$.

Definition 1.5.3. A *harmonic morphism* $\pi\colon G_1 \to G_2$ is a map of graphs $\pi\colon G_1 \to G_2$ and a degree assignment d_π on G_1 satisfying the following conditions:

- π is a surjective non-contracting map of graphs.
- For any $V \in V(G_1)$ and $H' \in r_{G_2}^{-1}(\pi(V)) \setminus \{\pi(V)\}$

$$d_\pi(V) = \sum_{H \in \pi^{-1}(H') \cap r_{G_1}^{-1}(V)} d_\pi(H).$$

If $\pi\colon G_1 \to G_2$ is a harmonic morphism, then the *local degree at* $V \in V(G_1)$ is the integer $d_\pi(V)$. In general, if $e \in E(G_1) \cup L(G_1)$, then we refer to $d_\pi(e)$ as the *weight of the edge (or leg)*.

Remark. Suppose $\pi\colon G_1 \to G_2$ is a harmonic morphism. We have previously remarked that a general map of graphs cannot map a leg to an edge, or an edge to a leg. In addition, the condition of π being non-contracting implies that it must map edges to edges, legs to legs, and vertices to vertices.

Harmonic morphisms have been vastly studied objects, and all the results in this section are well-known in the literature. Nonetheless, we provide proofs for these not only as a mean to impart familiarity with our notation but also to avoid bearing the reader with the tedious task of translation between different notations.

Lemma 1.5.4. *Let $\pi\colon G_1 \to G_2$ be a harmonic morphism. If G_2 is connected, then the following equality holds*

$$\sum_{V \in \pi^{-1}(A)} d_\pi(V) = \sum_{W \in \pi^{-1}(B)} d_\pi(W),$$

for any $A, B \in V(G_2)$.

Proof. Since G_2 is connected, it is sufficient to show the equality when there is $e \in E(G_2)$ with $\partial e = \{A, B\}$. Consider the subsets of $H(G_1)$

$$H(G_1)_A := \pi^{-1}\left(r_{G_2}^{-1}(A)\right) \cap H(G_1) = \{H \in H(G_1) : r_{G_1}(\pi(H)) = A\},$$
$$H(G_1)_B := \pi^{-1}\left(r_{G_2}^{-1}(B)\right) \cap H(G_1) = \{H \in H(G_1) : r_{G_1}(\pi(H)) = B\}.$$

By construction $\pi^{-1}(A) = r_{G_1}(H(G_1)_A)$ and $\pi^{-1}(B) = r_{G_1}(H(G_1)_B)$. Moreover, the involution ι_{G_1} restricts to a bijection $\iota_{G_1} : H(G_1)_A \to H(G_1)_B$. Since π is harmonic, it follows

that

$$\sum_{V\in\pi^{-1}(A)} d_\pi(V) = \sum_{H\in H(G_1)_A} d_\pi(H)$$
$$= \sum_{H\in H(G_1)_B} d_\pi(\iota_{G_1}(H)) = \sum_{H\in H(G_1)_B} d_\pi(H) = \sum_{V\in\pi^{-1}(B)} d_\pi(V).$$

□

Definition 1.5.5. The *degree* of a harmonic morphism $\pi\colon G_1 \to G_2$ with G_2 connected is defined as the integer

$$\deg\pi := \sum_{V\in\pi^{-1}(A)} d_\pi(V),$$

where $A \in V(G_2)$ is any vertex. It follows from the previous lemma, that this number is invariant with respect to the vertices of G_2. In addition, for an $e \in E(G_2)\cup L(G_2)$ the weights of the edges or legs in its fiber form a partition of $\deg\pi$. This unordered partition is called the *ramification profile of π at e*.

Example 1.5.6. We depict in Figure 1.9 a degree-5 harmonic morphism $\pi\colon G \to H$ between two trees. We have avoided the labelling of the legs and edges to unburden the depiction as much as possible. The coloring indicates the correspondence between the underlying map of graphs. The weights are written next to the corresponding edge or leg, whenever they are bigger than one.

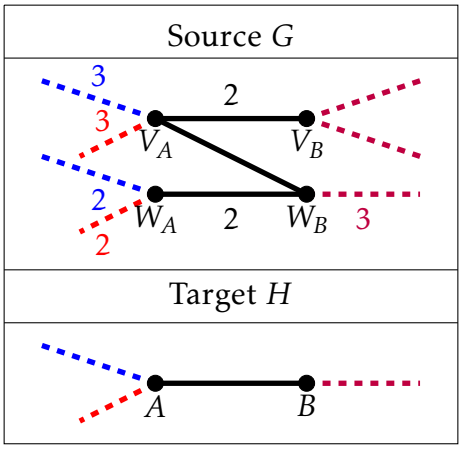


Figure 1.9: A degree 5 harmonic morphism between two trees

Definition 1.5.7. Given a harmonic morphism $\pi\colon G_1 \to G_2$, the *Riemann-Hurwitz number (RH number) of π at $V \in V(G_1)$* is defined as

$$r_\pi(V) := \mathrm{val}(V) - 2 - d_\pi(V)(\mathrm{val}(\pi(V)) - 2).$$

Proposition 1.5.8 (Riemann-Hurwitz equality)**.** *Suppose G_2 is connected and $\pi\colon G_1 \to G_2$ is a harmonic morphism. The genera $g(G_1)$ and $g(G_2)$ are related to the degree of π by:*

$$\#L(G_1) + 2(g(G_1) - 1) = \deg \pi \cdot (\#L(G_2) + 2(g(G_2) - 1)) + \sum_{V \in V(G_1)} r_\pi(V). \tag{1.6}$$

Proof. We begin by noting that

$$\sum_{V \in V(G_1)} r_\pi(V) = \sum_{V \in V(G_1)} (\operatorname{val} V - 2) - \sum_{V \in V(G_1)} d_\pi(V)(\operatorname{val} \pi(V) - 2). \tag{1.7}$$

Now, we using the fact that $\sum_{V \in V(G_1)} \operatorname{val} V = \#L(G_1) + 2\#E(G_1)$, we can express the first summand of the above equation as:

$$\sum_{V \in V(G_1)} (\operatorname{val} V - 2) = \#L(G_1) + 2\#E(G_1) - 2\#V(G_1) = \#L(G_1) + 2(g(G_1) - 1).$$

In particular, (1.7) yields the following equality:

$$\sum_{V \in V(G_1)} r_\pi(V) = \#L(G_1) + 2(g(G_1) - 1) - \sum_{V \in V(G_1)} d_\pi(V)(\operatorname{val} \pi(V) - 2). \tag{1.8}$$

For the second summand of (1.7), we observe that the set $V(G_1)$ is partitioned as $V(G_1) = \bigcup_{W \in V(G_2)} f^{-1}(W)$, and we can express this summand as follows

$$\begin{aligned}
\sum_{V \in V(G_1)} d_\pi(V)(\operatorname{val} \pi(V) - 2) &= \sum_{W \in V(G_2)} \sum_{V \in \pi^{-1}(W)} d_\pi(V)(\operatorname{val} W - 2) \\
&= \deg \pi \sum_{W \in V(G_2)} (\operatorname{val} W - 2) \\
&= \deg \pi (\#L(G_2) + 2(g(G_2) - 1)).
\end{aligned}$$

We can now substitute this in (1.8) to obtain

$$\sum_{V \in V(G_1)} r_\pi(V) = \#L(G_1) + 2(g(G_1) - 1) - \deg \pi (\#L(G_2) + 2(g(G_2) - 1)),$$

from which the desired equality follows. □

Example 1.5.9. We reprise Example 1.5.6 to illustrate the previous result. It can readily be computed from Figure 1.9 that

$$\begin{aligned}
r_\pi(V_A) &= -1, & r_\pi(W_A) &= -1, \\
r_\pi(V_B) &= 1, & r_\pi(W_B) &= 1.
\end{aligned}$$

Hence $\#L(G) = 7$, $\#L(H) = 3$, and we can attest the Riemann-Hurwitz equality with this harmonic morphism:

$$5 = 7 + 2(0-1) = 5(3 + 2(0-1)) + r_\pi(V_A) + r_\pi(W_A) + r_\pi(V_B) + r_\pi(W_B) = 5.$$

Our attention is now turned towards the interaction between harmonic morphisms and edge contractions of the target. We show that for an edge of the target graph whose fiber is a forest, the simultaneous contractions of this edge and its fiber yield a harmonic morphism and in this process the RH numbers are conserved. More precisely, the RH numbers of the new vertices are the sum of the RH numbers of the vertices that contract to them. This process is made explicit in the subsequent construction, and the conservation of RH numbers is shown in the proposition afterwards.

Construction 1.5.10. Suppose $\pi\colon G_1 \to G_2$ is a harmonic morphism, where both G_1 and G_2 are connected. Let $e \in E(G_2)$ be such that $\pi^{-1}(e)$ is a forest with corresponding connected components $T_1, \dots, T_m$. Since e is an edge, it necessarily follows that every tree T_i (with $1 \leq i \leq m$) is legless. Following our convention on notation from Definition 1.2.9, the graph $G_1/\pi^{-1}(e)$ has the set of vertices

$$V(G/\pi^{-1}(e)) = \left(V(G) \setminus \bigcup_{1 \leq i \leq m} V(T_i) \right) \cup \left\{ V_{T_1}, \dots, V_{T_m} \right\}.$$

Every T_i (with $1 \leq i \leq m$) is a tree, thus $g(G_1/\pi^{-1}(e)) = g(G_1)$. Let $p\colon G_2 \to G_2/e$ denote the corresponding edge contraction map. The map of graphs π induces a map of graphs

$$\pi_e\colon G_1/\pi^{-1}(e) \to G_2/e$$

given by:

$$\pi_e\colon F(G_1/\pi^{-1}(e)) \to F(G_2/e), H \mapsto \begin{cases} p(f(H)), & \text{if } H \neq V_{T_i}, \\ V_e, & \text{if } H = V_{T_i}. \end{cases}$$

Furthermore, the degree assignment d_π gives rise to the degree assignment

$$d_e\colon F(G_1/\pi^{-1}(e)) \to \mathbb{Z}_{>0}, H \mapsto \begin{cases} d_\pi(H), & \text{if } H \neq V_{T_i}, \\ \sum_{h \in E(T_i)} d_\pi(h), & \text{if } H = V_{T_i}. \end{cases}$$

Proposition 1.5.11. *Following the notation of Construction 1.5.10, the map fo graphs π_e and the degree assignment d_e give a harmonic morphism $\pi_e\colon G_1/\pi^{-1}(e) \to G_2/e$, such that for any $1 \leq i \leq m$*

$$r_{\pi_e}(V_{T_i}) = \sum_{V \in V(T_i)} r_\pi(V).$$

Proof. We only need to show the condition on harmonic morphisms for π_e at the ver-

tices $V_{T_1}, \dots, V_{T_m}$, because π is already a harmonic morphism. For this, we let $e \in E(G_2)$ be such that $\partial e = \{A, B\}$. Observe that for any $1 \leq i \leq m$ the incident vertices to any edge of T_i must lie over A or B. Since π is harmonic and T_i is a tree, it follows that

$$\sum_{h \in E(T_i)} d_\pi(h) = \sum_{V \in V(T_i) \cap \pi^{-1}(A)} d_\pi(V) = \sum_{V \in V(T_i) \cap \pi^{-1}(B)} d_\pi(V).$$

Let $H \in H(G_2/e)$ be such that $r_{G_2/e}(H) = V_e$. Hence, it also holds that $H \in F(G_2)$ and, without loss of generality, we may assume that $r_{G_2}(H) = A$. Now, the set of flags of $G_1/\pi^{-1}(e)$ incident to V_{T_i} and mapping to H correspond to the flags of G_1 incident to the vertices of T_i lying over A and mapping to H. This, in turn, implies that

$$\begin{aligned} \sum_{H' \in {\pi_e}^{-1}(H) \cap r^{-1}_{G_1/\pi^{-1}(e)}(V_{T_i})} d_e(H') &= \sum_{V \in V(T_i) \cap \pi^{-1}(A)} \left(\sum_{H' \in \pi^{-1}(H) \cap r^{-1}_{G_1}(V)} d_\pi(H') \right) \\ &= \sum_{V \in V(T_i) \cap \pi^{-1}(A)} d_\pi(V) \\ &= d_e(V_{T_i}), \end{aligned}$$

and thus shows that π_e is a harmonic morphism. It remains to show the equality concerning RH numbers. As before, let $1 \leq i \leq m$ be fixed, and let us keep the same notation. By construction we have that $\operatorname{val} V_e = \operatorname{val} A + \operatorname{val} B - 2$ and similarly

$$\operatorname{val} V_{T_i} = \sum_{V \in V(T_i)} \operatorname{val}(V) - 2\#E(T_i) = \sum_{V \in V(T_i)} \operatorname{val}(V) - 2\#V(T_i) + 2 = 2 + \sum_{V \in V(T_i)} (\operatorname{val} V - 2). \tag{1.9}$$

By definition, the RH number of V_{T_i} is

$$r_{\pi_e}(V_{T_i}) = \operatorname{val} V_{T_i} - 2 - d_e(V_{T_i})(\operatorname{val} V_e - 2),$$

and using the fact that $\operatorname{val} V_e = \operatorname{val} A + \operatorname{val} B - 2$, we can rewrite this as

$$r_{\pi_e}(V_{T_i}) = \operatorname{val} V_{T_i} - 2 - \sum_{V \in V(T_i) \cap \pi^{-1}(A)} d_e(V)(\operatorname{val} A - 2) - \sum_{V \in V(T_i) \cap \pi^{-1}(A)} d_e(V)(\operatorname{val} B - 2).$$

To finalize, we use (1.9) to rewrite $\operatorname{val} V_{T_i} - 2$, and employ the fact $\pi^{-1}(A) \cap V(T_i)$ and $\pi^{-1}(B) \cap V(T_i)$ partition $V(T_i)$, to obtain

$$r_{\pi_e}(V_{T_i}) = \sum_{V \in V(T_i) \cap \pi^{-1}(A)} r_\pi(V) + \sum_{V \in V(T_i) \cap \pi^{-1}(B)} r_\pi(V) = \sum_{V \in V(T_i)} r_\pi(V).$$

□

Example 1.5.12. In Figures 1.10 and 1.11 we depict the situation described in Construction 1.5.10. More precisely, Figure 1.10 depicts a harmonic morphism of degree 5 where all the RH numbers are zero. We follow an analogous convention to the previous examples. Namely, we avoid the labelling of edges, legs, and vertices, to keep notation as light as possible. The coloring, and location, indicates the correspondence under the harmonic morphism, and the edges to be contracted have been drawn as black dashed lines. The weights are written next to the corresponding edge or leg, whenever they are bigger than one. It is immediately observed that the degree of the harmonic morphism is 5, the source curve is of genus 2 and has 18 legs, and the target curve is a tree with 6 legs. We can thus attest once again the Riemann-Hurwitz equality since:

$$20 = 18 + 2(2-1) = 5(6 + 2(0-1)) + 0 = 20.$$

In Figure 1.11 the contraction of the dashed edges is depicted, and the newly obtained vertices are colored purple. As is expected, every RH number of this harmonic morphism is still zero.

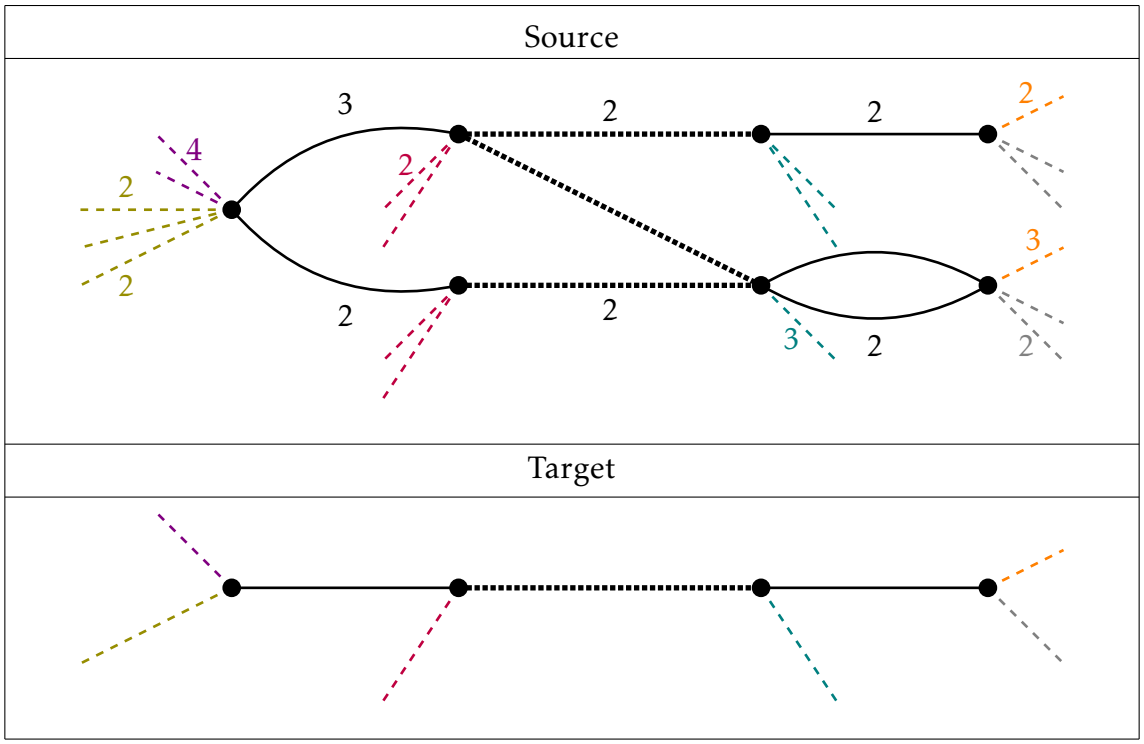


Figure 1.10: A harmonic morphism where an edge of the target (together with its fiber) is signaled for contraction

We close this section by introducing discrete admissible covers, which, as suggested by the title, are integral to this project. These are special kinds of harmonic morphisms where the RH numbers are not arbitrary. In subsequent sections we proceed as in the case of discrete graphs and construct moduli spaces for discrete admissible covers after

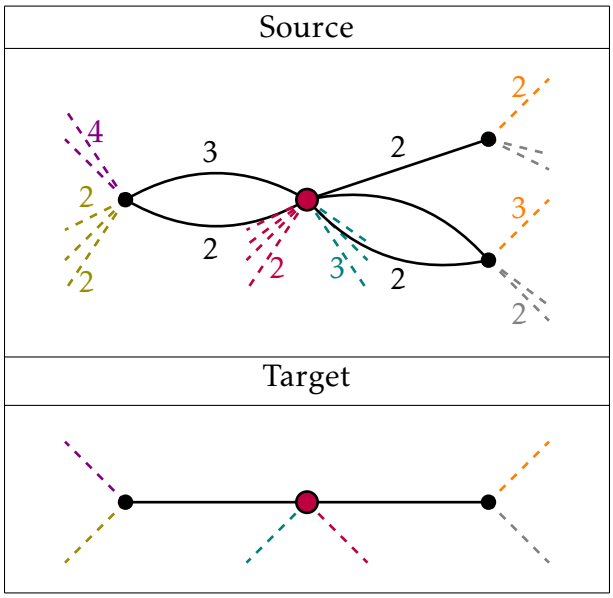


Figure 1.11: A harmonic morphism obtained from the previous one by contraction of the signaled edge of the target, where the obtained vertices are depicted in purple

fixing the genus, the marking, and the ramification above the marked legs. As will be apparent in later chapters, these moduli spaces exhibit a nice (and expected) *tropical behavior* (in the sense of balancing), and it is because of this restriction on the RH numbers that our study and manipulation of these spaces renders useful.

Definition 1.5.13. A harmonic morphism $\pi\colon G_1 \to G_2$ is a *discrete admissible cover* if for every vertex $V \in V(G_1)$ the RH number satisfies $r_\pi(V) = 0$.

It follows from Proposition 1.5.11 that the contraction described in Construction 1.5.10 applied to a discrete admissible cover produces a discrete admissible cover. In addition, the Riemann-Hurwitz formula shows that if $\pi\colon G_1 \to G_2$ is a discrete admissible cover, then

$$\#L(G_1) + 2(g(G_1) - 1) = \deg \pi(\#L(G_2) + 2(g(G_2) - 1)).$$

We note that the harmonic morphisms depicted in Figures 1.10 and 1.11 are both discrete admissible covers.

1.6 Categories and moduli spaces of discrete admissible covers

Throughout this section we will enforce the following notation:

Notation 1.6.1. Let $h, d, m \geq 0$ denote integers with $2h+m-2 > 0$, and let $\vec{\mu} = (\mu_1, \ldots, \mu_m)$ denote a vector of partitions of d. Additionally, set $n = \sum_{i=1}^{m} \ell(\mu_i)$, and let g be defined

by the equation

$$n + 2(g-1) = d \cdot (m + 2(h-1)).$$

We assume that g is an integer, which naturally imposes further conditions on d, m and $\vec{\mu}$.

Remark. By possibly trespassing on verbosity, as it may be apparent to the reader, we remark that this definition for g stems out of the Riemann-Hurwitz equality (1.6). Namely, these integers are intended to model the situation where there is a degree-d discrete admissible cover from a genus-g n-marked graph to a genus-h m-marked graph.

Definition 1.6.2. The *category of discrete admissible covers with ramification $\vec{\mu}$ of genus-h m-marked curves* is the category $\mathbb{AC}_{d,h}(\vec{\mu})$ given by the following data:

- The objects of $\mathbb{AC}_{d,h}(\vec{\mu})$ consist of degree-d discrete admissible covers $\pi\colon G \to H$ where:
  - H is an object of $\mathbb{G}_{h,m}$.
  - G is an object of $\mathbb{G}_{g,n}$ (it is immediate that $2g + n - 2 > 0$).
  - For every $1 \leq k \leq m$:
    $$\pi\left(\bigcup_{j=1}^{\ell(\mu_k)} \ell_{N+j}(G)\right) = \ell_k(H),$$
    where $N = \sum_{i<k} \ell(\mu_i)$.
  - For every $1 \leq k \leq m$ and every $1 \leq j \leq \ell(\mu_k)$, the weight of the $\sum_{i<k} \ell(\mu_i) + j$-th leg of G is μ_k^j.
- For two objects $\pi_1\colon G_1 \to H_1$ and $\pi_2\colon G_2 \to H_2$ of $\mathbb{AC}_{d,h}(\vec{\mu})$, the set of morphisms $\mathrm{Hom}_{\mathbb{AC}_{d,h}(\vec{\mu})}(\pi_1, \pi_2)$ consists of the tuples $f = (f_{\mathrm{src}}, f_{\mathrm{trgt}})$ where:
  - $f_{\mathrm{src}} \in \mathrm{Hom}_{\mathbb{G}_{g,n}}(G_1, G_2)$,
  - $f_{\mathrm{trgt}} \in \mathrm{Hom}_{\mathbb{G}_{h,m}}(H_1, H_2)$,
  - $\pi_2 \circ f_{\mathrm{src}} = f_{\mathrm{trgt}} \circ \pi_1$.
- Composition of morphisms is just composition of tuples of maps.

If $\pi\colon G \to H$ is an object of $\mathbb{AC}_{d,h}(\vec{\mu})$, then we set $\mathrm{src}(\pi) := G$ and $\mathrm{trgt}(\pi) := H$.

Remark. For an object π of $\mathbb{AC}_{d,h}(\vec{\mu})$, the notation $\mathrm{src}(\pi)$ and $\mathrm{trgt}(\pi)$ purports to convey the notion of "source" graph and "target" graph of π. It is also intended to facilitate notation in ensuing constructions.

Construction 1.6.3. The objects of the category $\mathbb{AC}_{d,h}(\vec{\mu})$ come with two natural projections:

- Onto $\mathbb{G}_{h,m}$ by forgetting the source and the cover, but keeping the target.
- Onto $\mathbb{G}_{g,n}$ by forgetting the target and the cover, but keeping the source.

More precisely, there are the following natural associations

$$\begin{aligned} \mathrm{src}\colon \mathrm{Obj}(\mathbb{AC}_{d,h}(\vec{\mu})) &\to \mathrm{Obj}(\mathbb{G}_{g,n}), \pi \mapsto \mathrm{src}(\pi), \\ \mathrm{trgt}\colon \mathrm{Obj}(\mathbb{AC}_{d,h}(\vec{\mu})) &\to \mathrm{Obj}(\mathbb{G}_{h,m}), \pi \mapsto \mathrm{trgt}(\pi). \end{aligned}$$

These are further improved to morphisms by setting for $f \in \mathrm{Hom}_{\mathbb{AC}_{d,h}(\vec{\mu})}(\pi_1, \pi_2)$ with $f = (f_{\mathrm{src}}, f_{\mathrm{trgt}})$:

$$\mathrm{src}(f_{\mathrm{src}}, f_{\mathrm{trgt}}) := f_{\mathrm{src}} \in \mathrm{Hom}_{\mathbb{G}_{g,n}}(G_1, G_2), \qquad \mathrm{trgt}(f_{\mathrm{src}}, f_{\mathrm{trgt}}) := f_{\mathrm{trgt}} \in \mathrm{Hom}_{\mathbb{G}_{h,m}}(H_1, H_2).$$

The above gives rise to functors:

$$\mathrm{src}\colon \mathbb{AC}_{d,h}(\vec{\mu}) \to \mathbb{G}_{g,n}, \qquad \mathrm{trgt}\colon \mathbb{AC}_{d,h}(\vec{\mu}) \to \mathbb{G}_{h,m}. \tag{1.10}$$

In addition, the product of these two functors gives rise to the functor

$$\mathrm{src} \times \mathrm{trgt}\colon \mathbb{AC}_{d,h}(\vec{\mu}) \to \mathbb{G}_{g,n} \times \mathbb{G}_{h,m}, \pi \mapsto (\mathrm{src}(\pi), \mathrm{trgt}(\pi)). \tag{1.11}$$

More interestingly, for an object $\pi\colon G \to H$ of $\mathbb{AC}_{d,h}(\vec{\mu})$ the corresponding projections give the following poic-morphisms:

$$\eta_{\mathrm{src},\pi}\colon \sigma_\pi \to \sigma_G, \qquad \eta_{\mathrm{trgt},\pi}\colon \sigma_\pi \to \sigma_H. \tag{1.12}$$

We now shift our attention to cones of metrics of a discrete admissible cover. The underlying idea is that a general harmonic morphism should describe an integral linear map between the underlying tropical curves, which forces a relation between the metrics of the edges. Namely, since edges are to map to edges, the metric of the target curve entirely determines that of the source curve. We make this explicit in the following definition with the notion of induced metric.

Definition 1.6.4. Suppose $\pi\colon G \to H$ is an admissible cover and δ' is a metric on H. The *induced metric* on G is the metric $\delta\colon E(G) \to \mathbb{R}_{>0}$ given by

$$\delta(e) = \frac{\delta' \pi(e)}{d_\pi(e)}.$$

It is clear that under these metrics the map induced by the cover between the corresponding metric spaces is integral linear when restricted to each interval. However, this notion of induced metrics might lead one to describe the relation established by the discrete admissible cover between the corresponding poics by a rational linear map, which does not yield in general a poic-morphism. For this reason we have to introduce the matrix associated to the cover, being the "minimal" integral matrix reflecting this relation, in the next definition.

Definition 1.6.5. Suppose $\pi\colon G \to H$ is an admissible cover. The *matrix associated to* π is the linear map $F_\pi : \mathbb{R}^{E(H)} \to \mathbb{R}^{E(G)}, \vec{v} \mapsto F_\pi(\vec{v})$, where for $h \in E(G)$ the h-coordinate of $F_\pi(\vec{v})$ is defined by

$$(F_\pi(\vec{v}))_h = \frac{\operatorname{lcm}\left((d_\pi(e))_{e\in\pi^{-1}(\pi(h))}\right)}{d_\pi(h)}(\vec{v})_{\pi(h)}, \tag{1.13}$$

and $\operatorname{lcm}(\bullet)$ denotes the least common multiple.

Example 1.6.6. In Figure 1.12 we depict an object of $\mathbb{AC}_{2,0}((2),(2),(2),(2))$. The weight of the middle edges of the source is 1, and the degree of every leg of the source is 2. We

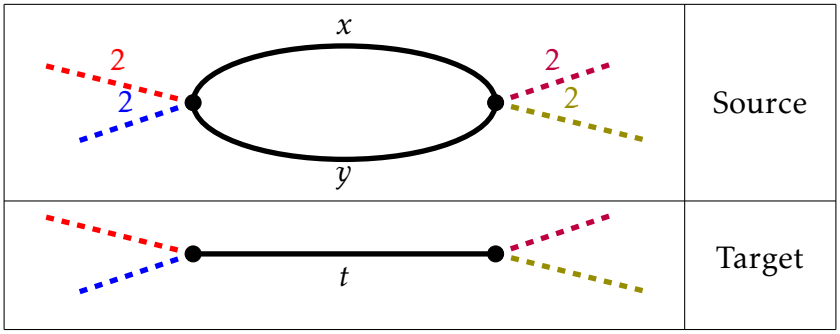


Figure 1.12: An object of $\mathbb{AC}_{2,0}((2),(2),(2),(2))$

remark that $\mathbb{AC}_{2,0}((2),(2),(2),(2))$ is a groupoid (every morphism is an isomorphism) and it has three isomorphism classes. To obtain representative objects of the other isomorphism classes, it suffices to permute the markings (colors in the picture) of the legs accordingly. The matrix associated to the cover is then simply

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Definition 1.6.7. Let $\pi\colon G \to H$ be an admissible cover. The *cone of metrics of* π is the cone σ_π given as the closure of the graph of the restriction of F_π to σ_H^0 in $\sigma_H \times \sigma_G$. In other words, the partially open polyhedral cone of $(N^{\sigma_H} \oplus N^{\sigma_G})_{\mathbb{R}}$ underlying the poic σ_π is given as the set of points

$$\sigma_\pi = \{(x, F_\pi(x)) : x \in F_\pi^{-1}(\sigma_G) \cap \sigma_H\} \subset \sigma_H \times \sigma_G.$$

The corresponding abelian group is just

$$N^{\sigma_\pi} := \{(v, F_\pi(v)) : v \in N^{\sigma_H}\} \subset N^{\sigma_H} \oplus F_\pi(N^{\sigma_H}).$$

By definition $\dim(\sigma_\pi) = \dim(\sigma_H)$, and in fact projection onto the first factor yields an isomorphism $\sigma_\pi \cong F_\pi^{-1}(\sigma_G) \cap \sigma_H$. This is succinctly put in the following commutative diagram:

$$\begin{array}{ccc} \sigma_\pi & \hookrightarrow & \sigma_H \times \sigma_G \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ F_\pi^{-1}(\sigma_G) \cap \sigma_H & \xrightarrow{F_\pi} & \sigma_G \\ \downarrow & & \downarrow \\ \mathbb{R}_{\geq 0}^{E(H)} & \xrightarrow{F_\pi} & \mathbb{R}_{\geq 0}^{E(G)}, \end{array}$$

where pr_i (for $i = 1, 2$) denotes the projection onto the first and second factor respectively.

Example 1.6.8. We explicitize the cone of metrics of Example 1.6.6. The cone of metrics of the target is the half line $\mathbb{R}_{\geq 0}$. The cone of metrics of the source can be given as the intersection of the non-negative orthant $\mathbb{R}^2_{\geq 0}$ and the poic ξ defined by the partially open integral cone of $\mathbb{R}^2$ given by $x + y > 0$. Clearly, the matrix only maps the relative interior of $\mathbb{R}_{\geq 0}$ to $\mathbb{R}^2_{\geq 0} \cap \xi$. In particular, the cone of metrics of the cover is the poic given by the partially open integral cone of $\left(\mathbb{Z}^2 \oplus \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle\right)_{\mathbb{R}} \cong \mathbb{R}^3$ defined as the set of points $\{(x, y, z) : x = y = z > 0\} \subset \mathbb{R}^2_{\geq 0} \times \mathbb{R}_{\geq 0}$.

Example 1.6.9. Let $h = 0$, $m = 5$, $\vec{\mu} = ((2), (2), (2), (2), (1, 1))$, $n = 6$ and $g = 1$. We make use of the colors to avoid overwhelming indexes. In Figure 1.13 we depict some objects of $\mathbb{AC}_{2,0}((2), (2), (2), (2), (1, 1))$, where we abuse notation slightly and ignore the source markings. We explicitly write their associated matrix and depict their corresponding cone of metrics as a subcone of that from the target tree (without illustrating the underlying lattice).

The introduction of the cone of metrics begs naturally for the interplay of these with morphisms between discrete admissible covers.

Construction 1.6.10. We follow our previously fixed Notation 1.6.1. Consider objects $\pi\colon G \to H$ and $\pi'\colon G' \to H'$ of $\mathbb{AC}_{d,h}(\vec{\mu})$ together with a morphism $f : \pi \to \pi'$. Recall from Lemma 1.2.11 that the morphism induces natural identifications $G/f_{\mathrm{src}}^{-1}(V(G')) \cong G'$ and $H/f_{\mathrm{trgt}}^{-1}(V(H')) \cong H'$. Moreover, since $\pi^{-1}(f_{\mathrm{trgt}}^{-1}(V(H))) \subset f_{\mathrm{src}}^{-1}(V(G))$, it follows that π gives rise to a map of graphs $\pi/(f_{\mathrm{src}}, f_{\mathrm{trgt}})\colon G/f_{\mathrm{src}}^{-1}(V(G')) \to H/f_{\mathrm{trgt}}^{-1}(V(H'))$. Together with the natural identifications $G/f_{\mathrm{src}}^{-1}(V(G')) \cong G'$ and $H/f_{\mathrm{trgt}}^{-1}(V(H')) \cong H'$, this

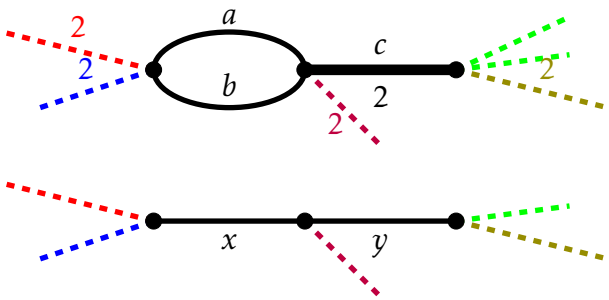
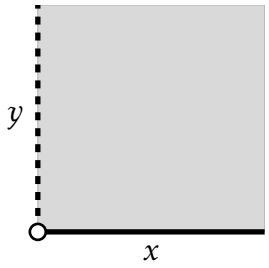
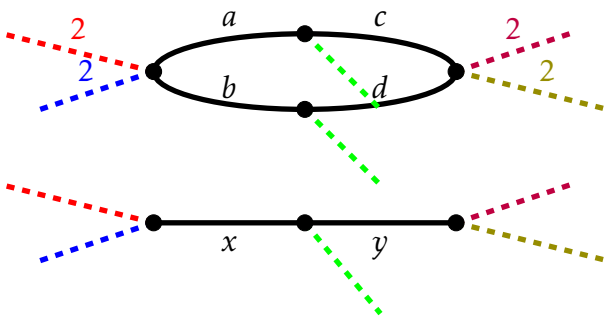
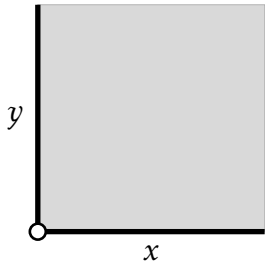
Object of $\mathbb{AC}_{2,0}((2),(2),(2),(2),(1,1))$	Ass. map	Cone of metrics
	$\begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$	
	$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \end{array}$	

Figure 1.13: Cones of metrics of some objects of $\mathbb{AC}_{2,0}((2),(2),(2),(2),(1,1))$

map yields a commutative diagram of maps of graphs

$$\begin{array}{ccc} G/f_{\mathrm{src}}^{-1}(V(G')) & \xrightarrow{\sim} & G' \\ {\scriptstyle \pi/(f_{\mathrm{src}},f_{\mathrm{trgt}})}\downarrow & & \downarrow{\scriptstyle \pi'} \\ H/f_{\mathrm{trgt}}^{-1}(V(H')) & \xrightarrow[\sim]{} & H' \end{array}$$

Therefore, the morphism $f = (f_{\mathrm{src}}, f_{\mathrm{trgt}})$ induces the face-embedding

$$\sigma_{f_{\mathrm{src}}} \times \sigma_{f_{\mathrm{trgt}}} \colon \sigma_{H'} \times \sigma_{G'} \to \sigma_H \times \sigma_G.$$

Restricting this face-embedding to σ_π, we obtain a face-embedding $\sigma_{(f_{\mathrm{src}},f_{\mathrm{trgt}})} \colon \sigma_\pi \to \sigma_{\pi'}$.

Lemma 1.6.11. *Following the notation of Construction 1.6.10, the cone of metrics construction gives rise to a functor*

$$\mathrm{AC}_{d,h,\vec{\mu}} \colon \mathbb{AC}^{\mathrm{op}}_{d,h}(\vec{\mu}) \to \mathrm{POIC}, \pi \mapsto \sigma_\pi.$$

Moreover, if π is an object of $\mathbb{AC}_{d,h}(\vec{\mu})$ and τ is a face of σ_π, then there exists a morphism $\pi \to \pi'$ of $\mathbb{AC}_{d,h}(\vec{\mu})$ unique up to isomorphism with $\mathrm{AC}_{d,h,\vec{\mu}}(\pi' \to \pi)$ isomorphic to this

face-embedding. Furthermore, the poic-morphisms (1.12) *define natural transformations*

$$\eta_{\mathrm{src}}\colon \mathrm{AC}_{d,h,\vec{\mu}} \to \mathrm{M}^{\mathrm{trop}}_{g,n} \circ \mathrm{src}, \qquad \eta_{\mathrm{trgt}}\colon \mathrm{AC}_{d,h,\vec{\mu}} \to \mathrm{M}^{\mathrm{trop}}_{h,m} \circ \mathrm{trgt}. \tag{1.14}$$

Proof. The definition of the cone of metrics of a cover and the functoriality of (1.3) imply directly that $\mathrm{AC}_{d,h,\vec{\mu}}$ is a functor. Similarly, the definition of cone of metrics of a cover and the properties expressed in Lemma 1.4.8 imply the analogous properties of $\mathrm{AC}_{d,h,\vec{\mu}}$. It is directly checked from the definition of the Hom-sets of the category $\mathbb{AC}_{d,h}(\vec{\mu})$ that the poic-morphisms (1.12) define the natural transformations (1.14). □

Independently of all the possible choices of d, h, m, and $\vec{\mu}$, the category $\mathbb{AC}_{d,h}(\vec{\mu})^{\mathrm{op}}$ has finitely many isomorphism classes. Thus, it is necessarily equivalent to a finite category, and hence the colimit of any functor from $\mathbb{AC}_{d,h}(\vec{\mu})^{\mathrm{op}}$ to Top is representable. So we have an analogous presentation of the sought-after moduli space of admissible covers.

Definition 1.6.12. Let d, h, m, $\vec{\mu}$, n and g be as in Definition 1.6.2. The *moduli space of discrete admissible covers with ramification $\vec{\mu}$ of genus-h m-marked curves* is the topological space

$$\mathcal{AC}^{\mathrm{trop}}_{d,h}(\vec{\mu}) := \mathrm{colim}_{\mathbb{AC}_{d,h}(\vec{\mu})^{\mathrm{op}}} \left|\mathrm{AC}_{d,h,\vec{\mu}}\right|.$$

Just as in the case of $\mathcal{M}^{\mathrm{trop}}_{g,n}$, the above construction shows that $\mathcal{AC}^{\mathrm{trop}}_{d,h}(\vec{\mu})$ can be constructed starting from a full set of representatives for $\mathbb{AC}_{d,h}(\vec{\mu})^{\mathrm{op}}$, taking the corresponding cones of metrics, and identifying points representing isomorphic covers. We treasure this presentation, as it not only emphasizes the combinatorial relations between cones of metrics given by face relations and isomorphic identifications, but also our construction of the associated cone of metrics allows us to take edge contractions with respect to the target curve while simultaneously controlling them with respect to the source curve. It is worth noting that the functors src and trgt together with the natural transofrmations (1.14) yield continuous maps

$$|\mathrm{src}|\colon \mathcal{AC}^{\mathrm{trop}}_{d,h}(\vec{\mu}) \to \mathcal{M}^{\mathrm{trop}}_{g,n}, \qquad |\mathrm{trgt}|\colon \mathcal{AC}^{\mathrm{trop}}_{d,h}(\vec{\mu}) \to \mathcal{M}^{\mathrm{trop}}_{h,m}.$$

Chapter 2

Poic-complexes and their cycles

This chapter is devoted to the introduction and description of (linear) poic-complexes, which are central to our enterprise. These objects can be regarded as a small generalization of partially open fans, and we define them in somewhat categorical terms. This perspective is mainly adopted for two reasons:

1. Our leading examples, and cases of interest, do not naturally involve finite categories. Instead, they are naturally presented in a broad manner, and hence it is more convenient to to handle them without having to make a choice of isomorphism classes or the like.

2. The next chapter deals with poic-fibrations, and we found it was cleaner and more pleasant to describe these in categorical terms.

This perspective comes at the expense of some notational complexity and additional technicalities, which may very well be subtle, but are nevertheless relevant. So for the sake of clarity and readability, we dedicate some space to establish and make these notions explicit. The proofs of some statements are left for the last chapter, as their spirit seemed more fitting for the general atmosphere thereof. It is worth noting that no additional difficulty arises from this perspective, thus we favor and, actually, indulge this viewpoint. Nevertheless, the underlying motivation for the introduction of poic-complexes is to discuss tropical cycles on them. This requires, in turn, the introduction of more structure: tropical cycles are not defined on a general poic-complex but rather on a linear poic-complex. Additionally, we explain how the moduli space of rational tropical curves is naturally an irreducible linear poic-complex, and close the chapter with the description of pushforwards of tropical cycles through proper morphisms. The latter construction is of utmost relevance in our initial outline.

2.1 Poic-complexes

We will regard any finite poset as a (finite) category in the usual way. That is, the objects of the category are simply the elements of the corresponding set, and the morphisms are determined by the order relation. Recall that a category is said to be:

- *thin*, if between any two objects there is at most one morphism,
- *essentially finite*, if it is equivalent to a category whose class of objects is a finite set.

Therefore, an essentially finite and thin category is equivalent to a finite poset, and any finite poset is equivalent to an essentially finite and thin category.

Definition 2.1.1. A *poic-complex* Φ is a functor

$$\Phi\colon C_\Phi \to \mathrm{POIC},$$

where C_Φ is an essentially finite thin category, satisfying:

1. The functor Φ maps morphisms of C_Φ to face-embeddings.
2. If q is an object of C_Φ and τ is a face of $\Phi(q)$, then there exists a unique up to isomorphism $p \to q$ in C_Φ, such that $\Phi(p \to q) \cong \tau \leq \Phi(q)$.

Notation 2.1.2. Suppose Φ is a poic-complex and q is an object of C_Φ. We will denote the dimension $\dim(\Phi(q))$ simply by $\dim(q)$. In addition, we avoid cumbersome notation and denote the lattice $N^{\Phi(q)}$ simply by N^q (or by N^q_Φ to emphasize the domain of q). Furthermore, if $p \to q$ is a morphism of C_Φ, then $\Phi(p \to q)$ is poic-morphism $(\Phi(p), N^p) \to (\Phi(q), N^q)$, which in particular carries an integral linear map $N^p \to N^q$. We abuse notation and denote this integral linear map just by $\Phi(p \to q)$.

Example 2.1.3. Consider a poic $\sigma = (\sigma, N^\sigma)$ and the following finite posets (where the order relation is face inclusion):

- The set $\underline{\sigma}$ of all faces of σ.
- The set $\partial\sigma$ of all proper faces of σ, namely, $\underline{\sigma}$ without σ.
- The one-point set $\underline{\sigma^0}$ consisting of the interior of σ.

Associating each cone to itself provides a poic-complex structure for each of these posets. We will denote these poic-complexes respectively by $\underline{\sigma}, \partial\sigma$ and $\underline{\sigma^0}$.

Example 2.1.4. In a similar vein to the previous example, let N be a finite rank lattice and suppose Σ is a (partially open) rational polyhedral fan of the vector space $V = N \otimes \mathbb{R}$. By definition Σ is a poset, and we can then regard this as a poic-complex, where to each cone $\sigma \in \Sigma$ we associate the poic $(\sigma, \mathrm{Lin}(\sigma) \cap N)$. At risk of oversimplifying notation, but fearing its overcomplexification, we will denote this poic-complex just by Σ.

Example 2.1.5. Let $n \geq 3$ be an integer, then Proposition 1.3.4 and Lemma 1.4.8 show that the functor $\mathrm{M}^{\mathrm{trop}}_{0,A}$ is actually a poic-complex. We will elaborate on this example later in Section 2.3.

Notation 2.1.6. When no confusion seems near, we will refer to objects of C_Φ as cones of Φ, and to the morphisms of C_Φ as morphisms of Φ. We denote the set of isomorphism classes of C_Φ by $[\Phi]$, and the isomorphism class of a cone p of Φ by $[p]$. The set of isomorphism classes of k-dimensional cones of Φ is denoted $[\Phi](k)$, and these form a partition of $[\Phi]$. The poic-complex Φ is called *pure of dimension n*, if every cone p of Φ appears as the source of a morphism to an n-dimensional cone of Φ.

Suppose Φ and Ψ are poic-complexes. Since the product of poics is a poic, the functors Φ and Ψ give rise to the functor

$$\Phi \times \Psi \colon C_\Phi \times C_\Psi \to \mathrm{POIC}, (q, s) \mapsto \Phi(q) \times \Psi(s).$$

Lemma 2.1.7. *If both Φ and Ψ are poic-complexes, then the product $\Phi \times \Psi$ is a poic-complex.*

Proof. We must check each of the conditions. Now, since C_Φ and C_Ψ are essentially finite, the category $C_\Phi \times C_\Psi$ is necessarily essentially finite. In addition, the definition of the product category implies that the set of morphisms between any two objects (q_1, s_1) and (q_2, s_2) of $C_\Phi \times C_\Psi$ corresponds to the product $\mathrm{Hom}_\Phi(q_1, s_2) \times \mathrm{Hom}_\Psi(q_1, s_2)$. Hence, it is either empty, or consists of a single element: that is, $C_\Phi \times C_\Psi$ is also thin. Let us now note that if σ and ξ are two poics, then the faces of $\sigma \times \xi$ consist of products of faces of both σ and ξ. Moreover, the product of any two faces of σ and ξ gives a face of $\sigma \times \xi$. This observation implies the necessary conditions on $\Phi \times \Psi$ to be a poic-complex. □

Naturally, there are objects of interest associated to a poic-complex.

Construction 2.1.8. Let Φ denote a poic-complex. There are the following full subcategories of C_Φ:

1. For a cone p of Φ, the category $C_\Phi(p,)$ consisting of morphisms $p \to q$ of C_Φ where maps are commuting triangles. Since C_Φ is thin, it follows that this is a full subcategory.

2. For a non-negative integer k, the full subcategory $C_\Phi(\leq k)$ of objects with cones of dimension at most k.

We let $\Phi(\leq k)$ denote the restriction of Φ to $C_\Phi(\leq k)$.

Remark. In general $\Phi|_{C_\Phi(p,)}$ is not a poic-complex, but almost, and this is to be expected. To be more precise, the underlying category is finite and thin, and the functor sends morphisms to face-embeddings. Therefore, what is missing is the second condition, since the not all faces of an object q of $C_\Phi(p,)$ necessarily cover p. This behaviour is unsurprising and expected, as this reflects the relationship between the *link* of a cone in a fan and the *star fan* of the cone in the quotient vector space.

Lemma 2.1.9. *Let Φ be a poic-complex, and let $k \geq 0$ be an integer. The functor $\Phi(\leq k)$ is a poic-complex.*

Proof. The category $C_\Phi(\leq k)$ must also be essentially finite and thin, because it is a full subcategory of an essentially finite thin category. The additional conditions of a poic-complex are obtained immediately from those of Φ. □

Definition 2.1.10. A poic-complex Υ is a *poic-subcomplex* of Φ, if C_Υ is a full subcategory of C_Φ and $\Upsilon = \Phi|_{C_\Upsilon}$.

Example 2.1.11. In the situation of Lemma 2.1.9 the poic-complex $\Phi(\leq k)$ is naturally a subcomplex of Φ.

Definition 2.1.12. Suppose Φ and Ψ are poic-complexes. A *morphism of poic-complexes* $F\colon \Phi \to \Psi$ consists of the following data:

- A functor $F\colon C_\Phi \to C_\Psi$.
- A natural transformation $\eta_F\colon \Phi \implies \Psi \circ F$.

These are subject to the additional condition: For an cone p of Φ, the image $\eta_{F,p}(\Phi(p))$ does not lie in a proper face of $\Psi(F(p))$.
Composition of morphisms of poic-complexes is defined as follows: Let Ω denote an additional poic-complex and $G\colon \Psi \to \Omega$ a morphism. Then $G \circ F\colon \Phi \to \Omega$ is given by:

- The functor $G \circ F\colon C_\Phi \to C_\Omega$.
- The natural transformation $\eta_{G\circ F}\colon \Phi \implies \Omega \circ (G \circ F)$, which is defined at a cone s of Φ by

$$\eta_{G\circ F,s} = \eta_{G,F(s)} \circ \eta_{F,s}.$$

Notation 2.1.13. Let $F\colon \Phi \to \Psi$ be a morphism of poic-complexes and let s be a cone of Φ. To simplify notation, we will omit η_F and denote the poic-morphism $\eta_{F,s}$ simply by F_s.

Example 2.1.14. Suppose N is a free abelian group of finite rank. The pair $(N_{\mathbb{R}}, N)$ is naturally a poic, and hence gives rise to the poic-complex $\underline{N_{\mathbb{R}}}$. If M is an additional free abelian group of finite rank, then under the used notation $\underline{N_{\mathbb{R}}} \times \underline{M_{\mathbb{R}}} = \underline{(N \oplus M)_{\mathbb{R}}}$. In addition, it is clear that any integral map $N \to M$ gives rise to a morphism of poic-complexes $\underline{N_{\mathbb{R}}} \to \underline{M_{\mathbb{R}}}$, and any morphism of poic-complexes $\underline{N_{\mathbb{R}}} \to \underline{M_{\mathbb{R}}}$ comes from a unique integral map $N \to M$.

Example 2.1.15. If Υ is a poic-subcomplex of Φ, then the natural inclusions define a morphism of poic-complexes $\Upsilon \to \Phi$.

Example 2.1.16. We seek to illustrate the importance of the natural transformation in Definition 2.1.12. Let $n \geq 0$ be an integer, and consider the poic $\mathbb{R}^n_{\geq 0}$. This poic has $\binom{n}{i}$ i-dimensional faces. More interestingly, for each $1 \leq i \leq n$ let ρ_i denote the ray of $\mathbb{R}^n_{\geq 0}$ generated by the standard basis vector e_i of $\mathbb{Z}^n \subset \mathbb{R}^n$. Then every face of $\mathbb{R}^n_{\geq 0}$ is given as the subcone generated by $\{\rho_i\}_{i \in I}$, where $I \subset \{1, \ldots, n\}$. For this example, given $I \subset \{1, \ldots, n\}$ we will let σ_I denote the face $\{\rho_i\}_{i \in I}$ of $\mathbb{R}^n_{\geq 0}$. This poic gives rise to the poic-complex $\partial\mathbb{R}^n_{\geq 0}$. Separately, consider $\mathbb{Z}^{n-1} \subset \mathbb{R}^{n-1}$, let $b_1, \ldots, b_{n-1}$ denote the standard basis vectors of $\mathbb{Z}^{n-1} \subset \mathbb{R}^{n-1}$, and set $b_n := -\sum_{j=1}^{n-1} b_j$. For $I \subsetneq \{1, \ldots, n\}$ let η_I denote the integral linear map

$$\eta_I \colon \sigma_I \to \mathbb{R}, \tag{2.1}$$

defined by $e_i \mapsto b_i$ for $i \in I$. We observe then that the constant functor $\partial\mathbb{R}^n_{\geq 0} \to \underline{\mathbb{R}^{n-1}}$ and the integral linear maps (2.1) define a morphism of poic-complexes $\partial\mathbb{R}^n_{\geq 0} \to \underline{\mathbb{R}^{n-1}}$. We observe that the functor in this example is rather trivial, whereas the natural transformation is the input carrying non-trivial data in the morphism of poic-complexes.

The data of poic-complexes with their morphisms and the compositions thereof forms a category. We denote this category by pCmplxs, and observe that the realization functor (1.2) on poics can be extended naturally to poic-complexes. Namely, let Φ be a poic-complex. Since C_Φ is essentially finite, the colimit of $|\bullet| \circ \Phi$ is representable in Top, and hence gives rise to a space $|\Phi|$. Properties of colimits show that from a morphism of poics $F \colon \Phi \to \Psi$, we obtain a continuous map $|F| \colon |\Phi| \to |\Psi|$ and this association gives rise to a functor

$$|\bullet| \colon \text{pCmplxs} \to \text{Top}. \tag{2.2}$$

2.2 Linear poic-complexes and Minkowski weights

Linear poic-complexes are poic-complexes with additional data that allows us to define Minkowski weights thereon. This is plausibly one of the most important definitions for this work, since tropical cycles are simply Minkowski weights in a subdivision of the

linear poic-complex. We first define what a weight on a general poic-complex is, and then define what Minkowski weights on a linear poic-complex are. In simple terms, a Minkowski weight is just a weight satisfying the balancing condition with respect to the underlying linear poic-complex structure.

Definition 2.2.1. Let Φ be a poic-complex and $k \geq 0$ an integer. A *k-dimensional weight ω on Φ* is a function

$$\omega \colon [\Phi](k) \to \mathbb{Z}.$$

We remark that $[\Phi](k)$ is by definition a finite set, therefore the k-dimensional weights on Φ are a free abelian group of finite rank. We will denote this group by $W_k(\Phi)$.

Construction 2.2.2. Suppose Υ is a subcomplex of a poic-complex Φ, and let $I \colon \Upsilon \to \Phi$ denote the morphism of poic-complexes given by the natural inclusions. Any weight $\omega \in W_k(\Upsilon)$ can be extended by zero to a weight on Φ and this extension by zero defines a map

$$I_!\omega \colon [\Phi](k) \to \mathbb{Z}, \sigma \mapsto \begin{cases} \omega([s]), & \text{if } [s] \in [\Upsilon](k), \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.2.3. *If Υ is a poic-subcomplex of Φ and $I \colon \Upsilon \to \Phi$ is the morphism of poic-complexes given by the natural inclusions, then extension by zero gives an injective linear map*

$$I_! \colon W_k(\Upsilon) \to W_k(\Phi).$$

Proof. Linearity of the map is clear. Since $[\Upsilon](k) \subset [\Phi](k)$, the injectivity follows directly. □

Definition 2.2.4. A *linear poic-complex* is a tuple (Φ, X) such that:

- Φ is a poic-complex.
- $X \colon \Phi \to \underline{(N_X)_{\mathbb{R}}}$ is a morphism of poic-complexes, where N_X is a free abelian group of finite rank.

We will denote a linear poic-complex (Φ, X) simply by Φ_X, and if the morphism is understood, then we will just omit it. Additionally, it is clear that if Φ_X is a linear poic-complex and Υ is a poic-subcomplex of Φ, then $(\Upsilon, X|_\Upsilon)$ is also a linear poic-complex. In this case, we say that $\Upsilon_{X|_\Upsilon}$ is a *linear poic-subcomplex* of Φ_X.

Construction 2.2.5. Suppose Φ_X and Ψ_Y are linear poic-complexes. Both morphisms $X \colon \Phi \to \underline{(N_X)_{\mathbb{R}}}$ and $Y \colon \Psi \to \underline{(N_Y)_{\mathbb{R}}}$ induce a morphism

$$X \times Y \colon \Phi \times \Psi \to \underline{(N_X)_{\mathbb{R}}} \times \underline{(N_Y)_{\mathbb{R}}} = \underline{(N_X \oplus N_Y)_{\mathbb{R}}}.$$

In particular, $\Phi \times \Psi_{X \times Y}$ is a linear poic-complex, where $N_{X\times Y} := N_X \oplus N_Y$.

Definition 2.2.6. A *morphism of linear poic-complexes* $\phi\colon \Phi_X \to \Psi_Y$ consists of:

- A morphism of poic-complexes $\phi\colon \Phi \to \Psi$,
- An integral linear map $\phi_{\text{int}}\colon N_X \to N_Y$.

These are additionally required to yield a commutative diagram of poic-complexes

$$\begin{array}{ccc} \Phi & \xrightarrow{\phi} & \Psi \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle Y} \\ (N_X)_{\mathbb{R}} & \xrightarrow[(\phi_{\text{int}})_{\mathbb{R}}]{} & (N_Y)_{\mathbb{R}}. \end{array}$$

The composition of morphisms of linear poic-complexes is defined in the obvious way by composition of pairs. Just as with poic-complexes, linear poic-complexes and their morphisms give rise to a category, which we denote by Lin-pCmplxs.

Notation 2.2.7. If Φ is a poic-complex and s is a cone of Φ with $\dim(s) = k-1$, we denote the set $[\Phi(s,)](k)$ by $\text{Star}^1_\Phi(s)$ (here the 1 stands for the difference of dimensions). If $s \to t$ is an object of $[\Phi(s,)](k)$, then we will denote its isomorphism class in $\text{Star}^1_\Phi(s)$ by $[s \to t]$.

As has been previously mentioned, Minkowski weights are weights that satisfy a balancing condition. The natural question is then: where does this balancing condition take place? The ensuing construction builds towards an answer to this question. Namely, the balancing condition takes place in multiple places, and these are just the codomains of the relative linear maps. In order to formulate this balancing condition we additionally require the notion of a normal vector, which we introduce in Definition 2.2.10.

Construction 2.2.8 (Relative linear maps)**.** Suppose Φ_X is a linear poic-complex and consider a morphism $s \to t$ of Φ, with $\Phi(s) \leq \Phi(t)$ a codimension 1 face. The quotient N^t/N^s is a rank 1 lattice and the morphism X induces a linear map

$$X_{s\to t}\colon N^t/N^s \to N_X/X_s(N^s). \tag{2.3}$$

We note that if t' is an object of Φ that is isomorphic to t, then there is a unique morphism $s \to t'$, because Φ is thin. More explicitly, if $t \to t'$ is the isomorphism, then the composition $s \to t \to t'$ is this unique morphism. This entire situation gives rise to the following commutative diagram of abelian groups:

$$\begin{array}{ccc} N^t & \xrightarrow[\sim]{\Phi(t\to t')} & N^{t'} \\ & {\scriptstyle \Phi(s\to t)}\nwarrow \quad \nearrow{\scriptstyle \Phi(s\to t')} & \\ & N^s & \end{array} \tag{2.4}$$

This diagram also shows that $\Phi(t \to t')$ additionally induces an isomorphism $N^t/N^s \to N^{t'}/N^s$.

Lemma 2.2.9. *Let Φ_X, $s \to t$, and $t \xrightarrow{\sim} t'$ be as in the previous construction. Then under the maps $X_{s\to t}$ and $X_{t'/s}$, the images of the generators of N^t/N^s and $N^{t'}/N^s$ lying correspondingly in the projections of $N^t \cap \Phi(t)$ and $N^{t'} \cap \Phi(t')$ coincide.*

Proof. Following the previous construction, we first observe that under $\Phi(t \to t')$ the underlying cone of $\Phi(t) \subset N^t_{\mathbb{R}}$ is mapped bijectively onto the underlying cone of $\Phi(t') \subset N^{t'}_{\mathbb{R}}$. In particular, the generator of N^t/N^s pointing towards $\Phi(t)$ is mapped to the generator of $N^{t'}/N^s$ pointing towards $\Phi(t')$. The commutative diagram (2.4) yields the following commutative diagram

$$\begin{array}{ccc} N^t/N^s & \xrightarrow{\quad\sim\quad} & N^{t'}/N^s \\ & {\scriptstyle X_{s\to t}} \searrow \quad \swarrow {\scriptstyle X_{t'/s}} & \\ & N/X_s(N^s) & \end{array},$$

which then shows that the images of these generators coincide in $N_X/X_s(N^s)$. □

Definition 2.2.10. Let Φ_X be a linear poic-complex and let s be an object of Φ. For an isomorphism class $[s \to t] \in \mathrm{Star}^1_\Phi(s)$ the *normal vector* $u^X_{[s\to t]} \in N_X/X_s(N^s)$ is defined as the image of the generator of N^t/N^s contained in (the projection of) $N^t \cap \Phi(t)$ under the linear map $X_{s\to t}$, where $s \to t$ is any object representing this isomorphism class (the previous lemma shows that this is well-defined).

Definition 2.2.11 (The balancing condition)**.** Suppose Φ_X is a linear poic-complex and let s be a cone of Φ with $\dim(s) = k-1$. We say that a k-dimensional weight $\omega \in W_k(\Phi)$ is *X-balanced at s*, if

$$\sum_{[s\to t]\in \mathrm{Star}^1_\Phi(s)} \omega([t]) u^X_{[s\to t]} = 0.$$

If it is X-balanced at every object of Φ of dimension $k-1$, then we will say that it is *X-balanced*.

A natural question to address after the previous definition is the independence up to isomorphism on the starting cone. This is made precise in the next lemma, which also implies that Minkowski weights of a linear poic-complex are a subset of the general weights, given by finitely many equations.

Lemma 2.2.12. *Let Φ_X be a linear poic-complex, and suppose that $s' \to s$ is an isomorphism of Φ with $\dim(s) = k-1$. A weight $\omega \in W_k(\Phi)$ is X-balanced at s if and only if it is X-balanced at s'.*

Proof. We remark that by precomposition, the isomorphism $s \to s'$ of C_Φ gives rise to an isomorphism of categories $C_\Phi(s',) \to C_\Phi(s,)$. Therefore, it also induces a bijection $\mathrm{Star}^1_\Phi(s') \to \mathrm{Star}^1_\Phi(s)$, and, more interestingly, it also gives rise to a commutative diagram

$$\begin{array}{ccc} N^{s'} & \xrightarrow[\sim]{\Phi(s'\to s)} & N^s \\ & {\scriptstyle X_{s'}} \searrow \quad \swarrow {\scriptstyle X_s} & \\ & N_X & \end{array}.$$

This diagram shows that $X_s(N^s) = X_{s'}(N^{s'})$, and naturally $N/X_s(N^s) = N/X_{s'}(N^{s'})$. Furthermore, if $s \to t$ is a morphism of Φ with $\dim(t) = k$, then $\Phi(s \to s')$ yields an isomorphism between $N^t/N^{s'}$ and N^t/N^s, and this isomorphism commutes with the maps induced by X_t:

$$N^t/N^{s'} \to N/X_{s'}(N^{s'}) = N^t/N^s, \qquad N^t/N^s \to N/X_s(N^s) = N/X_{s'}(N^{s'}).$$

This ultimately shows that $u^X_{[s\to t]} = u^X_{[s'\to t]}$, and it is then immediate that a weight $\omega \in W_k(\Phi)$ is X-balanced at s if and only if it is X-balanced at s'. □

Definition 2.2.13. For a linear poic-complex Φ_X, the set of *k-dimensional Minkowski weights* is the subset $M_k(\Phi_X)$ of $W_k(\Phi)$ consisting of the X-balanced weights $\omega \in W_k(\Phi)$.

Lemma 2.2.14. *Suppose Φ_X is a linear poic-complex. The set $M_k(\Phi_X)$ is an abelian subgroup of $W_k(\Phi)$. Furthermore, if Υ is a poic-subcomplex of Φ, with $I\colon \Upsilon \to \Phi$ the natural inclusion, then the map $I_!$ of Lemma 2.2.3 restricts to an injective linear map*

$$I_!\colon M_k(\Upsilon_{X|_\Upsilon}) \to M_k(\Phi_X),$$

for every non-negative integer k.

Proof. We kindly entrust the reader with the burden of proof for the first statement, and focus on the second. Let k be a non-negative integer, and observe that it is sufficient to show that the image of a Minkowski weight of $\Upsilon_{X|_\Upsilon}$ is a Minkowski weight of Φ_X, because linearity and injectivity would then directly follow from Lemma 2.2.3. Let $\omega \in M_k(\Upsilon_{X|_\Upsilon})$ and consider a cone s of Φ with $\dim(s) = k-1$. If s were not a cone of Υ, then there is nothing to check, as $I_!\omega$ would be trivial in $\mathrm{Star}^1_\Phi(s)$. So, we may assume that s is also a cone of Υ. Clearly, if $[s \to t] \in \mathrm{Star}^1_\Upsilon(s)$, then $u^{X|_\Upsilon}_{[s\to t]} = u^X_{[s\to t]}$. Since $I_!$ is extension by zero, it follows that

$$\sum_{[s\to t]\in \mathrm{Star}^1_\Phi(s)} I_!\omega([t])u^X_{[s\to t]} = \sum_{[s\to t]\in \mathrm{Star}^1_\Upsilon(s)} \omega([t])u^X_{[s\to t]}.$$

This sum vanishes, because $\omega \in M_k(\Upsilon_{X|_\Upsilon})$, and hence $I_!\omega \in M_k(\Phi_X)$. □

We close this section with a simple final construction, where we produce Minkowski weights on a product of linear poic-complexes, out of Minkowski weights in each factor.

Construction 2.2.15 (Cross product)**.** Suppose Φ_X and Ψ_Y are linear poic-complexes. For any $k \geq 0$, we have the following decomposition

$$[\Phi \times \Psi](k) = \bigcup_{i+j=k} [\Phi](i) \times [\Psi](j).$$

Let i and j be such that $k = i + j$. Observe that any $\omega \in M_i(\Phi_X)$ and $\eta \in M_j(\Psi_Y)$ give rise to a map

$$\omega \times \eta \colon [\Phi](i) \times [\Psi](j) \to \mathbb{Z}, (s,t) \mapsto \omega(s) \cdot \eta(t),$$

which after extending by zero gives a map

$$\omega \times \eta \colon [\Phi \times \Psi](k) \to \mathbb{Z}.$$

We call this weight the *cross product* of ω and η.

Lemma 2.2.16. *Following the notation of the previous construction: If $\omega \in M_i(\Phi_X)$ and $\eta \in M_j(\Psi_Y)$, then $\omega \times \eta \in M_{i+j}((\Phi \times \Psi)_{X \times Y})$. In addition, the cross product is a bilinear map*

$$M_i(\Phi_X) \otimes_{\mathbb{Z}} M_j(\Psi_Y) \to M_k((\Phi \times \Psi)_{X \times Y}), \omega \otimes \eta \mapsto \omega \times \eta.$$

Proof. We begin the proof by noting that for any cone $s \times t$ of $\Phi \times \Psi$, we have that $N^{s \times t} = N^s \oplus N^t$ and the linear map

$$(X \times Y)_{s \times t} \colon N^{s \times t} \to N_X \oplus N_Y = N_{X \times Y}$$

is the direct sum of X_s and Y_t. Now, any cone of $\Phi \times \Psi$ of dimension $(i + j - 1)$ is of the form $s \times t$, where:

- s is a cone of Φ.
- t is a cone of Ψ.
- Their dimensions add up to $i + j - 1$.

The definition of $\omega \times \eta$ implies that the only balancing to check is at cones of $\Phi \times \Psi$ that are of the form:

1. $p \times t$, where p is a cone of Φ of dimension $i - 1$ and t is a cone of Ψ of dimension j.
2. $s \times q$, where s is a cone of Φ of dimension i and q is a cone of Ψ of dimension $j - 1$.

Since both situations are analogous, we will only carry out explicitly the first one. Observe that the only subset of $\mathrm{Star}^1_{\Phi\times\Psi}(p\times t)$ where $\omega\times\eta$ does not vanish is

$$\{[p\times t\to s\times t]\colon [p\to s]\in \mathrm{Star}^1_\Phi(p)\}\subset \mathrm{Star}^1_{\Phi\times\Psi}(p\times t).$$

Now, for $[p\times t\to s\times t]\in \mathrm{Star}^1_{\Phi\times\Psi}(p\times t)$ the relative linear map

$$(X\times Y)_{p\times t\to s\times t}\colon N^{s\times t}/N^{p\times t}\to (N_{X\times Y})/(X\times Y)_{p\times t}(N^{p\times t})$$

is simply

$$\begin{pmatrix} X_{p\to s}\\ 0\end{pmatrix}\colon N^s/N^p\to N_X/X_p(N^p)\oplus N_Y/Y_t(N^t)=(N_{X\times Y})/(X\times Y)_{p\times t}(N^{p\times t}).$$

The normal vector $u^{X\times Y}_{[p\times t\to s\times t]}\in N_{X\times Y}/(X\times Y)_{p\times t}(N^{p\times t})$ is consequently just $\begin{pmatrix} u^X_{[p\to s]}\\ 0\end{pmatrix}$, and finally:

$$\begin{aligned}\sum_{[p\times t\to s\times t]\in\mathrm{Star}^1_{\Phi\times\Psi}(p\times t)}(\omega\times\eta)([p\times t\to s\times t])u^{X\times Y}_{[p\times t\to s\times t]} &= \sum_{[p\to s]\in\mathrm{Star}^1_\Phi(p)}\eta([t])\omega([s])u^{X\times Y}_{[p\times t\to s\times t]}\\ &=\eta([t])\begin{pmatrix}\sum_{[p\to s]\in\mathrm{Star}^1_\Phi(p)}\omega([p\to s])u^X_{[p\to s]}\\ 0\end{pmatrix}\\ &=\begin{pmatrix}0\\0\end{pmatrix}.\end{aligned}$$

This shows that $\omega\times\eta\in M_k(\Phi\times\Psi_{X\times Y})$, and the bilinearity is immediate from the definition. □

2.3 Moduli space of rational tropical curves

We will expand on Example 2.1.5. For this, let A be a set with $\#A\geq 3$, and we explain, as in [GKM09], how the topological space $\mathcal{M}^{\mathrm{trop}}_{0,A}$ can be embedded as an $(\#A-3)$-dimensional fan through the distance map. By construction, a point of this space consists of an A-marked metric tree Γ.

Notation 2.3.1. Given a tree T of $\mathbb{G}_{0,A}$ and a $\delta\in\sigma_T$, we let $|(T,\delta)|$ denote the metric tree defined by T with the legnths prescribed by δ (where length 0 means that the corresponding edge is contracted).

Let $\binom{A}{2}$ denote the set of 2-element subsets of A and consider the vector space $\mathbb{R}^{\binom{A}{2}}$,

together with the linear map

$$M_A : \mathbb{R}^A \to \mathbb{R}^{\binom{A}{2}}, (x_i)_{i\in A} \mapsto (x_i + x_j)_{\{i,j\}\subset A}.$$

We denote the cokernel of M_A by Q_A. It is shown in Theorem 4.2 of [SS04] that the distance map

$$\mathrm{dist}\colon \mathcal{M}^{\mathrm{trop}}_{0,A} \to Q_A, \Gamma \mapsto (\mathrm{dist}_\Gamma(\ell_i(\Gamma)), \ell_j(\Gamma))_{\{i,j\}\subset A} \tag{2.5}$$

embeds the space $\mathcal{M}^{\mathrm{trop}}_{0,A}$ as a simplicial fan pure of dimension $(\#A - 3)$, where the k-dimesional cones are given by the combinatorial types of A-marked trees with k-edges. These combinatorial types agree with the isomorphism classes of our category $\mathbb{G}_{0,A}$. Furthermore, this fan is rational with respect to the lattice $N_{\mathrm{dist}(A)}$ generated by the vectors v_I, where $I \subset A$ with $1 < \#I < \#A - 1$, given by the image of the metric tree with a unique bounded edge of length 1 having the marked legs with labels in I on one side of the edge and the marked legs with labels in $A\backslash I$ on the other. In our terminology, the distance map (2.5) defines a morphism of poic-complexes

$$\mathrm{dist}_A\colon \mathrm{M}^{\mathrm{trop}}_{0,A} \to \underline{\left(N_{\mathrm{dist}_A}\right)_{\mathbb{R}}} \tag{2.6}$$

given by the trivial functor and the natural transformation η_{dist_A} defined at a tree T of $\mathbb{G}_{0,A}$ by

$$\eta_{\mathrm{dist}_A,T}\colon \sigma_T \to \left(N_{\mathrm{dist}_A}\right)_{\mathbb{R}}, \delta \mapsto \left(\mathrm{dist}_{|(T,\delta)|}(|\ell_i(T)|, |\ell_j(T)|\right)_{\{i,j\}\subset A}.$$

Definition 2.3.2. The morphism (2.6) defines a linear poic-complex structure on $\mathrm{M}^{\mathrm{trop}}_{0,A}$. We call this linear poic-complex *the linear poic-complex of A-marked trees*. We will always consider $\mathrm{M}^{\mathrm{trop}}_{0,A}$ as a linear poic-complex in this way, therefore we will avoid cumbersome subscript notation and simply denote this linear poic-complex by $\mathrm{M}^{\mathrm{trop}}_{0,A}$.

Example 2.3.3. If $n = 4$, then we can explicitly depict $\mathcal{M}^{\mathrm{trop}}_{0,4}$ embedded in $\mathbb{R}^2 \cong Q_4$ through the distance map (2.5) as a 1-dimensional fan. The fan consists of 3 rays and the origin. These 3 rays correspond to the 1-dimensional isomorphism classes of discrete graphs illustrated in Figure 2.1. The lattice $N_{\mathrm{dist}(4)}$ is generated by the set of vectors $\{v_{1,2}, v_{1,3}, v_{1,4}\}$, which satisfy the linear relation

$$v_{1,2} + v_{1,3} + v_{1,4} = 0.$$

Furthermore, these vectors are the generators of the rays, and hence we can depict this fan as the tropical line $\mathbb{R}^2 \cong Q_2$, see Figure 2.2, where we have made explicit the isomorphism class corresponding to each ray.

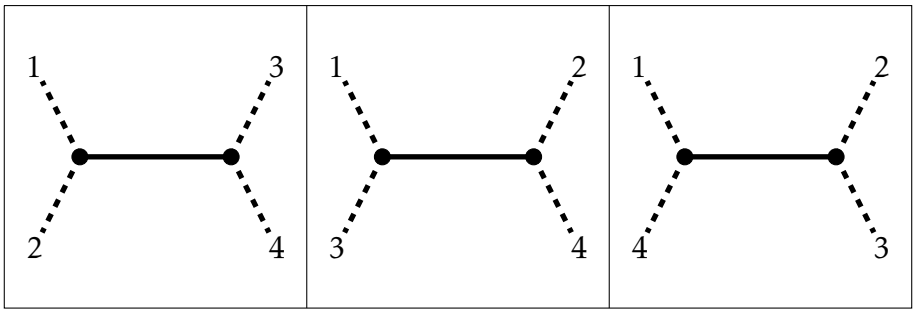


Figure 2.1: Isomorphism classes of $\mathbb{G}_{0,4}$ of dimension 1

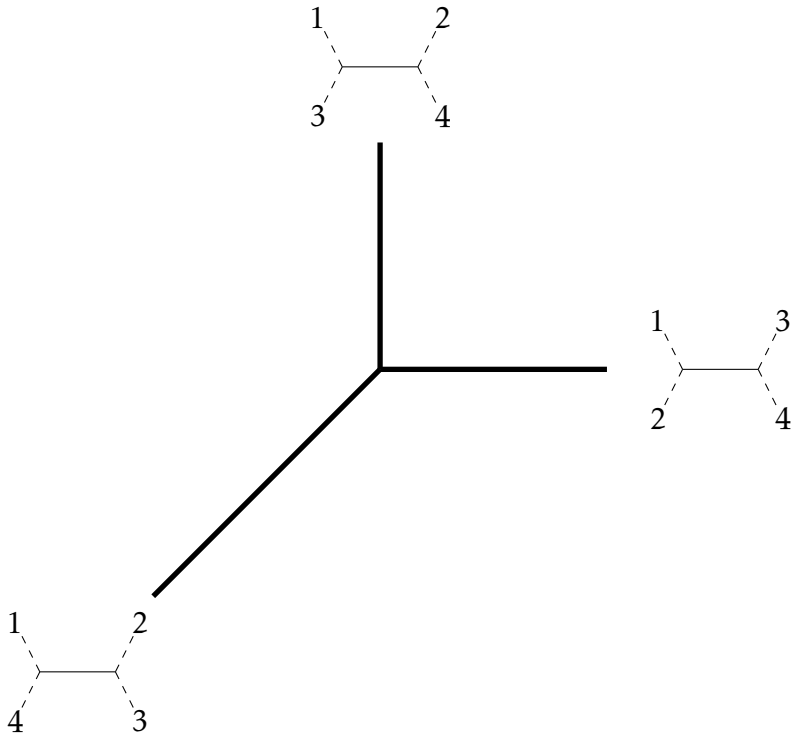


Figure 2.2: Depiction of $\mathcal{M}_{0,4}^{\text{trop}}$

Example 2.3.4. If $n = 5$, then $\mathcal{M}_{0,5}^{\text{trop}}$ is a 2-dimensional fan in a 5-dimensional space. Of course it is not possible to depict this fan, but we can provide a close representation by looking at the graph defined by the intersection of this fan with the corresponding sphere. In other words, we can depict the graph whose vertices correspond to the rays of the fan, and whose edges correspond to the 2-dimensional cones joining the rays. In this case, we have 15 2-dimensional cones and 10 rays. The obtained graph is not planar (it is the famous Petersen graph), but toroidal, so it is best to depict it in a torus. We do this in Figure 2.3, where the square surrounding the graph is to be interpreted as the fundamental domain of the corresponding torus, and the edges going through the boundary have been color coded for reading convenience. We have labelled the vertices of the graph by the combinatorial types of the underlying graphs, where we have only marked two legs (that entirely determine the type).

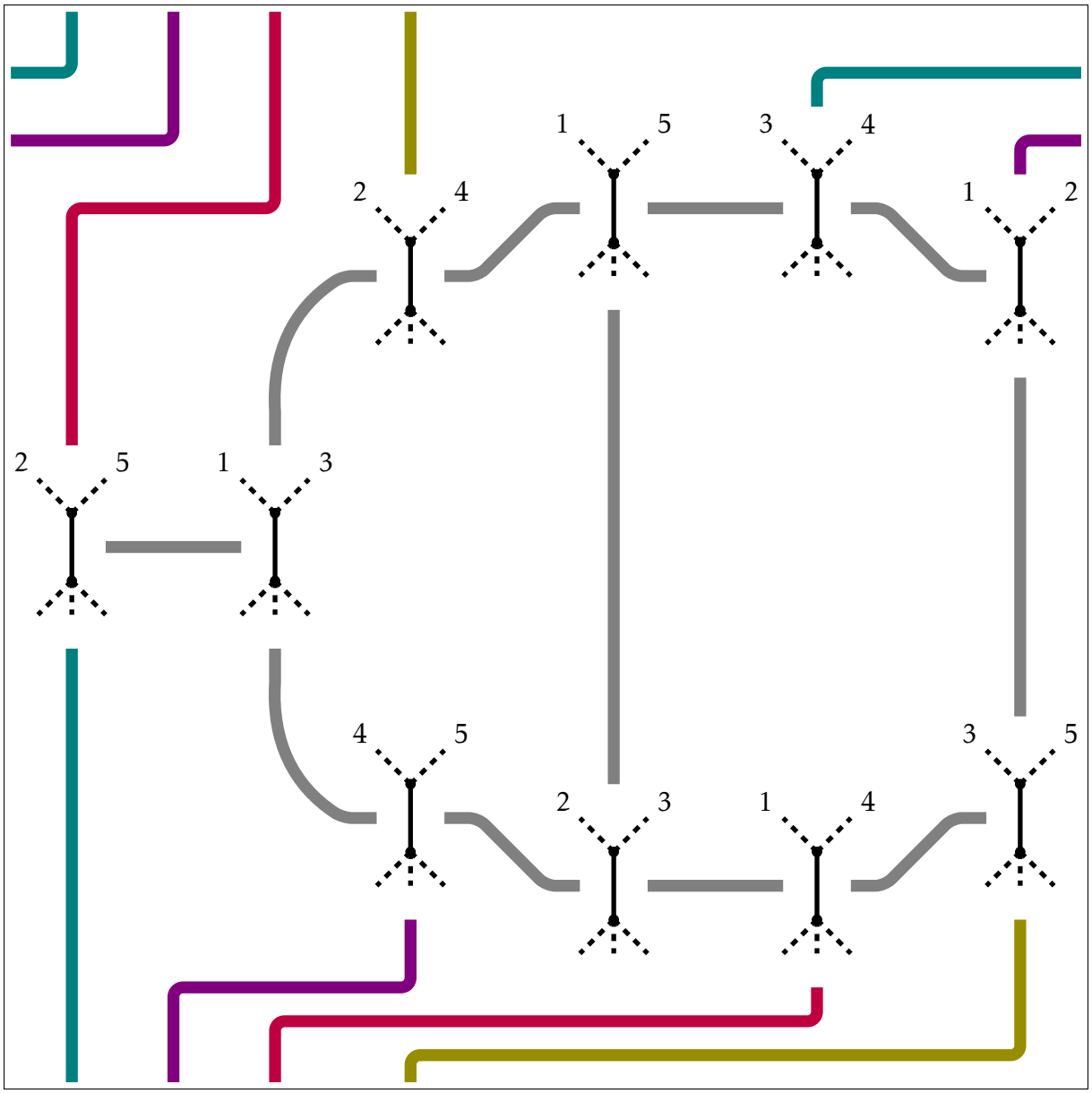


Figure 2.3: The toroidal graph obtained from $\mathcal{M}_{0,5}^{\mathrm{trop}}$

2.4 Subdivisions and tropical cycles

Subdivisions are a special kind of morphisms of poic-complexes that we employ to define tropical cycles on a linear poic-complex. The picture to have in mind is that of subdivisions of a partially open fan in a given vector space. However, there are a few remarks in order due to our different perspective. We do show much later in Section 6.2 that our definition is equivalent to this initial picture. In other words, the reader can rest assured and just imagine subdivisions in the usual way.

Let N be a free abelian group of finite rank. The following notions will be indiscriminately employed throughout this and, possibly, the ensuing sections:

- A closed convex polyhedral cone σ of $N_{\mathbb{R}}$ is called *simplicial* if $\dim \sigma$ coincides

with the number of 1-dimensional faces.

- A partially open polyhedral cone of $N_{\mathbb{R}}$ is called *simplicial* if its closure is simplicial.
- A poic is *simplicial* if its underlying partially open polyhedral cone is simplicial.
- A poic-complex is *simplicial* if the underlying partially open polyhedral cone of every cone is simplicial.

Definition 2.4.1. Let Φ be a poic-complex. A *subdivision* $S\colon \Upsilon \to \Phi$ is a morphism of poic-complexes $S\colon \Upsilon \to \Phi$, such that:

- The underlying functor $S\colon C_\Upsilon \to C_\Phi$ is essentially surjective, and the underlying integral linear maps of the natural transformation are injective. Furthermore, if t is a cone of Υ such that $\dim(t) = \dim(S(t))$, then the underlying integral linear map of $S_t\colon \Upsilon(t) \to \Phi(S(t))$ is an isomorphism.
- For every cone s of Φ, the relative interior of $\Phi(s)$ is the disjoint union of the images of the relative interiors of the cones p of Υ with $S(p) \cong s$, and the images of these relative interiors characterize these cones of Υ up to isomorphism. So if they coincide, then they are isomorphic in C_Υ.
- For every cone t of Υ and every $s \to S(t)$ of Φ, there exists $q \to t$ in Υ with $S(q \to t) \cong s \to S(t)$.

Construction 2.4.2. Let $k \geq 0$ be an integer. A subdivision $S\colon \Upsilon \to \Phi$ gives rise to the map

$$S^*\colon W_k(\Phi) \to W_k(\Upsilon), \omega \mapsto S^*\omega, \tag{2.7}$$

where

$$S^*\omega\colon [\Upsilon](k) \to \mathbb{Z}, [p] \mapsto \begin{cases} \omega([S(p)]), & \text{if } [S(p)] \in [\Phi](k), \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.4.3. *The map* (2.7) *is injective and linear.*

Proof. Linearity is clear from the definition, so we only focus on the injectivity. Consider $\omega \in W_k(\Phi)$ such that $S^*\omega = 0$ and let $[s] \in [\Phi](k)$. Since S is essentially surjective, there exists a cone p of Υ such that $[S(p)] = [s]$. Note that $\dim(p) \leq k$, because the underlying linear map of S_p is injective. Now, the second condition of Definition 2.4.1 shows that if $\dim(p) < k$, then there exists a morphism $p \to q$ of Υ such that $\dim(q) = k$, and $[S(q)] = [s]$. But $S^*\omega = 0$ implies that $\omega([s]) = \omega([S(q)]) = 0$ and since $[s] \in [\Phi](k)$ was arbitrary, it necessarily follows that $\omega = 0$. In particular, S^* is injective. □

Lemma 2.4.4. *Suppose Φ_X is a linear poic-complex, and let $S\colon \Upsilon \to \Phi$ be a subdivision. Then $S^*\colon W_k(\Phi) \to W_k(\Upsilon)$ restricts to an injective linear map*

$$S^*\colon M_k(\Phi_X) \to M_k(\Upsilon_{X\circ S}).$$

Proof. We have already shown that $S^*\colon W_k(\Phi) \to W_k(\Upsilon)$ is injective and linear, therefore we just need to show that if $\omega \in M_k(\Phi_X)$, then $S^*\omega \in M_k(\Upsilon_{X\circ S})$. For this, consider a cone p of Υ with $\dim(p) = k-1$. Then $\dim(S(p)) \geq k-1$, and the following two situations arise: Either $\dim(S(p)) = k-1$, or $\dim(S(p)) \geq k$.

- Suppose that $\dim(S(p)) = k-1$. If $p \to q$ represents a class of $\mathrm{Star}^1_\Upsilon(p)$, then $\dim(S(q)) = k$ and $S(p \to q)$ represents a class of $\mathrm{Star}^1_\Upsilon(S(p))$. So in this case, the functor S gives a map:

$$S\colon \mathrm{Star}^1_\Upsilon(p) \to \mathrm{Star}^1_\Phi(S(p)), [p \to q] \mapsto [S(p) \to S(q)]. \tag{2.8}$$

  We claim that this map is a bijection. Surjectivity follows from the third condition, while injectivity is a bit more delicate but is a consequence of the second condition. In detail, suppose that $p \to q$ and $p \to q'$ are in Υ and represent classes of $\mathrm{Star}^1_\Upsilon(p)$, with $[S(p) \to S(q)] = [S(p) \to S(q')]$. Then there exists a cone s of Φ with $\dim(s) = k$ and $S(q) \cong s \cong S(q')$. Now, the images of their relative interiors in $\Phi(s)^0$ are either disjoint or the same. If they were disjoint, then the closures (relative to $\Phi(s)$) of these relative interiors would consist of two k-dimensional cones that intersect at two proper faces: a $(k-1)$-dimensional one lying in the boundary of the closure of $\Phi(s)^0$, and the other lying in the interior. Therefore, they are necessarily the same, and hence they must also be isomorphic in Υ. Thus (2.8) is a bijection.
  It is clear from the latter that if $[p \to q] \in \mathrm{Star}^1_\Upsilon(p)$, then $\omega([S(p) \to S(q)]) = S^*\omega([p \to q])$. In addition, the subdivision morphism induces an isomorphism between the corresponding lattices and the associated normal vectors coincide. Therefore, balancing around p follows from balancing around $S(p)$.

- Suppose now that $\dim(S(p)) > k-1$. Then either the dimension increases by 1 or by more. If $\dim(S(p)) = k$, then $\Upsilon(p)$ is a $(k-1)$-dimensional cone subdividing a k-dimensional cone. Now, the second condition shows that $\mathrm{Star}^1_\Upsilon(p)$ consists of two classes, and the third condition implies that these have opposite normal vectors, and in this case, $S^*\omega$ is a constant function on $\mathrm{Star}^1_\Upsilon(p)$, so the balancing condition holds around such p. If $\dim(S(\tau)) > k$ then, by construction, $S^*\omega$ is trivially zero on $\mathrm{Star}^1_\Upsilon(p)$ and balancing around p trivially holds.

In conclusion: $S^*\omega$ is a k-dimensional Minkowski weight of $\Upsilon_{X\circ S}$. □

Definition 2.4.5. Let Φ be a poic-complex. The *category of subdivisions of* Φ is the category $\mathrm{Subd}(\Phi)$ whose class of objects consists of subdivisions $S\colon \Upsilon \to \Phi$ and morphisms are commuting subdivision morphisms. In other words, a morphism $Q\colon S' \to S$, where $S'\colon \Upsilon' \to \Phi$ and $S\colon \Upsilon \to \Phi$ are subdivisions, is a subdivision $Q\colon \Upsilon' \to \Upsilon$ such that $S \circ Q = S'$.

Lemma 2.4.6. *For any poic-complex* Φ *there exists a subdivision* $S\colon \Phi' \to \Phi$ *such that the category* $\mathrm{Subd}(\Phi')$ *is essentially small.*

Proof. We kindly refer the reader to Section 6.4 for a proof, and content ourselves momentarily with an explanation of why just state that $\mathrm{Subd}(\Phi')$ is essentially small and not $\mathrm{Subd}(\Phi)$. In simple terms, because we do not need $\mathrm{Subd}(\Phi)$ to be essentially small. For an arbitrary subdivison $S\colon \Phi' \to \Phi$ the inclusion functor $\mathrm{Subd}(\Phi') \to \mathrm{Subd}(\Phi)$ is final (see XI.3 of [ML78]): Any subdivision of Φ can be subdivided to a subdivision of Φ', and any two subdivisions have a common subdivision. We only use this lemma to show that defining the tropical cycles of a linear poic-complex as a colimit of Minkowski weights of subdivisions makes sense. Since the inclusion $\mathrm{Subd}(\Phi') \to \mathrm{Subd}(\Phi)$ is final, this colimit can be computed at $\mathrm{Subd}(\Phi')$. □

Lemma 2.4.7. *Suppose* Φ_X *is a linear poic-complex, and let* $k \geq 0$ *be an integer. Minkowski weights give rise to a functor*

$$M_k(\bullet_X)\colon \mathrm{Subd}(\Phi)^{\mathrm{op}} \to \mathrm{Ab}, S\colon \Upsilon \to \Phi \mapsto M_k(\Upsilon_{X \circ S}), \tag{2.9}$$

whose colimit is representable.

Proof. It follows from Lemma 2.4.6 that there is a subdivision $S\colon \Phi' \to \Phi$ such that $\mathrm{Subd}(\Phi')$ is an essentially small category. The inclusion functor $\mathrm{Subd}(\Phi') \to \mathrm{Subd}(\Phi)$ is final, since any subdivision of Φ can be further subdivided to a subdivision of Φ'. Therefore, this colimit can be computed at $\mathrm{Subd}(\Phi')$ and at an equivalent small category thereof. Since small colimits of abelian groups are representable, it follows that the colimit of $M_k(\bullet_X)$ is representable. □

Definition 2.4.8. For a linear poic-complex Φ_X, the *group of tropical k-cycles* $Z_k(\Phi_X)$ is defined as the colimit of (2.9).

In simple terms, if Φ_X is a linear poic-complex, then a tropical k-cycle on Φ_X is a k-dimensional Minkowski weight on a subdivision. The representability of the colimit grants us that this is well-defined and, in fact, makes sense.

Lemma 2.4.9. *Suppose* Υ *is a poic-subcomplex of* Φ, Φ_X *is a linear poic-complex, and let* $I\colon \Upsilon \to \Phi$ *denote the natural inclusion. Then extension by zero gives rise to an injective map*

$$I_!\colon Z_k(\Upsilon_{X|_\Upsilon}) \to Z_k(\Phi_X).$$

Proof. This is immediate from Lemma 2.2.14 and the definition of tropical cycles. □

Notation 2.4.10. We say that a linear poic-complex Φ_X is pure of dimension n if the poic-complex Φ is pure of dimension n.

Definition 2.4.11. A linear poic-complex Φ_X pure of dimension n is called *irreducible* if $Z_n(\Phi_X)$ is free of rank 1.

Example 2.4.12. For a poic σ, we claim that any linear poic-complex structure on $\underline{\sigma^0}$ is irreducible. This follows immediately, because in any subdivision of $\underline{\sigma^0}$ the star of any cone of codimension 1 is of size two and the corresponding normal vectors are related by a change of sign, and any subdivision of $\underline{\sigma^0}$ is connected in codimension 1.

Example 2.4.13. We can use the previous example to show that any complete rational fan is irreducible. Let N be a free abelian group of finite rank and let Φ be a complete rational fan in $N \otimes_{\mathbb{Z}} \mathbb{R}$. Then Φ is a subdivision of the poic $\underline{N_{\mathbb{R}}}$. Since $\underline{N_{\mathbb{R}}}$ is equal to the poic given by its relative interior, the above example shows that $\underline{N_{\mathbb{R}}}$ is irreducible. Therefore, Φ must also be irreducible.

Lemma 2.4.14. *If Φ_X is a linear poic-complex pure of dimension n, then the natural inclusion $M_n(\Phi_X) \hookrightarrow Z_n(\Phi_X)$ is an isomorphism. In particular, Φ_X is irreducible if and only if $M_n(\Phi_X)$ is free of rank* 1.

Proof. The lemma is equivalent to showing that for a subdivision $S\colon \Phi' \to \Phi$, the induced map $S^*\colon M_n(\Phi_X) \to M_n(\Phi'_{X\circ S})$ is surjective. A weight $\omega' \in M_n(\Phi'_{X\circ S})$ is of the form $S^*\omega$ with $\omega \in M_n(\Phi_X)$ if and only if for every class $[s] \in [\Phi](n)$ the weight ω' is constant on the classes $[p] \in [\Phi'](n)$ with $[S(p)] = [s]$. As was shown in Example 2.4.12, the linear poic-complex $\underline{\sigma^0}$ is irreducible for any poic σ. Since ω' restricted to the $[p] \in [\Phi'](n)$ with $[S(p)] = [s]$ gives a top-dimensional cycle of the poic $\underline{\Phi(s)^0}$, it follows that ω' is necessarily constant and S^* is surjective. □

Corollary 2.4.15. *If Φ_X is an irreducible linear poic-complex pure of dimension n, then $M_n(\Phi_X)$ is free of rank* 1.

Proof. Let $S\colon \Phi' \to \Phi$ be a subdivision with Φ' simplicial. Then Φ' is also irreducible, and it follows from Lemma 2.4.14 that $M_n(\Phi'_{X\circ S})$ is free of rank 1. From Lemma 2.4.4 we know that $S^*\colon M_n(\Phi_X) \to M_n(\Phi'_{X\circ S})$ is injective. Hence, $M_n(\Phi_X)$ is also free of rank 1. □

Example 2.4.16. Let A be a finite set with $\#A \geq 3$. Following the notation of Section 2.3, we remark that it is shown in [AK06] that the simplicial fan $\mathcal{M}^{\mathrm{trop}}_{0,A}$ is a coarsening of a matroidal fan of the same support. Therefore, Lemma 2.4.14 and Proposition 5.2 of [AHK18] imply that the linear poic-complex $\mathrm{M}^{\mathrm{trop}}_{0,A}$ is irreducible.

Similarly to the structure of Section 2.2, we now turn our focus towards cross products. We show that the product of irreducible linear poic-complexes is also irreducible, and describe cross-products of tropical cycles.

Proposition 2.4.17. *Suppose Φ_X and Ψ_Y are linear poic-complexes, where Φ is pure of dimension n and Ψ is pure of dimension m. If both are irreducible then $(\Phi \times \Psi)_{X\times Y}$ is also irreducible and, moreover, the cross product induces an isomorphism*

$$Z_n(\Phi_X) \otimes_{\mathbb{Z}} Z_m(\Psi_Y) \to Z_{n+m}((\Phi \times \Psi)_{X\times Y}).$$

Proof. Following Lemma 2.4.14, it will suffice to show that the cross product induces an isomorphism

$$M_n(\Phi_X) \otimes_{\mathbb{Z}} M_m(\Psi_Y) \to M_{n+m}(\Phi \times \Psi_{X\times Y}).$$

This same lemma shows that the irreducibility of Φ_X and Ψ_Y implies that both $M_n(\Phi_X)$ and $M_m(\Psi_Y)$ are free of rank 1. Let us consider respective generators $\omega \in M_n(\Phi_X)$ and $\eta \in M_m(\Psi_Y)$. The cross product map is clearly injective, and we just have to show that any weight $\theta \in M_{n+m}((\Phi \times \Psi)_{X\times Y})$ is an integral multiple of $\omega \times \eta$. Since any cone of Φ (resp. Ψ) is at most n-dimensional (resp. m-dimensional), it follows that any class of $[\Phi \times \Psi](n+m)$ must be of the form $[s \times t]$ with $[s] \in [\Phi](n)$ and $[t] \in [\Psi](m)$. Let $\theta \in M_{n+m}((\Phi \times \Psi)_{X\times Y})$ be arbitrary, and consider $[t] \in [\Psi](m)$. An argument analogous to the proof to Lemma 2.2.16 shows that

$$\omega_{\theta,t} \colon [\Phi](n) \to \mathbb{Z}, [s] \mapsto \theta([s \times t])$$

is actually an n-dimensional Minkowski weight of Φ_X. From the irreducibility of Φ_X, it follows that $\omega_{\theta,t} = \lambda_t \cdot \omega$, for some integer λ_t. Again, as $\theta \in M_{n+m}((\Phi \times \Psi)_{X\times Y})$, a similar argument to that of the proof of Lemma 2.2.16 shows that

$$\lambda \colon [\Psi](m) \to \mathbb{Z}, [t] \mapsto \lambda_t$$

is an m-dimensional Minkowski weight of Ψ_Y. As above, irreducibility of Ψ_Y implies that there is an integer $\ell \in \mathbb{Z}$, such that $\lambda_t = \ell \cdot \eta([t])$ for all $[t] \in [\Psi](m)$. Thus, we can conclude from the above that $\theta = \ell \cdot \omega \times \eta$. □

We close this section with a brief digression towards rational Minkowski weights. These are relevant in Section 4, and seem like a good closure for this section.

Definition 2.4.18. Suppose Φ is a poic-complex and let $k \geq 0$ be an integer. The *group of rational weights on* Φ is defined simply as $W_k(\Phi)_{\mathbb{Q}} := W_k(\Phi) \otimes_{\mathbb{Z}} \mathbb{Q}$. If Φ_X is additionally a linear poic-complex, we analogously define:

- The *group of k-dimensional rational Minkowski weights of* Φ_X as

$$M_k(\Phi_X)_{\mathbb{Q}} := M_k(\Phi_X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

- The *group of k-dimensional rational tropical cycles* as

$$Z_k(\Phi_X)_{\mathbb{Q}} := Z_k(\Phi_X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

It may come as a surprise that we are not defining the groups of rational tropical cycles as a colimit, but the next lemma shows that these two are equivalent.

Lemma 2.4.19. *Suppose Φ_X is a linear poic-complex and $k \geq 0$ is an integer. The natural maps $M_k(\Phi'_{X\circ S})_{\mathbb{Q}} \to Z_k(\Phi_X)_{\mathbb{Q}}$, where $S\colon \Phi' \to \Phi$ is a subdivision, induce an isomorphism*

$$\varinjlim_{\mathrm{Subd}(\Phi)} M_k(\bullet_X)_{\mathbb{Q}} \to Z_k(\Phi_X)_{\mathbb{Q}}.$$

Proof. Observe that the functor

$$M_k(\bullet_X)_{\mathbb{Q}}\colon \mathrm{Subd}(\Phi)^{\mathrm{op}} \to \mathrm{Ab}$$

is by definition naturally isomorphic to $M_k(\bullet_X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Hence, the standard commutativity of tensor products with colimits implies that these natural isomorphisms induce an isomorphim

$$\varinjlim_{\mathrm{Subd}(\Phi)} M_k(\bullet_X)_{\mathbb{Q}} \to Z_k(\Phi_X)_{\mathbb{Q}}.$$

This isomorphism factors through the natural maps $M_k(\bullet_X) \otimes_{\mathbb{Z}} \mathbb{Q} \to Z_k(\Phi_X) \otimes_{\mathbb{Z}} \mathbb{Q}$, and hence the lemma follows. □

We remark that the above definition has the same outcome as considering $\mathbb{Q}$-valued functions in Definitions 2.2.1 and 2.2.11. This direction would define a functor with the same properties regarding subdivisions, and the analogous colimit would necessarily be isomorphic to the rational tropical cycles defined above.

2.5 Pushforwards

The last section of this chapter is devoted to the description of pushforward of weights through morphisms of (linear) poic-complexes. To be more precise, consider a morphism of poic-complexes $P\colon \Phi \to \Psi$. We would like to pushforward weights of Φ to weights of Ψ, and if Φ_X and Ψ_Y are linear poic-complexes, we would like to pushforwards Minkowski weights of Φ_X to Minkowski weights of Ψ_Y. Sadly there are multiple

reasons for this to fail. Concerning general weights, it may happen that Ψ is too coarse in general (weights of a subdivision are not weights of the poic-complex). Secondly, a general morphism does not relate the face relations of the target with that of the source, and these face relations are essential for the definition of Minkowski weights. In practical terms, this means that we must define new classes of subdivisions and morphisms that solve these issues.

We deal with the first situation, namely that Ψ is too coarse. In this case, we need to pass to a subdivison of Ψ where this pushforward can be introduced. On these grounds is that we introduce the notion of a P-fine subdivision of Ψ, making rigorous what is meant by Ψ being subdivided enough.

Definition 2.5.1. Suppose $P : \Phi \to \Psi$ is a poic-morphism. A subdivision $S\colon \Psi' \to \Psi$ is called *P-fine (or fine for the morphism P)*, if for every cone s of Φ for which the linear part of $P_s\colon \Phi(s) \to \Psi(P(s))$ is injective, there exists a cone t of Ψ' such that:

- There is an isomorphism $f\colon P(s) \xrightarrow{\sim} S(t)$ of Ψ.
- The images of $\Psi'(t)^0$ and $\Phi(s)^0$ under the linear parts of S_t and $\Psi(f)\circ P_s$ coincide.

It is clear that in this situation $\dim(t) = \dim(s)$ and such a t is unique up to isomorphism. Furthermore, we write $(P_S)_*([s]) = [t]$ for the classes of s and t satisfying the above conditions.

Construction 2.5.2. Suppose $P : \Phi \to \Psi$ is a poic-morphism, $S\colon \Psi' \to \Psi$ is a P-fine subdivision, and let k be a non-negative integer. The *pushforward of $\omega \in W_k(\Phi)$ along P* is the weight

$$(P_S)_*\omega\colon [\Psi'](k) \to \mathbb{Z}, [t] \mapsto \sum_{\substack{[p]\in[\Phi](k),\\ (P_S)_*([p])=[t]}} \omega([p])[S_t N^t\colon P_p N^p]. \tag{2.10}$$

Lemma 2.5.3. *Following the notation of the above construction, the pushforward* (2.10) *gives a linear map:*

$$(P_S)_*\colon W_k(\Phi) \to W_k(\Psi').$$

Proof. This is clear from the construction. □

We now turn our attention towards the case of Minkowski weights of linear poic-complexes. Here we have to introduce the notion of a (weakly) proper morphism, because (as has been mentioned earlier) an arbitrary morphism of poic-complexes does not automatically reflect the face relations of the target. In fact, it is quite unreasonable to expect that the pushforward of a Minkowski weight would yield a Minkowski weight in a general setting. A simple example exhibiting this problematic is the natural inclusion $\underline{\sigma^0} \hookrightarrow \underline{\sigma}$ for a poic σ. For instance, if σ is closed, then there are no top dimensional cycles for $\underline{\sigma}$, but $\underline{\sigma^0}$ is irreducible.

Definition 2.5.4. We say that a morphism $P\colon \Phi \to \Psi$ is *weakly proper* if it satisfies the following lifting property: For every cone s of Φ and every face τ in the closure of $P_s(\Phi(s))$ in $\Psi(P(s))$, there exists $q \to s$ of Φ with the same codimension which under the morphism P gives this face τ. We say that P is *proper*, if for every subdivision $S\colon \Phi' \to \Phi$, the composition $P \circ S$ is weakly proper. In other words, the lifting property holds for every sub-poic of $\Phi(s)$.

This lifting property of a weakly proper morphism liberates us from the issue between the face relations on the target and the face relations of the source. In simple terms, a morphism is weakly proper if every face of the image of an arbitrary cone of the source actually arises as the image of an original face. It is immediate to check that the inclusion $\underline{\sigma^0} \hookrightarrow \underline{\sigma}$, where σ is a poic, is a weakly proper morphism if and only if $\sigma = \sigma^0$. The difference between weakly proper and proper comes from taking subdivisions, and perhaps is best illustrated with a simple example.

Example 2.5.5. Consider $\mathbb{R}^2$ with the standard lattice $\mathbb{Z}^2$ and let $\mathrm{pr}_2\colon \mathbb{R}^2 \to \mathbb{R}$ denote the integral linear map given by projection onto the second coordinate. We have the (linear) poic-complex $\underline{H_{\mathrm{pr}_2>0}}$, and the map pr_2 induces a morphism of (linear) poic-complexes $\underline{H_{\mathrm{pr}_2>0}} \to \underline{\mathbb{R}}$. This morphism is (vacuosly) weakly proper but it is definitely not proper. Indeed, any 1-dimensional subcone of $\underline{H_{\mathrm{pr}_2>0}}$ different from the coordinate axis shows that this morphism cannot be proper.

The next lemma covers an initial desired situation, and its proof basically follows that of Proposition 2.25 of [GKM09].

Lemma 2.5.6. *Suppose Ψ_Y is a linear poic-complex and $P\colon \Phi \to \Psi$ is a weakly proper morphism of poic-complexes. If $S : \Psi' \to \Psi$ is a P-fine subdivision, then the pushforward map $(P_S)_*\colon W_k(\Phi) \to W_k(\Psi')$ restricts to a linear map*

$$(P_S)_*\colon M_k(\Phi_{Y\circ P}) \to M_k(\Psi'_{Y\circ S}),$$

for any integer $k \geq 0$.

Remark. A few words may be in order before the proof. A natural direction to start the proof might be to first assume that Ψ is itself already P-fine and show the statement in this case. Then, the next step would then be to use this to show the general statement. Now, this strategy would basically require either the construction of a morphism $P'\colon \Phi \to \Psi'$ satisfying $S \circ P' = P$, or to just apply the same argument as before being cautious of the subdivision. The latter option seemed more appealing in its efficiency to us, and hence is the one we adopt. Furthermore, to avoid repetition we just prove the general case.

Proof. Let $\omega \in M_k(\Phi_{Y\circ P})$ and consider a cone q of Ψ' of dimension $k-1$. We intend to show that $(P_S)_*\,\omega$ is balanced at q. For this, we first observe that if there is a cone s of Φ and $q \to t$ of Ψ' such that $(P_S)_*([s]) = [t]$ and $[q \to t] \in \mathrm{Star}^1_{\Psi'}(q)$, then weakly properness of P implies that there exists a morphism $p \to s$ of Φ with:

- $[p] \in [\Phi](k-1)$,
- $[P(p)] = [S(q)]$,
- $(P_S)_*([p]) = [q]$.

Consequently, we can assume without loss of generality that the cone q of Ψ' is such that its class $[q] \in [\Psi'](k-1)$ is of the form $(P_S)_*([s])$ for some $[s] \in [\Phi](k-1)$.
Consider a morphism $q \to t$ of Ψ' with $\dim(t)-\dim(q) = 1$, and let $p \to s$ be a morphism of Φ with $(P_S)_*([p]) = [q]$, $(P_S)_*([s]) = [t]$, and $\dim(s) - \dim(p) = 1$. This whole set-up yields the following commutative diagram (we emphasize the domains of the cones):

$$\begin{array}{ccccc}
N^p_\Phi & \hookrightarrow & N^s_\Phi & \twoheadrightarrow & N^s_\Phi/N^p_\Phi \\
\downarrow{\scriptstyle P_p} & & \downarrow{\scriptstyle P_s} & & \downarrow \\
S_q(N^q_{\Psi'}) & \hookrightarrow & S_t(N^t_{\Psi'}) & \twoheadrightarrow & S_t(N^t_{\Psi'})/S_q(N^q_{\Psi'}) \\
\downarrow & & \downarrow & & \downarrow \\
S_q(N^q_{\Psi'})/P_p(N^p_\Phi) & \hookrightarrow & S_t(N^t_{\Psi'})/P_s(N^s_\Phi) & \twoheadrightarrow & S_t(N^t_{\Psi'})/(S_q(N^q_{\Psi'}) + P_s(N^s_\Phi)).
\end{array}$$

The bottom row shows that

$$[S_t(N^t_{\Psi'})\colon P_s(N^s_\Phi)] = [S_q(N^q_{\Psi'})\colon P_p(N^p_\Phi)] \cdot [S_t(N^t_{\Psi'})\colon S_q(N^q_{\Psi'}) + P_s(N^s_\Phi)]. \tag{2.11}$$

After composition with the morphism $Y\colon \Psi \to \underline{(N_Y)_{\mathbb{R}}}$, the previous diagram shows that P induces a integral linear map

$$N_Y/(Y\circ P)_p(N^p_\Phi) \to N_Y/(Y\circ S)_q(N^q_{\Psi'}), \tag{2.12}$$

and the vector $u^{Y\circ P}_{[p\to s]}$ is mapped through (2.12) to $[S_t(N^t_{\Psi'})\colon S_q(N^q_{\Psi'}) + P_s(N^s_\Phi)] \cdot u^{Y\circ S}_{[q\to t]}$. We now observe that the sum $\sum_{[p\to s]\in \mathrm{Star}^1_\Phi(p)} \omega([s]) u^{Y\circ P}_{[p\to s]}$ necessarily vanishes (in the quotient $N_Y/(Y\circ P)_p(N^p)$), because $\omega \in M_k(\Phi_{Y\circ P})$. Therefore, following this previous discussion, after multiplying by a factor of $[S_q(N^q_{\Psi'})\colon P_p(N^p_\Phi)]$ and using (2.11), we obtain the following equation in the quotient $N_Y/(Y\circ S)_q(N^q_{\Psi'})$:

$$\sum_{[p\to s]\in \mathrm{Star}^1_\Phi(p)} \omega([s])[S_{(P_S)_*(s)}(N^{(P_S)_*(s)}_{\Psi'})\colon P_s(N^s_\Phi)] u^{Y\circ S}_{[q\to (P_S)_*(s)]} = 0.$$

To finalize, we sum all these equations with respect to all the $[p] \in [\Phi](k-1)$ such that $(P_S)_*([p]) = [t]$, and ultimately obtain:

$$0 = \sum_{\substack{[p]\in[\Phi](k-1),\\ (P_S)_*([p])=[q]}} \sum_{[p\to s]\in \mathrm{Star}^1_\Phi(p)} \omega([s])[S_{(P_S)_*(s)}(N_{\Psi'}^{(P_S)_*(s)})\colon P_s(N_\Phi^s)]u^{Y\circ S}_{[q\to(P_S)_*(s)]} =$$
$$\sum_{[q\to t]\in \mathrm{Star}^1_{\Psi'}(q)} \sum_{\substack{[s]\in\Phi(k),\\ (P_S)_*([s])=[t]}} \omega([s])[S_t(N_{\Psi'}^t)\colon P_s(N_\Phi^s)]u^{Y\circ S}_{[q\to t]} = \sum_{[q\to t]\in \mathrm{Star}^1_{\Psi'}(q)} ((P_S)_*\omega)([t])u^{Y\circ S}_{[q\to t]}.$$

This shows balancing for $(P_S)_*\omega$ at the cone q of Ψ'. Since the latter was arbitrary, it follows that $(P_S)_*\omega \in M_k(\Psi'_{Y\circ S})$. □

We close this section by extending the previous notions and constructions to the case of linear poic-complexes, and more generally to tropical cycles. In the latter, we are eventually required to show that any poic-complex has a subdivision that is fine for a given morphism. Similar to previous instances, we postpone the proof of this statement to Section 6.4 and kindly demand patience (and trust) from the reader.

Definition 2.5.7. A morphism of linear poic-complexes $\phi\colon \Phi_X \to \Psi_Y$ is called *weakly proper* (resp. *proper*), if the morphism of poic-complexes $\phi\colon \Phi \to \Psi$ is weakly proper (resp. proper).

Remark. We observe the necessity of properness to pushforward general tropical cycles of linear poic-complexes by following the proof of Lemma 2.5.6 with respect to the morphism $\underline{H_{\mathrm{pr}_2>0}} \to \underline{\mathbb{R}}$ of Example 2.5.5. The linear poic-complex $\underline{H_{\mathrm{pr}_2>0}}$ has several 1-dimensional tropical cycles given by 1-dimensional subcones of $\underline{H_{\mathrm{pr}_2>0}}$. None of these can be pushforwarded to an actual tropical cycle of $\underline{\mathbb{R}}$.

Construction 2.5.8. Suppose that $\phi\colon \Phi_X \to \Psi_Y$ is a proper morphism of linear poic-complexes. Observe that the map ϕ_{int} induces a linear map

$$(\phi_{\mathrm{int}})_*\colon M_k(\Phi_X) \to M_k(\Phi_{Y\circ\phi}),$$

and if $S\colon \Psi' \to \Psi$ is ϕ-fine, then ϕ induces the pushforward map

$$(\phi_S)_*\colon M_k(\Phi_{Y\circ\phi}) \to M_k(\Psi'_{Y\circ S}).$$

Collectively, ϕ and S give rise to the *pushforward map* of Minkowski weights

$$(\phi_S)_*\colon M_k(\Phi_X) \to M_k(\Psi'_{Y\circ S}) \tag{2.13}$$

given as the composition $(\phi_S)_* \circ (\phi_{\mathrm{int}})_*$.

To extend the previous constructions to tropical cycles, the following lemma is necessary.

Lemma 2.5.9. *If $P\colon \Phi \to \Psi$ is a morphism, then there exists a P-fine subdivision $Q\colon \Psi' \to \Psi$.*

Proof. As has previously been mentioned, a proof of this lemma can be found in Section 6.4. □

Construction 2.5.10. Suppose $\phi\colon \Phi_X \to \Psi_Y$ is a proper morphism between linear poic-complexes. We remark that if $R\colon \Phi' \to \Phi$ is a subdivision, then $\phi \circ S\colon \Phi' \to \Phi$ is also proper. Now, Lemma 2.5.9 shows that there exists a $\phi \circ R$-fine subdivision $S\colon \Psi' \to \Psi$, and we thus get the pushforward map

$$((\phi \circ R)_S)_*\colon M_k(\Phi'_{X\circ R}) \to M_k(\Psi'_{Y\circ S}).$$

Moreover, taking multiple subdivisions gives rise to multiple commutative diagrams, and composition with the standard map $M_k(\Psi'_{Y\circ S}) \to Z_k(\Psi_Y)$ shows then that these maps give rise to a linear map

$$\phi_*\colon Z_k(\Phi_X) \to Z_k(\Psi_Y), \tag{2.14}$$

which we call the *pushforward map of ϕ*. If ϕ is only weakly proper, then we simply obtain the *weak pushforwardforward map of ϕ*

$$\phi_*\colon M_k(\Phi_X) \to Z_k(\Psi_Y). \tag{2.15}$$

We close the section, with a small example of pushforwards.

Example 2.5.11. Consider the linear poic-complexes $\underline{\mathbb{R}^3_{>0}}$ and $\underline{\mathbb{R}^2_{>0}}$, with the morphism of linear poic-complexes $\mathrm{pr}\colon \underline{\mathbb{R}^3_{>0}} \to \underline{\mathbb{R}^2_{>0}}$ given by projection onto the first two coordinates, which is a proper morphism of poic-complexes. We consider the subdivision of $\underline{\mathbb{R}^3_{>0}}$ depicted in Figure 2.4, where the ray in the middle is generated by the vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$. The 2-dimensional poics in this subdivision are given by the intersection of $\mathbb{R}^3_{>0}$ with the cones generated by the ray in the middle and each of the coordinate axes. Let ω denote the 2-dimensional constant weight with value 1. It can be readily seen that this is a Minkowski weight and hence defines a 2-cycle of $\underline{\mathbb{R}^3_{>0}}$.
For the pushforward of ω through pr, we consider the subdivision of $\underline{\mathbb{R}^2_{>0}}$ depicted on Figure 2.5 where the middle ray is generated by the vector $\begin{pmatrix}1\\1\end{pmatrix}$. This subdivision

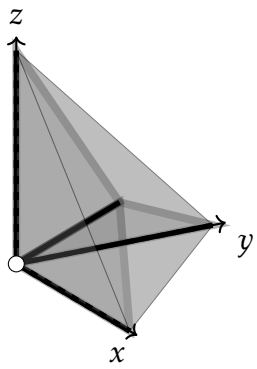


Figure 2.4: Subdivision of $\underline{\mathbb{R}^3_{>0}}$

is pr-fine for the subdivision of $\underline{\mathbb{R}^3_{>0}}$ that we have considerd. In this case $\mathrm{pr}_*\omega$ is the constant weight with value 1, which is also a Minkowski weight. We observe that this pushforward "kills" whatever contribution there was along the 2-dimensional cone of this subdivision of $\underline{\mathbb{R}^3_{>0}}$ that is perpendicular to the xy-plane, because the restriction of the linear map π is not injective in this cone.

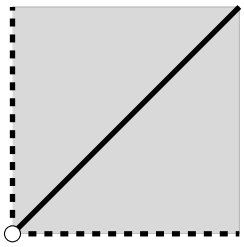


Figure 2.5: Subdivision of $\underline{\mathbb{R}^2_{>0}}$

2.6 Closing technicalities on pushforwards

We can use a small generalization of the notion of weakly properness, that applies to very specific cases. Following the notation of the previous subsection, let $P\colon \Phi \to \Psi$ denote a morphism of poic-complexes, and let $k \geq 0$ be an integer.

Definition 2.6.1. We say that P is weakly proper in dimension k, if it satisfies the lifting property of Definition 2.5.4 at facets of k-dimensional cones of Φ. More precisely, if for every k-dimensional cone s of Φ and every facet τ in the closure of $P_s(\Phi(s))$ in $\Psi(P(s))$, there exists a facet $q \to s$ of Φ which under the morphism P gives the facet τ.

Notation 2.6.2. In the eventual case that Φ is pure of dimension n and ϕ is weakly proper of dimension n, then we just say that ϕ is weakly proper in top dimension.

In this case, following the proof of Lemma 2.5.6 verbatim yields the following:

Lemma 2.6.3. *Let $k \geq 0$ be an integer, and suppose Ψ_Y is a linear poic-complex. If $P\colon \Phi \to \Psi$ is a morphism of poic-complexes that is weakly proper in dimension k, and $S : \Psi' \to \Psi$ is a P-fine subdivision, then the pushforward map $(P_S)_*\colon W_k(\Phi) \to W_k(\Psi')$ restricts to a linear map*

$$(P_S)_*\colon M_k(\Phi_{Y\circ P}) \to M_k(\Psi'_{Y\circ S}).$$

In the case of linear poic-complexes, we follow the blueprint of (weakly-)proper morphisms for linear poic-complexes.

Definition 2.6.4. Suppose $k \geq 0$ is an integer. A morphism of linear poic-complexes $\phi\colon \Phi_X \to \Psi_Y$ is called *weakly proper in dimension k*, if the morphism of poic-complexes $\phi\colon \Phi \to \Psi$ is weakly proper in dimension k.

If a morphism of linear poic-complexes $\phi\colon \Phi_X \to \Psi_Y$ is weakly proper in dimension k, then we can apply Lemma 2.5.9 to obtain a P-fine subdivision $S\colon \Psi' \to \Psi$, to thus obtain the linear map

$$(\phi_S)_*\colon M_k(\Phi_X) \to M_k(\Psi'_{Y\circ S}),$$

and, just as before, obtain the *weak pushforward in dimension k*

$$(\phi)_*\colon M_k(\Phi_X) \to Z_k(\Psi_Y). \tag{2.16}$$

Remark. These technicalities, however terse and unpalatable, arise in our endeavors in Section 3.7 and Chapter 4.

Example 2.6.5. A simple situation where these technicalities arise is the following. Suppose Φ_X is a linear poic-complex, and let $k \geq 0$ be an integer such that $[\Phi](k) \neq \varnothing$. Then if Φ'_X denotes the linear poic-complex given by removing every cone of Φ of dimension strictly less than $k-1$ and modifying the corresponding poics accordingly, then the natural inclusions define a morphism of linear poic-complexes $\Phi'_X \to \Phi_X$ that is weakly proper in dimension k (also in any dimension greater than k). In this case there is an isomorphism between the groups of k-dimensional Minkowski weights (also for any dimension greater than k), but we cannot pushforward lower-dimensional Minkowski weights.

Chapter 3

Poic-fibrations and their cycles

In this chapter we introduce the core notion of our project: poic-fibrations. Before we discuss this, we need to introduce poic-spaces. These are objects closely related to poic-complexes that come with some non-trivial automorphisms. Perhaps a better suited name for these would be that of *generalized poic-complexes*, as their relationship with poic-complexes is of the same nature as that of generalized cone complexes and cone complexes [ACP15]. At this point we have already encountered multiple examples of non-trivial poic-spaces in nature: the moduli spaces of tropical curves of a fixed genus and marking, and the moduli spaces of discrete admissible covers of a fixed degree, genera, and with specific ramification profiles above the legs of the target. These two classes of spaces are our original motivation and lie at the heart of our endeavors. Our overall goal is to describe tropical cycles in general poic-spaces, but the presence of non-trivial automorphisms at the cones forbids such a general introduction. We circumvent this difficulty by shifting our attention towards poic-fibrations and placing ourselves in a relative situation. That is, instead of in poic-spaces we seek to describe cycles relative to special kind of morphisms, which we call poic-fibrations. A poic-fibration is a morphism between a poic-complex and a poic-space with nice local properties and a lifting property concerning face relations and automorphisms. When the source is a linear poic-complex, we are able to introduce tropical cycles of the poic-fibration (or tropical cycles equivariant with respect to the poic-fibration) as a subgroup of cycles of the source. The path we follow is entirely analogous to that of linear poic-complexes. Namely, we introduce subdivisions and weights of a subdivision (that exhibit the suggested equivariance with respect to the poic-fibration), and define the equivariant tropical cycles as the colimit over the subdivisions (here some care has to be taken, as we must restrict to subdivisions that are compatible with the poic-fibration). We also discuss the spanning tree fibrations, and the extended spanning tree fibrations, which are our poic-fibrations of interest to describe cycles in our motivating classes of examples.

3.1 Poic-spaces

Definition 3.1.1. A *poic-space* $\mathcal{X}$ is a functor $\mathcal{X}\colon C_{\mathcal{X}} \to \mathrm{POIC}$, where $C_{\mathcal{X}}$ is an essentially finite category with finite hom-sets, subject to the following conditions:

1. The functor $\mathcal{X}$ maps morphisms of $C_{\mathcal{X}}$ to face-embeddings.
2. If x is an object of $C_{\mathcal{X}}$ and τ is a face of $\mathcal{X}(x)$, then there exists a unique up to isomorphism $w \to x$ in $C_{\mathcal{X}}$ with $\mathcal{X}(w \to x) \cong \tau \leq \mathcal{X}(x)$.

If $\mathcal{Y}$ is an additional poic-space, a *morphism of poic-spaces* $F\colon \mathcal{X} \to \mathcal{Y}$ consists of the following data:

- A functor $F\colon C_{\mathcal{X}} \to C_{\mathcal{Y}}$.
- A natural transformation $\eta_F\colon \mathcal{X} \implies \mathcal{Y} \circ F$.

These are subject to the following condition: for an object x of $C_{\mathcal{X}}$, the cone $\eta_{F,x}(\mathcal{X}(x))$ does not lie in a proper face of $\mathcal{Y}(F(x))$. Composition of morphisms is defined analogously to that of poic-complexes.

It is clear that both conditions on the definition of poic-spaces are analogous to those of poic-complexes. The main difference between these two notions lies in the morphisms and, more precisely, in the groups of automorphisms of objects in the category. Namely, we are no longer required to have a thin category, so in principle there could be various other (auto)morphisms. Naturally, every poic-complex is a poic-space by itself.

Notation 3.1.2. We follow the same notational conventions as with poic-complexes. If $\mathcal{X}$ is a poic-space, then:

- We will refer to the objects of $C_{\mathcal{X}}$ as the cones of $\mathcal{X}$.
- We will denote the set of isomorphism classes of $C_{\mathcal{X}}$ by $[\mathcal{X}]$. This set is partitioned by the dimensions of its cones, and, for an integer $k \geq 0$, $[\mathcal{X}](k)$ denotes the set of isomorphism classes of k-dimensional cones.
- We say that $\mathcal{X}$ is pure of dimension n if every cone x of $\mathcal{X}$ appears as the source of a morphism to an n-dimensional cone of $\mathcal{X}$.
- For an integer $k \geq 0$, the *group $W_k(\mathcal{X})$ of k-dimensional weights on $\mathcal{X}$* is just the abelian group of functions $[\mathcal{X}](k) \to \mathbb{Z}$.

Analogously to pCmplxs, poic-spaces and their morphisms form a category, that we denote by pSpcs. As can be readily observed, pCmplxs is a full subcategory of

pSpcs. Similarly to the case (2.2) of poic-complexes, the realization functor 1.2 can be extended naturally to poic-spaces, and we thus obtain:

$$|\bullet|\colon \mathrm{pSpcs} \to \mathrm{Top}, \mathcal{X} \mapsto \mathrm{colim}|\mathcal{X}|. \tag{3.1}$$

If $\mathcal{X}$ is a poic-space, then we refer to $\mathrm{colim}|\mathcal{X}|$ as the *realization of* $\mathcal{X}$.

Example 3.1.3. We remark that Lemma 1.4.8 shows that the functor $\mathrm{M}^{\mathrm{trop}}_{g,A}$ (where $2g+\#A-2>0$) is a poic-space. In Example 1.4.10 we have already carried out an illustrated description of the realization of $\mathrm{M}^{\mathrm{trop}}_{1,3}$.

Example 3.1.4. Analogously, Lemma 1.6.11 readily shows that the functor $\mathrm{AC}_{d,h}(\vec{\mu})$ (where h, d, and $\vec{\mu}$ are as in Notation 1.6.1) is a poic-space.

3.2 Poic-fibrations and their Minkowski weights

Definition 3.2.1. Suppose Φ is a poic-complex and $\mathcal{X}$ is a poic-space. A morphism of poic-spaces $\pi\colon \Phi \to \mathcal{X}$ is a *poic-fibration*, if:

1. π is essentially surjective.
2. For any cone p of Φ the map $\eta_{\pi,p}\colon \Phi(p) \to \mathcal{X}(\pi(p))$ induces an isomorphism between their relative interiors $\Phi(p)^0$ and $\mathcal{X}(\pi(p))^0$.
3. For any object p of Φ and any morphism $f\colon \pi(p) \to x$ in $\mathcal{X}$, there exists a morphism $h\colon p \to q$ in Φ and an isomorphism $g\colon \pi(q) \to x$ such that $f = g \circ \pi(h)$, where h is unique up to isomorphism under p, and g is unique up to isomorphism over x.

A morphism of poic-fibrations is simply a morphism of pairs yielding a commutative square. If Φ_X is additionally a linear poic-complex, then we say that $\pi\colon \Phi_X \to \mathcal{X}$ is a linear poic-fibration.

Remark. We chose to name these morphisms poic-fibrations because of the underlying lifting property of the functor. More precisely, the accurate term for these kinds of functors is that of *discrete Street opfibrations*, which means that the functor between the opposite categories is a *discrete Street fibration*[1] (see [Str81]). These kinds of fibrations were previously studied by Street and are just a categorification of the usual notion of a Grothendieck fibration. However, we deliberately choose to strain ourselves from this notation, simply call them poic-fibrations, and explicitize the corresponding lifting property, with the aim of emphasizing its combinatorial content relevant to our framework.

[1] Here the adjective discrete means that the fibers of the Street fibration are (finite) sets.

Example 3.2.2. We exhort the reader to look in Section 3.5 for a family of examples, here we carry out explicitly the case of $g = 1$ and $n = 2$. We have already considered this case in Example 1.4.6 when illustrating the cones of metrics. For the poic-complex we take the product $\Phi = \mathrm{M}^{\mathrm{trop}}_{0,4} \times \underline{\mathbb{R}^1_{>0}}$, and for the poic-space we take $\mathcal{X} = \mathrm{M}^{\mathrm{trop}}_{1,2}$. The morphism $\pi\colon \Phi \to \mathcal{X}$ is then given as specified in Figure 3.1. It is readily checked that this morphism is a poic-fibration, and we depict it in Figure 3.2.

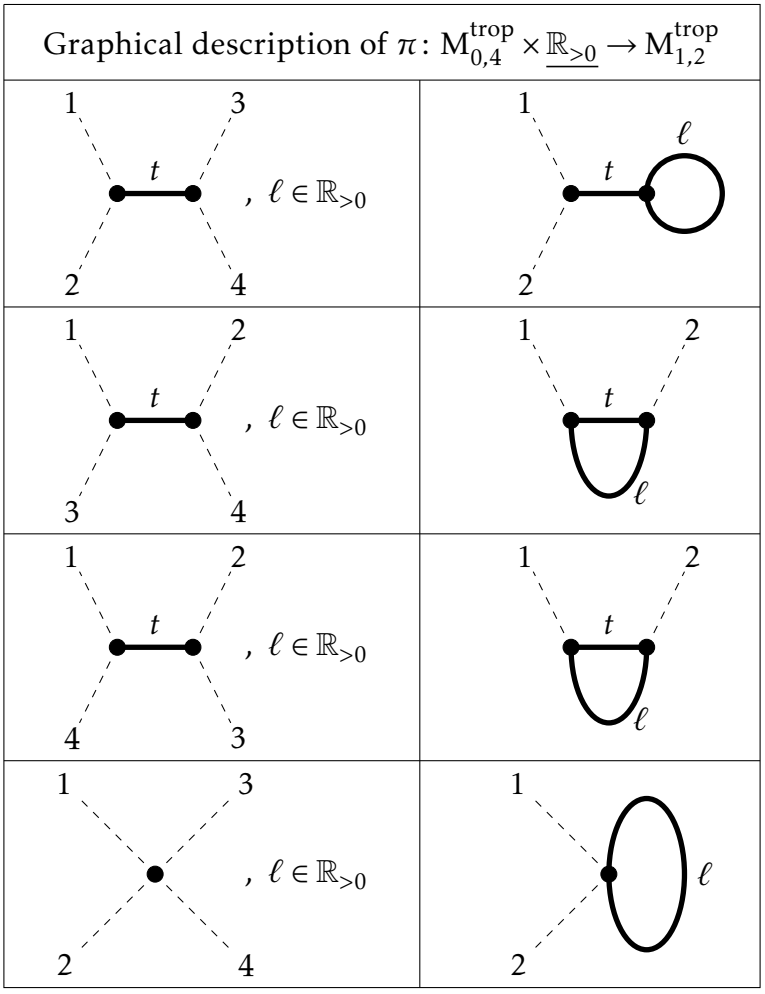


Figure 3.1: The poic-fibration of Example 3.2.2

The following two examples will be our working examples throughout the chapter. At the moment they may seem dull and uninteresting, but they will prove useful for illustrative purposes in subsequent constructions.

Example 3.2.3. Consider the poic-space $\mathcal{X}$ given by the single poic $\mathbb{R}^3_{>0}$ but with set of automorphisms:

$$\left\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\right\}.$$

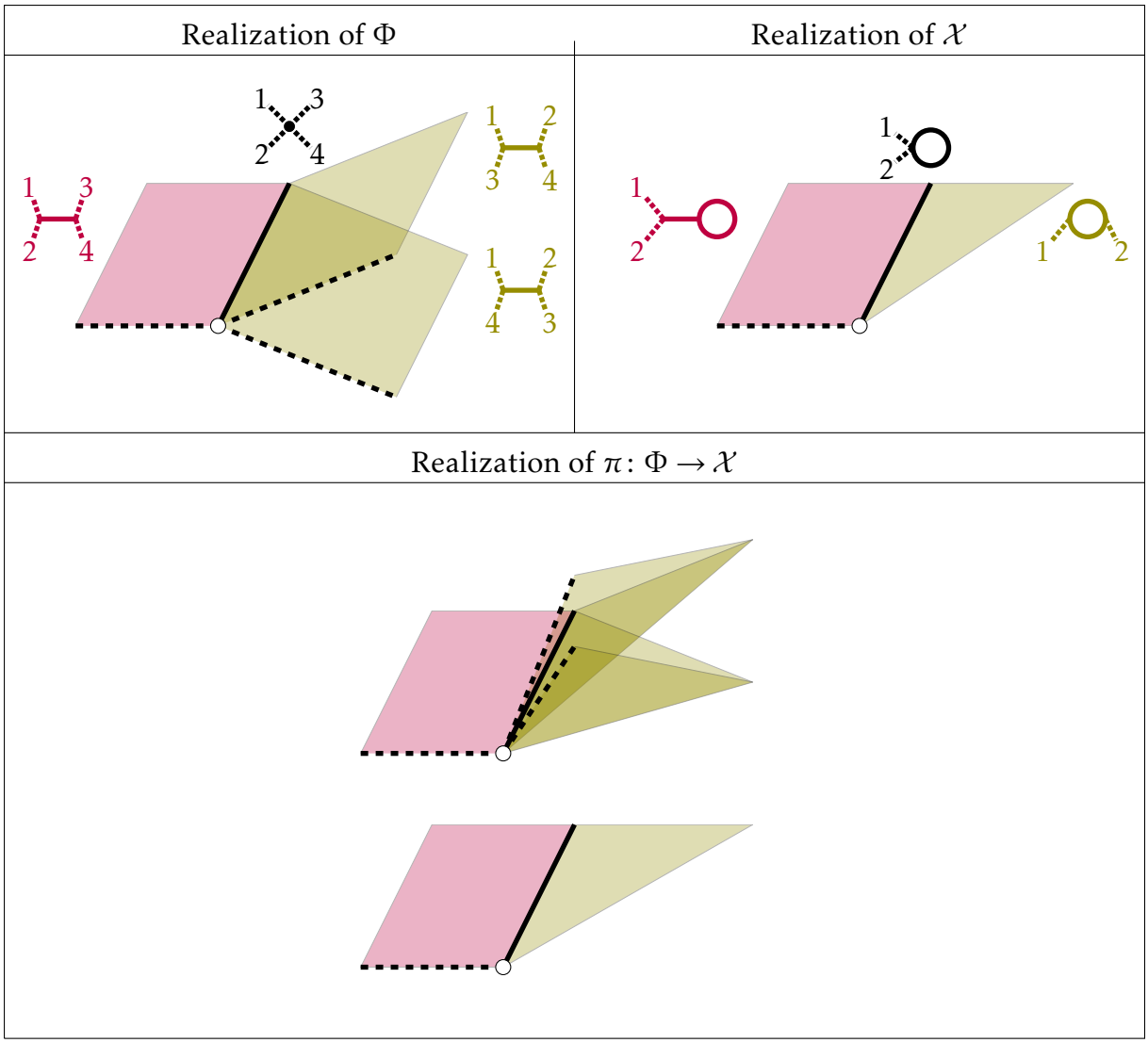


Figure 3.2: Example of a poic-fibration

As poic-complex Φ we just take two disjoint copies of $\mathbb{R}^3_{>0}$, and for the poic-fibration we just consider the identity maps. It can immediately be checked that this, trivially, defines a poic-fibration. We depict both Φ and $\mathcal{X}$ in Figure 3.3 by looking at the corresponding slices given by $x + y + z = 1$. There is nothing special about the two copies of $\mathbb{R}^3_{>0}$ that appear in this example. In fact, we can proceed in the same way and consider any positive number of finite copies of $\mathbb{R}^3_{>0}$ and produce similarly spirited poic-fibrations. The reason for our choice of two copies in this example is for illustrative purposes of future objects of interest.

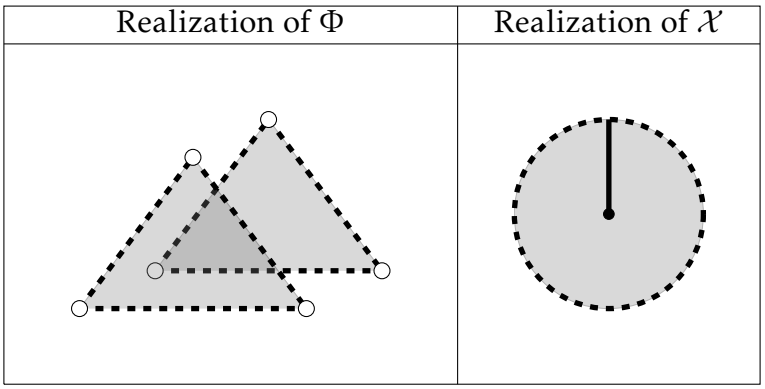


Figure 3.3: A working example of a (linear) poic-fibration

Example 3.2.4. Consider the poic-space $\mathcal{X}$ given by the single poic $\mathbb{R}^3_{>0}$ but with the following group of automorphisms:

$$\left\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\right\}.$$

As a poic-complex Φ we just take $\underline{\mathbb{R}^3_{>0}}$, and for the morphism of poic-spaces we just take the identity morphism of the underlying poics. This is trivially a poic-fibration, and we depict the realizations of both $\mathcal{X}$ and Φ in Figure 3.4 by looking at the corresponding slices given by $x + y + z = 1$. In this case, what we obtain for $\mathcal{X}$ is the quotient of a 2-simplex where we have identified one half with the other (so it looks as if it has been chopped at half). We follow the same conventions when depicting faces and vertices that have been removed.

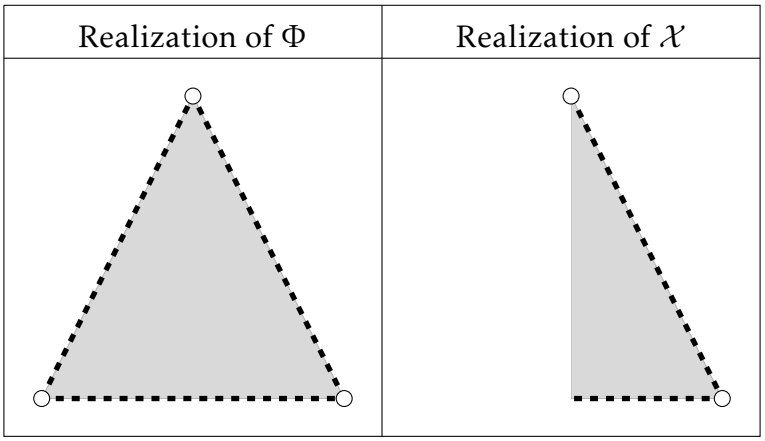


Figure 3.4: An additional working example of a poic-fibration

The following lemma attests the congenial local behaviour within face relations implicit in poic-fibrations.

Lemma 3.2.5. *Let Φ be a poic-complex, and $\mathcal{X}$ be a poic-space. Suppose $\pi : \Phi \to \mathcal{X}$ is a poic-fibration, with s and s' cones of Φ. Then an isomorphism $f : \pi(s) \to \pi(s')$ of $\mathcal{X}$ induces a bijection*

$$\pi_f : \operatorname{Star}^1_\Phi([s]) \to \operatorname{Star}^1_\Phi([s']). \tag{3.2}$$

Proof. Let $i : s \to t$ be a morphism of Φ such that $[s \to t] \in \operatorname{Star}^1_\Phi([s])$. Then $\pi(i) \circ f^{-1} : \pi(s') \to \pi(t)$, and the third condition of a poic-fibration implies that there exists $i' : s' \to t'$ of Φ and an isomorphism $f' : \pi(t') \to \pi(t)$ of $\mathcal{X}$, such that $\pi(i) \circ f^{-1} = f' \circ \pi(i')$. It is clear then that $i' : s' \to t'$ is a face-embedding of codimension 1, and the uniqueness up to isomorphism then shows that $\pi_f([s \to t]) := [s' \to t']$ is well defined and injective. Since the sets are finite and the same argument applied to f^{-1} shows that $\pi_{f^{-1}}$ is also injective, it follows that these maps must be bijections. Furthermore, the naturality and the uniqueness up to isomorphism show that π_f and $\pi_{f^{-1}}$ are inverse to each other. □

Suppose $\pi : \Phi_X \to \mathcal{X}$ is a linear poic-fibration. The first and second condition imply that the poic-fibration π induces a surjective map of sets

$$[\pi] : [\Phi](k) \to [\mathcal{X}](k).$$

This map, in turn, induces an injective linear map $[\pi]^* : W_k(\mathcal{X}) \to W_k(\Phi)$.

Definition 3.2.6. Let $\pi : \Phi_X \to \mathcal{X}$ be a linear poic-fibration. A *k-dimensional π-equivariant Minkowski weight* ω is a weight $\omega \in M_k(\Phi_X)$ such that $\omega \in [\pi]^*(W_k(\mathcal{X}))$. The set of π-equivariant Minkowski weights is denoted by $M_k(\mathcal{X}_{\pi,X})$, and is clearly an abelian subgroup of $M_k(\Phi_X)$.

Remark. If $\pi : \Phi_X \to \mathcal{X}$ is a linear poic-fibration, then a k-dimensional Minkowski weight $\omega \in M_k(\Phi_X)$ is π-equivariant if and only if it is stable under the fibers of $[\pi]$. More precisely, $\omega \in M_k(\Phi_X)$ is π-equivariant if and only if $\omega([s]) = \omega([t])$ for all pairs $[s], [t] \in [\Phi](k)$ with $[\pi(s)] = [\pi(t)]$. This situation turns more interesting when dealing with Minkowski weights of special subdivisions.

We close this section with a non-example with the aim of conveying to a greater extent the whole harmony of the three conditions of a poic-fibration.

Example 3.2.7. Consider an arbitrary poic-complex Φ, and let C'_Φ denote the category obtained from C_Φ having the same class of objects but removing all non-isomorphisms (this makes sense and is unproblematic because the category is thin). We let $\operatorname{Dis}(\Phi)$ (as in disassembled) denote the poic-complex given by

$$\operatorname{Dis}(\Phi) : C'_\Phi \to \mathrm{POIC}, p \mapsto \Phi(p)^0.$$

This clearly defines a poic-complex and, furthermore, the natural functor $C'_\Phi \to C_\Phi$ and the natural subcone inclusions $\Phi(p)^0 \hookrightarrow \Phi(p)$ define a morphism of poic-complexes $\mathrm{Diss}(\Phi) \to \Phi$. This morphism satisfies the first two conditions of a poic-fibration, but not the third. We would expect in this situation that the Minkowski weights equivariant with respect to $\mathrm{Dis}(\Phi) \to \Phi$ will relate, or even compute, the Minkowski weights of Φ, but clearly this would seldom be the case.

3.3 Subdivisions of poic-fibrations and their tropical cycles

The way we define tropical cycles of a poic-fibration is to proceed just as in the case of poic-complexes: by subdivisions of the source. Naturally, an arbitrary subdivision will not be useful, as it will not reflect any compatibility with the poic-fibration (more precisely, the isomorphisms of the target). This prompts us to introduce the notion of subdivisions that are compatible with the poic-fibration. For this, we are due to start with some constructions.

Construction 3.3.1. Suppose Φ is a poic-complex, s is a cone of Φ, $S\colon \Phi' \to \Phi$ is a subdivision, and t is a cone of Φ' such that $S(t) \cong s$. By definition, the induced linear map

$$S_t\colon \Phi'(t) \to \Phi(s)$$

is injective, and $S_t(\Phi'(t))$ does not lie in a proper face of $\Phi(s)$. Therefore, the intersection $S^0(t) := S_t(\Phi'(t)) \cap \Phi(s)^0$ is a non-empty (partially open) subcone of $\Phi(s)^0$. We let $S^{-1}(s)$ denote the following set of partially open subcones of $\Phi(s)^0$

$$S^{-1}(s) := \{S^0(t) : t \text{ is a cone of } \Phi' \text{ with } S(t) \cong s\}.$$

Lemma 3.3.2. *Suppose Φ, s, S, and Φ' are as in the previous construction. The set $S^{-1}(s)$ is closed under face relations, and hence the tautological association gives rise to a poic-complex (which we will denote in the same way). This poic-complex is furthermore a subdivision of $\underline{\Phi(s)^0}$.*

Proof. Let t be as above, that is t is a cone of Φ' with $S(t) \cong s$. Observe that a face of $S^0(t)$ must come from a face of $\Phi'(t)$, and hence comes from a morphism $t' \to t$ of Φ'. Since S is a subdivision, necessarily $S(t') \cong s$ and therefore $S^0(t') \in S^{-1}(s)$. Therefore, this set is closed under face relations, and it must also be a subdivision of $\underline{\Phi(s)^0}$, because S is a subdivision of Φ. □

In simple terms, the subdivisions previously introduced in the construction are the local subdivisions we would "see" at each poic of Φ. With these in mind, we can eas-

ily express the compatibility of a subdivision with respect to a poic-fibration. More precisely, an isomorphism of the poic-space allows us to transport these subdivisions between the fibers of the poic-fibration. The idea then is that a subdivision is compatible if these local subdivisions are stable under this transport. We express and record this notion in the following definition.

Definition 3.3.3. Let $\pi : \Phi \to \mathcal{X}$ be a poic-fibration. A subdivision $S\colon \Phi' \to \Phi$ is called *π-compatible* if: For any cones p and q of Φ together with an isomorphism $f\colon \pi(p) \to \pi(q)$ of $\mathcal{X}$, the subdivisions $\mathcal{X}(f)\left(\pi_p\left(S^{-1}(p)\right)\right)$ and $\pi_q\left(S^{-1}(q)\right)$ of $\underline{\mathcal{X}(\pi(q))}^0$ coincide. Here, $\mathcal{X}(f)\left(\pi_p\left(S^{-1}(p)\right)\right)$ denotes the subdivision of $\mathcal{X}(\pi(q))^0$ arising from the subdivision $\pi_p\left(S^{-1}(p)\right)$ of $\mathcal{X}(\pi(p))^0$ through the bijective linear map $\mathcal{X}(f)$.

We state an essential lemma before illustrating some examples of the previous definition.

Lemma 3.3.4. *Suppose $\pi\colon \Phi \to \mathcal{X}$ is a poic-fibration. If $S : \Phi' \to \Phi$ is a subdivision, then there exists a subdivision $S' : \Phi'' \to \Phi'$ such that $S \circ S' : \Phi'' \to \Phi$ is π-compatible. In other words, π-compatible subdivison are final as subdivisions of Φ.*

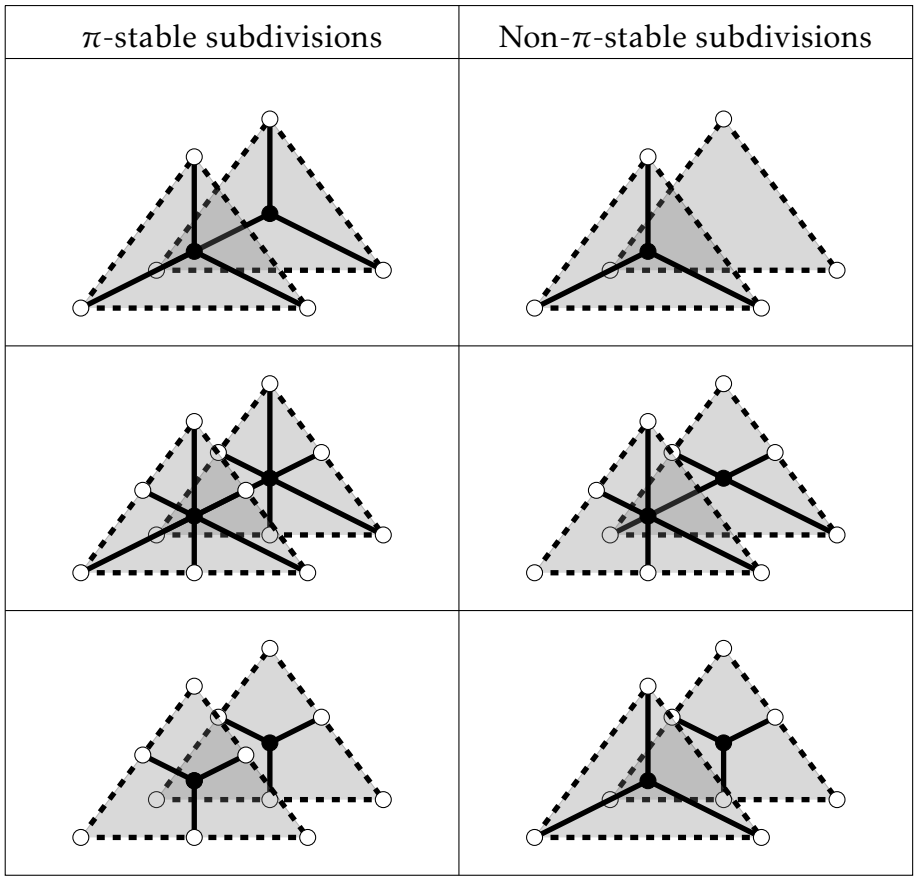
Proof. The proof of this lemma is technical, and, quite frankly, somewhat perpendicular to the sentiment of the section (and the whole chapter). We postpone it for later in Section 6.4. □

Example 3.3.5. Following example 3.2.3, we depict in Figure 3.3.5 several π-stable subdivisions and non-π-stable subdivisions. The new ray introduced in each cone, depicted by the black dot, is the ray given by $x = y = z > 0$ at the corresponding cone.

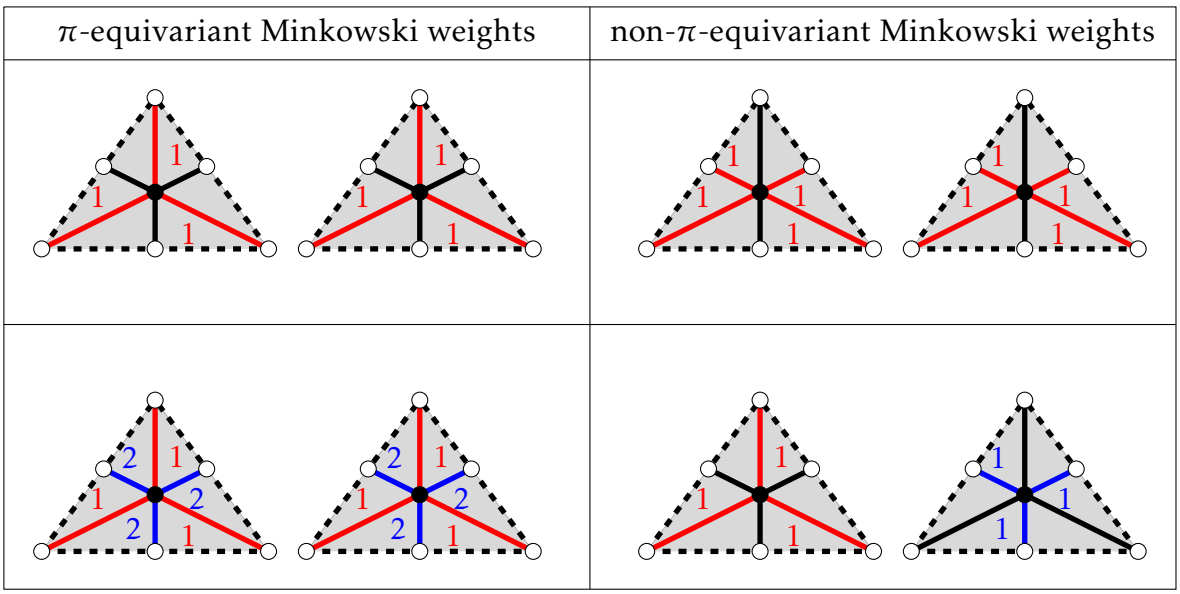
Notation 3.3.6. In what follows, we will let $\pi : \Phi_X \to \mathcal{X}$ denote a linear poic-fibration and $S : \Phi' \to \Phi$ a π-compatible subdivision.

Construction 3.3.7. If p and q are cones of Φ, then an isomorphism $f\colon \pi(p) \to \pi(q)$ of $\mathcal{X}$ induces a bijective map $[S^{-1}(p)] \to [S^{-1}(q)]$ that preserves the underlying dimensions. Here we are using the square brackets to emphasize that are regarding the sets $[S^{-1}(p)]$ and $[S^{-1}(q)]$ as subsets of $[\Phi']$. This poses no issue and can be done, because S is a subdivision map and Φ' is a poic-complex. Therefore, by trivially extending the previous bijection, we obtain a permutation of $[\Phi']$ which we denote by b_f that preserves the dimension of the isomorphism classes of the cones.

Definition 3.3.8. Let p and q be cones of Φ with $f\colon \pi(p) \to \pi(q)$ an isomorphism of $\mathcal{X}$. A weight $\omega \in M_k(\Phi'_{X\circ S})$, where $k \geq 0$ is an integer, is called *f-stable* if $\omega = \omega \circ b_f$. The weight ω is called a *π-equivariant Minkowski weight* if it is f-stable for every triple (p, q, f) as above. We denote the set of π-equivariant Minkowski weights of $S\colon \Phi' \to \Phi$ by $M_k(\mathcal{X}_{\pi, X\circ S})$.



Example 3.3.9. We use the poic-complex Φ of Example 3.2.3 and a stable subdivision of it from Example 3.3.5 to depict a couple of 2-dimensional π-equivariant Minkowski weights and non-π-equivariant Minkowski weights. We do these depictions in Figure 3.3.9, here we separate the disjoint cones of Φ to make the weight assignment explicit.



Lemma 3.3.10. *Following Notation 3.3.6, being a π-equivariant Minkowski weight of $S : \Phi' \to \Phi$ is equivalent to being a Minkowski weight that is invariant under a finite num-*

ber of permutations of $[\Phi'](k)$. *In particular, the set* $M_k(\mathcal{X}_{\pi,X\circ S})$ *is an abelian subgroup of* $M_k(\Phi'_{X\circ S})$.

Proof. We observe that both $M_k(\mathcal{X}_{\pi,X\circ S})$ and $M_k(\Phi'_{X\circ S})$ are subsets of $W_k(\Phi')$, and the permutations of $[\Phi'](k)$ act on this latter group. Therefore, the second statement follows directly from the first (intersection of subgroups is a subgroup). To show the first statement we proceed with the following observation: If s and t are objects of $\mathcal{X}$, then the set of isomorphisms $s \to t$ is a torsor (namely, the action is free and transitive) under the automorphisms of t. The poic-fibration $\pi\colon \Phi \to \mathcal{X}$ induces a surjective map (that preserves the dimensions of the cones)

$$\pi_*\colon [\Phi] \to [\mathcal{X}],$$

so the fibers of π_* partition $[\Phi]$. For each $[x] \in [\mathcal{X}]$ let $s_{[x]}$ denote a cone of Φ with $\pi(s_{[x]})$ representing[2] the class $[x] \in [\mathcal{X}]$. Consider a full set of representatives[3] R of $[\Phi]$ that includes the previously chosen $s_{[x]}$. For each $r \in R$ fix an isomorphism $f_r\colon \pi(r) \to \pi(s_{[\pi(r)]})$ of $\mathcal{X}$. Then a weight $\omega \in M_k(\Phi'_{X\circ S})$ is a π-equivariant Minkowski weight if and only if

- ω is f_r-stable, for every $r \in R$,
- for every $[x] \in [\mathcal{X}]$ the weight ω is f-stable, for every $f \in \mathrm{Aut}_{\mathcal{X}}(\pi(s_{[x]}))$.

Thus the first statement follows. □

Lemma 3.3.11. *Let* $S'\colon \Phi'' \to \Phi'$ *be a subdivision such that the composition* $\Phi'' \to \Phi$ *is* π*-compatible. The map* $(S')^*\colon M_k(\Phi'_{X\circ S}) \hookrightarrow M_k(\Phi''_{X\circ S\circ S'})$ *maps* π*-equivariant Minkowski weights to* π*-equivariant Minkowski weights.*

Proof. Consider a π-equivariant Minkowski weight $\omega \in M_k(\mathcal{X}_{\pi,X\circ S})$. We want to show that $(S')^*\omega \in M_k(\mathcal{X}_{\pi,X\circ S'\circ S})$. Let p and q be objects of Φ with $f : \pi(p) \to \pi(q)$ an isomorphism of $\mathcal{X}$. Observe first that $(S'\circ S)^{-1}(p)$ is a subdivision of $S^{-1}(p)$ (resp. $(S'\circ S)^{-1}(q)$ is a subdivision of $S^{-1}(q)$), and furthermore we have the following commutative diagram

$$\begin{array}{ccc} \pi_p((S'\circ S)^{-1}(p)) & \xrightarrow{\mathcal{X}(f)} & \pi_q((S'\circ S)^{-1}(q)) \\ \downarrow{\scriptstyle S'} & & \downarrow{\scriptstyle S'} \\ \pi_p(S^{-1}(p)) & \xrightarrow[\mathcal{X}(f)]{} & \pi_q(S^{-1}(q)) \end{array}.$$

[2]This is a strong instance of the axiom of choice, often called the axiom of global choice. We fully endorse it, as we believe that we can choose out of every non-trivial isomorphism class a representing object. For another instance where we use this, and some more words on this matters, see (apologies for the huge forward referencing) the footnote at the proof of Lemma 6.1.4.

[3]Same as the previous footnote.

From this it follows that $(S')^*(\omega \circ b_f) = ((S')^*\omega) \circ b_f$, and hence $(S')^*\omega$ is also π-compatible. □

Construction 3.3.12. Let π-Subd(Φ) denote the full subcategory of Subd(Φ) given by π-compatible subdivisions of Φ. Then the π-equivariant Minkowski weights give rise to the functor

$$M_k(\mathcal{X}_{\pi,X\circ\bullet})\colon \pi\text{-Subd}(\Phi)^{\mathrm{op}} \to \mathrm{Ab}.$$

Just as in the case of tropical cycles, the colimit of this functor is representable, and thus the following definition.

Definition 3.3.13. The *group of π-equivariant tropical k-cycles of* $\pi\colon \Phi_X \to \mathcal{X}$ is defined as

$$Z_k(\mathcal{X}_{\pi,X})\colon = \varinjlim M_k(\mathcal{X}_{\pi,X\circ\bullet}).$$

Lemma 3.3.14. *Suppose $\pi\colon \Phi_X \to \mathcal{X}$ is a linear poic-fibration. The natural inclusions $M_k(\mathcal{X}_{\pi,X\circ S}) \subset M_k(\Phi'_{X\circ S})$, where $S : \Phi' \to \Phi$ is a π-compatible subdivision, give rise to an injective linear map*

$$\pi_k^*\colon Z_k(\mathcal{X}_{\pi,X}) \to Z_k(\Phi_X).$$

Proof. It follows from Lemma 3.3.4 that any subdivision of Φ can be refined into a π-compatible subdivision and that, subsequently, $Z_k(\Phi_X)$ can be computed as the direct limit over the π-compatible subdivisions. Hence Lemma 3.3.11 shows that there is a map

$$\pi^*\colon Z_k(\mathcal{X}_{\pi,X}) \to Z_k(\Phi_X), \tag{3.3}$$

and since the maps

$$M_k(\mathcal{X}_{\pi,X\circ S}) \hookrightarrow M_k(\Phi'_{X\circ S})$$

are always injective, it follows that (3.3) is also injective. □

We can extend the notion of irreducibility from linear poic-complexes naturally to linear poic-fibrations:

Definition 3.3.15. A linear poic-fibration $\pi\colon \Phi_X \to \mathcal{X}$, with Φ pure of dimension n, is said to be *irreducible* if $Z_n(\mathcal{X}_{\pi,X})$ is free of rank 1.

Remark. If Φ_X is pure of dimension n and irreducible, then a linear poic-fibration $\pi\colon \Phi_X \to \mathcal{X}$ is either irreducible or has trivial n-cycles.

3.4 Pushforwards for poic-fibrations

Similarly to section 2.5 we want to pushforward weights of linear poic-fibrations through morphisms. As expected, in order to pushforward cycles of a linear poic-fibration

through a morphism, we need to impose additional conditions on the morphism. Recall that a morphism of poic-fibrations is defined simply as a morphism of pairs yielding a commutative square. However, up to this point, we have not dealt with any of these morphisms. Hence, in the interest of readability and clarity, we first dedicate some space to fix and introduce notation.

Consider two poic-fibrations $\pi\colon \Phi \to \mathcal{X}$ and $\rho\colon \Psi \to \mathcal{Y}$. For simplicity, we say that a morphism of poic-fibrations $\mathfrak{f}\colon \pi \to \rho$ consists of:

- A morphism of poic-complexes $\mathfrak{f}^{\mathrm{c}}\colon \Phi \to \Psi$, where the superscript c stands for complexes (as in poic-complexes).
- A morphism of poic-spaces $\mathfrak{f}^{\mathrm{s}}\colon \mathcal{X} \to \mathcal{Y}$, where the superscript s stands for spaces (as in poic-spaces).
- These are subject to the condition that the following is a commutative diagram of poic-spaces:

$$\begin{array}{ccc} \Phi & \xrightarrow{\mathfrak{f}^{\mathrm{c}}} & \Psi \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\rho} \\ \mathcal{X} & \xrightarrow[\mathfrak{f}^{\mathrm{s}}]{} & \mathcal{Y}. \end{array}$$

If, in addition, $\pi\colon \Phi_X \to \mathcal{X}$ and $\rho\colon \Psi_Y \to \mathcal{Y}$ are linear poic-fibrations, then we require $\mathfrak{f}^{\mathrm{c}}$ to also be a morphism linear poic-complexes $\mathfrak{f}^{\mathrm{c}}\colon \Phi_X \to \Psi_Y$.

Definition 3.4.1. Suppose that $\pi\colon \Phi \to \mathcal{X}$ and $\rho\colon \Psi \to \mathcal{Y}$ are poic-fibrations. Let $k \geq 0$ be an integer. A morphism of poic-fibrations $\mathfrak{f}\colon \pi \to \rho$ is called *weakly proper in dimension k* (resp. weakly proper, resp. *proper*) if:

1. $\mathfrak{f}^{\mathrm{c}}$ is a weakly proper in dimension k (resp. weakly proper, resp. proper) morphism of poic-complexes.
2. $\mathfrak{f}^{\mathrm{s}}$ satisfies the following lifting property: if $f\colon s \to t$ is an isomorphism of $\mathcal{Y}$, then for every cone s' of $\mathcal{X}$ with $\mathfrak{f}^{\mathrm{s}}(s') = s$ there exists a unique isomorphism $f'\colon s' \to t'$ of $\mathcal{X}$ with $\mathfrak{f}^{\mathrm{s}}(f') = f$ (in particular $\mathfrak{f}^{\mathrm{s}}(t') = t$).

The following proposition and its proof are quite technical, while notation seems ornate and opulent. However, this result forms the basis for the definition of the pushforward of equivariant (Minkowski weights) tropical cycles of a linear poic-fibration through a (weakly) proper morphism. So in an effort to alleviate the aforementioned difficulties, we seek to provide bit of context and explanations of what goes on. Suppose we are given linear poic-fibrations $\pi\colon \Phi_X \to \mathcal{X}$ and $\rho\colon \Psi_Y \to \mathcal{Y}$ with $\mathfrak{f}\colon \pi \to \rho$ a morphism of linear poic-fibrations, and let $k \geq 0$ be an integer. We want to understand how to pushforward k-dimensional π-equivariant (Minkowski weights) tropical

cycles through $\mathfrak{f}$ to ρ-equivariant tropical k-cycles of ρ. By definition, a π-equivariant tropical k-cycle of π is just a π-equivariant Minkowski weight $\omega \in M_k(\mathcal{X}_{\pi, X\circ S})$, where $S\colon \Phi' \to \Phi$ is a π-compatible subdivision. From this subdivision we get the morphism of linear poic-complexes $\mathfrak{f}^c \circ S\colon \Phi' \to \Psi$. If $\mathfrak{f}^c \circ S$ is weakly proper in dimension k, then we can pushforward ω through this composition. But, as the reader may recall, this forces us to consider a $(S \circ \mathfrak{f}^c)$-fine subdivision of Ψ, say $T\colon \Psi' \to \Psi$. In this case, we obtain from (2.13) that $((S \circ \mathfrak{f}^c)_T)_*\omega \in M_k(\Psi'_{Y\circ T})$. Unfortunately, we are not done yet, since the subdivision $T\colon \Psi' \to \Psi$ is not necessarily compatible with the poic-fibration. This means that we cannot say (and it is not even defined!) that $((S \circ \mathfrak{f}^c)_T)_*\omega$ is ρ-equivariant. For this reason, we have to further consider a subdivision $T'\colon \Psi'' \to \Psi'$ such that $\Psi'' \to \Psi$ is ρ-compatible, and show that $(T')^*((S \circ \mathfrak{f}^c)_T)_*\omega$ is ρ-equivariant.

Proposition 3.4.2. *Suppose $k \geq 0$ is an integer, $\pi\colon \Phi_X \to \mathcal{X}$ and $\rho\colon \Psi_Y \to \mathcal{Y}$ are linear poic-fibrations, $\mathfrak{f}\colon \pi \to \rho$ is a morphism of poic-fibrations, $S\colon \Phi' \to \Phi$ is a π-stable subdivision, and $T\colon \Psi' \to \Psi$ is a $(S \circ \mathfrak{f}^c)$-fine subdivision. Suppose that $\mathfrak{f}^c \circ S$ is weakly proper in dimension k, $\omega \in M_k(\mathcal{X}_{\pi, X\circ S})$, and $T'\colon \Psi'' \to \Psi'$ is such that $T' \circ T$ is ρ-stable. Then $(T')^*((S \circ \mathfrak{f}^c)_T)_*\omega$ is ρ-equivariant.*

Remark. Although only a special case is applied later in the text, we present this Proposition in its full generality for the sake of completeness. Briefly wallowing in forward referencing, we make explicit the results where (and the way in which) it will be applied in the future. This proposition will be used in Theorem 4.3.6, and the special case is when S is the identity map and $\mathfrak{f}$ is weakly proper in top dimension.

Proof. As we have mentioned previously, it is clear that $((S \circ \mathfrak{f}^c)_T)_*\omega \in M_k(\Psi'_{Y\circ T})$, and therefore $(T')^*((S \circ \mathfrak{f}^c)_T)_*\omega \in M_k(\Psi'_{Y\circ T\circ T'})$. By definition of the map $(T')^*$ we can assume for the sake of simplicity, and without loss of generality, that the subdivision $T\colon \Psi' \to \Psi$ is itself ρ-compatible (recall that with a subdivision map, the new equations to check either correspond to previous ones or are just constant weights for opposite direction vectors). With this simplification, we just need to check that $((S \circ \mathfrak{f}^c)_T)_*\omega$ is ρ-equivariant.
For this, suppose p and q are objects of Ψ and $g\colon \rho(p) \to \rho(q)$ is an isomorphism of $\mathcal{Y}$, we must check then that $((S \circ \mathfrak{f}^c)_T)_*\omega \circ b_g = ((S \circ \mathfrak{f}^c)_T)_*\omega$. As before, we regard $[T^{-1}(p)]$ and $[T^{-1}(q)]$ as subsets of $[\Psi']$, and let p' and q' be k-dimensional cones of Ψ' such that $[p'] \in [T^{-1}(p)]$ and $[q'] \in [T^{-1}(q)]$ with $b_g([p']) = [q']$. Consider the following subsets of $[\Phi](k)$:

$$(S \circ \mathfrak{f}^c)^{-1}([p']) := \{[s] : ((S \circ \mathfrak{f}^c)_T)_*[s] = [p']\},$$
$$(S \circ \mathfrak{f}^c)^{-1}([q']) := \{[s] : ((S \circ \mathfrak{f}^c)_T)_*[s] = [q']\}.$$

Both of these sets are partitioned by the fibers of $(\mathfrak{f}^s)^{-1}([\rho(T(p'))]) = (\mathfrak{f}^s)^{-1}([\rho(T(p'))])$.

More precisely, let y be an object of $\mathcal{Y}$ with $[y] = [\rho(p)] = [\rho(q)]$, then following the notation,

$$[y] = [\rho(T(p'))] = [\rho(T(q'))],$$

and we obtain the partitions:

$$(S \circ \mathfrak{f}^c)^{-1}([p']) = \bigsqcup_{[x] \in (\mathfrak{f}^s)^{-1}([y])} \{[s] : ((S \circ \mathfrak{f}^c)_T)_*[s] = [p'] \text{ and } \pi([s]) = [x]\}, \tag{3.4}$$

$$(S \circ \mathfrak{f}^c)^{-1}([q']) = \bigsqcup_{[x] \in (\mathfrak{f}^s)^{-1}([y])} \{[s] : ((S \circ \mathfrak{f}^c)_T)_*[s] = [q'] \text{ and } \pi([s]) = [x]\}. \tag{3.5}$$

The condition on lifting isomorphisms of $\mathfrak{f}^s$ implies that b_f gives rise to a bijection

$$\widehat{b_f} \colon (S \circ \mathfrak{f}^c)^{-1}([p']) \to (S \circ \mathfrak{f}^c)^{-1}([q']),$$

such that:

- $\widehat{b_f}$ preserves the partitions induced by $(\mathfrak{f}^s)^{-1}([y])$.
- If $[s] \in (S \circ \mathfrak{f}^c)^{-1}([p'])$, then:
  - We have equality between the weights $\omega([s]) = \omega(\widehat{b_f}([s]))$, because ω is a π-equivariant Minkowski weight.
  - We have equality between the indices $\left[T_{p'}N^{p'} : \mathfrak{f}^c_s N^s\right] = \left[T_{q'}N^{q'} : \mathfrak{f}^c_{\widehat{b_f}(s)} N^{\widehat{b_f}(s)}\right]$, because these are related by integral linear isomorphisms.

It follows then that $((S \circ \mathfrak{f}^c)_T)_*\omega \in M_k(\mathcal{Y}_{\rho, Y \circ T})$, and hence the proposition. □

We close this section with the construction of pushforwards of equivariant tropical cycles.

Construction 3.4.3. Suppose that $\pi \colon \Phi_X \to \mathcal{X}$ and $\rho \colon \Psi_Y \to \mathcal{Y}$ are linear poic-fibrations, and $\mathfrak{f} \colon \pi \to \rho$ is a proper morphism of linear poic-fibrations. Observe that if $S \colon \Phi' \to \Phi$ is a π-compatible subdivision, then Lemma 2.5.9 shows that there is a $(S \circ \mathfrak{f}^c)$-fine subdivision, and Lemma 3.3.4 shows that there is a subdivision $T' \colon \Psi'' \to \Psi'$ such that $T \circ T'$ is ρ-compatible. Therefore, starting from a π-compatible subdivision, it is always possible to place ourselves in the situation of Proposition 3.4.2. Since the morphism is proper, it follows from this same proposition that $\mathfrak{f}$ induces a linear map

$$(T')^*((S \circ \mathfrak{f}^c)_T)_* \colon M_k(\mathcal{X}_{\pi, X \circ S}) \to M_k(\mathcal{Y}_{\pi, Y \circ T \circ T'}),$$

which together with the natural inclusion $M_k(\mathcal{Y}_{\rho, B \circ T \circ T'})$ give rise to a linear map

$$(\mathfrak{f}_S)_* \colon M_k(\mathcal{X}_{\pi, X \circ S}) \to Z_k(\mathcal{Y}_{\rho, Y}).$$

Just as in the case of linear poic-complexes, taking subsequent subdivisions will give rise to commutative diagrams, so that these maps $(\mathfrak{f}_S)_*$ actually give rise to a linear map

$$\mathfrak{f}_*\colon Z_k(\mathcal{X}_{\pi,X}) \to Z_k(\mathcal{Y}_{\rho,Y}). \tag{3.6}$$

We call (3.6) the *pushforward map of* $\mathfrak{f}$. We additionally remark that this whole situation also gives rise to the following commutative diagram

$$\begin{array}{ccc} Z_k(\mathcal{X}_{\pi,X}) & \xrightarrow{\mathfrak{f}_*} & Z_k(\mathcal{Y}_{\rho,Y}) \\ \big\downarrow & & \big\downarrow \\ Z_k(\Phi_X) & \xrightarrow[\mathfrak{f}^c_*]{} & Z_k(\Psi_Y). \end{array}$$

Separately, if $\mathfrak{f}$ is just weakly proper in dimension k, then from Proposition 3.4.2 we obtain a linear map

$$(\mathfrak{f})_*\colon M_k(\mathcal{X}_{\pi,X}) \to Z_k(\mathcal{Y}_{\rho,Y}), \tag{3.7}$$

which we call the *weak pushforward in dimension k of* $\mathfrak{f}$.

This section is closed with a very simple example building on the previous Examples 3.2.4 and 2.5.11.

Example 3.4.4. Let $\mathcal{X}$, Φ, and π denote correspondingly the poic-space, linear poic-complex, and linear poic-fibration of Example 3.2.4. We observe that the subdivision of $\underline{\mathbb{R}^3_{>0}}$ depicted in Figure 2.4 is π-stable. Now, similarly to Example 3.2.4 let Ψ denote the linear poic-complex given by $\underline{\mathbb{R}^2_{>0}}$ and let $\mathcal{Y}$ denote the poic-space given by the single poic $\mathbb{R}^2_{>0}$ but with the following group of automorphisms

$$\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\}.$$

As the poic-fibration $\rho\colon \Psi \to \mathcal{Y}$ we consider just the identity morphism between the corresponding underlying poics. Furthermore, the subdivision of Ψ previously depicted in Figure 2.5 is also ρ-stable. We also observe that the projection $\mathrm{pr}\colon \Phi \to \Psi$ is a morphism of linear poic-complexes, and this same underlying linear map defines a morphism of poic-spaces $\mathcal{X} \to \mathcal{Y}$. Altogether these morphisms define a morphism of linear poic-fibrations $\mathfrak{pr}\colon \pi \to \rho$. The 2-dimensional tropical cycle of Φ that we have described in Example 2.5.11 is a π-equivariant tropical cycle, and its pushforward is a ρ-equivariant tropical cycle. It is perhaps worth noting that an analogous procedure replacing Φ, $\mathcal{X}$ and π with the ones from Example 3.2.3 does not make sense, since then the projection $\mathbb{R}^3 \to \mathbb{R}^2$ that we have considered will not induce a morphism between the corresponding poic-spaces. Nonetheless, the cycle of Φ from Example 2.5.11

is still equivariant with respect to the poic-fibration in this case and its pushforward is ρ-equivariant.

3.5 The spanning tree fibration

In this section, we will describe a poic-fibration over the poic-space $\mathrm{M}^{\mathrm{trop}}_{g,A}$ (here A is a finite set and $g \geq 0$ with $2g+\#A-2 > 0$). For this we let $\mathbb{g}$ denote the set $\{1,\dots,g,1^*,\dots,g^*\}$, where i^* is just a symbol for $1 \leq i \leq g$. The underlying combinatorial idea for this poic-fibration is to build a genus-g A-marked discrete graph out of a $(A \sqcup \mathbb{g})$-marked tree by joining the i- and i^*-legs into a new edge. This idea will actually produce a poic-fibration, which will denote by $\mathrm{st}_{g,A}$, where the st stands for *spanning tree*. We carefully carry out the constructions.

We will proceed in several steps. First, we will define the functor underlying the morphism of the poic-fibration. Then we will specify the linear poic-complex for the domain of the morphism, the underlying category is already known, but the (linear) poic-complex has to be defined. Afterward, we will upgrade the previously constructed functor to a morphism of poic-spaces, which then will be shown to be a poic-fibration. We do remark that we have previously depicted a case of the constructions in this section in Example 3.2.2.

Construction 3.5.1. Consider an object T of $\mathbb{G}_{0,A\sqcup\mathbb{g}}$. The graph T has its own involution map ι_T, which among its fixed points contains the legs $L(T)$ of T (here for the sake of simplicity, we regard $L(T) \subset F(T)$). Consider the following map

$$\iota_{\mathrm{st}_{g,A}(T)} \colon F(T) \to F(T), h \mapsto \begin{cases} \iota_T(h), & h \notin L(T), \\ \ell_i, & h = \ell_i\, (i \in A), \\ \ell_{i^*}, & h = \ell_i\, (1 \leq i \leq g), \\ \ell_i, & h = \ell_{i^*}\, (1 \leq i \leq g). \end{cases}$$

This is an involution of $F(T)$, which together with the root map r_T and the marking on the A-legs gives rise to a connected A-marked graph $\mathrm{st}_{g,A}(T)$ of genus g.

Notation 3.5.2. In case $A = \{1,\dots,n\}$, instead of looking at $\mathbb{g}$, we look at the set $\{n+1,\dots,n+2g\}$. So that $A \sqcup \mathbb{g} = \{1,\dots,n+2g\}$, and the functor in this case is given by joining the $(n+i)$-th leg with the $(n+i+g)$-th leg for $1 \leq i \leq g$.

Proposition 3.5.3. *The above construction gives rise to a functor*

$$\mathrm{st}_{g,A} \colon \mathbb{G}_{0,A\sqcup\mathbb{g}} \to \mathbb{G}_{g,A},$$

which is essentially surjective.

Proof. Since contraction of a tree edge becomes contraction of an edge of the graph, the functoriality readily follows. Essential surjectivity can be shown as follows. Let G be any connected genus-g A-marked graph, and let T denote a spanning tree of G. For $1 \leq i \leq g$ let $e_1, \dots, e_g$ denote edges of G that do not lie in T, and let $\partial e_i = \{A_i, B_i\}$. Let T_G denote the tree obtained from T by grafting at A_i a marked leg ℓ_i, and at B_i a marked leg ℓ_{i^*}. Then tree T_G is connected, has $(A \sqcup \mathbb{g})$-marked legs, and by construction $\mathrm{st}_{g,A}(T_G) \cong G$. □

We now continue with an example of this functor, and then proceed with the construction of the (linear) poic-complex structure on $\mathbb{G}_{0,A\sqcup\mathbb{g}}$. In simple terms, this linear poic-complex structure simply is the product of $\mathrm{M}^{\mathrm{trop}}_{0,A\sqcup\mathbb{g}}$ with the constant functor $\mathbb{R}^g_{>0}$. We begin with a construction, and not a definition, because we take the product with the constant functor and not with the linear poic-complex $\underline{\mathbb{R}^g_{>0}}$ (so it technically is not a product). The difference is quite mild, as the underlying category of the latter linear poic-complex is just a point, but nonetheless relevant (it may also convey an inappropriate picture). Before continuing with an example, we also refer the reader to Figure 3.1 (ignoring the metric if necessary), for a depiction of this functor in the case of $g = 1$ and $n = 2$.

Example 3.5.4. In the case of $g = 2$ and $n = 0$ the functor $\mathrm{st}_{2,0}$ can be represented vertically between isomorphism classes, excluding the morphisms, as depicted in Figure 3.5.

Construction 3.5.5. Let T be any graph of $\mathbb{G}_{0,A\sqcup\mathbb{g}}$. We denote by ST_T the poic $\sigma_T \times \mathbb{R}^g_{>0}$, and observe that this defines a functor

$$\mathrm{ST}_{g,A} \colon \mathbb{G}^{\mathrm{op}}_{0,A\sqcup\mathbb{g}} \to \mathrm{POIC}, \tag{3.8}$$

which is, in particular, a poic-complex. In addition, the first projection and the $\mathrm{dist}_{A\sqcup\mathbb{g}}$ morphism give rise to a morphism of poic-complexes

$$D_{g,A} : \mathrm{ST}_{g,A} \to \underline{\left(N_{\mathrm{dist}_{A\sqcup\mathbb{g}}}\right)_{\mathbb{R}}}. \tag{3.9}$$

Remark. Following the previous notation, since colimits commute with finite products, it is clear that $\mathrm{colim}\left|\mathrm{ST}_{g,A}\right| \cong \mathcal{M}^{\mathrm{trop}}_{0,A\sqcup\mathbb{g}} \times \mathbb{R}^g_{>0}$.

Definition 3.5.6. The morphism of poic-complexes (3.9) makes $\mathrm{ST}_{g,A}$ into a linear poic-complex. We call this the *linear poic-complex of spanning trees of genus-g A-marked tropical curves*. In the spirit of simplicity and since we will always consider $\mathrm{ST}_{g,A}$ as a linear poic-complex in this way, we will just denote this linear poic-complex by $\mathrm{ST}_{g,A}$.

We now seek to upgrade our previously constructed functor into a morphism of poic-spaces. For this, we need to introduce and define the corresponding morphisms

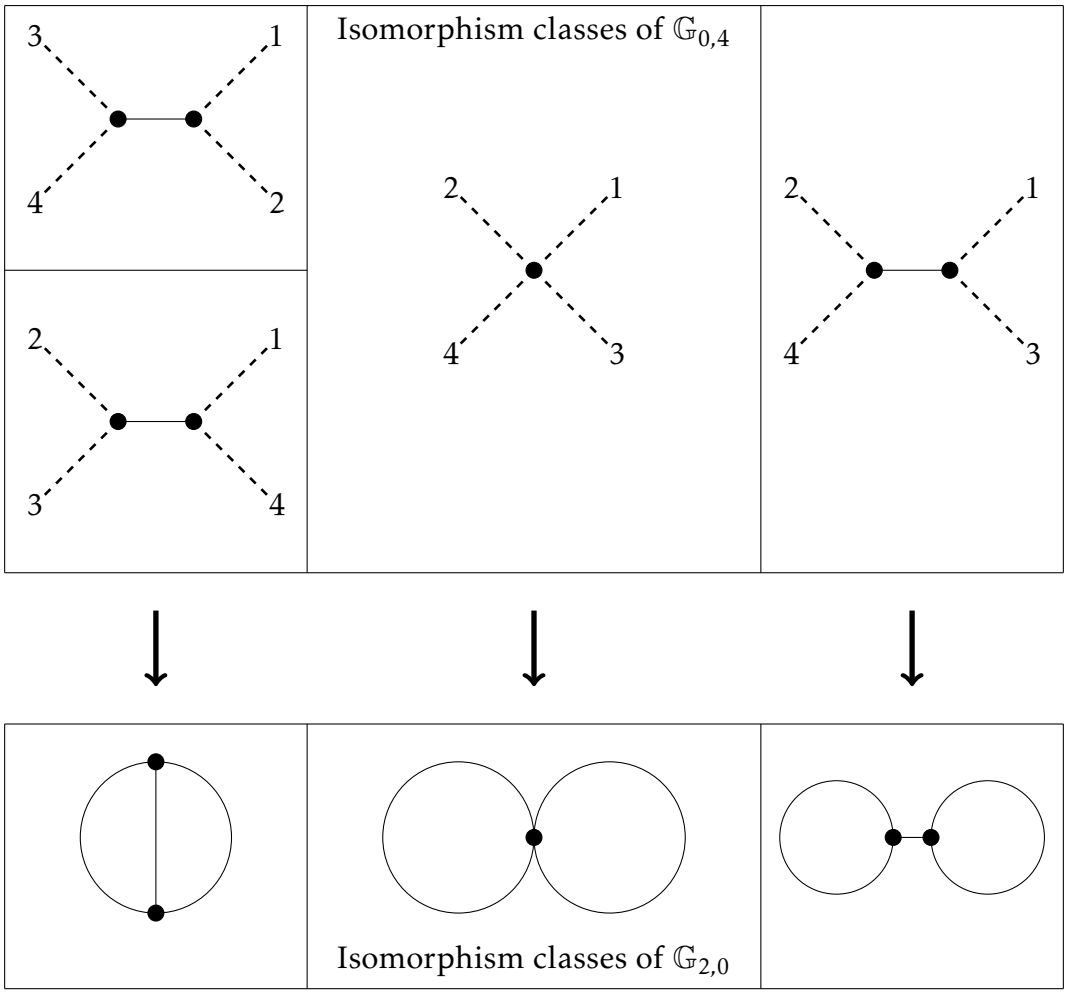


Figure 3.5: The functor $\mathrm{st}_{2,0}$ between the corresponding isomorphism classes

between the poics coming from the poic-complex and the poic-space. The underlying idea is that the $\mathbb{R}^g$-coordinates determine the lengths of the additional g edges.

Construction 3.5.7. Let T be any graph of $\mathbb{G}_{0,A\sqcup \mathrm{g}}$. For $1 \leq i \leq g$ let e_i denote the edge of $\mathrm{st}_{g,A}(T)$ given by the orbit $\{\ell_i, \ell_{i^*}\}$. The graph $\mathrm{st}_{g,A}(T)$ has the set of edges

$$E(\mathrm{st}_{g,A}(T)) = E(T) \cup \{e_1, \ldots, e_g\},$$

and hence the cone σ_T can be identified with a face of $\sigma_{\mathrm{st}_{g,A}(T)}$. Furthermore, the poic $\mathbb{R}^g_{>0}$ corresponds to the relative interior of a face of $\sigma_{\mathrm{st}_{g,A}(T)}$ by identifying the ith-coordinate with the e_i-coordinate (corresponding to the edge e_i). Let

$$\eta_{\mathrm{st}_{g,A},T} \colon \mathrm{ST}_T \to \sigma_{\mathrm{st}_{g,A}(T)} \tag{3.10}$$

denote the poic-morphism given by the product of the previosuly described ones.

Proposition 3.5.8. *The poic-morphisms* (3.10) *give rise to a natural transformation*

$$\eta_{\mathrm{st}_{g,A}} \colon \mathrm{ST}_{g,A} \to \mathrm{M}^{\mathrm{trop}}_{g,A} \circ \mathrm{st}_{g,A},$$

which together with $\mathrm{st}_{g,A}$ *gives a linear poic-fibration* $\mathrm{st}_{g,A} \colon \mathrm{M}^{\mathrm{trop}}_{0,A\sqcup \mathrm{g}} \to \mathrm{M}^{\mathrm{trop}}_{g,A}$.

Proof. The essential surjectivity of the functor has already been established in Proposition 3.5.3. For a tree T of $\mathbb{G}_{0,A\sqcup \mathrm{g}}$, the map $\eta_{\mathrm{st}_{g,A},T}$ identifies the relative interiors of the poics ST_T and $\sigma_{\mathrm{st}_{g,A},T}$. Therefore, the third condition is also fulfilled. It remains to show the second condition of a poic-fibration. By definition of the structure of the morphisms of $\mathbb{G}_{g,A}$, it is sufficient to show the lifting condition in the following set-up: Let T be an object of $\mathbb{G}_{0,A\sqcup \mathrm{g}}$, G an object of $\mathbb{G}_{g,A}$, and consider a morphism $\mathrm{ST}_{g,A}(T) \to G$ given by: the contraction of an edge $e \in E(G)$ and a bijective map of graphs $f \colon \mathrm{st}_{g,A}(T) \to G/e$. Let $V \in V(T)$ denote the vertex of T such that $f(V) = V_e$. Consider the object T' of $\mathbb{G}_{0,A\sqcup \mathrm{g}}$ obtained by resolving V in the same way as $V_e \in V(G/e)$ to G and let e' denote the resolving edge. So that there is a natural bijective map of graphs $g \colon T \to T'/e'$, and the map f can be extended to a bijective map of graphs

$$f' \colon \mathrm{st}_{g,A}(T') \to G,$$

by additionally specifying $e' \to e$. Then $e' \in E(G)$ and the bijective map of graphs $g \colon T \to T'$ define a morphism $T \to T'/e'$ that satisfies the required condition. □

Definition 3.5.9. We call the linear poic-fibration of Proposition 3.5.8 the *spanning tree fibration of genus-g A-marked graphs*.

We close this section by showing some essential properties of thesepoic-fibrations.

Proposition 3.5.10. *The poic-complex* $\mathrm{ST}_{g,A}$ *is pure of dimension* $3g + \#A - 3$ *and the poic-fibration* $\mathrm{st}_{g,A}$ *is irreducible.*

Proof. Observe that $\#(A \sqcup \mathrm{g}) = 2g + \#A$, hence the linear poic-complex $\mathrm{M}^{\mathrm{trop}}_{0,A\sqcup \mathrm{g}}$ is pure of dimension $2g + \#A - 3$, and $\mathrm{ST}_{g,A}$ is pure of dimension $2g + \#A - 3 + g = 3g + \#A - 3$. Irreducibility of the poic-fibration follows from that of $\mathrm{M}^{\mathrm{trop}}_{0,A\sqcup \mathrm{g}}$, and the fact that the constant weight 1 is $\mathrm{st}_{g,A}$-equivariant. □

3.6 Forgetting the marking

In this section we construct several weakly proper morphisms concerning the spanning tree fibrations. To be more precise, suppose A is a a non-empty finite set, let $a \in A$, and let $g \geq 0$ be an integer with $2g + \#A - 2 > 1$. For the sake of notational simplicity, we will denote the set $A \backslash \{a\}$ just by $A \backslash a$. We will construct a proper morphism of

linear poic-fibrations $\mathrm{st}_{g,A} \to \mathrm{st}_{g,A\backslash a}$, given by forgetting the a-marked leg. We proceed in the simplest way, by first carrying out this construction meticulously for the poic-spaces $\mathrm{M}^{\mathrm{trop}}_{g,A}$ and $\mathrm{M}^{\mathrm{trop}}_{g,A\backslash a}$. Then we proceed with the linear poic-complexes $\mathrm{M}^{\mathrm{trop}}_{0,A}$ and $\mathrm{M}^{\mathrm{trop}}_{0,A\backslash a}$, and finally we fit all this together for the linear poic-fibrations $\mathrm{st}_{g,A}$ and $\mathrm{st}_{g,A\backslash a}$. In the case of rational curves, these forgetting the marking morphisms have readily been studied and we record an important well-known property that we will use in the following chapter.

Construction 3.6.1. Let G be an object of $\mathbb{G}_{g,A}$. We will construct an object $\mathrm{ft}_a G$ of $\mathbb{G}_{g,A\backslash a}$ given by forgetting the a-leg of G. Let $V_a \in V(G)$ denote the vertex incident to $\ell_a(G)$.

1. If $\mathrm{val}(V_a) > 3$, then set $F(\mathrm{ft}_a G) := F(G)\backslash \ell_a(G)$. The root, involution, and marking maps of G restrict to respective root, involution and marking maps of $F(\mathrm{ft}_a G)$, and hence define an $A\backslash a$-marked graph $\mathrm{ft}_a G$.

2. If $\mathrm{val}(V_a) = 3$ with V_a incident to two edges, then $r_G^{-1}(V_a) = \ell_a \cup \{f_1, f_2, V_a\}$. In this case, we set $F(\mathrm{ft}_a G) = F(G)\backslash r_G^{-1}(V_a)$ and $r_{\mathrm{ft}_a G} = r_G$. To define $\iota_{\mathrm{ft}_a G}$ it is only necessary to specify where to map $\iota_G(f_1)$ and $\iota_G(f_2)$, for which we simply put

$$\iota_{\mathrm{ft}_a G}(\iota_G(f_1)) = \iota_G(f_2), \qquad \iota_{\mathrm{ft}_a G}(\iota_G(f_2)) = \iota_G(f_1).$$

   With the above, the triple $(F(\mathrm{ft}_a G), r_{\mathrm{ft}_a G}, \iota_{\mathrm{ft}_a G})$ defines an $A\backslash a$-marked graph $\mathrm{ft}_a G$.

3. If $\mathrm{val}\, V_a = 3$ with V_a incident to an additional leg, then $r_G^{-1}(V_a) = \ell_a \cup \{l, f, V_a\}$ with $\{l\} \in L(G)$. In this case we set $F(\mathrm{ft}_a G) = F(G)\backslash (\ell_a \cup \{f, \iota_G(f), V_a\})$ and $\iota_{\mathrm{ft}_a G} = \iota_G$. To define $r_{\mathrm{ft}_a G}$ it is only necessary to specify where to map l, for which we simply set $r_{\mathrm{ft}_a G}(l) = r_G(\iota_G(f))$. With this, the triple $(F(\mathrm{ft}_a(G)), r_{\mathrm{ft}_a(G)}, \iota_{\mathrm{ft}_a G})$ defines an $A\backslash a$-marked graph $\mathrm{ft}_a G$.

Let H denote an additional object of $\mathbb{G}_{g,A}$, and let $g\colon G \to H$ denote a morphism thereof. We define a morphism $\mathrm{ft}_a g\colon \mathrm{ft}_a G \to \mathrm{ft}_a H$ as follows. Let $W_a \in V(H)$ denote the vertex $\partial \ell_a(H)$. Observe first that we must only study the behavior of g around $g^{-1}(W_a)$, and as such we can assume without loss of generality that the inverse image of any other vertex is a single vertex. Namely, any edge contraction goes to W_a and nowhere else. We proceed as above and separate into the same three cases:

1. Suppose that $\mathrm{val}\, V_a > 3$. Then $\mathrm{val}\, W_a \geq \mathrm{val}\, V_a > 3$, and in this case g directly induces $\mathrm{ft}_a g\colon \mathrm{ft}_a G \to \mathrm{ft}_a H$.

2. Suppose that $\mathrm{val}\, V_a = 3$ with V_a incident to two edges, so that, as before, $r_G^{-1}(V_a) = \ell_a \cup \{f_1, f_2, V_a\}$. If $r_G^{-1}(V_a) \subset g^{-1}(W_a)$, then g directly induces $\mathrm{ft}_a g\colon \mathrm{ft}_a G \to \mathrm{ft}_a H$. So suppose not:

- If $f_1 \notin g^{-1}(W_a)$ but $f_2 \in g^{-1}(W_a)$ (or equivalently $f_2 \notin g^{-1}(W_a)$ but $f_1 \in g^{-1}(W_a)$), then $\iota_G(f_1) \notin (W_a)$ and $\iota_G(f_2) \in g^{-1}(W_a)$ and we define $\mathrm{ft}_a\, g$ as follows. We only change g at the elements $\iota_G(f_1), \iota_G(f_2) \in F(\mathrm{ft}_a\, G)$ and set

$$\mathrm{ft}_a\, g(\iota_G(f_1)) = \iota_H(g(f_1)), \qquad \mathrm{ft}_a\, g(\iota_G(f_2)) = \iota_H(f_1).$$

- If both $f_1, f_2 \notin g^{-1}(W_a)$, then $\iota_G(f_1), \iota_G(f_2) \notin g^{-1}(W_a)$. We set

$$\mathrm{ft}_a\, g(\iota_G(f_i)) = \iota_H(g(f_i)), \qquad \text{for } i = 1, 2,$$

  and let $\mathrm{ft}_a\, g$ agree with g everywhere else.

3. Suppose that $\mathrm{val}\, V_a = 3$ with V_a incident to an additional leg. So that, as before, $r_G^{-1}(V_a) = \ell_a \cup \{l, f, V_a\}$, with $\{l\} \in L(G)$. In this case g directly induces $\mathrm{ft}_a\, g \colon \mathrm{ft}_a\, G \to \mathrm{ft}_a\, H$.

We now introduce a poic-morphism $\eta_{\mathrm{ft}_a, G} \colon \sigma_G \to \sigma_{\mathrm{ft}_a\, G}$. Observe that there are the following possibilities for $E(\mathrm{ft}_a\, G)$:

1. If $\mathrm{val}(V_a) > 3$, then $E(G) = E(\mathrm{ft}_a\, G)$. In this case we set $\eta_{\mathrm{ft}_a, G} = \mathrm{Id}_{\mathbb{R}_{\geq 0}^{E(G)}}$.

2. If $\mathrm{val}(V_a) = 3$ with V_a incident to two edges, then $r_G^{-1}(V_a) = \ell_a \cup \{f_1, f_2, V_a\}$, and $e = \{\iota_G(f_1), \iota_G(f_2)\}$ defines an edge of $E(\mathrm{ft}_a\, G)$. For $i = 1, 2$, let $e_i \in E(G)$ denote the edge given by $\{f_i, \iota_G(f_i)\}$. In this case

$$E(\mathrm{ft}_a\, G) = E(G) \backslash \{e_1, e_2\} \cup \{e\},$$

   and we let $\eta_{\mathrm{ft}_a, G} \colon \sigma_G \to \sigma_{\mathrm{ft}_a\, G}$ be the map given by

$$\eta_{\mathrm{ft}_a, G}(\delta) := \begin{cases} \delta(h) = \delta(h), & \text{if } h \in E(\mathrm{ft}_a\, G) \backslash \{e\}, \\ \delta(h) = \delta(e_1) + \delta(e_2), & \text{if } h = e. \end{cases}$$

3. If $\mathrm{val}\, V_a = 3$ with V_a incident to an additional leg, then $r_G^{-1}(V_a) = \ell_a \cup \{l, f, V_a\}$ with $\{l\} \in L(G)$. The set $\{f, \iota_G(f)\}$ defines an edge $e' \in E(G)$ and $E(\mathrm{ft}_a\, G) = E(G) \backslash e'$. In this case, we let $\eta_{\mathrm{ft}_a, G} \colon \sigma_G \to \sigma_{\mathrm{ft}_a\, G}$ denote the face-embedding given by $E(\mathrm{ft}_a\, G) \subset E(G)$.

Lemma 3.6.2. *The previous construction gives rise to a functor* $\mathrm{ft}_a \colon \mathbb{G}_{g,A}^{\mathrm{op}} \to \mathbb{G}_{g,A\backslash a}^{\mathrm{op}}$ *and a natural transformation* η_{ft_a}*, that define a morphism of poic-spaces*

$$\mathrm{ft}_a : \mathrm{M}_{g,A}^{\mathrm{trop}} \to \mathrm{M}_{g,A\backslash a}^{\mathrm{trop}}. \tag{3.11}$$

Furthermore, if $b \in A\backslash a$ is an additional element, then $\mathrm{ft}_a \circ \mathrm{ft}_b = \mathrm{ft}_b \circ \mathrm{ft}_a$. If $g = 0$, then ft_a is a proper morphism of poic-complexes. If $g > 0$, then ft_a satisfies the lifting property of Definition 3.4.1.

Proof. The functoriality of ft_a, and the commutativity $\mathrm{ft}_a \circ \mathrm{ft}_b = \mathrm{ft}_b \circ \mathrm{ft}_a$, follow from the construction, and the reader is entrusted with the corresponding details. We show that η_{ft_a} is a natural transformation. Let $g\colon G \to H$ be a morphism of $\mathbb{G}_{g,A}$ (that is $f \in \mathrm{Hom}_{\mathbb{G}_{g,A}^{\mathrm{op}}}(H, G)$), we want to show that the following diagram is commutative:

$$\begin{array}{ccc} \sigma_H & \xrightarrow{\sigma_{g,A}(g)} & \sigma_G \\ {\scriptstyle \eta_{\mathrm{ft}_a,H}}\downarrow & & \downarrow{\scriptstyle \eta_{\mathrm{ft}_a,G}} \\ \sigma_{\mathrm{ft}_a H} & \xrightarrow[\sigma_{g,A\backslash a}(\mathrm{ft}_a g)]{} & \sigma_{\mathrm{ft}_a G} \end{array} \tag{3.12}$$

Let $V_a \in V(G)$ be the incident vertex to ℓ_a. The only interesting case is when $\mathrm{val}\, V_a = 3$ and V_a is incident to two edges. As before, we have that $r_G^{-1}(V_a) = \ell_a \cup \{f_1, f_2, V_a\}$ and the set $e = \{\iota_G(f_1), \iota_G(f_2)\} \subset F(\mathrm{ft}_a G)$ is an edge. Any morphism of $\mathbb{G}_{g,A}$ factors as a composition of multiple single edge contractions and automorphisms, so it suffices to study these particular cases.

- Suppose g is an automorphism of G. Since it preserves the A-marking, it follows that $g(\{f_1, f_2\}) = \{f_1, f_2\}$. Commutativity with the involution ι_G implies that $\mathrm{ft}_a(g)$ must fix the new edge of $\mathrm{ft}_a G$, and therefore (3.12) is commutative.

- Suppose g is a single edge contraction $G \to G/h$, where $h \in E(G)$. If $V_a \notin \partial h$, then e is also an edge of $\mathrm{ft}_a G/h$. In this case, both maps $\sigma_{g,A}(g)$ and $\sigma_{g,A\backslash a}(\mathrm{ft}_a g)$ preserve this e-coordinate, and thus the commutativity of (3.12). If $V_a \in \partial h$, we can assume without loss of generality that $h = \{f_1, \iota_G(f_1)\}$. In this case, $e' = \{f_2, \iota_G(f_2)\}$ is an edge of both $\mathrm{ft}_a G/h$ and G. Additionally, $\mathrm{ft}_a g : \mathrm{ft}_a G \to \mathrm{ft}_a G/h$ is an isomorphism of $\mathbb{G}_{g,A\backslash a}$, where the edge e is mapped to the edge e'. The poic-morphism $\sigma_{g,A}(g)$ is the face-embedding where the h-coordinate is trivial, the map $\eta_{\mathrm{ft}_a,H}$ is the identity, and the map $\sigma_G \xrightarrow{\eta_{\mathrm{ft}_a,G}} \sigma_{\mathrm{ft}_a,G}$ is given as follows at a $\delta \in \sigma_G$:

$$\left(\eta_{\mathrm{ft}_a,G}(\delta)\right)_t := \begin{cases} \delta(t), & t \neq e', \\ \delta(e') + \delta(h) & t = e', \end{cases}$$

where $t \in E(\mathrm{ft}_a G/h)$ is an edge. From the latter, the commutativity of (3.12) follows.

Now, if $g = 0$, then we have linear maps between closed convex polyhedral cones, therefore $\mathrm{ft}_a\colon \mathrm{M}_{0,A}^{\mathrm{trop}} \to \mathrm{M}_{0,A\backslash a}^{\mathrm{trop}}$ is proper in the sense of Definition 2.5.4. For $g > 0$, the

lifting property of ft_a readily follows, since an isomorphism between the target will define an isomorphism from the source (flags of the target are a subset of those of the source). □

Notation 3.6.3. Following Lemma 3.6.2, for an arbitrary subset $I \subset A$ we let ft_I denote the composition of all the ft_i for $i \in I$.

We now proceed with the case of rational curves. This is a relevant step due to the role of rational curves in the underlying linear poic-complex structure of $\mathrm{ST}_{g,A}$. After the construction we state, and record, Lemma 2.3 of [GMO17] (and the reader is referred thereto for its proof).

Construction 3.6.4. Let us briefly consider an additional set C with an element $c \in C$ and $\#C \geq 4$. Let $p'_c\colon \mathbb{R}^C \to \mathbb{R}^{C\setminus c}$ and $p_c\colon \mathbb{R}^{\binom{C}{2}} \to \mathbb{R}^{\binom{C\setminus c}{2}}$ denote the natural projection maps respectively given by forgetting the c-coordinate and forgetting any coordinate containing c. These fit into the commutative diagram of integral linear maps

$$\begin{array}{ccc} \mathbb{R}^C & \xrightarrow{M_C} & \mathbb{R}^{\binom{C}{2}} \\ {\scriptstyle p'_c}\downarrow & & \downarrow{\scriptstyle p_c} \\ \mathbb{R}^{C\setminus c} & \xrightarrow[M_{C\setminus c}]{} & \mathbb{R}^{\binom{C\setminus c}{2}}. \end{array}$$

Therefore, p_c gives rise to a linear integral map $p_c\colon Q_C \to Q_{C\setminus c}$, and in particular to a morphism of poic-complexes $p_c\colon \underline{\left(N_{\mathrm{dist}_C}\right)_{\mathbb{R}}} \to \underline{\left(N_{\mathrm{dist}_{C\setminus c}}\right)_{\mathbb{R}}}$. Furthermore, we observe that $p_c \circ \mathrm{dist}_C = \mathrm{dist}_{C\setminus c} \circ \mathrm{ft}_c$, so that by letting $(\mathrm{ft}_c)_{\mathrm{int}} := p_c$, we obtain a morphism of linear poic-complexes

$$\mathrm{ft}_c\colon \mathrm{M}^{\mathrm{trop}}_{0,C} \to \mathrm{M}^{\mathrm{trop}}_{0,C\setminus c}. \tag{3.13}$$

We also remark that if $\#C > 4$ and $d \in C\setminus c$ is an additional element, then just as in the second statement of Lemma 3.6.2 we have that $p_d \circ p_c = p_c \circ p_d$. In particular, for $I \subset C$ we define p_I in the same way as ft_I. Namely, p_I is the composition of all the p_i for $i \in I$.

Lemma 3.6.5. *A vector $\vec{v} \in (N_{\mathrm{dist}_A})_{\mathbb{R}}$ is zero if and only if $p_{I^c}(\vec{v}) = 0$ for every $I \subset A$ with $\#I = 4$.*

Proof. As was mentioned above, this is Lemma 2.3 of [GMO17]. □

To close this section we extend the forgetting the marking morphism to the linear poic-complex in the source of the spanning tree fibration. In this case, the combinatorics work perfectly fine but the natural transformations between the cones require to be changed. In addition, we only obtain a weakly proper map.

Construction 3.6.6. Let T be an object of $\mathbb{G}_{0,\mathrm{g}\sqcup A}$ and consider the morphism of poics $\eta_{\mathfrak{ft}^{\mathrm{c}}_a,T}\colon \mathrm{ST}_{g,A}(T) \to \mathrm{ST}_{g,A\setminus a}(\mathrm{ft}_a T)$ given by the linear map

$$\begin{pmatrix} \eta_{\mathrm{ft}_a,T} & 0 \\ M(T,a) & \mathrm{Id}_{g\times g} \end{pmatrix}\colon \sigma_T \times \mathbb{R}^g_{>0} \to \sigma_{\mathrm{ft}_a T} \times \mathbb{R}^g_{>0},$$

where $M(T,a)$ is the $g \times \#E(T)$-matrix such that for $1 \le i \le g$ and $e \in E(T)$ its (i,e)-entry is

$$M(T,a)_{i,e} = \begin{cases} 1, & \text{if } \mathrm{val}\, r_T(\ell_a(T)) = 3 \text{ and } \partial\ell_a(T) = \partial\ell_i(\mathrm{ft}_a T) \subset \partial e, \\ 1, & \text{if } \mathrm{val}\, r_T(\ell_a(T)) = 3 \text{ and } \partial\ell_a(T) = \partial\ell_{i^*}(\mathrm{ft}_a T) \subset \partial e, \\ 0, & \text{else.} \end{cases}$$

In simple terms, this matrix records how the $\mathbb{R}^g_{>0}$-coordinates must be modified when forgetting a leg. Namely, if T is an object of $\mathbb{G}_{0,\mathrm{g}\sqcup A}$ and the leg $\ell_a(T)$ is incident to a 3-valent vertex that is also incident to one of the $\ell_i(T)$ or $\ell_{i^*}(T)$ legs, then the length of the other incident edge must be added to the corresponding $\mathbb{R}^g_{>0}$-coordinate of $\mathrm{ft}_a T$ in $\mathrm{ST}_{g,A\setminus a}(\mathrm{ft}_a T)$.

Proposition 3.6.7. *The poic-morphisms of Construction 3.6.6 define a morphism of poic-complexes* $\mathfrak{ft}^{\mathrm{c}}_a\colon \mathrm{ST}_{g,A} \to \mathrm{ST}_{g,A\setminus a}$. *Furthermore, letting* $(\mathfrak{ft}^{\mathrm{c}}_a)_{\mathrm{int}} := p_a$ *makes* $\mathfrak{ft}^{\mathrm{c}}_a$ *into a morphism of linear poic-complexes between the corresponding linear poic-complex structures. In addition, if* $\mathfrak{ft}^{\mathrm{s}}$ *denotes the morphism of poic-spaces* $\mathrm{ft}_a\colon \mathrm{M}^{\mathrm{trop}}_{g,A} \to \mathrm{M}^{\mathrm{trop}}_{g,A\setminus a}$, *then the tuple* $\mathfrak{ft}_a = (\mathfrak{ft}^{\mathrm{c}}_a, \mathfrak{ft}^{\mathrm{s}}_a)$ *defines a weakly proper morphism of linear poic-fibrations*

$$\mathfrak{ft}_a : \mathrm{st}_{g,A} \to \mathrm{st}_{g,A\setminus a}. \tag{3.14}$$

Proof. Let $g\colon T \to T'$ be a morphism of $\mathbb{G}_{0,\mathrm{g}\sqcup A}$ (that is $f\colon T' \to T$ in $\mathbb{G}^{\mathrm{op}}_{0,\mathrm{g}\sqcup A}$), we have to show that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{ST}_{T'} & \xrightarrow{\iota_f \times \mathrm{Id}} & \mathrm{ST}_T \\ {\scriptstyle \eta_{\mathfrak{ft}^{\mathrm{c}}_a}}\downarrow & & \downarrow{\scriptstyle \eta_{\mathfrak{ft}^{\mathrm{c}}_a}} \\ \mathrm{ST}_{\mathrm{ft}_a T'} & \xrightarrow[\iota_{\mathrm{ft}_a f}\times \mathrm{Id}]{} & \mathrm{ST}_{\mathrm{ft}_a T} \end{array}.$$

This is analogous to the proof of Lemma 3.6.2 and the commutativity of diagram (3.12), so we omit it. A weakly proper morphism of linear poic-fibrations must satisfy three properties, on which we will now elaborate. We begin with the first of these. This follows directly from the fact that an edge contraction after forgetting the marking actually comes from an edge contraction of the original graph (perhaps best understood through illustration), so that facets after forgetting a marked leg arise from facets before forgetting the marked leg. The second of these properties has already been shown

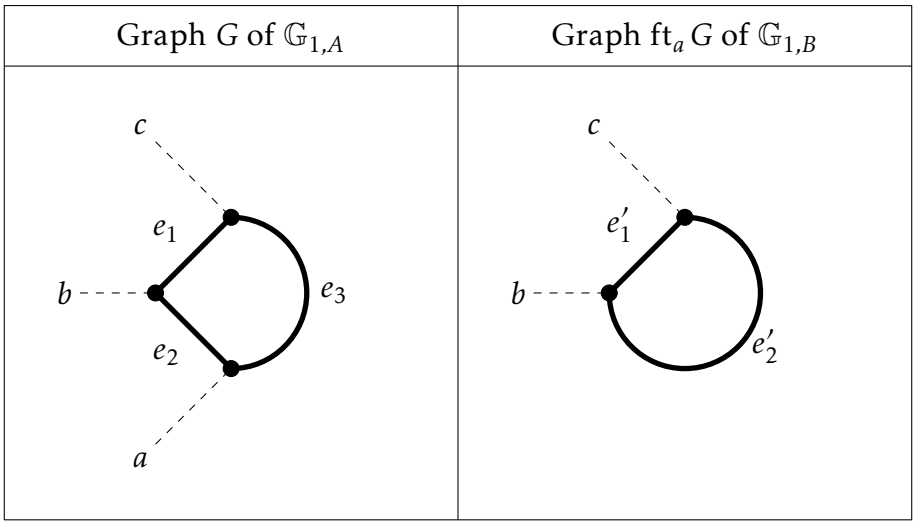


Figure 3.6: Graphs of Example 3.6.8

in Lemma 3.6.2, and the third is just the equality $\mathfrak{ft}^{\mathrm{s}} \circ \mathrm{st}_{g,A} = \mathrm{st}_{g,A\setminus a} \circ \mathfrak{ft}^{\mathrm{c}}$. This can be readily seen from our constructions, and worked (as everything throughout this section) on a case-by-case basis depending on the incidence of the a-leg. □

Remark. In the case of trivial genus, we have already explained that we actually obtain a proper morphism of linear poic-complexes. However, for positive genus the partial openness in this situation moves us to weak properness. We illustrate in Example 3.6.8 a case where the morphism fails to be proper.

Example 3.6.8. Suppose $A = \{a, b, c\}$, $B = \{b, c\}$, and let $g = 1$. We consider the morphism $\mathrm{ft}_a \colon \mathrm{st}_{1,A} \to \mathrm{st}_{1,B}$. Consider the graphs G of $\mathbb{G}_{g,A}$ and $\mathrm{ft}_a G$ of $\mathbb{G}_{g,B}$ depicted in Figure 3.6, and consider the trees T and $\mathrm{ft}_a T$, lying in $\mathrm{st}^{-1}_{g,A}(G)$ and $\mathrm{st}^{-1}_{g,B}(\mathrm{ft}_a G)$ correspondingly, depicted in Figure 3.7. Then the morphism of poics $\mathrm{ST}_{1,A}(T) \to \mathrm{ST}_{1,B}(\mathrm{ft}_a T)$ is given explicitly as the morphism of poics:

$$F \colon \mathbb{R}^2_{\geq 0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}, (x, y, \ell) \mapsto (x, y + \ell).$$

Then if we consider the subcone σ of $\mathbb{R}^2_{\geq 0} \times \mathbb{R}_{>0}$ given by

$$\sigma := \{(x, y, \ell) \in \mathbb{R}^2_{\geq 0} \times \mathbb{R}_{>0} : x = \ell\},$$

then $F(\sigma)$ is the subcone of $\mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ given by

$$F(\sigma) = \{(s, t) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} : t - s \geq 0, s > 0\}.$$

However, the closure of $F(\sigma)$ relative to $\mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ contains the facet given by $s = 0$, which does not arise as a face of σ.

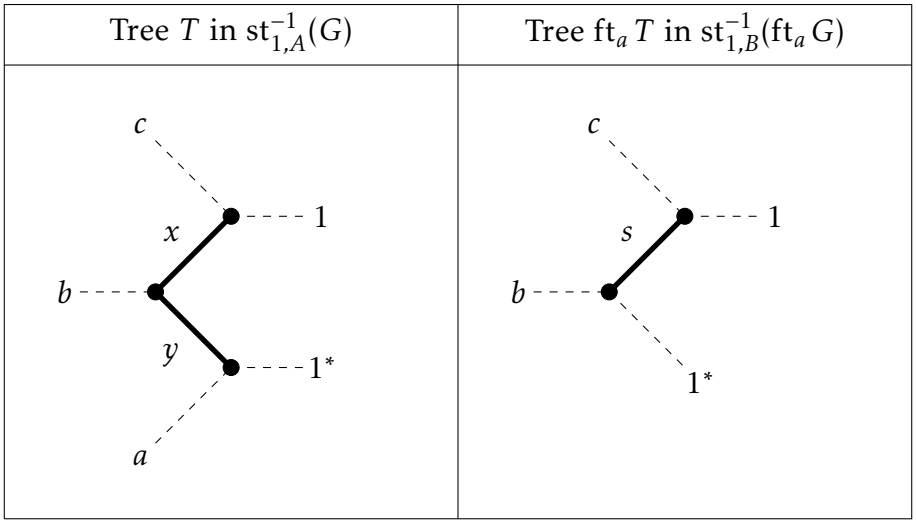


Figure 3.7: Trees of Example 3.6.8

3.7 The extended spanning tree fibrations

We now describe several linear poic-fibrations, with the aim of describing tropical cycles in the moduli space of discrete admissible covers. These constructions are of similar nature to that of the spanning tree fibrations, in the sense that we follow a similar combinatorial blueprint: we first describe the underlying functor, then the corresponding natural transformation to upgrade it to a morphism of poic-spaces, and finally show the properties of a poic-fibration. The underlying idea behind our constructions is to pullback our spanning tree fibrations through the product of the source and target morphisms. This is the reason why we call them extended spanning tree fibrations, as we *extend* the spanning tree fibrations to the case of admissible covers. There is a combinatorial caveat due to partial openness: we have to prescribe where do we want to regard the source from. More precisely, we would not be interested in the source of a given admissible cover, but rather on a graph obtained from forgetting several markings of the source. Now, as we have readily seen in the previous section, these forgetting the marking morphisms are not proper in general because of the partial openness. This phenomenon propagates to the case at hand, in the sense that, if we are interested in forgetting several markings, then the underlying combinatorics must be governed by that of the graphs with the fewer markings. This is what eventually leads us to the introduction of J-marked extended admissible covers and their corresponding extended spanning tree fibrations.

We set the following notation for the whole section (basically recalling Notation 1.6.1):

Notation 3.7.1. Suppose $h, d, m \geq 0$, are integers with $2h + m - 2 > 0$, let $\vec{\mu} = (\mu_1, \ldots, \mu_m)$

be a vector of partitions of d, set $n = \sum_{i=1}^{m} \ell(\mu_i)$, and let g be given by

$$n + 2(g-1) = d \cdot (m + 2(h-1)).$$

Just as before, we are assuming that g is an integer and all the further restrictions this implies.

Now, observe that the product of linear poic-fibrations is a linear poic-fibration. So the linear poic-fibrations $\mathrm{st}_{g,n}$ and $\mathrm{st}_{h,m}$ give a linear poic-fibration

$$\mathrm{st}_{g,n} \times \mathrm{st}_{h,m} \colon \mathrm{ST}_{g,n} \times \mathrm{ST}_{h,m} \to \mathrm{M}^{\mathrm{trop}}_{g,n} \times \mathrm{M}^{\mathrm{trop}}_{h,m}.$$

Our main idea to get a poic-fibration over $\mathrm{AC}_{d,h,\vec{\mu}}$ is to pullback this linear poic-fibration through the functor $\mathrm{src} \times \mathrm{trgt} \colon \mathbb{AC}_{d,h}(\vec{\mu}) \to \mathbb{G}_{g,n} \times \mathbb{G}_{h,m}$. Due to our formalism, this requires some care and constructions.

Definition 3.7.2. Let h, m, d, $\vec{\mu}$, n and g be as above (that is, as in Notation 3.7.1). The *category of extended admissible covers of degree d to genus-h m-marked graphs and ramification $\vec{\mu}$* is the category $\mathbb{EAC}_{d,h}(\vec{\mu})$ given by the following data:

- The objects are the triples (T_G, T_H, π), where T_G is an object of $\mathbb{G}_{0,n+2g}$, T_H is an object of $\mathbb{G}_{0,m+2h}$, and $\pi : \mathrm{st}_{g,n}(T_G) \to \mathrm{st}_{h,m}(T_H)$ is an object of $\mathbb{AC}_{d,h}(\vec{\mu})$.
- For two objects (T_G, T_H, π) and $(T_{G'}, T_{H'}, \pi')$ the set of morphisms is defined as the subset

  $$\mathrm{Hom}_{\mathbb{EAC}_{d,h}(\vec{\mu})}((T_G, T_H, \pi), (T_{G'}, T_{H'}, \pi')) \subset \mathrm{Hom}_{\mathbb{G}_{0,g+2n}}(T_G, T_{G'}) \times \mathrm{Hom}_{\mathbb{G}_{0,h+2m}}(T_H, T_{H'})$$

  consisting of tuples $(f_{\mathrm{src}}, f_{\mathrm{trgt}})$ such that $(\mathrm{st}_{g,n}(f_{\mathrm{src}}), \mathrm{st}_{h,m}(f_{\mathrm{trgt}})) \in \mathrm{Hom}_{\mathbb{AC}_{d,h}(\vec{\mu})}(\pi, \pi')$.
- Composition of morphisms is just composition of tuples.

From the above, it is clear that the category $\mathbb{EAC}_{d,h}(\vec{\mu})$ is thin and has finitely many isomorphism classes. Furthermore, it comes naturally equipped with three functors

$$\mathrm{est}_{d,h,\vec{\mu}} \colon \mathbb{EAC}_{d,h}(\vec{\mu}) \to \mathbb{AC}_{d,h}(\vec{\mu}), \qquad (T_G, T_H, \pi) \mapsto \pi, \tag{3.15}$$

$$\mathrm{src} \colon \mathbb{EAC}_{d,h}(\vec{\mu}) \to \mathbb{G}_{0,2g+n}, \qquad (T_G, T_H, \pi) \mapsto T_G, \tag{3.16}$$

$$\mathrm{trgt} \colon \mathbb{EAC}_{d,h}(\vec{\mu}) \to \mathbb{G}_{0,2h+m}, \qquad (T_G, T_H, \pi) \mapsto T_H. \tag{3.17}$$

Construction 3.7.3. Let (T_G, T_H, π) be an object of $\mathbb{EAC}_{d,h}(\vec{\mu})$, and for simplicity, let us set $G = \mathrm{st}_{g,n}(T_G)$ and $H = \mathrm{st}_{h,m}(T_H)$. The poic-fibrations $\mathrm{st}_{g,n}$ and $\mathrm{st}_{h,m}$ give a poic-morphism

$$\eta_{\mathrm{st}_{h,m},T_H} \times \eta_{\mathrm{st}_{g,n},T_G} : \mathrm{ST}_{T_H} \times \mathrm{ST}_{T_G} \to \sigma_H \times \sigma_G, \tag{3.18}$$

which is an embedding. On the other hand, the admissible cover $\pi : G \to H$ gives the poic σ_π which is naturally a subcone of $\sigma_H \times \sigma_G$.

Definition 3.7.4. Following Construction 3.7.3, the *extended cone of metrics* of the triple (T_G, T_H, π) is the poic $\mathrm{EST}_{(T_G,T_H,\pi)}$ defined as the inverse image of σ_π under the map (3.18). In particular, it naturally comes with an inclusion

$$\eta_{\mathrm{est}_{d,h,\vec{\mu}},(T_G,T_H,\pi)} : \mathrm{EST}_{(T_G,T_H,\pi)} \to \sigma_\pi, \tag{3.19}$$

and poic-morphisms given by the projection maps

$$\eta_{\mathrm{src},(T_G,T_H,\pi)} \colon \mathrm{EST}_{(T_G,T_H,\pi)} \to \mathrm{ST}_{T_G}, \tag{3.20}$$

$$\eta_{\mathrm{trgt},(T_G,T_H,\pi)} \colon \mathrm{EST}_{(T_G,T_H,\pi)} \to \mathrm{ST}_{T_H}. \tag{3.21}$$

In addition, the natural projections, $\mathrm{ST}_{T_G} \times \mathrm{ST}_{T_H} \to \mathrm{ST}_{T_G}$ and $\mathrm{ST}_{T_G} \times \mathrm{ST}_{T_H} \to \mathrm{ST}_{T_G}$, together with the poic-morphisms, $(D_{g,n})_{T_G}$ and $(D_{h,m})_{T_H}$, define a poic-morphism

$$(D_{d,h,\vec{\mu}})_{(T_G,T_H,\pi)} \colon \mathrm{EST}_{(T_G,T_H,\pi)} \to \underline{\left(N_{\mathrm{dist}_{2g+n}} \oplus N_{\mathrm{dist}_{2h+m}}\right)_{\mathbb{R}}}. \tag{3.22}$$

Remark. For a triple (T_G, T_H, π) as above, the relative interior of the cone $\mathrm{EST}_{(T_G,T_H,\pi)}$ corresponds to all possible metrics on T_G and T_H that are related under π, and the faces of this cone correspond to the possible contractions of both T_G and T_H that are related through π.

Proposition 3.7.5. *The association*

$$\mathrm{EST}_{d,h,\vec{\mu}} : \mathbb{EAC}_{d,h}(\vec{\mu})^{\mathrm{op}} \to \mathrm{POIC}, (T_G, T_H, \pi) \mapsto \mathrm{EST}_{(T_G,T_H,\pi)} \tag{3.23}$$

is a poic-complex, the poic-morphisms (3.22) *define a morphism of poic-complexes*

$$D_{d,h,\vec{\mu}} \colon \mathrm{EST}_{d,h,\vec{\mu}} \to \underline{\left(N_{\mathrm{dist}_{2g+n}} \oplus N_{\mathrm{dist}_{2h+m}}\right)_{\mathbb{R}}}, \tag{3.24}$$

and the poic-morphisms (3.19) *define a natural transformation* $\eta_{\mathrm{est}_{d,h,\vec{\mu}}} \colon \mathrm{EST}_{d,h,\vec{\mu}} \to \mathrm{AC}_{d,h,\vec{\mu}}$ *which together with* (3.15) *(and the linear poic-complex structure given by* (3.24)*) defines a linear poic-fibration*

$$\mathrm{est}_{d,h,\vec{\mu}} : \left(\mathrm{EST}_{d,h,\vec{\mu}}\right)_{D_{d,h,\vec{\mu}}} \to \mathrm{AC}_{d,h,\vec{\mu}}. \tag{3.25}$$

Furthermore, the poic-morphisms (3.20) *and* (3.21) *define corresponding natural transformations, which together with the integral linear maps given by the natural projections define*

morphisms of linear poic-complexes

$$\mathrm{src}\colon \mathrm{EST}_{d,h,\vec{\mu}} \to \mathrm{ST}_{g,n}, \tag{3.26}$$

$$\mathrm{trgt}\colon \mathrm{EST}_{d,h,\vec{\mu}} \to \mathrm{ST}_{h,m}, \tag{3.27}$$

where src *is weakly proper.*

Proof. We have previously mentioned that the category $\mathbb{EAC}_{d,h}(\vec{\mu})$ is thin and essentially finite. The association $\mathrm{EST}_{d,h,\vec{\mu}}$ is a functor, because $\mathrm{ST}_{g,n}$, $\mathrm{ST}_{h,m}$, and $\mathrm{AC}_{d,h,\vec{\mu}}$ are functors. The definition of the functor $\mathrm{EST}_{d,h,\vec{\mu}}$ at triples, together with the fact that $\mathrm{ST}_{g,n}$, $\mathrm{ST}_{h,m}$, and $\mathrm{AC}_{g,h,\vec{\mu}}$ are also poic-complexes, implies that $\mathrm{EST}_{d,h,\vec{\mu}}$ also satisfies the second condition of (and is therefore) a poic-complex.
From the definition of the inclusion (3.19), and the fact that $\eta_{\mathrm{st}_{h,m}}$ and $\eta_{\mathrm{st}_{g,n}}$ are natural transformations, it follows that $\eta_{\mathrm{est}_{d,h,\vec{\mu}}}$ is also a natural transformation. Therefore, the tuple $\mathrm{est}_{d,h,\vec{\mu}} = (\mathrm{est}_{d,h,\vec{\mu}}, \eta_{\mathrm{est}_{d,h,\vec{\mu}}})$ is a morphism of poic-spaces. The essential surjectivity of the functor $\mathrm{est}_{d,h,\vec{\mu}}$ follows from that of $\mathrm{st}_{g,n}$ and $\mathrm{st}_{h,m}$. This category and this functor are obtained by pulling back the functor underlying the (linear) poic-fibration $\mathrm{st}_{h,m} \times \mathrm{st}_{g,n}$ through $\mathbb{AC}_{d,h}(\vec{\mu}) \to \mathbb{G}_{h,m} \times \mathbb{G}_{g,n}$. Thus, it can be readily checked that they preserve the required lifting properties.
It only remains to check the condition of poic-fibrations on relative interiors. For this, let (T_G, T_H, π) be an object of $\mathbb{EAC}_{d,h}(\vec{\mu})$. Since both $\eta_{\mathrm{st}_{h,m},T_H}$ and $\eta_{\mathrm{st}_{g,n},T_G}$ induce an isomorphism between the corresponding relative interiors, this also holds for their product and, by definition, for $\eta_{\mathrm{est}_{d,h,\vec{\mu}}}$, because it is the restriction of this product to a dense subcone. As $D_{g,n}$ and $D_{h,m}$ are morphisms of poic-complexes, it is clear that the poic-morphisms (3.22) define the morphism of poic-complexes (3.24). The statement concerning the poic-morphisms (3.20) and (3.21) is analogous to that of $\mathbb{AC}_{d,h}(\vec{\mu})$, so we omit it.
To finalize, we remark that the weak properness of src follows from the following facts: the only possible edge contractions of the source come from edge contractions of the target, and every face of $\mathrm{EST}_{(T_G,T_H,\pi)}$ comes from one of ST_{T_H}. In this way, a facet of the closure of $\eta_{\mathrm{src},(T_G,T_H,\pi)}$ actually arises from a facet of $\mathrm{EST}_{(T_G,T_H,\pi)}$. □

Definition 3.7.6. Let h, m, d, $\vec{\mu}$, g, and n be as in our standing notation for the section: Notation 3.7.1. We call $\mathrm{EST}_{d,h,\vec{\mu}}$, with the linear poic-complex structure (3.24), the *linear poic-complex of extended admissible covers of degree d with ramification $\vec{\mu}$ to m-marked genus-h graphs*, and the poic-fibration (3.25) the *extended spanning tree fibration of discrete admissible covers of genus-h m-marked graphs with ramification $\vec{\mu}$.*

After we have constructed the extended spanning tree fibrations of discrete admissible covers we look at the source and target morphisms. Both define morphisms of linear poic-fibrations, but only the source morphism is weakly proper in the top

dimension. This is due to the cone of metrics of the cover and our impossibility of contracting non-trivial cycles of a graph. In other words, it may very well be the case that the fiber over a subtree of the target discrete graph has positive genus and thus we are not allowed to perform this deformation. This is not an issue for our present endeavors, and later in Section 5.2 we explain a way of going around this.

Proposition 3.7.7. *The morphisms of linear poic-complexes* (3.26) *and* (3.27)*, together with the morphisms of poic-spaces* $\mathrm{src}\colon \mathrm{AC}_{d,h,\vec{\mu}} \to \mathrm{M}^{\mathrm{trop}}_{g,n}$ *and* $\mathrm{trgt}\colon \mathrm{AC}_{d,h,\vec{\mu}} \to \mathrm{M}^{\mathrm{trop}}_{h,m}$ *define morphisms of linear poic-fibrations:*

$$\mathfrak{src}\colon \mathrm{est}_{d,h,\vec{\mu}} \to \mathrm{st}_{g,n}, \qquad \mathfrak{trgt}\colon \mathrm{est}_{d,h,\vec{\mu}} \to \mathrm{st}_{h,m}, \tag{3.28}$$

where $\mathfrak{src}^{\mathrm{c}}$ *and* $\mathfrak{src}^{\mathrm{s}}$ *are the corresponding* src *morphisms, and* $\mathfrak{trgt}^{\mathrm{c}}$ *and* $\mathfrak{trgt}^{\mathrm{s}}$ *are the corresponding* trgt *morphisms. Furthermore, the morphism of linear poic-complexes*

$$\mathfrak{src}\colon \mathrm{est}_{d,h,\vec{\mu}} \to \mathrm{st}_{g,n} \tag{3.29}$$

is weakly proper.

Proof. The commutativity relations can be readily seen from the definitions. We have already established the weak properness of $\mathfrak{src}^{\mathrm{c}}$ in Proposition 3.7.5. Hence, it only remains to establish the lifting property of both $\mathfrak{src}^{\mathrm{s}}$ and $\mathfrak{trgt}^{\mathrm{s}}$. These are a direct consequence of the following amazing facts: isomorphisms of $\mathbb{G}_{g,n}$ define degree-1 discrete admissible covers, and the degree of a composition of harmonic morphisms is the product of the degrees. In other words, if $\pi\colon G \to H$ is an object of $\mathbb{AC}_{d,h}(\vec{\mu})$ and $f\colon G' \to G$ is an isomorphism then $\pi \circ f$ is also an object of $\mathbb{AC}_{d,h}(\vec{\mu})$ and (f, Id_H) is an isomorphism $\pi \to \pi \circ f$ of $\mathbb{AC}_{d,h}(\vec{\mu})$. □

Remark. The non-properness of $\mathfrak{trgt}$ is due to our impossibility of contracting cycles. Namely, it is possible that above an edge of the target there is a piece of the source graph with positive genus (see for instance Figure 1.13). This situation can be avoided by extending our category to that of stable graphs and permitting non-trivial genus functions on the vertices, however this forces new constructions on both the spanning tree fibrations, as well as the extended spanning tree fibrations. The case of the spanning tree fibration follows verbatim. However, the case of the extended spanning tree fibration requires more care and is, perhaps, a matter of future endeavors.

We now generalize the previous construction as follows. Let h, d, m, $\vec{\mu}$, g, and n be as in Notation 3.7.1, and let $J \subset \{1,\dots,n\}$ be arbitrary. We will construct a linear poic-complex $\mathrm{EST}^J_{d,h,\vec{\mu}}$ together with a poic-fibration $\mathrm{est}^J_{d,h,\vec{\mu}}\colon \mathrm{EST}^J_{d,h,\vec{\mu}} \to \mathrm{AC}_{d,h,\vec{\mu}}$, in the same fashion as $\mathrm{EST}_{d,h,\vec{\mu}}$, but by pulling back $\mathrm{est}_{g,J} \times \mathrm{est}_{h,m}$ through $\mathrm{ft}_{J^c} \circ \mathfrak{src} \times \mathfrak{trgt}$. The motivation underlying such a generalization is the construction of these

poic-complexes in the same vein as $\mathrm{EST}_{d,h,\vec{\mu}}$, but having a morphism of linear poic-complexes $\mathfrak{src}\colon \mathrm{est}^J_{d,h,\vec{\mu}} \to \mathrm{st}_{g,J}$ that is weakly proper in top dimension.

Definition 3.7.8. Let h, m, d, $\vec{\mu}$, n, and g be as in Notation 3.7.1, and let $J \subset \{1,\dots,n\}$ be arbitrary. The *category of J-marked extended admissible covers of degree d to genus-h m-marked graphs and ramification $\vec{\mu}$* is the category $\mathbb{EAC}^J_{g,h}(\vec{\mu})$ given by the following data:

- The objects are the triples (T_s, T_t, π), where T_s is an object of $\mathbb{G}_{0,J\sqcup \mathrm{g}}$, T_t is an object of $\mathbb{G}_{0,m+2h}$, and π is an object of $\mathbb{AC}_{d,h}(\vec{\mu})$ such that
$$\mathrm{ft}_{J^c}(\mathrm{src}(\pi)) = \mathrm{st}_{g,J}(T_s), \qquad \mathrm{trgt}(\pi) = \mathrm{st}_{h,m}(T_t).$$
- For two objects (T_s, T_t, π) and (T_s', T_t', π') the set of morphisms is defined as the subset of $\mathrm{Hom}_{\mathbb{G}_{0,J\sqcup \mathrm{g}}}(T_t, T_t') \times \mathrm{Hom}_{\mathbb{G}_{0,h+2m}}(T_s, T_s') \times \mathrm{Hom}_{\mathbb{AC}_{d,h}(\vec{\mu}}(\pi, \pi')$ consisting of the triples (f_s, f_t, f) such that
$$\mathrm{ft}_{J^c}(f_{\mathrm{src}}) = \mathrm{st}_{g,J}(f_s), \qquad f_{\mathrm{trgt}} = \mathrm{st}_{h,m}(f_t). \tag{3.30}$$
- Composition of morphisms is just composition of triples.

It is clear that this category is essentially finite. By construction, this category comes naturally equipped with three functors:

$$\mathrm{est}^J_{d,h,\vec{\mu}}\colon \mathbb{EAC}^J_{d,h}(\vec{\mu}) \to \mathbb{AC}_{d,h}(\vec{\mu}), \qquad (T_s, T_t, \pi) \mapsto \pi, \tag{3.31}$$
$$\mathrm{src}\colon \mathbb{EAC}^J_{d,h}(\vec{\mu}) \to \mathbb{G}_{0,J\sqcup \mathrm{g}}, \qquad (T_s, T_t, \pi) \mapsto T_s, \tag{3.32}$$
$$\mathrm{trgt}\colon \mathbb{EAC}^J_{d,h}(\vec{\mu}) \to \mathbb{G}_{0,2h+m}, \qquad (T_s, T_t, \pi) \mapsto T_t. \tag{3.33}$$

Lemma 3.7.9. *Following the notation of Definition 3.7.8, the category $\mathbb{EAC}^J_{d,h}(\vec{\mu})$ is thin.*

Proof. Let (T_s, T_t, π) and (T_s', T_t', π') be two objects of $\mathbb{EAC}^J_{d,h}(\vec{\mu})$, and consider two morphisms
$$(f_s, f_t, f), (f_s', f_t', f') \in \mathrm{Hom}_{\mathbb{EAC}^J_{d,h}(\vec{\mu})}((T_s, T_t, \pi), (T_s', T_t', \pi')). \tag{3.34}$$
Since $\mathbb{G}_{0,2h+m}$ and $\mathbb{G}_{0,J\sqcup \mathrm{g}}$ are thin, it follows that $f_t = f_t'$ and $f_s = f_s'$. In particular, $f_{\mathrm{trgt}} = f'_{\mathrm{trgt}}$. Moreover, $\mathrm{ft}_{I^c}(f_{\mathrm{src}}) = \mathrm{ft}_{I^c}(f_{\mathrm{src}})$, implies that $\mathrm{ft}_{J\cup J^c}(f_{\mathrm{src}}) = \mathrm{ft}_{J\cup J^c}(f_{\mathrm{src}})$. Since f_{src} and f'_{src} must preserve the n-marking, the previous equality forces $f_{\mathrm{src}} = f'_{\mathrm{src}}$. □

Construction 3.7.10. Let (T_s, T_t, π) be an object of $\mathbb{EAC}^J_{d,h}(\vec{\mu})$, and for simplicity, let us set $G = \mathrm{st}_{g,J}(T_s)$ and $H = \mathrm{st}_{h,m}(T_t)$. The poic-fibrations $\mathrm{st}_{g,J}$ and $\mathrm{st}_{h,t}$ give a poic-morphism
$$\eta_{\mathrm{st}_{h,m},T_t} \times \eta_{\mathrm{st}_{g,J},T_s} : \mathrm{ST}_{T_t} \times \mathrm{ST}_{T_s} \to \sigma_H \times \sigma_G, \tag{3.35}$$

which is an embedding. On the other hand, the admissible cover $\pi : G \to H$ gives the poic σ_π, and forgetting the marking yields a morphism

$$\mathrm{Id} \times \eta_{\mathrm{ft}_{J^c},\mathrm{src}(\pi)} \colon \sigma_\pi \to \sigma_H \times \sigma_G \tag{3.36}$$

Definition 3.7.11. Following Construction 3.7.10 and in the same spirit of Definition 3.7.4, the *extended cone of metrics* of the triple (T_s, T_t, π) is the poic $\mathrm{EST}^J_{(T_s,T_t,\pi)}$ defined as the inverse image of $\mathrm{Id} \times \eta_{\mathrm{ft}_{J^c},\mathrm{src}(\pi)}(\sigma_\pi)$ under the map (3.35). We can now follow the blueprint outlined in Definition 3.7.4. Namely, this poic naturally comes with an inclusion

$$\eta_{\mathrm{est}^J_{d,h,\vec{\mu}},(T_s,T_t,\pi)} : \mathrm{EST}^J_{(T_s,T_t,\pi)} \to \mathrm{Id} \times \mathrm{ft}_{J^c}(\sigma_\pi), \tag{3.37}$$

as well as poic-morphisms given by the projection maps

$$\eta_{\mathrm{src},(T_s,T_t,\pi)} \colon \mathrm{EST}^J_{(T_s,T_t,\pi)} \to \sigma_{T_s}, \tag{3.38}$$

$$\eta_{\mathrm{trgt},(T_s,T_t,\pi)} \colon \mathrm{EST}^J_{(T_s,T_t,\pi)} \to \sigma_{T_t}. \tag{3.39}$$

Just as before, the natural projections, $\mathrm{ST}_{T_s} \times \mathrm{ST}_{T_t} \to \mathrm{ST}_{T_t}$ and $\mathrm{ST}_{T_s} \times \mathrm{ST}_{T_t} \to \mathrm{ST}_{T_s}$, together with the poic-morphisms, $(D_{g,J})_{T_s}$ and $(D_{h,m})_{T_t}$, define a poic-morphism

$$(D^J_{d,h,\vec{\mu}})_{(T_s,T_t,\pi)} \colon \mathrm{EST}^J_{(T_s,T_t,\pi)} \to \underline{\left(N_{\mathrm{dist}_{J\sqcup \mathrm{g}}} \oplus N_{\mathrm{dist}_{2h+m}}\right)_{\mathbb{R}}}. \tag{3.40}$$

Remark. Similar to our previous situation, for a triple (T_s, T_t, π) as above, the relative interior of the cone $\mathrm{EST}^J_{(T_s,T_t,\pi)}$ corresponds to all possible metrics on T_s and T_t that are related under π and forgetting the marking. Additionally, the faces of this cone correspond to the possible contractions of both T_s and T_t that are related through π, which are governed by contractions of the target.

Similar to Proposition 3.7.5, we obtain the following, where the only difference is with respect to the weak properness of the src morphism.

Proposition 3.7.12. *The association*

$$\mathrm{EST}^J_{d,h,\vec{\mu}} : \mathbb{EAC}^J_{d,h}(\vec{\mu})^{\mathrm{op}} \to \mathrm{POIC}, (T_s, T_t, \pi) \mapsto \mathrm{EST}_{(T_s,T_t,\pi)} \tag{3.41}$$

is a poic-complex. The poic-morphisms (3.40) *define a morphism of poic-complexes*

$$D^J_{d,h,\vec{\mu}} \colon \mathrm{EST}^J_{d,h,\vec{\mu}} \to \underline{\left(N_{\mathrm{dist}_{J\sqcup \mathrm{g}}} \oplus N_{\mathrm{dist}_{2h+m}}\right)_{\mathbb{R}}}, . \tag{3.42}$$

The poic-morphisms (3.37) *define a natural transformation* $\eta_{\mathrm{est}^J_{d,h,\vec{\mu}}} \colon \mathrm{EST}^J_{d,h,\vec{\mu}} \to \mathrm{AC}_{d,h,\vec{\mu}}$ *which together with* (3.31) *(and the linear poic-complex structure given by* (3.42)*) defines a*

linear poic-fibration

$$\mathrm{est}^J_{d,h,\vec{\mu}} \colon \left(\mathrm{EST}^J_{d,h,\vec{\mu}}\right)_{D^J_{d,h,\vec{\mu}}} \to \mathrm{AC}_{d,h,\vec{\mu}}. \tag{3.43}$$

The poic-morphisms (3.38) *and* (3.39) *define corresponding natural transformations, which together with the integral linear maps given by the natural projections define morphisms of linear poic-complexes*

$$\mathrm{src} \colon \mathrm{EST}^J_{d,h,\vec{\mu}} \to \mathrm{ST}_{g,J}, \tag{3.44}$$

$$\mathrm{trgt} \colon \mathrm{EST}^J_{d,h,\vec{\mu}} \to \mathrm{ST}_{h,m}, \tag{3.45}$$

where src *is weakly proper in top dimension.*

Proof. The proof of Proposition 3.7.5 carries over verbatim except for the last argument concerning weak properness. Consider an object (T_s, T_t, π) of $\mathbb{EAC}^J_{d,h}(\vec{\mu})$. In this case, we only obtain weak properness in the top dimension because the forgetting the marking morphism gives further positivity conditions between several edges of the target tree T_t. This is in contrast with the previous situation, where we were not forgetting the marking and the induced positivity conditions only influenced a single edge of the target tree (namely, T_H in the notation of the proof of Proposition 3.7.5). □

Definition 3.7.13. Suppose h, m, d, μ, n, g, and J are as in Definition 3.7.8. We call $\mathrm{EST}^J_{d,h,\vec{\mu}}$, with the linear poic-complex structure (3.42), the *linear poic-complex of J-marked extended admissible covers of degree d with ramification $\vec{\mu}$ to m-marked genus-h graphs*, and the poic-fibration (3.43) the *extended spanning tree fibration of J-marked discrete admissible covers of genus-h m-marked graphs with ramification $\vec{\mu}$.*

Remark. Of course, if $J = \{1, \dots, n\}$, then $\mathrm{EST}^J_{d,h,\vec{\mu}} = \mathrm{EST}_{d,h,\vec{\mu}}$ and $\mathrm{est}^J_{d,h,\vec{\mu}} = \mathrm{est}_{d,h,\vec{\mu}}$.

Just as before, the proof of Proposition 3.7.7 shows the analogous statement in this context. We record this as a proposition for later referencing:

Proposition 3.7.14. *The morphisms of linear poic-complexes* (3.44) *and* (3.45)*, together with the morphisms of poic-spaces* $\mathrm{ft}_{J^c} \circ \mathrm{src} \colon \mathrm{AC}_{d,h,\vec{\mu}} \to \mathrm{M}^{\mathrm{trop}}_{g,J}$ *and* $\mathrm{trgt} \colon \mathrm{AC}_{d,h,\vec{\mu}} \to \mathrm{M}^{\mathrm{trop}}_{h,m}$ *define morphisms of linear poic-fibrations:*

$$\mathfrak{src} \colon \mathrm{est}^J_{d,h,\vec{\mu}} \to \mathrm{st}_{g,J}, \qquad \mathfrak{trgt} \colon \mathrm{est}^J_{d,h,\vec{\mu}} \to \mathrm{st}_{h,m}, \tag{3.46}$$

where $\mathfrak{src}^{\mathrm{c}}$ *and* $\mathfrak{src}^{\mathrm{s}}$ *are the corresponding* src *morphisms, and* $\mathfrak{trgt}^{\mathrm{c}}$ *and* $\mathfrak{trgt}^{\mathrm{s}}$ *are the corresponding* trgt *morphisms. Additionally, the morphism of linear poic-complexes* $\mathfrak{src} \colon \mathrm{est}_{d,h,\vec{\mu}} \to \mathrm{st}_{g,n}$ *is weakly proper in top dimension.*

We close this section with an example highlighting the differences in our constructions.

Example 3.7.15. Let $\pi\colon G \to H$ denote the object of $\mathbb{AC}_{2,0}((2),(2),(2),(2),(1,1))$ depicted in Figure 3.8. We have already seen this cover in Figure 1.13, where in Example 1.4.6 we described its associated cone of metrics.

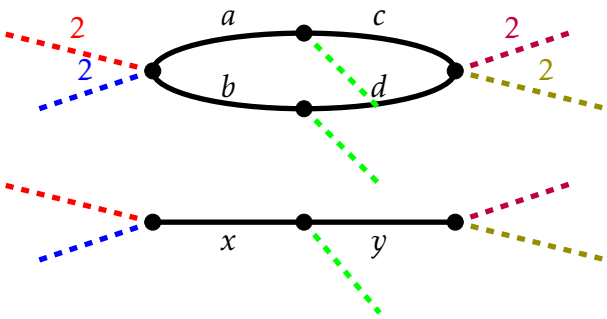


Figure 3.8: An object of $\pi\colon G \to H$ of $\mathbb{AC}_{2,0}((2),(2),(2),(2),(1,1))$.

If T_G denotes the tree of $\mathbb{G}_{0,6\sqcup\mathbb{1}}$ (we are being lax with notation to avoid overbearing the figures and the example) depicted in Figure 3.9, then the triple (T_G, H, π) is an object of $\mathbb{EAC}_{2,0}((2),(2),(2),(2),(1,1))$.

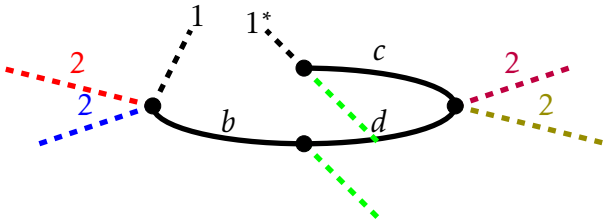


Figure 3.9: An object T_G of $\mathbb{G}_{0,6\sqcup\mathbb{1}}$ with $\mathrm{st}_{1,6}(T_G) \cong G$.

If now J denotes the subset of the markings of G consisting of the red, blue, olive, and purple markings, then the triple (T_t, H, π) is an object of $\mathbb{EAC}^J_{2,0}((2),(2),(2),(2),(1,1))$ where T_s is the tree depicted in Figure 3.10. This tree satisfies $\mathrm{st}_{1,J}(T_s) \cong \mathrm{ft}_{J^c}(G)$, where the isomorphism is suggestively given by the corresponding depictions.

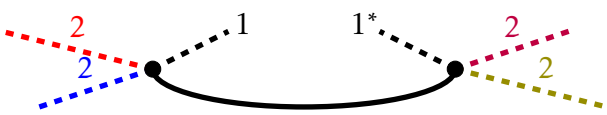


Figure 3.10: An object T_t of $\mathbb{G}_{0,J\sqcup\mathbb{1}}$ with $\mathrm{st}_{1,J}(T_s) \cong \mathrm{ft}_{J^c} G$.

We close by depicting the corresponding extended cones of metrics of the triples in Figure 3.11. We remark that for the triple (T_G, H, π) we have that the length on the edge x of the tree H cannot vanish, whereas the extended cone of metrics of (T_s, H, π) coincides with the associated cone of metrics of the cover π.

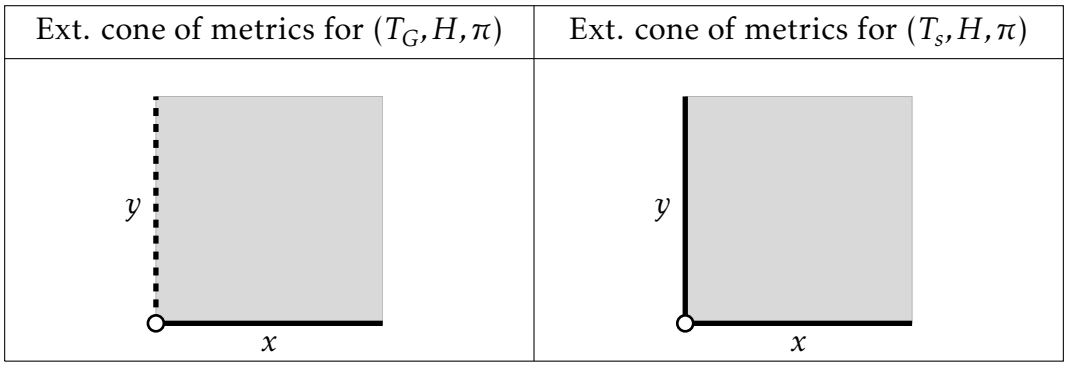


Figure 3.11: Extended cones of metrics of the triples

Chapter 4

Tropical cycles of discrete admissible covers

Let d, h, m, $\vec{\mu}$, n, and g be as in Notation 3.7.1, and let $J \subset \{1,\dots,n\}$. In this chapter, we prove that the usual weight assignment on discrete admissible covers (see [CMR16]) defines an $\mathrm{est}^J_{d,h,\vec{\mu}}$-equivariant Minkowski weight. The proof of this general result occupies most of the chapter, and requires an additional algebrogeometric input. Afterward, we focus on a particular application of this theorem involving previous constructions. More precisely, we explain how after fixing d, h, m, μ, and J, we can use this theorem and our framework to recover the enumerative results of [VDL22] concerning the Catalan many morphisms over a general tropical curve. We also take this space to set notation (and simply reference to it) for the rest of the chapter:

Notation 4.0.1. Suppose $h, d, m \geq 0$, are integers with $2h + m - 2 > 0$, let $\vec{\mu} = (\mu_1, \dots, \mu_m)$ be a vector of partitions of d, set $n = \sum_{i=1}^m \ell(\mu_i)$, let $J \subset \{1,\dots,n\}$, and let g be given by

$$n + 2(g-1) = d \cdot (m + 2(h-1)).$$

As usual, it is assumed that g is an integer together with all the further conditions this implies.

4.1 The standard weight

Consider the extended spanning tree fibration of J-marked discrete admissible covers of genus-h m-marked graphs with ramification $\vec{\mu}$ from Definition 3.7.13

$$\mathrm{est}^J_{d,h,\vec{\mu}} : \mathrm{EST}^J_{d,h,\vec{\mu}} \to \mathrm{AC}_{d,h,\vec{\mu}}.$$

We remark that from the definition (3.41) of $\mathrm{EST}^J_{d,h,\vec{\mu}}$ it follows that every cone of this poic-complex is of dimension at most $m+3(h-1)$. We define the standard weight on the

$m+3(h-1)$-dimensional cones, following the usual assignment as in Definition 22 of [CMR16] (it is important to remark that in (W3) only the internal edges are considered, see their Theorem 34 for instance).

Definition 4.1.1. For a cover $\pi\colon G \to H$ of $\mathbb{AC}_{d,h}(\vec{\mu})$ the *standard weight* $\omega^J_{d,h,\vec{\mu}}(\pi)$ *of the cover* is the number given by

$$\omega^J_{d,h,\vec{\mu}}(\pi) = \frac{\left(\prod_{e\in E(G)} d_\pi(e)\right)\cdot\left(\prod_{V\in V(G)} H(V)\cdot \mathrm{CF}(V)\right)}{\prod_{h\in E(H)} \operatorname{lcm}\left((d_\pi(e))_{e\in\pi^{-1}(h)}\right)}, \tag{4.1}$$

where for $V \in V(G)$:

- $H(V)$ denotes the local rational connected Hurwitz number. Namely, around V the cover induces integer partitions partitions $\lambda_1,\dots,\lambda_k$ of $d_\pi(V)$ and therefore $H(V) = H_{0\to 0}(\lambda_1,\dots,\lambda_k)$[1].
- $\mathrm{CF}(V)$ is the product of factors of $k!$ for each k-tuple of adjacent edges or legs of the same weight that map to the same edge of the target.

The *standard weight* $\omega^J_{d,h,\vec{\mu}}$ of $\mathrm{EST}^J_{d,h,\vec{\mu}}$ is the rational weight

$$\omega^J_{d,h,\vec{\mu}}\colon [\mathbb{EAC}^J_{d,h}(\vec{\mu})](m+3(h-1)) \to \mathbb{Q}, [(T_s,T_t,\pi)] \mapsto \omega^J_{d,h,\vec{\mu}}(\pi).$$

It is clear that if $\pi \cong \pi'$, then $\omega^J_{d,h,\vec{\mu}}(\pi) = \omega^J_{d,h,\vec{\mu}}(\pi')$. This means that

$$\omega^J_{d,h,\vec{\mu}} \in [\mathrm{est}^J_{d,h,\vec{\mu}}]^* W_{m+3(h-1)}(\mathrm{AC}_{d,h,\vec{\mu}}).$$

Notation 4.1.2. If $J=\{1,\dots,n\}$, then we simply denote $\omega^J_{d,h,\vec{\mu}}$ by $\omega_{d,h,\vec{\mu}}$.

What is not clear, and is the matter of the following section, is that this standard weight is a Minkowski weight. But it is, and this theorem, which we now state, is the pinnacle of our constructions. It is perhaps a good moment to emphasize that this is independent of the degree, the genus of the target, and the ramification profiles. Also, we are by no means approaching any result concerning the corresponding existence problem, which translates in our terms to the categories $\mathbb{AC}_{d,h}(\vec{\mu})$ and $\mathbb{EAC}_{d,h}(\vec{\mu})$ being empty or not. On the contrary, we are just showing that this weight assignment (which might be trivial[2] if the categories are empty) defines a Minkowski weight. In the last section of this chapter, we handle a specific case in which these categories are non-empty.

[1] This Hurwitz number $H_{0\to 0}(\lambda_1,\dots,\lambda_k)$ is just the product of $\frac{1}{d!}$ with the number of isomorphism classes of branched coverings $f\colon \mathbb{P}^1 \to \mathbb{P}^1$, where the branching locus consists of m distinct points $p_1,\dots,p_m$ and the ramification profile of f at p_i is λ_i (for $1 \le i \le m$).

[2] and in particular, trivially balanced ☺

Theorem 4.1.3. *The standard weight $\omega^J_{d,h,\vec{\mu}}$ is an $\mathrm{est}^J_{d,h,\vec{\mu}}$-equivariant Minkowski weight.*

The proof is postponed until the end of the next section. It is essentially split into two parts: a local deformation picture, and a local-to-global balancing argument. More precisely, the balancing of the standard weight comes from studying how an admissible cover can be deformed through edge contractions and extensions of the base graph. We first restrict ourselves to the behavior of a discrete admissible cover around a fixed edge of the target graph. From this, we obtain multiple admissible covers (by genus-0 curves) of a 3-valent 4-marked tree with a single edge. Then the contraction and extension of this single edge can be studied at each of these (our local deformation picture), and the global behavior of the cover can be understood out of these local cases (local-to-global balancing). As it turns out, the most intricate part corresponds to the local deformation picture which demands an algebrogeometric input. We also take this space to mention that in Figure 1 the deformation involved in the balancing of this standard weight has already been depicted.

4.2 Balancing of the standard weight

Let $d \geq 0$ be an integer and consider integer partitions $\alpha, \beta, \gamma, \delta \vdash d$, such that

$$\ell(\alpha) + \ell(\beta) + \ell(\gamma) + \ell(\delta) = 2(d+1). \tag{4.2}$$

For the time being we just consider $J = \{1, \ldots, 2(d+1)\}$, and focus on the linear poic-fibration $\mathrm{est}_{d,0,(\alpha,\beta,\gamma,\delta)}$. Later, in Theorem 4.2.6 and its proof, the remaining cases are established from this case.

The condition given by the equation (4.2) has several implications:

- If $\pi\colon G \to T$ is an object of $\mathbb{AC}_{d,h}(\alpha,\beta,\gamma,\delta)$, then $g(G) = 0$.
- The category $\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)$ coincides with $\mathbb{AC}_{d,h}(\alpha,\beta,\gamma,\delta)$, and the poic-fibration $\mathrm{est}_{d,0,(\alpha,\beta,\gamma,\delta)}$ is just the identity functor.
- The category $\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)$ has an initial object. This can be represented as the cover of a graph defined by a single 4-valent vertex from a graph defined by single vertex with valency $2(d+1)$, where the first $\ell(\alpha)$-marked legs lie above the first leg, the next $\ell(\beta)$-marked legs lie above the second leg, the next $\ell(\gamma)$-marked legs lie above the third leg, and the last $\ell(\delta)$-legs lie above the fourth leg.

In this case, the morphism of poic-complexes $D_{d,0,(\alpha,\beta,\gamma,\delta)}$ (3.24) presents this linear poic-complex $\mathrm{EST}_{d,0,(\alpha,\beta,\gamma,\delta)}$ as a 1-dimensional fan in the underlying vector space of $\mathcal{M}^{\mathrm{trop}}_{0,2(d+1)} \times \mathcal{M}^{\mathrm{trop}}_{0,4}$. Let $\Theta \in [\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)](1)$, with $(G_\Theta, H_\Theta, \pi_\Theta)$ an object of

$\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)$ representing this isomorphism class. The integral generator of the ray given by Θ is $u_\theta = (u_{\mathrm{src}(\Theta)}, u_{\mathrm{trgt}(\Theta)})$, where

- $u_{\mathrm{trgt}(\Theta)}$ is the integral generator of the ray in $\mathcal{M}_{0,4}^{\mathrm{trop}}$ given by the isomorphism class $[H_\Theta]$,
- $u_{\mathrm{src}(\Theta)}$ is the integral generator of the ray generated by $F_\pi(u_{\mathrm{trgt}(\Theta)})$, where F_π is the matrix (1.13) associated to the corresponding admissible cover π (this is invariant with respect to the isomorphism class).

Therefore, to show the balancing of the fundamental cycle $\omega_{d,0,(\alpha,\beta,\gamma,\delta)}$ it is necessary to show the equation:

$$\sum_{\Theta\in[\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)](1)} \omega_{d,0,(\alpha,\beta,\gamma,\delta)}(\Theta)\cdot u_\Theta = 0. \tag{4.3}$$

Example 4.2.1. Let $d = 4$, $\alpha = (3,1)$, $\beta = (2,1,1)$, $\gamma = (1,1,1,1)$ and $\delta = (4)$. In Figure 4.1 an initial object of $\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)$ is depicted, where colors on edges indicate the morphism, colored numbers indicate the weights, trivial weights on edges are omitted, and (for the sake of simplicity) the marking of the source is also omitted. In Figure 4.2 four different objects of $\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)$ are depicted, following the same conventions as previously.

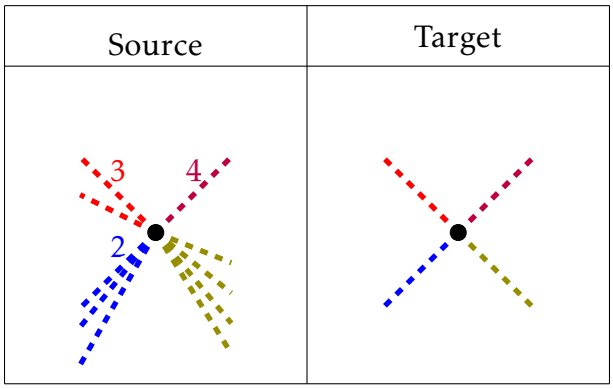


Figure 4.1: An initial object of $\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)$

It turns out that proving (4.3) is quite difficult, and managed to resist several of our combinatorial efforts. Similar situations have already been studied in [GMO17], [GO17], and [BM15] (and our approach to prove the balancing follows their spirit), but the general case of (4.3) resisted its phrasing (or interpretation) under these previous works. We do remark that it is feasible to establish (4.3) for some special cases: low degrees, or special choices of ramification profiles. However, for arbitrary degrees and ramification profiles, the underlying combinatorics become too wild. Our deus

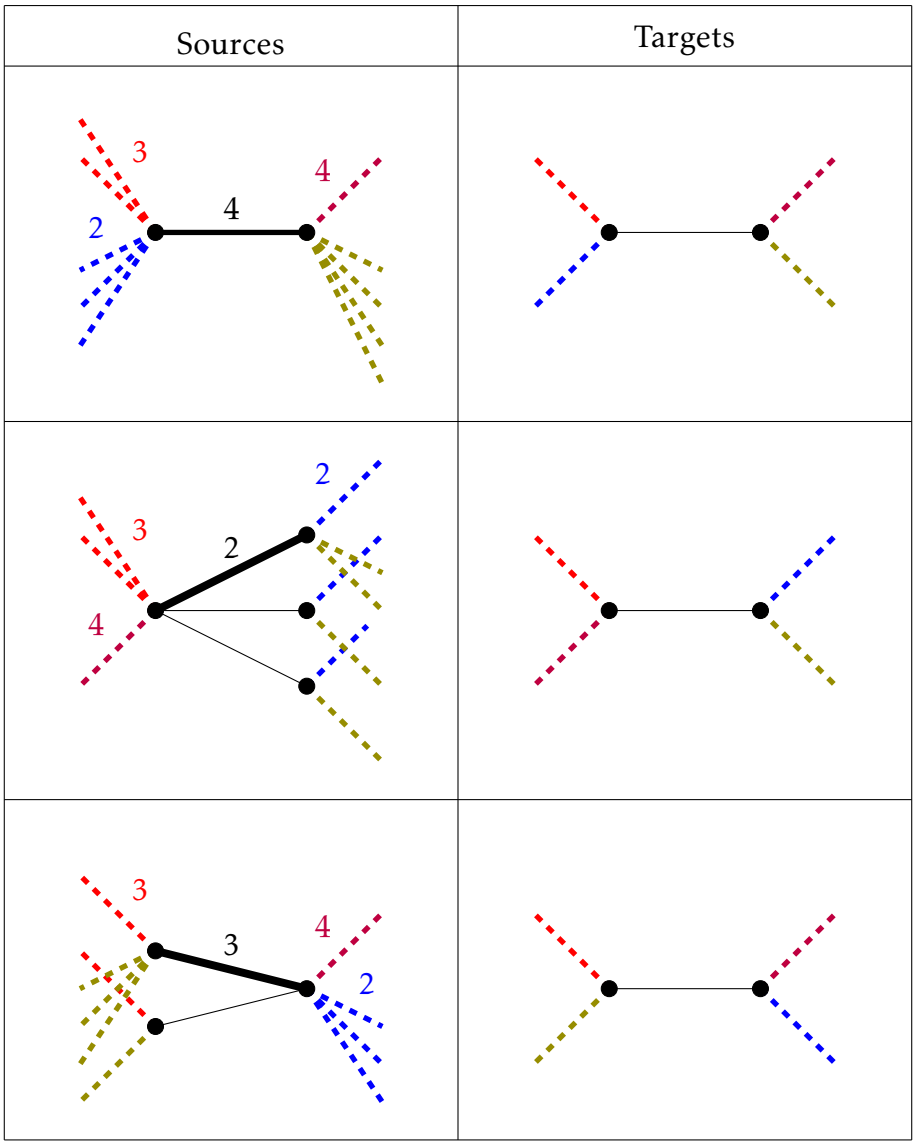


Figure 4.2: Different objects of $\mathbb{EAC}_{4,0}(\alpha,\beta,\gamma,\delta)$

ex machina comes from Jun Li's degeneration formula in an analogous algebrogeometric situation. Namely, we look at (relative) stable maps of degree d from rational curves to $\mathbb{P}^1$ with 4 marked points, where the ramification above these marked points is prescribed by our partitions, and how these degenerate when two of the marked points come together into a new branch. This set-up fits under the deep and more general machinery of [Li01] and [Li02] (see also [Li04] for an outline of the developments of both references), and we apply unapologetically the results therefrom for our purposes. More specifically, we seek to use the degeneration formula from [Li02] to our advantage[3]. For the reader's convenience, we recall the relevant facts and set-up

[3]Is this the only way? Probably not, but perhaps any other similarly spirited approach will involve proving or using an analog of the degeneration formula (be it directly or indirectly) for the situation at hand. The formula is convenient because we are bound to compare contributions coming from all the possible degenerations of some morphisms, and this phenomenon will be present in any other way. As the perennial adage says: As above, so below.

under the exact notation of [Li01] and [Li02], and provide exact references thereto. As ground field we take the complex numbers. Consider $\mathbb{P}^1$ with a closed point $0 \in \mathbb{P}^1$, and a flat and projective family $W \to \mathbb{P}^1$ such that:

- The fiber W_t over any closed $t \in \mathbb{P}^1$ with $t \neq 0$ is isomorphic to $\mathbb{P}^1$.
- The fiber over $0 \in \mathbb{P}^1$ is the union of two $\mathbb{P}^1$'s transversally intersecting at a point. For notational simplicity we denote this fiber by W_0 with the point of intersection P. In addition, we denote the corresponding $\mathbb{P}^1$'s by Y_1 and Y_2, and denote the point P lying in Y_i by $P_i \in Y_i$.

Let $\mathfrak{W}$ denote the stack of expanded degenerations of $W/\mathbb{P}^1$ (Section 1.2 and Definition 1.9 of [Li01]), and let $\mathfrak{M}(\mathfrak{W}, \Gamma)$, with $\Gamma = (0, 2(d+1), d)$, denote the moduli stack of stable morphisms to $\mathfrak{W}$ (see Section 3.1 of loc. cit.) of degree d from $2(d+1)$-marked rational curves. In other words, 0 accounts for the genus, $2(d+1)$ for the number of markings, and d for the degree (it should be $d \cdot$ [the fiber] but we omit this notation). It is shown in loc. cit. that this is a Deligne-Mumford stack that is separated and proper over $\mathbb{P}^1$ (Theorem 3.10 of loc. cit.). In addition, when $t \neq 0$ the fiber products $\mathfrak{M}(\mathfrak{W}, \Gamma) \times_{\mathbb{P}^1} \{t\}$ are naturally isomorphic to the moduli stack of stable morphisms to W_t of the prescribed topological type. The more interesting situation arises at the closed point $0 \in \mathbb{P}^1$, in which case the fiber product

$$\mathfrak{M}(\mathfrak{W}_0, \Gamma) := \mathfrak{M}(\mathfrak{W}, \Gamma) \times_{\mathbb{P}^1} \{0\}$$

is shown to have a perfect obstruction theory (Theorem 2.5 of [Li02]) and carries a well defined virtual fundamental cycle. It is shown in Theorem 3.15 of loc. cit. that this cycle satisfies the following degeneration formula

$$[\mathfrak{M}(\mathfrak{W}_0, \Gamma)]^{\mathrm{virt}} = \sum_{\eta \in \overline{\Omega}} \frac{m(\eta)}{|\mathrm{Eq}(\eta)|} \cdot \Phi_{\eta_*}[\mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \Gamma_1)]^{\mathrm{virt}} \times [\mathfrak{M}(\mathfrak{Y}_2^{\mathrm{rel}}, \Gamma_2)]^{\mathrm{virt}}, \tag{4.4}$$

where the notation is as follows:

- The Γ_i denote topological types, which are specified by admissible graphs (Definition 4.6 of [Li01]) for (Y_i, P_i) (for $i = 1, 2$). An admissible graph (not to be confused with our previous discrete graphs) Γ_i for (Y_i, P_i) is simply a finite collection of vertices, legs and roots (legs and roots are line segments with only one end attached to a vertex), with:
  1. An ordering of the legs, an ordering and weight on the roots, and two weight functions on the set of vertices

$$g : V(\Gamma_i) \to \mathbb{Z}_{\geq 0}, \qquad b : V(\Gamma_i) \to H_2 Y_i,$$

where b assigns algebraic homology classes to vertices of Γ_i. In our case of interest, the function g is constant with value 0, and the functions b are determined by an integer (the degree).

2. The graphs Γ_i are relatively connected. This means, by definition, that either $V(\Gamma_i)$ is a single element, or each vertex in $V(\Gamma_i)$ has at least one root attached to it.

The admissible graphs Γ_1 and Γ_2 appearing in the formula are not arbitrary. These are given by the $\eta \in \overline{\Omega}$ (we will shortly explain what this means), thus they have some compatibility conditions. We depict in Figure 4.5 some admissible graphs for the situation in Example 4.2.2 which is the algebraic counterpart to our previous Example 4.2.1.

- The set $\bar{\Omega}$ is the quotient of the set Ω, which we now describe, by an equivalence relation, which we describe shortly afterwards. The set Ω consists of the triples (Γ_1, Γ_2, I), where:

  1. Γ_1 and Γ_2 are two admissible graphs that have the same number of roots and such that the weights of their j-th roots coincide. In addition, our case of interest requires the total number of legs adds to $2(d+1)$ and the total sum of the degrees is d.
  2. The graph obtained by joining all the j-th roots is a connected tree (the tree condition is due to our case of interest) and has no roots itself.
  3. The I in the triple is an ordering on the set given by the legs of both Γ_1 and Γ_2. It is assumed to extend the ordering of the legs of each Γ_i.

  The equivalence relation is induced from the natural action of the symmetric groups on the roots. More precisely, let $\eta \in \Omega$. If the graphs of η have r roots, then any permutation $\sigma \in S_r$ defines a new element η^σ by reordering the roots according to σ. For $\eta_1, \eta_2 \in \Omega$, we say $\eta_1 \sim \eta_2$ if $\eta_1 = \eta_2^\sigma$ for some σ. Finally, the set $\bar{\Omega}$ is the set of equivalence classes Ω/S_r. For an $\eta \in \overline{\Omega}$, the number $m(\eta)$ is just the product of the weights of the roots and $\mathrm{Eq}(\eta)$ denotes the stabilizer in S_r of any representative of this class.

- The notation $\mathcal{Y}_i^{\mathrm{rel}}$ refers to the stack of expanded relative pairs of (Y_i, P_i) (see Section 4.1 and Definition 4.4 of [Li01]).

- The cycle $[\mathfrak{M}(\mathcal{Y}_i^{\mathrm{rel}}, \Gamma_i)]^{\mathrm{virt}}$ refers to the virtual fundamental cycle (Proposition 3.9 of [Li02]) of $\mathfrak{M}(\mathcal{Y}_i^{\mathrm{rel}}, \Gamma_i)$, the moduli stack of relative stable morphisms to $\mathcal{Y}_i^{\mathrm{rel}}$ of topological type Γ_i (Definition 4.9 of [Li01]). This is also a separated proper Deligne-Mumford stack (Proposition 4.10 of loc. cit.).

- The notation Φ_η, for $\eta = (\Gamma_1, \Gamma_2, I) \in \overline{\Omega}$ as above, refers to the morphism defined by the root glueing construction (joining together the j-th roots of the underlying admissible graphs and reordering the legs according to I) applied to the corresponding (families) maps (Definition 4.11 and (4.10) of loc. cit.):

$$\Phi_\eta : \mathfrak{M}\left(\mathcal{Y}_1^{\mathrm{rel}}, \Gamma_1\right) \times \mathfrak{M}\left(\mathcal{Y}_2^{\mathrm{rel}}, \Gamma_2\right) \longrightarrow \mathfrak{M}(\mathcal{W}_0, \Gamma),$$

  where $\Gamma = (0, 2(d+1), d)$ is the topological type that we have been considering. It is shown in Proposition 4.13 of loc. cit. that this map is finite étale of pure degree $\#\mathrm{Eq}(\eta)$. The image $\mathfrak{M}(\mathcal{Y}_1^{\mathrm{rel}} \sqcup \mathcal{Y}_2^{\mathrm{rel}}, \Gamma)$ is the substack obtained by the glueing process. With the image stack, the degeneration formula takes the form (Corollary 3.13 of [Li02])

$$[\mathfrak{M}(\mathcal{W}_0, \Gamma)]^{\mathrm{virt}} = \sum_{\eta \in \overline{\Omega}} m(\eta) \cdot [\mathfrak{M}(\mathcal{Y}_1^{\mathrm{rel}} \sqcup \mathcal{Y}_2^{\mathrm{rel}}, \eta)]^{\mathrm{virt}}. \tag{4.5}$$

The moduli stack $\mathfrak{M}(\mathcal{W}, \Gamma)$ does not contain as a substack the usual moduli space of stable maps $\overline{M}_{0,2(d+1)}(W, d)$, where d refers to $d \cdot [\text{the fiber}]$. However, because of our setting we can fix our attention towards a common substack and aim to use (4.5) in order to gather information about our balancing weights. We fix distinct sections s_α, s_β, s_γ, and s_δ of $W \to \mathbb{P}^1$ such that

$$Q_\alpha := s_\alpha \times_{\mathbb{P}^1} \{0\}, Q_\beta := s_\beta \times_{\mathbb{P}^1} \{0\} \in Y_1 \text{ and } Q_\gamma := s_\gamma \times_{\mathbb{P}^1} \{0\}, Q_\delta := s_\delta \times_{\mathbb{P}^1} \{0\} \in Y_2$$

(also distinct from the P_i), and inside the moduli space $\overline{M}_{0,2(d+1)}(W, d)$ we consider the subspace (substack) $\overline{M}_{0,(\alpha,\beta|\gamma,\delta)}(W, d)$ given by all stable maps $\mathcal{C} = (C, x_1, \ldots, x_{2(D+1)}, \pi)$, where

$$\pi^* s_\alpha = \sum_{i=1}^{\ell(\alpha)} \alpha_i x_i, \qquad \pi^* s_\beta = \sum_{i=1}^{\ell(\beta)} \beta_i x_{\ell(\alpha)+i},$$
$$\pi^* s_\gamma = \sum_{i=1}^{\ell(\gamma)} \gamma_i x_{\ell(\alpha)+\ell(\beta)+i}, \qquad \pi^* s_\delta = \sum_{i=1}^{\ell(\delta)} \delta_i x_{\ell(\alpha)+\ell(\beta)+\ell(\gamma)+i},$$

and $\alpha = (\alpha_i)$, $\beta = (\beta_i)$, $\gamma = (\gamma_i)$ and $\delta = (\delta_i)$.

This is a 1-dimensional moduli subspace (substack), and it is also a substack of $\mathfrak{M}(\mathcal{W}, \Gamma)$ from before. Our conditions on the partitions and special points force that there are no contracted components to take into account in the expanded degenerations. Indeed, in this specific context the expanded degenerations would add chains of $\mathbb{P}^1$'s but our conditions make that the new added components do not carry any special

points different from the two nodes where other components are glued to. This is impossible because the curve has to have genus 0 and must also be stable.
We denote the fiber product $\overline{M}_{0,(\alpha,\beta|\gamma,\delta)}(W,d) \times_{\mathbb{P}^1} \{0\}$ by $\overline{M}_{0,(\alpha,\beta|\gamma,\delta)}(W_0,d)$, which is zero dimensional. Additionally, the underlying combinatorics of an isomorphism class $\Theta \in [\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)](1)$ determines a subset of points $\overline{M}_\Theta \subset \overline{M}_{0,(\alpha,\beta|\gamma,\delta)}(W_0,d)$, whenever the legs of the target tree with ramification α and β share a common vertex. In fact, these subsets form a partition of $\overline{M}_{0,(\alpha,\beta|\gamma,\delta)}(W_0,d)$. Similar constructions and considerations give rise to the analogous subspaces (substacks) $\overline{M}_{0,(\alpha,\gamma|\beta,\delta)}(W_0,d)$ and $\overline{M}_{0,(\alpha,\delta|\beta,\gamma)}(W_0,d)$.

Example 4.2.2. In Example 4.2.1, the figures 4.1 and 4.2 showed examples of some isomorphism classes of $\mathbb{EAC}_{d,h}((3),(2,1^2),(1^4),(4))$. In figures 4.3 and 4.4 we depict their algebraic counterparts. Additionally, we depict in Figure 4.5 admissible graphs (Definition 4.6 of [Li01]) corresponding to each of the depicted cases in Figure 4.4. In this case, the legs are colored accordingly to the marked points they represent, and the roots are black edges. We omit all the labels to avoid overbearing the depictions.

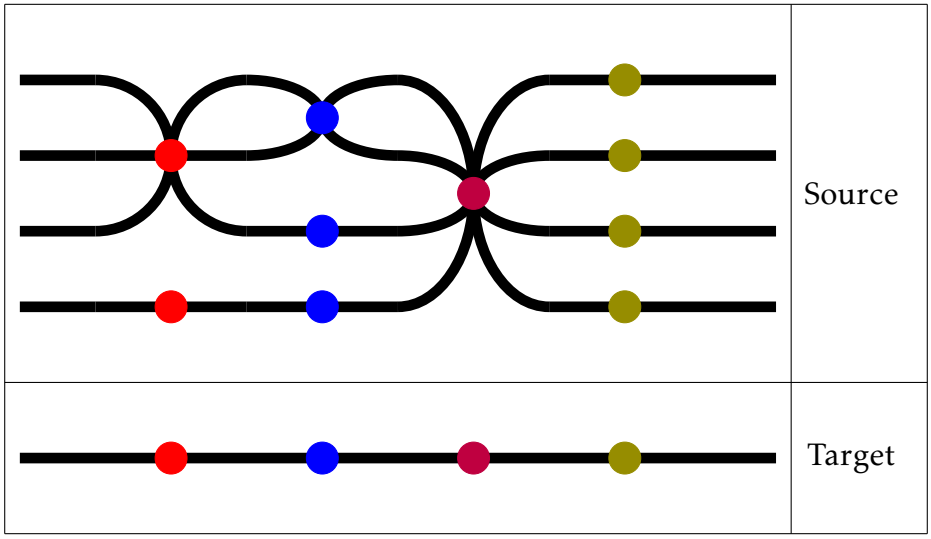


Figure 4.3: Algebraic counterpart of Figure 4.1

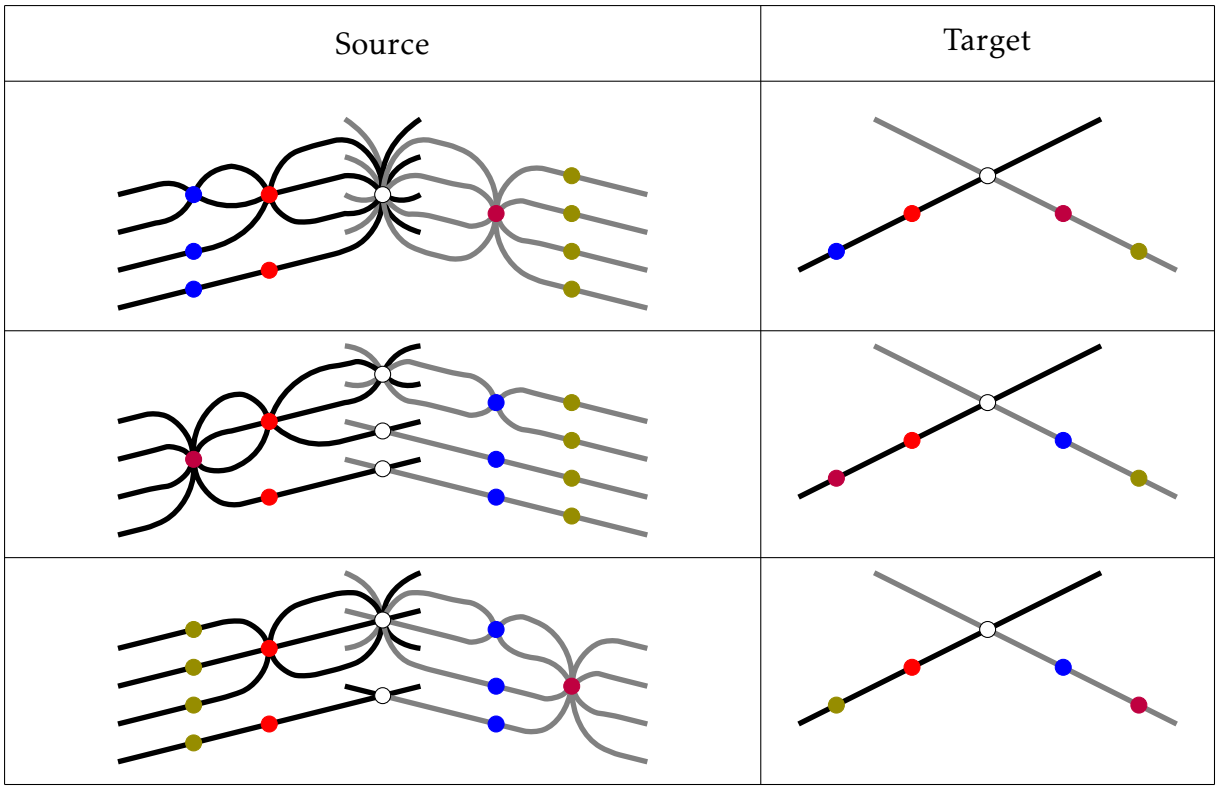


Figure 4.4: Depiction of the algebraic counterpart (without keeping track of the marking) of Figure 4.2. We use colors to specify which components of the source curves lie over which component of the target curve.

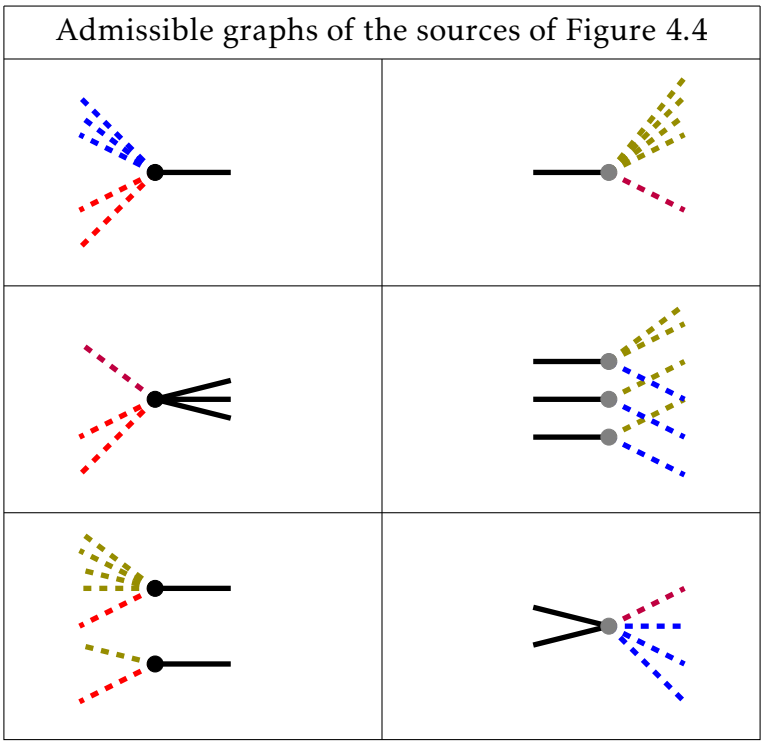


Figure 4.5: Admissible graphs of the sources of Figure 4.4

Notation 4.2.3. For a general subset $I = \{i, j, k, l\}$ of $\{1, \ldots, 2(d+1)\}$ we have the forgetful morphism $\mathrm{ft}_{I^c} : \mathfrak{M}_{0,2(d+1)}(\mathfrak{W}_0, d) \to \overline{M}_{0,I}$, whose restriction to $\overline{M}_{0,(\alpha,\beta|\gamma,\delta)}(W_0, d)$ will be denoted identically.

Lemma 4.2.4. *Let $\pi\colon G \to H$ be an object of $\mathbb{EAC}_{d,h}(\alpha, \beta, \gamma, \delta)$ with isomorphism class $\Theta \in [\mathbb{EAC}_{0,4}(\alpha, \beta, \gamma, \delta)](1)$ and such that the legs with ramification α and β share a common vertex. For a subset of distinct points $I = \{i, j, k, l\}$ of $\{1, \ldots, 2(d+1)\}$, the order of $\mathcal{C}_\Theta \in \overline{M}_\Theta$ at the pull-back by ft_{I^c} of the divisor $(ij|kl)$ of $\overline{M}_{0,I}$ is*

$$\mathrm{ord}_{\mathcal{C}_\Theta} \mathrm{ft}^*_{I^c}(ij|kl) = \prod_{V \in V(G)} \mathrm{CF}(V) \cdot \prod_{e \in E(G)} d_\pi(e) \cdot \sum_{e \in G(ij|kl)} \frac{1}{d_\pi(e)},$$

where $G(ij|kl)$ denotes the set of edges of G between the tuples of legs $\{i, j\}$ and $\{k, l\}$ (that is, $G(ij|kl)$ consists of the edges lying in the intersection of: the path from i to k, the path from i to l, the path from j to k, and the path from j to l).

Notation 4.2.5. In the proof of this lemma we will do several computations with cross-ratios, for which we use the following notation and convention. For $a, b, c, d \in \mathbb{P}^1$, we let $\lambda(a, b, c, d) \in \mathbb{P}^1$ denote the point given by the image of d under the unique automorphism of $\mathbb{P}^1$ that sends the points a, b, c to $0, 1, \infty \in \mathbb{P}^1$ respectively. More explicitly:

$$\lambda(a, b, c, d) = \frac{(b-c)}{(b-a)} \cdot \frac{(d-a)}{(d-c)}.$$

We take this space to also recall the identities:

$$\lambda(a, b, c, d) = \lambda(a, b, e, d) \cdot \lambda(e, b, c, d), \tag{4.6}$$

$$\lambda(a, b, c, d) = \lambda(b, a, d, c). \tag{4.7}$$

Proof. Observe that if $G(ij|kl)$ is empty, then the order is zero and the sum is emptily indexed, hence also zero. So we have to show the lemma when $G(ij|kl)$ is non-empty. We prove the statement about the order by several reductions. Let $\mathcal{C}_\Theta = (C_\Theta, x_1, \ldots, x_{2(D+1)}, \pi_\Theta) \in \overline{M}_\Theta$, and observe that by definition C_Θ with the marked points is a stable curve of genus 0. This means that C_Θ is a tree of $\mathbb{P}^1$'s and the set of edges of $G(ij|kl)$ determines a set of nodes of C_Θ. We observe that $G(ij|kl)$ actually defines a path (of the source graph), and we order this set of edges so that the induced orientation is directed from i or j to k or l. In terms of C_Θ, the set $G(ij|kl)$ determines a chain of projective lines $C_1, \ldots, C_N$ such that

- C_i and C_{i+1} intersect at a node that has weight d_i,
- $C_i \cap C_j$ is non-trivial if and only if $j = i \pm 1$,

- if C_i maps to Y_1 (respectively Y_2), then C_{i+1} maps to Y_2 (respectively Y_1). This is due to the non-emptiness of $G(ij|kl)$.

We depict this situation in Figure 4.6, where we assume that the marked red points lie over Q_α, the olive points lie over Q_β, the purple lie over Q_γ and the blue over Q_δ. We

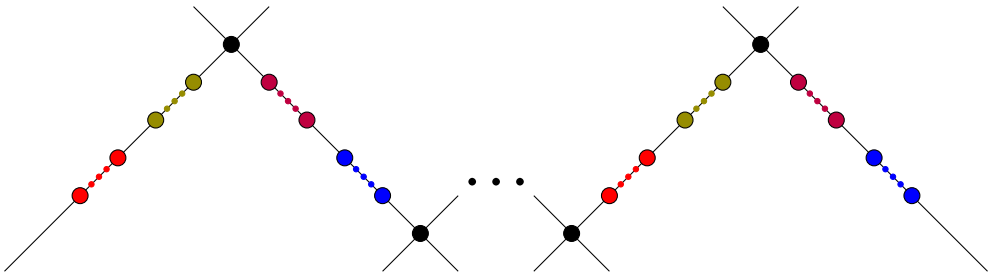


Figure 4.6: Depiction of our situation with a chain of projective lines determined by the edges $G(ij|kl)$

apply the relations (4.6) and (4.7) several times to understand how each node of this chain contributes to the cross ratio $\lambda(i,j,k,l)$.

1. We can assume without loss of generality that:

   - The points i, j and the node determined by the first edge of $G(ij|kl)$ lie on the same component.
   - The points k, l and the node determined by the last edge of $G(ij|kl)$ lie on the same component.

   Indeed, suppose m is an additional point of C_Θ. An application of (4.7), (4.6), and (4.7) (twice) shows that

   $$\lambda(i,j,k,l) = \lambda(i,j,k,m) \cdot \lambda(i,m,k,l). \tag{4.8}$$

   Now, if we assume that

   - i and m (or j and m) lie in the same component as the node given by the first edge of $G(ij|kl)$,
   - j (or i) lies in a different component that is connected to this component at a different node,

   then $\lambda(i,j,k,m)$ is a unit. From this our claim follows.

2. The cross ratio $\lambda(i,j,k,l)$ can be expressed as

   $$\lambda(i,j,k,l) = (\text{unit}) \cdot \prod_{s=1}^{N-1} \lambda(x_s, y_s, x_{s+1}, y_{s+1}), \tag{4.9}$$

where $x_1 = i$, $y_1 = j$, $x_N = k$ and $y_N = l$, and for each $1 \leq s \leq N-1$ the points x_s and y_s are distinct marked points of the component C_s.
To see this claim suppose that m and n are two additional distinct points of C_Θ. From (4.8) we obtain that

$$\lambda(i,j,k,l) = \lambda(i,j,k,n) \cdot \lambda(i,n,k,l).$$

We can now apply (4.6) to $\lambda(i,j,k,n)$ and $\lambda(i,n,k,l)$ and obtain

$$\lambda(i,j,k,l) = (\lambda(i,j,m,n) \cdot \lambda(m,j,k,n)) \cdot (\lambda(i,n,m,l) \cdot \lambda(m,n,k,l)). \tag{4.10}$$

An additional application of (4.7) to $\lambda(m,j,k,n)$ shows that we obtain

$$\lambda(i,j,k,l) = \lambda(i,j,m,n) \cdot \lambda(i,n,m,l) \cdot \lambda(j,m,n,k) \cdot \lambda(m,n,k,l), \tag{4.11}$$

We observe that if

- i and j lie in the same component,
- m and n lie in the same component different from the previous one,
- k and l lie in the same component different from the previous two,

then both $\lambda(j,m,n,k)$ and $\lambda(l,m,n,i)$ are units. Now, we can just fix $x_1 = i$, $y_1 = j$, $x_N = k$ and $y_N = l$, and proceed inductively to define the remaining points. Namely, after x_i and y_i have been defined, we just let x_{i+1} and y_{i+1} be distinct marked points of the ensuing component.

3. Following the notation of the previous item, we can further assume in (4.9) that for $1 \leq s \leq N$ the points x_s and y_s lie over different fibers of the marked points. Indeed, let x' denote an additional point in the component C_s. The relation (4.6) shows that (without loss of generality $s \leq N-1$):

$$\lambda(x_s, y_s, x_{s+1}, y_{s+1}) = \lambda(x', y_s, x_{s+1}, y_{s+1}) \cdot \lambda(x_s, y_s, x', y_{s+1}). \tag{4.12}$$

We observe that $\lambda(x_s, y_s, x', y_{s+1})$ is a unit, and hence our claim. We depict this situation in Figure 4.7.

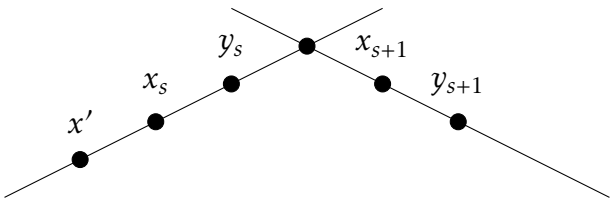


Figure 4.7: Situation of the third item.

In conclusion, we are interested then in the contributions to the order of vanishing that come from the nodes because of (4.9) and the last item. Hence, we restrict ourselves to compute the order of vanishing of $\lambda(i,j,k,l)$ where

- the points i and j lie on the same component but on different fibers,
- the points k and l lie on a different component and on different fibers.

Now, we apply Proposition 1.1 of [Vak00] (or 3.7 in the published version) showing that deformations are local around special loci, so that we can content ourselves with the following very special case: Let $\alpha = (d)$, $\beta = (1^d)$, $\gamma = (d)$ and $\delta = (1^d)$, consider $\mathbb{P}^1$ with the four marked points 0, 1, ∞, and t. We look at the family of smooth covers $(C, x_{t1}, \ldots, x_{td}, x_\infty, x_0, x_{11}, \ldots, x_{1d}, \pi)$ of $\mathbb{P}^1$ of degree D satisfying:

$$\pi^*0 = d \cdot x_0, \qquad \pi^*1 = \sum_i x_{1i},$$
$$\pi^*\infty = d \cdot x_\infty, \qquad \pi^*t = \sum_i x_{ti}.$$

On the source curve we set $x_0 = 0$ and $x_\infty = \infty$, and parameterize this family by $\pi([z_0 : z_1]) = [\lambda z_0^d : z_1^d]$, with $\lambda \in \mathbb{C}^*$. This makes $\{x_{1s}\}$ be the dth-roots of λ and $\{x_{ts}\}$ be the dth roots of $\lambda \cdot t$. Our cover of interest arises when $t \mapsto 0$, and we have to compute the order of vanishing of the cross-ratio $\lambda(x_0, x_{1q}, x_\infty, x_{tp})$ for arbitrary p and q. But we can readily see that $\lambda(x_0, x_{1q}, x_\infty, x_{tp}) = \frac{x_{tp}}{x_{1q}}$, which is a dth root of t, and therefore it vanishes[4] at $t = 0$ with order $\frac{1}{d}$.

To finish the lemma we look at the following form (Corollary 3.13 of [Li02]) of the degeneration formula (4.5)

$$[\mathfrak{M}_{0,2(d+1)}(\mathfrak{W}_0, d)]^{\mathrm{virt}} = \sum_{\eta \in \overline{\Omega}} m(\eta)[\mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}} \sqcup \mathfrak{Y}_2^{\mathrm{rel}}, \eta)]^{\mathrm{virt}}, \tag{4.13}$$

There are multiple $\eta \in \overline{\Omega}$ that contribute to $\mathrm{ord}_{\mathcal{C}_\Theta} \mathrm{ft}^*_{I^c}(ij|kl)$, namely the different $\eta \in \overline{\Omega}$ that glue to this curve. For any of these, the factor $m(\eta)$ coincides (by definition) with $m(\eta) = \prod_{e \in E(G)} d_\pi(e)$, and the above procedure shows that each η glueing to this curve contributes

$$m(\eta) \sum_{e \in G(ij|kl)} \frac{1}{d_\pi(e)}. \tag{4.14}$$

[4] A more formal execution of the computation would involve actually constructing the moduli space of these covers and performing the same computation. More specifically, our parameter λ is not a parametrization of this moduli space because we must also keep track of the markings of the $\{x_{1i}\}$ and the $\{x_{ti}\}$. Such moduli space would cover of our parametrization wtih λ, and when taking the automorphisms into account we would obtain this order of $\frac{1}{d}$. Our willingness to sacrifice formality and our general preference for the other argument not only stem out of its straightforwardness and simplicity, but also because of its obvious direct answer to an age-old question: How can an algebraic vanishing order be $\frac{1}{d}$? (Because it comes from a dth root)

On the other hand, the number of η's glueing to $\mathcal{C}_\Theta$ is the product $\prod_{V\in V(G)}\mathrm{CF}(V)$, because we have to account the reordering possibilities of the legs (with their weights) of the admissible graphs of η, and the (non-simultaneous) permutation of the roots that have the same weight. This gives the equation we needed to show for the lemma. □

All the ingredients for the final result are finally at our disposition.

Theorem 4.2.6. *Let $J\subset\{1,\dots,2(d+1)\}$. The standard weight $\omega^J_{d,0,(\alpha,\beta,\gamma,\delta)}$ is a Minkowski weight.*

Proof. As we have previously mentioned, we first do the case of $J=\{1,\dots,2(d+1)\}$, and use it to establish the others. This means that we focus on the weight $\omega_{d,0,(\alpha,\beta,\gamma,\delta)}$ and show that (4.3) holds. This is an equation in the space $\mathcal{M}^{\mathrm{trop}}_{0,2(d+1)}\times\mathcal{M}^{\mathrm{trop}}_{0,4}$. Therefore, we check it in the $\mathcal{M}^{\mathrm{trop}}_{0,4}$-coordinates (target) and the $\mathcal{M}^{\mathrm{trop}}_{0,2(d+1)}$-coordinates (source).

- For the $\mathcal{M}^{\mathrm{trop}}_{0,4}$-coordinates, let $\mathrm{CF}(\alpha,\beta,\gamma,\delta)$ denote the combinatorial factor of the vertex of an initial object of $\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)$. In the underlying vector space of $\mathcal{M}^{\mathrm{trop}}_{0,4}$ the equation $v_{\{1,2\}}+v_{\{1,3\}}+v_{\{1,4\}}=0$ holds (following the notation of Section 2.3), and to show our equation of interest we will make use of a result from [CMR16] (as well as their notation). An object $\pi\colon G\to H$ of $\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)$ corresponds to a combinatorial type of loc. cit. and furthermore (following the notation of loc. cit.)
$$\#\,\mathrm{Aut}_0(\pi)=\frac{\mathrm{CF}(\alpha,\beta,\gamma,\delta)}{\prod_{V\in V(G)}\mathrm{CF}(V)}. \tag{4.15}$$
In the sum
$$\sum_{\Theta\in[\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)](1)}\omega_{d,0,(\alpha,\beta,\gamma,\delta)}(\Theta)\cdot\mathrm{trgt}(u_\Theta), \tag{4.16}$$
the vector $v_{\{1,2\}}$ appears with factor given by the weighted sum
$$\sum\omega_{d,0,(\alpha,\beta,\gamma,\delta)}(\Theta), \tag{4.17}$$
where the index Θ runs over the isomorphism classes $[\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)](1)$ where the legs of the target tree with ramification α and β share a common vertex. If we divide (4.17) by $\mathrm{CF}(\alpha,\beta,\gamma,\delta)$, then because of (4.15) it follows from Theorem 2 of loc. cit. and its proof (Section 5.4.1), that we obtain $H_{0\to0}(\alpha,\beta,\gamma,\delta)$ (where $H_{0\to0}(\alpha,\beta,\gamma,\delta)$ denotes the corresponding rational quadruple Hurwitz number). More succinctly, the vector $v_{\{1,2\}}$ appears in the sum (4.16) with factor $H_{0\to0}(\alpha,\beta,\gamma,\delta)\cdot\mathrm{CF}(\alpha,\beta,\gamma,\delta)$ Analogously, the vector $v_{\{1,3\}}$ appears with factor $H_{0\to0}(\alpha,\gamma,\beta,\delta)\cdot\mathrm{CF}(\alpha,\beta,\gamma,\delta)$, and the vector $v_{\{1,4\}}$ appears with factor $H_{0\to0}(\alpha,\delta,\beta,\gamma)\cdot\mathrm{CF}(\alpha,\beta,\gamma,\delta)$. Since
$$H_{0\to0}(\alpha,\beta,\gamma,\delta)=H_{0\to0}(\alpha,\gamma,\beta,\delta)=H_{0\to0}(\alpha,\delta,\beta,\gamma),$$

it follows that

$$\sum_{\Theta\in[\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)](1)} \varpi_{d,0,(\alpha,\beta,\gamma,\delta)}(\Theta)\cdot\mathrm{trgt}(u_\Theta) = H_{0\to 0}(\alpha,\beta,\gamma,\delta)\cdot(v_{\{1,2\}}+v_{\{1,3\}}+v_{\{1,4\}}) = 0.$$

- For the $\mathcal{M}^{\mathrm{trop}}_{0,2(d+1)}$-coordinates, we will use Lemma 3.6.5. Let $I=\{i,j,k,l\}$ be an arbitrary subset of $\{0,\dots,2(d+1)\}$, and consider a triple (T_G,T_H,π) of $\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)$ with isomorphism class $\Theta\in[\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)](1)$. An edge $e\in E(G)$ gives a partition of the legs of G into two sets, say A and B, which by definition must each be of size ≥ 2 and therefore (following the notation of Section 2.3 for the vectors generating the lattice $N_{\mathrm{dist}_{2(d+1)}}$) $v_A=v_B$. We let v_e denote this common vector, and observe that the definition of the associated matrix to π (1.13) implies that

$$\varpi_{d,0,(\alpha,\beta,\gamma,\delta)}(\Theta)\cdot\mathrm{src}(u_\Theta) = \sum_{e\in E(G)}\left(\prod_{V\in V(G)}\mathrm{CF}(V)\cdot\prod_{h\in E(G)}d_\pi(h)\right)\frac{1}{d_\pi(e)}\cdot v_e.$$

Hence, it follows from Lemma 4.2.4 that $v_{\{i,j\}}$ appears in the sum

$$\sum_{\Theta\in[\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)](1)} \varpi_{d,0,(\alpha,\beta,\gamma,\delta)}(\Theta)\cdot\mathrm{ft}_{I^c}\,\mathrm{src}(u_\Theta) \tag{4.18}$$

with a factor of $\deg\mathrm{ft}^*_{I^c}(ij|kl)$. This same argument, as well as Lemma 4.2.4, hold for the other splittings of I, so that the sum 4.18 contains the vectors $v_{\{i,k\}}$ and $v_{\{i,l\}}$ with corresponding factors of $\deg\mathrm{ft}^*_{I^c}(ik|jl)$ and $\deg\mathrm{ft}^*_{I^c}(il|jk)$. Since the divisors $(ij|kl)$, $(ik|jl)$, and $(il|jk)$ are linearly equivalent, it follows then that

$$\sum_{\Theta\in[\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)](1)} \varpi_{d,0,(\alpha,\beta,\gamma,\delta)}(\Theta)\cdot\mathrm{ft}_{I^c}\,\mathrm{src}(u_\Theta) = \deg\mathrm{ft}^*_{I^c}(ij|kl)\cdot\left(v_{\{i,j\}}+v_{\{i,k\}}+v_{\{i,l\}}\right) = 0. \tag{4.19}$$

We apply Lemma 3.6.5 to conclude that

$$\sum_{\Theta\in[\mathbb{EAC}_{d,h}(\alpha,\beta,\gamma,\delta)](1)} \varpi_{d,0,(\alpha,\beta,\gamma,\delta)}(\Theta)\cdot\mathrm{src}(u_\Theta) = 0.$$

Now, for $J\subsetneq\{1,\dots,2(d+1)\}$ we remark that, in this special case, forgetting the marking is a proper morphism of linear poic-complexes, since the linear poic-complexes that we are dealing with can be considered as 1-dimensional closed fans in a vector space. Then we can just push forward through the respective forgetting the marking morphism (namely forgetting the whole J^c) and we are done, since the lattice indexes that arise are always 1. □

We conclude this section with a proof of Theorem 4.1.3. This result is central to our

whole enterprise, and afterward, we will seek to apply it for our initial purpose. We restate the theorem for the reader's convenience.

Theorem 4.1.3. *The standard weight $\omega^J_{d,h,\vec{\mu}}$ is an $\mathrm{est}^J_{d,h,\vec{\mu}}$-equivariant Minkowski weight.*

Proof. Similar to the case of Lemma 4.2.6, we first study the initial case of $J = \{1,\dots,n\}$, and subsequently explain how to deduce the others from this. Just as in Theorem 4.2.6, the balancing of $\omega_{d,h,\vec{\mu}}$ follows from the balancing in both coordinates $\mathcal{M}^{\mathrm{trop}}_{0,m+2h} \times \mathbb{R}^h_{>0}$ and $\mathcal{M}^{\mathrm{trop}}_{0,n+2g} \times \mathbb{R}^g_{>0}$. Moreover, by default it just involves the $\mathcal{M}^{\mathrm{trop}}_{0,m+2h}$- and $\mathcal{M}^{\mathrm{trop}}_{0,n+2g}$-coordinates, since we are never contracting any edges parameterized by the $\mathbb{R}^h_{>0}$- or the $\mathbb{R}^g_{>0}$-coordinates.

- For the balancing along the $\mathcal{M}^{\mathrm{trop}}_{0,m+2h} \times \mathbb{R}^h_{>0}$-coordinates we have to keep in mind that we are only contracting edges of the target graph whose fiber is acyclic. Then we proceed exactly as in Lemma 4.2.6 with $\mathcal{M}^{\mathrm{trop}}_{0,4}$-coordinates, just with several components. Since the argument is exactly the same, we rather omit it.

- The balancing along the $\mathcal{M}^{\mathrm{trop}}_{0,n+2g}$-coordinates demands a more careful examination. Consider a triple (T_G, T_H, π) of $\mathbb{EAC}_{d,h}(\vec{\mu})$ of codimension 1, and let $G = \mathrm{st}_{g,n}(T_G)$ and $H = \mathrm{st}_{h,m}(T_H)$. The triple being of codimension 1 means that H has a unique 4-valent vertex, which we will denote by W and let $\pi^{-1}(W) = \{V_1,\dots,V_k\} \subset V(G)$. Without loss of generality we will refer to the triple simply by π, and additionally assume that for any isomorphism class $\Theta \in \mathrm{Star}^1(\pi)$ there is given a representative object $(T_{G,\Theta}, T_{H,\Theta}, \pi_\Theta)$ with $G_\Theta := \mathrm{st}_{g,n}(T_{G,\Theta})$ and $H_\Theta := \mathrm{st}_{h,m}(T_{H,\Theta})$. For each $1 \le j \le k$ and each $T_{G,\Theta}$, we let $T_{\Theta,j}$ denote the connected legless subtree of G_Θ that contracts to V_j.

  Just as in the proof of Lemma 4.2.6, any edge $e \in E(T_{\Theta,j})$ gives a partition of the legs of G into two sets, say $A \sqcup B = L(G)$, which by definition must each be of size ≥ 2 and therefore $v_A = v_B$ (again following the notation of the vectors generating the lattice $N_{\mathrm{dist}(n+2g)}$ of Section 2.3). We let v_e denote this common vector, and observe that (just as in the previous proof) for $\Theta \in \mathrm{Star}^1(\pi)$ we have

$$\omega_{d,h,\vec{\mu}}(\Theta)\mathrm{src}(u_\Theta) = K \cdot \sum_{t=1}^{k} \sum_{e \in E(T_{\Theta,t})} \left(\prod_{V \in V(T_{\Theta,t})} \mathrm{CF}(V) \cdot \prod_{h \in E(T_{\Theta,t})} d_\pi(h) \right) \frac{1}{d_\pi(e)} v_e, \quad (4.20)$$

  where K is a factor coming from $\omega_{d,h,\vec{\mu}}(\Theta)$ but common along all the elements of $\mathrm{Star}^1(\pi)$ as it pertains to the vertices and edges not in the subtrees $T_{\Theta,j}$ (for $1 \le t \le k$) (we prefer to just call it K to lessen the burden of an already complex notation). Therefore, to show balancing it is sufficient to show that for each $1 \le$

$t \leq k$, the following equation holds in the corresponding quotient space

$$\sum_{\Theta \in \mathrm{Star}^1(\pi)} \sum_{e \in E(T_{\Theta,t})} \left(\prod_{V \in V(T_{\Theta,t})} \mathrm{CF}(V) \cdot \prod_{h \in E(T_{\Theta,t})} d_\pi(h) \right) \frac{1}{d_\pi(e)} v_e = 0. \tag{4.21}$$

Again, we seek to apply the useful Lemma 3.6.5, and hence we let $I = \{i, j, k, l\} \subset \{1, \ldots, n\}$ be arbitrary. Now, observe that we can turn each $T_{\Theta,t}$ to the case of Theorem 4.2.6 by letting the other edges and legs of $T_{G,\Theta}$ that are adjacent to vertices of $T_{\Theta,t}$ be legs and keeping their corresponding weights given by the cover π_Θ. If we apply the forgetting-the-marking morphism ft_{I^c} to (4.21), then we arrive at the same equation (4.19) (and its vanishing) or we obtain a contribution that lies in the subspace we are modding out. Since I and $1 \leq t \leq k$ were arbitrary, we obtain the equation (4.21) and hence $\omega_{d,h,\vec{\mu}}$ is an equivariant $\mathrm{est}_{d,h,\vec{\mu}}$-Minkowski weight.

To finalize, we consider an arbitrary $J \subsetneq \{1, \ldots, n\}$ and explain the balancing of $\omega^J_{d,h,\vec{\mu}}$. Again, we must consider the $\mathcal{M}^{\mathrm{trop}}_{0,2h+m}$- and $\mathcal{M}^{\mathrm{trop}}_{0,J\sqcup \mathrm{g}}$-coordinates. The balancing along the first coordinates is identical to the previous situation (namely, the same as in Lemma 4.2.6 with $\mathcal{M}^{\mathrm{trop}}_{0,4}$-coordinates), so we focus on balancing along the $\mathcal{M}^{\mathrm{trop}}_{0,J\sqcup \mathrm{g}}$-coordinates. We remark that the forgetful morphism ft_{J^c} yields a morphism of poic-complexes $\mathrm{EST}_{d,h}(\vec{\mu}) \to \mathrm{EST}^J_{d,h}(\vec{\mu})$, and at isomorphism classes we have the equality $\omega^J_{d,h,\vec{\mu}} \circ \mathrm{ft}_{J^c} = \omega_{d,h,\vec{\mu}}$. We do observe that this morphism is neither necessarily proper nor weakly-proper (in any dimension). The idea is to proceed in exactly the same manner as before, keeping track of the edges and legs that are being forgotten.

Consider an object (T_s, T_t, π) of $\mathbb{EAC}^J_{d,h}(\vec{\mu})$ of codimension 1. The triple being of codimension 1 means that $\mathrm{trgt}(\pi)$ has a unique 4-valent vertex, say W, and let $\pi^{-1}(W) = \{V_1, \ldots, V_k\} \subset V(\mathrm{src}(\pi))$. In contrast to the previous case (J being the whole set $\{1, \ldots, n\}$), we do not have to consider the deformation around every V_i, but only around those that after applying ft_{J^c} to $\mathrm{src}(\pi)$ lie on an actual edge of T_s. Let us assume without loss of generality that these are all the V_i with $i \leq k'$ for a $k' \leq k$.

Just as before, we simply refer to the triple (T_s, T_t, π) by π and assume that for any isomorphism class $\Theta \in \mathrm{Star}^1(\pi)$ there is given a representative object $(T_{s,\Theta}, T_{t,\Theta}, \pi_\Theta)$. It is worth noting that if T_G is an object of $\mathbb{G}_{0,2g+n}$ with $\mathrm{st}_{g,n}(T_G) = \mathrm{src}(\pi)$ and $\mathrm{ft}_{J^c}(T_G) = T_s$, then there is an explicit bijection between $\mathrm{Star}^1((T_s, T_t, \pi))$ and $\mathrm{Star}^1((T_G, T_t, \pi))$. For each $1 \leq j \leq k'$ and each $T_{s,\Theta}$, we let $T_{\Theta,j}$ denote the connected legless subtree of $\mathrm{src}(\pi_\Theta)$ that contracts to V_j. Proceeding in the exact same way as above, for $\Theta \in \mathrm{Star}^1(\pi)$ we

obtain the following analog of (4.20) in this situation

$$\omega^J_{d,h,\vec{\mu}}(\Theta)\mathrm{src}(u_\Theta) = K \cdot \sum_{t=1}^{k'} \sum_{e\in E(T_{\Theta,t})} \left(\prod_{V\in V(T_{\Theta,t})} \mathrm{CF}(V) \cdot \prod_{h\in E(T_{\Theta,t})} d_\pi(h) \right) \frac{1}{d_\pi(e)} \mathrm{ft}_{J^c} v_e, \quad (4.22)$$

where once again K is a factor coming from $\omega^J_{d,h,\vec{\mu}}(\Theta)$ but common along all the elements of $\mathrm{Star}^1(\pi)$ as it pertains to the vertices and edges not in the subtrees $T_{\Theta,j}$ (for $1 \le t \le k'$). At the risk of being too repetitive, we observe, just as in the previous case, that for balancing it is sufficient to show that for each $1 \le t \le k'$, the following equation holds in the corresponding quotient space

$$\sum_{\Theta\in\mathrm{Star}^1(\pi)} \sum_{e\in E(T_{\Theta,t})} \left(\prod_{V\in V(T_{\Theta,t})} \mathrm{CF}(V) \cdot \prod_{h\in E(T_{\Theta,t})} d_\pi(h) \right) \frac{1}{d_\pi(e)} \mathrm{ft}_{J^c} v_e = 0. \quad (4.23)$$

But we obtain this equation immediately from what we have already argued after (4.21). More precisely, for $I = \{i,j,k,l\} \subset \{1,\dots,n\}$ we have that $\mathrm{ft}_{I^c} \circ \mathrm{ft}_{J^c} = \mathrm{ft}_{I^c\cup J^c}$, then the only non-trivial case that arises is when $I \subset J$ and in this case $\mathrm{ft}_{I^c} \circ \mathrm{ft}_{J^c} = \mathrm{ft}_{I^c}$. Thus, it follows from the vanishing of (4.23), that $\omega^J_{d,h,\vec{\mu}}$ is a $\mathrm{est}^J_{d,h,\vec{\mu}}$-equivariant Minkowski weight. □

4.3 How many covers of trees?

In this section we apply our results to the case of discrete admissible covers of trees with simple ramification over the legs. More precisely, we will consider the following set-up: Let $g, r \ge 0$ be integers with $g + r \equiv 0 \mod 2$, and let

$$\begin{aligned} &h = 0, && m = 3\cdot g + r, && d = \frac{g+r}{2} + 1, \\ &\vec{\mu} = (\mu_1,\dots,\mu_m), \text{ with } \mu_i = (2,1^{d-2}), && \text{for all } 1 \le i \le m, \\ &n = (d-1)\cdot 3g = \frac{3}{2}(g+r)g, && J = \{(i-1)(d-1)+1\}_{i=1}^{r}. \end{aligned} \quad (4.24)$$

We can readily observe that $\mathrm{EST}^J_{d,h,\vec{\mu}}$ is of dimension $m - 3 = 3g + r - 3$, which is equal to the dimension of $\mathrm{ST}_{g,J}$. Since $\mathfrak{src}$ is weakly proper in top dimension, Proposition 3.4.2 shows that we can pushforward the $\mathrm{est}^J_{d,h,\vec{\mu}}$-equivariant Minkowski weight $\omega^J_{d,h,\vec{\mu}}$ through $\mathfrak{src}$ to produce a top dimensional $\mathrm{st}_{g,J}$-equivariant tropical cycle. It follows from Proposition 3.5.10 that $\mathrm{st}_{g,J}$ is an irreducible poic-fibration, which implies that this pushforward is an scalar multiple of the fundamental cycle. It turns out that it is a non-trivial integral multiple of $C_{\frac{g+r}{2}}$, the $\left(\frac{g+r}{2}\right)$th Catalan number. For the unfamiliar

reader, we recall that the nth Catalan number is given as

$$C_n := \frac{1}{n+1}\binom{2n}{n}, \tag{4.25}$$

and we will make use of the well-known fact that C_n coincides with the number of *Dyck paths* of length $2n$ [OEI24, A000108]. We first bring forward some combinatorial notations and definitions, as well as the statement of a (combinatorial) lemma whose proof we postpone to the end of the section. Subsequently, we apply our machinery and compute the aforementioned pushforward.

We consider $\mathrm{EST}^J_{d,h,\vec{\mu}}$ and $\mathrm{AC}_{d,h}(\vec{\mu})$. As it might be evident from (4.24), this choice of J is motivated mainly by $\mathrm{st}_{g,J}$. To be more precise, for an object π of $\mathbb{AC}_{d,h}(\vec{\mu})$, we want to keep the first r-marked legs of $\mathrm{src}(\pi)$ that have weight 2.

Definition 4.3.1. Suppose T is an object of $\mathbb{G}_{h,m}$ and let $V \in V(T)$. The *leg valency* of V is defined as

$$\mathrm{legval}(V) := \#\{\ell \in L(T) : r_T(\ell) = V\}. \tag{4.26}$$

We say that an edge $e \in E(T)$ is a *leaf* if there is $V \in \partial e$ with $\mathrm{val}(V) = 3$ and $\mathrm{legval}(V) = 2$.

We postpone the proof of the following lemma to the end of the section. The structure of the cover in Lemma 4.3.2 is depicted in Figure 4.8, where the legs are color coded to depict the correspondence of the cover π.

Lemma 4.3.2. *Following the notation of* (4.24)*, let π be an object of $\mathbb{AC}_{d,h}(\vec{\mu})$ of top dimension and such that the composition $\left(\eta_{\mathrm{ft}_{J^c},\mathrm{src}(\pi)} \circ F_\pi\right)$ is invertible, where F_π is as in* (1.13)*. If $W \in V(\mathrm{src}(\pi))$ is such that*

- *W is also a vertex of $\mathrm{ft}_J(\mathrm{src}(\pi))$,*
- *$\mathrm{val}_{\mathrm{ft}_{J^c}(\mathrm{src}(\pi))}(W) = 3$,*
- *W is incident to a loop and an edge in $\mathrm{ft}_{J^c}(\mathrm{src}(\pi))$,*

then $\pi(W)$ is incident to a leaf of $\mathrm{trgt}(\pi)$ and $\mathrm{legval}(\pi(W)) = 1$. Furthermore, if e_1 denotes the leaf of $\mathrm{trgt}(\pi)$ that $\pi(W)$ is incident to, V denotes the other vertex of $\mathrm{trgt}(\pi)$ incident to e_1, and e_2 denotes the edge of $\mathrm{trgt}(\pi)$ that is not incident to $\pi(W)$, then we have the following:

- *The ramification profile above e_1 is (1^d).*
- *The ramification profile above V is $(2, 1^{d-2})$.*
- *The ramification profile above e_2 is $(2, 1^{d-2})$.*

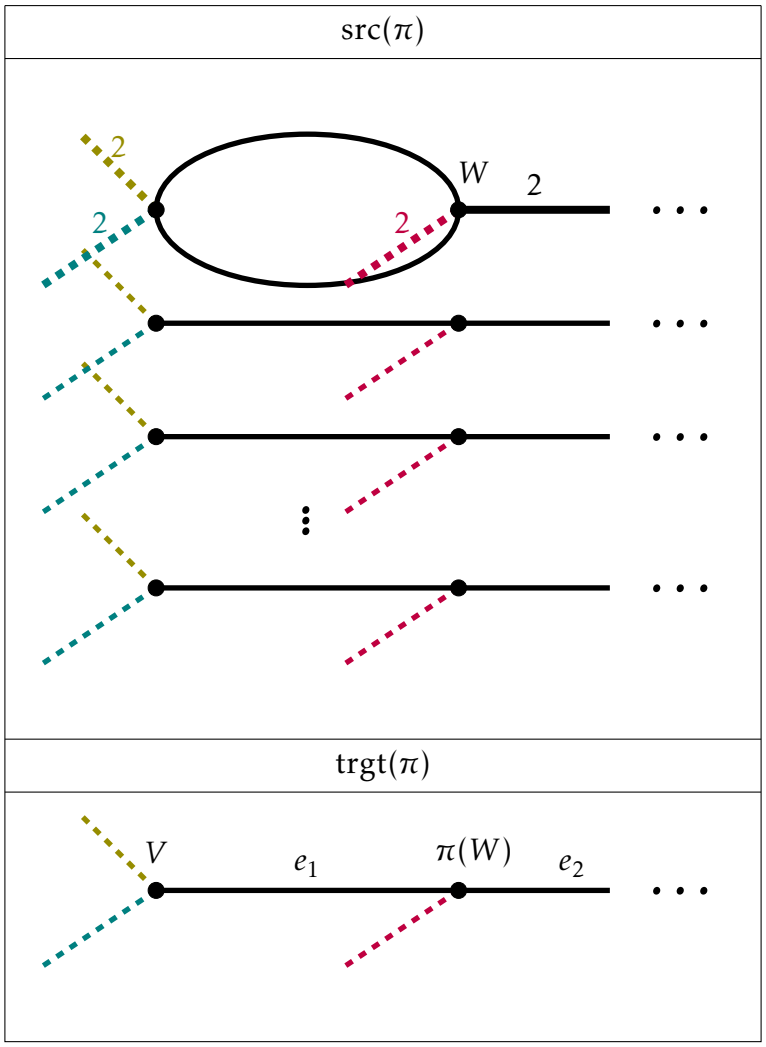


Figure 4.8: The cover π of Lemma 4.3.2

Notation 4.3.3. For the sake of simplicity, when $r > 0$ we set $j_i = (i-1)(d-1)+1$ for $1 \leq i \leq r$. With this notation we have that $J = \{j_1 < j_2 < \dots j_r\}$.

Definition 4.3.4. Suppose $r > 0$ in the notation of (4.24). The genus-g J-marked discrete graph[5] $O_{g,J}$ (we illustrate $O_{g,J}$ for $g = 4$ and $r = 2$ in Figure 4.9) is the object of $\mathbb{G}_{g,J}$ given by the following discrete graph:

- It has vertices $V_1, \dots, V_g$, $W_0, \dots, W_{g-3}$, and $C_1, \dots, C_r$. We set $C_{r-1} = C_r$ if $g = 1$.
- There are g loops l_i, for $1 \leq i \leq g$, with $\partial l_i = \{V_i\}$.
- There are edges $e_0, \dots, e_{g-4}$ with $\partial e_i = \{W_i, W_{i+1}\}$ for $0 \leq i \leq g-4$.
- There is an edge h'_1 with $\partial h'_1 = \{V_1, C_1\}$. If $g > 2$, there are edges $h_1, h_2, \dots, h_g$ with $\partial h'_1 = \{V_1, C_1\}$, $\partial h_1 = \{C_r, W_0\}$, $\partial h_i = \{V_i, W_{i-2}\}$ for $2 \leq i \leq g-1$, and $\partial h_g = \{V_g, W_{g-3}\}$. If $g = 2$, there is an edge h_1 with $\partial h_1 = \{C_r, V_2\}$.

[5]The notation with the letter O comes from the resemblance of this graph to an olive branch.

- There are edges $c_1, \ldots, c_{r-2}$ with $\partial c_i = \{C_i, C_{i+1}\}$ for $0 < i < r-1$. If $g > 1$, then there is an additional edge c_{r-1} with $\partial c_{r-1} = \{C_{r-1}, C_r\}$.
- There are legs[6] $L_{j_1}, \ldots, L_{j_r}$ with $\partial L_{j_i} = C_i$ for $1 \leq i \leq r$.

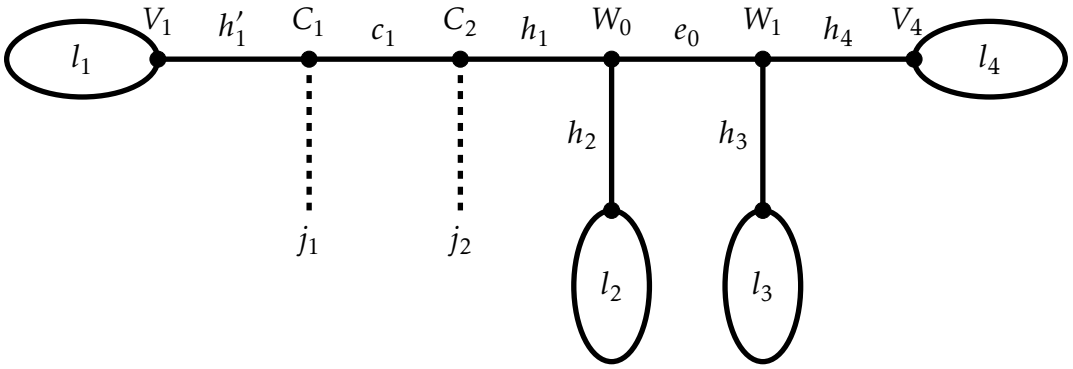


Figure 4.9: Depiction of $O_{g,J}$ for $g = 4$ and $r = 2$

Definition 4.3.5. Suppose $r = 0$ in the notation of (4.24), so that $J = \varnothing$. The genus-g discrete graph O_g (we illustrate O_g for $g = 6$ in Figure 4.10) is the object of $\mathbb{G}_{g,0}$ given by the following discrete graph:

- It has vertices $V_1, \ldots, V_g$ and $W_0, \ldots, W_{g-3}$.
- There are g loops l_i, $1 \leq i \leq g$, with $\partial l_i = \{V_i\}$.
- There are edges $e_0, \ldots, e_{g-4}$ with $\partial e_i = \{W_i, W_{i+1}\}$ for $0 \leq i \leq g-4$.
- There are edges $h_1, \ldots, h_g$, with $\partial h_1 = \{W_0, V_1\}$, $\partial h_i = \{V_i, W_{i-2}\}$ for $2 \leq i \leq g-1$, and $\partial h_g = \{V_g, W_{g-3}\}$.

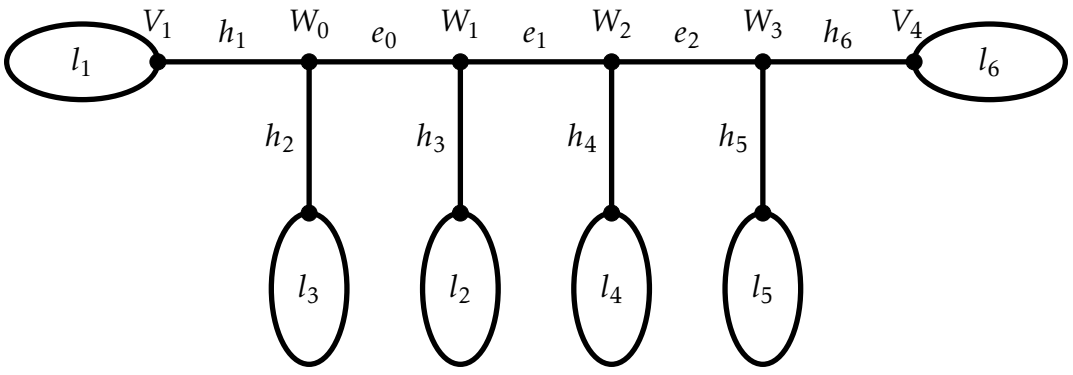


Figure 4.10: Depiction of O_6

[6]We apologize for notational inconsistency. Throughout the document the legs of a graph have always been denoted by ℓ, but this is quite similar to the way we are denoting loops. So in the spirit of clarity, we decided to denote the legs in these very special cases by L.

We now deal with some generalities concerning the permutation of the markings, and make use of the set-up (4.24). The group $(S_{d-2})^m$ acts naturally on $\mathrm{AC}_{d,h}(\vec{\mu})$ as follows: For an element $(\alpha_1,\dots,\alpha_m)\in(S_{d-2})^m$, we examine the following association

$$\vec{\alpha}\cdot(\bullet)\colon \mathbb{AC}_{d,h}(\vec{\mu})\to\mathbb{AC}_{d,h}(\vec{\mu}),\pi\mapsto\vec{\alpha}\cdot(\pi),$$

where $\vec{\alpha}\cdot(\pi)$ is the discrete admissible cover obtained by letting α_i permute the $(d-2)$ marked legs of weight 1 of $\mathrm{src}(\pi)$ in the fiber of the ith marked leg of T_t. At morphisms this association is defined as the morphism between the targets obtained by the corresponding permutations of the marked legs. This actually gives rise to a functor, and we let $\eta_{\vec{\alpha}}$ denote the natural transformation $\mathrm{AC}_{d,h}(\vec{\mu})\implies\mathrm{AC}_{d,h}(\vec{\mu})\circ\phi_{\vec{\alpha}}$ given by the linear maps corresponding to the permutations of the marked legs. In other words, this defines an automorphism $\vec{\alpha}\cdot(\bullet)$ of the poic-space $\mathrm{AC}_{d,h}(\vec{\mu})$. It can be further observed that we actually obtain an action of $(S_{d-2})^m$ on $\mathrm{AC}_{d,h}(\vec{\mu})$.
Similarly, the group S_m acts naturally on $\mathrm{AC}_{d,h}(\vec{\mu})$. Namely, if π is an object of $\mathbb{AC}_{d,h}(\vec{\mu})$ and $\beta\in S_m$, then we let $\beta(\pi)$ denote the object of $\mathbb{AC}_{d,h}(\vec{\mu})$ obtained by letting β permute the marking of the target tree $\mathrm{trgt}(\pi)$, and correspondingly permuting the marking of the source graph $\mathrm{src}(\pi)$. Just as in the previous paragraph, the analogous constructions actually give rise to an action of S_m on the poic-space $\mathrm{AC}_{d,h}(\vec{\mu})$.

Theorem 4.3.6. *Suppose g, h, d, m, $\vec{\mu}$, n, and J are as in* (4.24)*. Then*

$$\mathfrak{src}_*\omega^J_{d,h,\vec{\mu}}=\left((3g)!\cdot\left(\left(\frac{g+r}{2}-1\right)!\right)^{3g+r}\cdot C_{\frac{g+r}{2}}\right)\cdot[\mathrm{st}_{g,J}]. \tag{4.27}$$

Proof. We have readily observed that under these conditions if $\mathfrak{src}_*\omega^J_{d,h,\vec{\mu}}$ is non-trivial, then it is a top dimensional cycle and hence a rational multiple of the fundamental cycle $[\mathrm{st}_{g,J}]$. To show that it is non-trivial we will compute its multiplicity. For this, our attention is shifted towards a particular cone, and we count the covers in the fiber with the correct multiplicity. We have to keep in mind that this is a pushforward, hence, in principle, the cone where we count should be a cone of a ($\mathfrak{src}^{\mathrm{c}}$-fine) subdivision. Nevertheless, we place ourselves at the very special case of $O_{g,J}$ where we do not need to subdivide. This computation basically follows that of [VDL22] section 13.2.
Consider a cover π of $\mathbb{AC}_{d,h}(\vec{\mu})$ with $\mathrm{ft}_{J^c}(\mathrm{src}(\pi))\cong O_{g,J}$. It follows from Lemma 4.3.2 that $\mathrm{src}(\pi)$ must have the edges of $h_1',h_2,h_3,\dots,h_6$ with weight 2 (the condition on the matrix being invertible means that they give non-trivial contributions in the pushforward). We abuse notation and denote the corresponding edges of $\mathrm{src}(\pi)$ in the same way as those of $O_{g,J}$. The vanishing of RH numbers forces that:

(a) $d_\pi(c_1)-d_\pi(h_1')=\pm1$,

(b) $d_\pi(c_i)-d_\pi(c_{i-1})=\pm1$ for $2\le i\le r-1$,

(c) $d_\pi(h_1) - d_\pi(c_{r-1}) = \pm 1$,

(d) $d_\pi(e_0) - d_\pi(h_1) = \pm 1$,

(e) $d_\pi(e_i) - d_\pi(e_{i-1}) = \pm 1$ for $1 \leq i \leq g-4$,

(f) $d_\pi(h_g) - d_\pi(e_{g-4}) = \pm 1$.

The previous items, Lemma 4.3.2, and the vanishing of RH numbers imply that that

(i) The ramification profiles at $\pi(h_1'), \pi(h_2), \pi(h_3), \ldots, \pi(h_g)$ are $(2, 1^{d-2})$.

(ii) The ramification profile at $\pi(c_i)$ is $(d_\pi(c_i), 1^{d-d_\pi(c_i)})$ for $1 \leq i \leq r-1$.

(iii) The ramification profile at $\pi(e_i)$ is $(d_\pi(e_i), 1^{d-d_\pi(e_i)})$ for $0 \leq i \leq g-4$.

These last conditions allow us to classify the isomorphism class of the target tree T_t up to permutation of the marking. It should not come as a surprise that it is tightly related (or similar) to that of the graphs $O_{g,J}$ or O_g. The structure of the tree is as follows:

- If $J \neq \varnothing$, the target tree is isomorphic, up to permutation of the marking, to the tree $T_{g,J}$ specified as follows (we depict this tree for $g = 3$ and $r = 1$ in Figure 4.11):
  - It has vertices $B_1, \ldots, B_g, B_1', \ldots, B_g', M_0, \ldots, M_{g-3}$, and $R_1, \ldots, R_r$.
  - There are g edges b_i, for $1 \leq i \leq g$, with $\partial b_i = \{B_i, B_i'\}$.
  - There are edges $m_0, \ldots, m_{g-4}$ with $\partial m_i = \{M_i, M_{i+1}\}$ for $0 \leq i \leq g-4$.
  - There are edges q_1', q_1, q_2, $\ldots$, q_g with $\partial q_1' = \{B_1, R_1\}$, $\partial q_1 = \{R_r, M_0\}$, $\partial q_i = \{B_i, M_{i-2}\}$ for $2 \leq i \leq g-1$, and $\partial q_g = \{B_g, M_{g-3}\}$.
  - There are edges $p_1, \ldots, p_{r-1}$ with $\partial p_i = \{R_i, R_{i+1}\}$ for $1 \leq i \leq r-1$.
  - There are legs $L_1, \ldots, L_{3g+r}$, with:
    - $\partial L_i = R_i$, for $1 \leq i \leq r$.
    - $\partial L_{r+3\cdot i-2} = \partial L_{r+3\cdot i-1} = \{B_i'\}$ and $\partial L_{r+3\cdot i} = \{B_i\}$, for $1 \leq i \leq g$.
- If $J = \varnothing$, the target tree is isomorphic, up to permutation of the marking, to the tree $T_{g,\varnothing}$ specified as follows:
  - It has vertices $B_1, \ldots, B_g, B_1', \ldots, B_g'$, and $M_0, \ldots, M_{g-3}$.
  - There are g edges b_i, $1 \leq i \leq g$, with $\partial b_i = \{B_i, B_i'\}$.
  - There are edges $m_0, \ldots, m_{g-4}$ with $\partial m_i = \{M_i, M_{i+1}\}$ for $0 \leq i \leq g-4$.
  - There are edges $q_1, \ldots, q_g$, with $\partial q_1 = \{M_0, B_1\}$, $\partial q_i = \{B_i, M_{i-2}\}$ for $2 \leq i \leq g-1$, and $\partial q_g = \{B_g, M_{g-3}\}$.

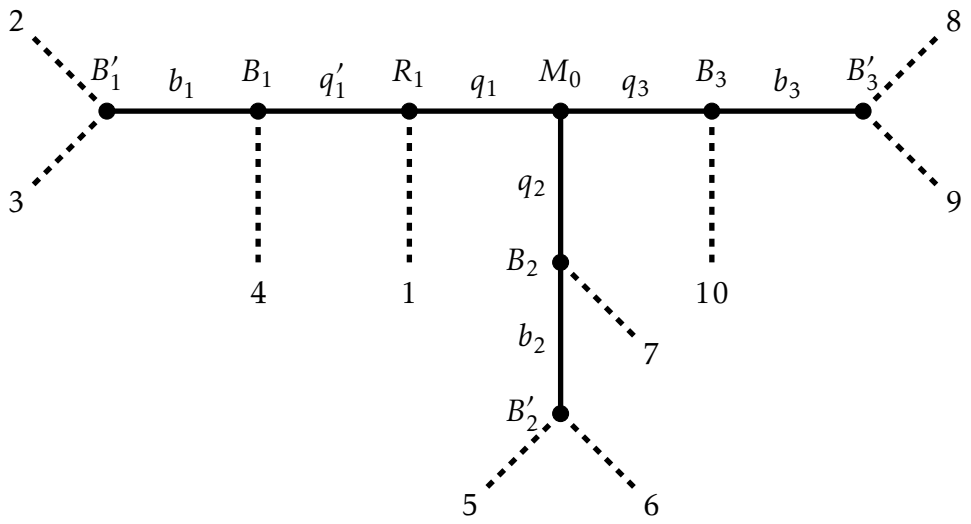


Figure 4.11: Depiction of $T_{g,J}$ for $g = 4$ and $r = 2$

– There are legs $L_1, \ldots, L_{3g}$, with $\partial L_{3\cdot i-2} = \partial L_{3\cdot i-1} = \{B_i'\}$ and $L_{3\cdot i} = \{B_i\}$ for $1 \leq i \leq g$.

We observe that if $T_t \cong T_{g,J}$, then under the cover π we have that:

(1) The image of h_1' corresponds to q_1', and furthermore the image of $\pi(h_i)$ corresponds to q_i for $1 \leq i \leq g$.

(2) The image of c_i corresponds to p_i for $1 \leq i \leq r-1$.

(3) The image of e_i corresponds to m_i for $0 \leq i \leq g-4$.

(4) The image of the loop l_i corresponds to b_i for $1 \leq i \leq g$.

These previous correspondences fix the behavior of the cover π at the vertices.
Coming back to the multiplicity, we observe that due to the markings of the legs of $\mathrm{src}(\pi)$, different sequences of weights for the edges c_i (with $1 \leq i \leq r-1$) and the edges e_k (for $0 \leq k \leq g-4$), satisfying the conditions (a) to (f), will give rise to different isomorphism classes that contribute to the computation of the multiplicity. More precisely, if we fix the isomorphism class of T_s and of T_t, then different sequences of weights give rise to non-isomorphic covers. From this we obtain directly the Catalan factor of $C_{\frac{g+r}{2}}$. Indeed, given weights $d(h_1')$, $d(c_1)$, $d(c_2)$, …, $d(c_{r-1})$, $d(e_0)$, …, $d(e_{g-4})$, and $d(h_g)$ that satisfy the set of conditions (a) to (f) and $d(h_1') = d(h_g) = 2$, then we obtain a Dyck path from 0 to $g+r$ by considering the ensuing points of $\mathbb{Z}^2$:

$$(0,0), (1, d(h_1')-1), (2, d(c_1)-1), \ldots, (r, d(c_{r-1})-1),), (r+1, d(h_1)-1),$$
$$(r+2, d(e_0)-1), \ldots, (r+g-2, d(e_{g-4})-1), (r+g-1, d(h_g)-1), (r+g, 0).$$

Since the weights are always positive, it follows that this path never goes below the x-axis. Therefore, it defines a Dyck path from 0 to $g+r$. Vice versa, any Dyck path from 0 to $g+r$ gives in this precise manner a set of weights that gives rise to an isomorphism class that we count. We depict in Figure 4.12 a full-set of representatives up to permutation of the marking of the isomorphism classes of the discrete admissible covers π subject to (4.24) with $J = \{j_1, j_2, j_3, j_4\}$ and $\mathrm{ft}_{J^c}\,\mathrm{src}(\pi) \cong O_{4,J}$. We remark that, in this case, $C_{\frac{2+4}{2}} = 5$. In the figure, the legs of the sources are color coded to depict the correspondence of the respective cover, and we omit labels of the legs to avoid overbearing the Figure.

The symmetric group S_g acts on $\mathbb{G}_{0,\mathrm{g}\sqcup J}$ by permuting the g-marked legs accordingly. Namely, if T is a tree of $\mathbb{G}_{0,\mathrm{g}\sqcup J}$ and $\sigma \in S_g$, then $\ell_{\sigma(i)}(T)) = \ell_i(\sigma(T))$ and $\ell_{\sigma(i^*)}(T)) = \ell_{i^*}(\sigma(T))$ for $1 \le i \le g$. The isomorphism classes of trees of $\mathbb{G}_{0,\mathrm{g}\sqcup J}$ whose image is isomorphic to $O_{g,J}$ are all related by permutations of S_g. We fix one object T_O of $\mathbb{G}_{0,\mathrm{g}\sqcup J}$ with $\mathrm{st}_{g,J}(T_O) \cong O_{g,J}$ and proceed to count (with multiplicity) the isomorphism classes of the triples of $\mathbb{EAC}^J_{d,h}(\vec{\mu})$ whose image under the morphism $\mathfrak{src}^{\mathrm{c}}$ is in the isomorphism class of T_O. We first observe that if (T_s, T_t, π) is an object of $\mathbb{EAC}^J_{d,h}(\vec{\mu})$ with $\mathfrak{src}^{\mathrm{c}}(T_s, T_t, \pi) \cong T_O$, then the matrix corresponding to the map

$$\eta_{\mathrm{src},(T_s,T_t,\pi)} \colon \mathrm{EST}_{(T_s,T_t,\pi)} \to \mathrm{ST}_{T_s}$$

is square (because the dimensions coincide) and, furthermore, diagonal. In fact, it coincides with the composition $\left(\eta_{\mathrm{ft}_{J^c},\mathrm{src}(\pi)} \circ F_\pi\right)$ up to reordering of columns and rows. Furthermore, we have that the only non-trivial entries are the ones corresponding to the loops (all others cancel due to the ramification profiles) which are 2. This shows that the index of this map is 2^g.

We now study the contribution of the weight $\omega^J_{d,h,\vec{\mu}}$. It follows from Lemma 4.3.2 that the ramification profiles above the leaves of T_t are always (1^d), and we also have the conditions (i) to (iii) on ramification profiles above the other edges. This shows that in this case

$$\omega^J_{d,h,\vec{\mu}}(\pi) = \prod_{V \in \mathrm{src}(\pi)} H(V) \cdot \mathrm{CF}(V). \tag{4.28}$$

Since the RH numbers are trivial, the conditions (i) to (iii) on the ramification profiles at the edges of T_t imply that there are the following possibilities for partitions of $d_\pi(V)$ at a vertex $V \in V(\mathrm{src}(\pi))$:

(1) If $d_\pi(V) \ge 2$, then we will see $(2, 1^{d_\pi(V)-2})$, $(d_\pi(V)-1, 1)$, and $(d_\pi(V))$.

(2) If $d_\pi(V) = 1$, then we will just have (1), (1), and (1).

We observe that in (1) we obtain the following triple rational Hurwitz number

(a) If $d_\pi(V) = 2$, then $H_{0\to 0}((2),(1,1),(2)) = \frac{1}{2}$.

(b) If $d_\pi(V) \geq 3$, then $H_{0\to 0}((2,1^{d_\pi(V)-2}),(d_\pi(V)-1,1),(d_\pi(V)) = 1$. Indeed, this is best seen by observing that this Hurwitz number corresponds to the number of transpositions and $(d_\pi(V)-1)$-cycles that multiply to a $d_\pi(V)$-cycle in $S_{d_\pi(V)}$ divided by $d_\pi(V)!$. After fixing the $(d_\pi(V)-1)$-cycle, we obtain that there are $(d_\pi(V)-1)$ transpositions that when multiplied with the fixed $(d_\pi(V)-1)$-cycle give a $d_\pi(V)$-cycle. Since the number of $(d_\pi(V)-1)$-cycles is $\frac{d_\pi(V)!}{(d_\pi(V)-1)}$, it follows that $H_{0\to 0}((2,1^{d_\pi(V)-2}),(d_\pi(V)-1,1),(d_\pi(V)) = 1$.

Of course in (2) the rational triple Hurwitz number is also 1. At the vertices where we forcibly run into case (a) from above, this fraction is canceled by the factor of $\mathrm{CF}(V)$ that is coming from the two edges with weight 1 (this is the case, for instance, at all the $V_1,\dots,V_g$). The factors $\mathrm{CF}(V)$ of the vertices V arising in case (b) are relevant when computing the number of isomorphism classes of marked discrete graphs realizing to the isomorphism class of the given T_s. Namely, after marking the target tree T_t, the markings of each weight-1 leg on the source can be permuted along the same fiber. This is the action of S_{d-2}^m that we have previously described, and acting by an $\vec{\alpha} \in S_{d-2}^m$ may not necessarily produce new isomorphism classes. The action of the permutations $\vec{\alpha}$ is reflected locally at the vertices of $\mathrm{src}(\pi)$ (or T_s) by the combinatorial factors $\mathrm{CF}(\bullet)$, and the only vertices of interest are those with local degree bigger than 1 that are not lying in the fibers of the $B_1,\dots,B_g,B_1',\dots,B_g'$ (these are vertices where case (a) above applies and have nothing to do with the action of S_{d-2}^m). Our conditions (i) to (iii) on ramification profiles, the vanishing of the RH numbers, and Lemma 4.3.2 imply that these vertices with local degree bigger than 1 not in the fiber of the B_i and B_i' $(1 \leq i \leq g)$ are exactly the $C_1,\dots,C_r,W_0,\dots,W_{g-3}$. We have already explained the behavior of the $\mathrm{CF}(\bullet)$ at the V_i for $1 \leq i \leq g$, so we focus on the others. If all these had weight ≤ 2, then we would have $\#S_{d-2}^m = ((d-2)!)^m$ possibilities. If there are vertices with higher weights, then we have to divide by the stabilizer, which is given by the factors from $\mathrm{CF}(\bullet)$. These cancel out with the remaining $\mathrm{CF}(V)$ in (4.28), and all together this shows that we obtain the factor of $((d-2)!)^m$. More succinctly, we are counting the orbits of the S_{d-2}^m action as many times as the remaining factors $\mathrm{CF}(V)$ in (4.28), which turn out to be the size of the corresponding stabilizer.

To finish, we just have to understand what happens when we permute the marking of the base tree. In other words, we study the previously defined action of S_m. Observe that when permuting the marking of the legs of T_t (and change π correspondingly) we will only stay in the same isomorphism class of $\mathbb{EAC}_{d,h}^J(\vec{\mu})$ if and only if the permutation switches legs incident to the same vertex and leaves the first r legs fixed (because of the J-marked legs). This shows that there are $\frac{(m-r)!}{2^N}$ different permutations of the marking that produce different isomorphism classes, where N is the number of vertices of T_t with leg valency 2. It is clear that, in this case, $N = g$, and therefore we have

shown that

$$\mathfrak{src}_*\omega^J_{d,h,\vec{\mu}} = \left(\frac{(m-r)!}{2^g}\cdot((d-2)!)^m\cdot 2^g\cdot C_{\frac{g+r}{2}}\right)[\mathrm{st}_{g,J}] = \left((3g)!\cdot\left(\left(\frac{g+r}{2}-1\right)!\right)^{3g+r}\cdot C_{\frac{g+r}{2}}\right)[\mathrm{st}_{g,J}]$$

□

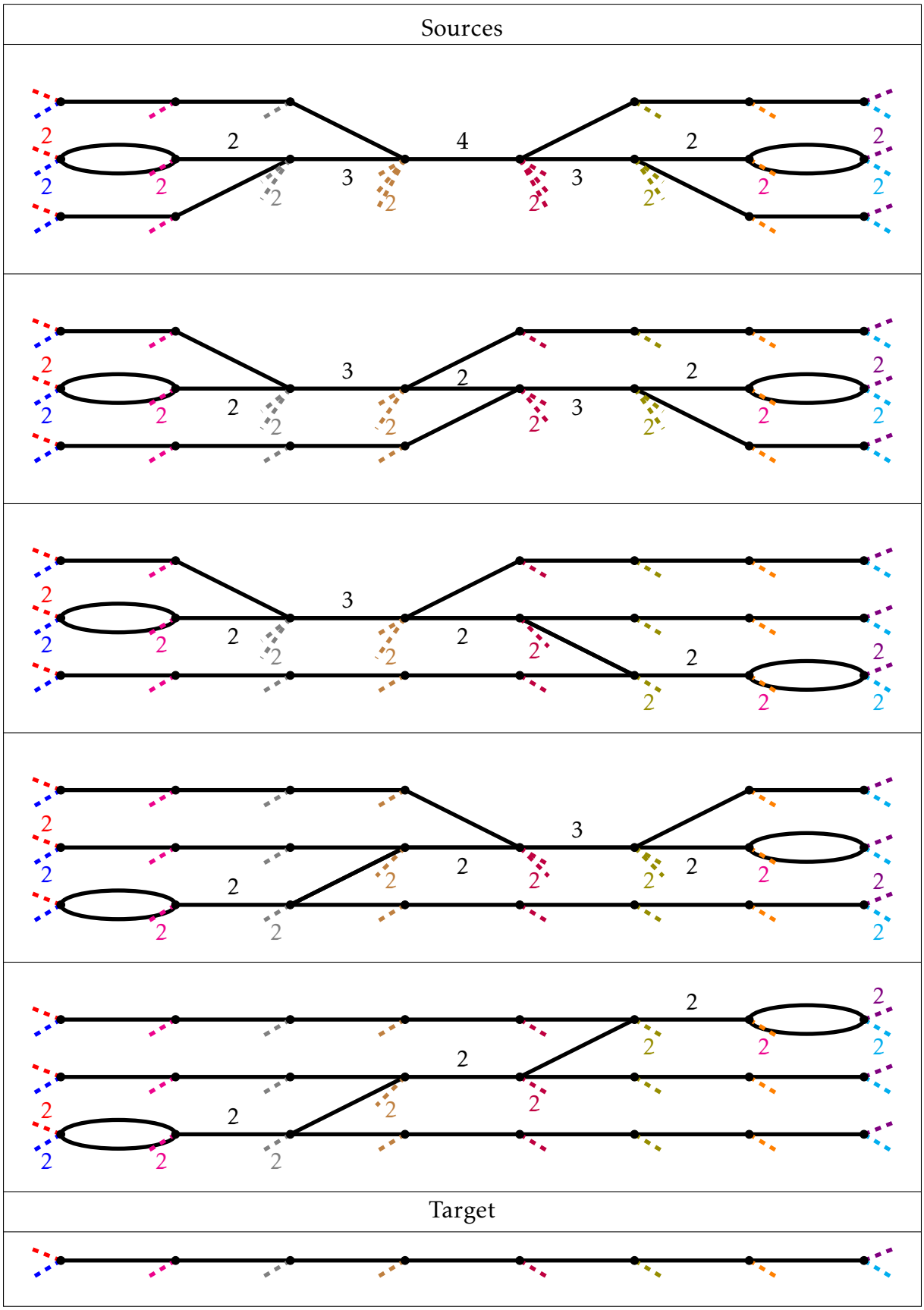


Figure 4.12: Discrete admissible covers from the proof of Theorem 4.3.6

By further studying the actions of S_m and S_{d-2}^m, it is possible to just keep the Catalan factor $C_{\frac{g+r}{2}}$ in (4.27). We make this more precise in Theorem 4.3.9 and for this purpose, we introduce a multiplicity for objects of $\mathbb{EAC}_{d,h}^J(\vec{\mu})$, an equivalence relation generated by the actions of these groups, and then we prove the corresponding enumerative statement.

Definition 4.3.7. Let (T_s, T_t, π) be an object of $\mathbb{EAC}_{d,h}^J(\vec{\mu})$. Its *multiplicity* is the number

$$\operatorname{mult}(T_s, T_t, \pi) := \frac{\omega_{d,0,\vec{\mu}}^J(\pi) \cdot \left|\det(\eta_{\mathrm{ft}_{J^c},\mathrm{src}(\pi)} \circ F_\pi)\right|}{\mathrm{HS}(\pi) \cdot \mathrm{VS}(\pi)}, \tag{4.29}$$

where $\mathrm{VS}(\pi)$ and $\mathrm{HS}(\pi)$ are the sizes of the stabilizers of the action of $(S_{d-2})^m$ and S_m correspondingly.

We intend to use this multiplicity to count covers. However, we want to identify triples (T_s, T_t, π) and (T_s', T_t', π') where the covers π and π' are related by the action of S_m and S_{d-2}^m, and $\mathrm{ft}_{J^c}(T_s) \cong \mathrm{ft}_{J^c}(T_s')$. This leads us to the notion of J-marked discrete admissible covers.

Definition 4.3.8. The *set of J-marked discrete admissible covers* is the set of equivalence classes of $[\mathbb{AC}_{d,h}(\vec{\mu})]$ under the following relation: $[\pi] \sim [\pi']$ if there exist $\vec{\alpha} \in (S_{d-2})^m$ and $\beta \in S_m$ such that $[\pi] = \vec{\alpha}(\beta([\pi']))$ and $\mathrm{ft}_{J^c}(\mathrm{src}([\pi])) = \mathrm{ft}_{J^c}(\mathrm{src}([\pi']))$. It is clear that the multiplicity (4.29) gives a well-defined function on the set of J-marked discrete admissible covers.

We are finally ready to state and prove our previously advertised enumerative statement. We make use therein of the notion of tropical modifications of a tropical curve. A tropical modification of a tropical curve Γ is simply a tropical curve Γ' that is obtained by grafting several rational tropical curves (metric trees possibly with other marked legs) to vertices of Γ. In the context of the theorem, these tropical modifications arise as points in the fibers of forgetting the marking morphisms. To be much more precise, given a genus-g J-marked tropical curve $\Gamma \in |\mathrm{M}_{g,J}^{\mathrm{trop}}|$, the fiber $|\mathrm{ft}_{J^c}|^{-1}(\Gamma)$ consists of tropical modifications of Γ.

Theorem 4.3.9. *Suppose g, h, d, m, $\vec{\mu}$, n, and J are as in* (4.24)*. Then for a generic genus-g J-marked tropical curve Γ, there are $C_{\frac{g+r}{2}}$ many degree-d J-marked discrete admissible covers (counted with multiplicity induced from* (4.29)*) that have a tropical modification of Γ as a source.*

Proof. Following Theorem 4.3.6 we take the geometric realization of $\mathfrak{src}\colon \mathrm{EST}_{d,h}^J(\vec{\mu}) \to \mathrm{ST}_{g,J}$, and apply Corollary 2.36 of [GKM09] (the argument follows verbatim). This shows that the fiber of a generic point $p \in \left|\mathrm{ST}_{g,J}\right|$ has $\left((m-r)! \cdot ((d-2)!)^m \cdot C_{\frac{g+r}{2}}\right)$-many

points counted with their respective multiplicity. By definition, if $q \in |\mathfrak{src}^c|^{-1}(p)$ and lies in the realization of the cone of a triple (T_s, T_t, π), then this multiplicity is

$$\omega^J_{d,h,\vec{\mu}}(\pi) \cdot \left[N^{\mathrm{ST}_{T_s}} : \mathfrak{src}^c \left(N^{\mathrm{EST}^J_{(T_s,T_t,\pi)}} \right) \right] = \omega^J_{d,h,\vec{\mu}}(\pi) \cdot \left| \det \left(\eta_{\mathrm{ft}_{J^c},\mathrm{src}(\pi)} \circ F_\pi \right) \right|. \tag{4.30}$$

We first consider the action of $(S_{d-2})^m$, which can be naturally extended to $\mathrm{EST}^J_{d,h}(\vec{\mu})$ (and hence to its realization). The action restricts to an action on the fiber $|\mathfrak{src}^c|^{-1}(p)$ preserving the corresponding multiplicities, hence if we divide by the stabilizers we can remove the factor $((d-2)!)^m$. By definition of the action, if a point $q \in |\mathfrak{src}^c|^{-1}(p)$ lies in the realization of the cone of a triple (T_s, T_t, π), then the size of the stabilizer if $\mathrm{VS}(\pi)$. Let G_p denote the subgroup of S_m such that:

- $G_p(\{1,\dots,r\}) = \{1,\dots,r\}$.
- The induced action of G_p on J and its extension to $\mathrm{ST}_{g,J}$ leaves the point p fixed.

Similarly to the case of $(S_{d-2})^m$, the action of S_m extends naturally to an action of G_p on $\mathrm{EST}^J_{d,h}(\vec{\mu})$. By construction it restricts to an action on the fiber $|\mathfrak{src}^c|^{-1}(p)$, and the analogous argument shows that if we divide by the stabilizers then we can remove the factor $(m-r)!$. From the definition of G_p, the S_m-stabilizer of π coincides with the G_p-stabilizer, and hence the size of the stabilizer if $\mathrm{HS}(\pi)$. It now follows from (4.30) that counting the points in the fiber with the multiplicity induced from (4.29) gives precisely $C_{\frac{g+r}{2}}$. □

Remark. In the next section we explain how the enumerative result of [VDL22] follows from our previous theorem in the case of $J = \varnothing$. We additionally show that our multiplicity in this case coincides with the Vargas-Draisma multiplicity (Definition 13.18 of [VDL22]).

We close this section with the proof of Lemma 4.3.2. This is a multi-step process, which we now build toward. The results and methods are simple, but the proofs are somewhat lengthy.

Lemma 4.3.10. *Following the notation of* (4.24)*, let π be an object of top dimension of $\mathbb{AC}_{d,h}(\vec{\mu})$ with $\left(\eta_{\mathrm{ft}_{J^c},\mathrm{src}(\pi)} \circ F_\pi\right)$ invertible. If V is a vertex of $\mathrm{trgt}(\pi)$ with $\mathrm{legval}(V) = 2$, then the fiber $\pi^{-1}(V)$ is given as follows:*

- *There is a unique vertex $\hat{V}$ with $d_\pi(\hat{V}) = 2$.*
- *All the other vertices have local degree* 1.

Furthermore, if $e \in E(\mathrm{trgt}(\pi))$ is the unique edge with $V \in \partial e$, then the ramification profile of e is (1^d).

Proof. Observe that every vertex of $\mathrm{trgt}(\pi)$ must be 3-valent, because π is of top dimension. Let ℓ_i and ℓ_j denote the marked legs of $\mathrm{trgt}(\pi)$ that are incident to V. Suppose $W \in \pi^{-1}(V)$ and consider partitions $\lambda_i, \lambda_j, \lambda \vdash d_\pi(W)$ such that

- λ_i is the partition given by the weights of the marked legs incident to W in the fiber of ℓ_i.
- λ_j is the partition given by the weights of the marked legs incident to W in the fiber of ℓ_j.
- λ is the partition given by the weights of the edges incident to W in the fiber of e.

Since the ramification profile of ℓ_i and ℓ_j is $(2, 1^{d-2})$, it follows that there are only two possibilities for λ_i and λ_j: either $(2^{d_\pi(W)}, 1^{d_\pi(W)-2})$ or $(1^{d_\pi(W)})$. By an exhaustive argument we will show that the only cases are:

$(*)$ $d_\pi(W) = 2$, $\lambda_i = (2)$, $\lambda_j = (2)$, and $\lambda = (1,1)$.

$(**)$ $d_\pi(W) = 1$, $\lambda_i = (1)$, $\lambda_j = (1)$, and $\lambda = (1)$.

We consider all the possible cases:

- Suppose that both $\lambda_i = \lambda_j = (1^{d_\pi(W)})$. The vanishing of the RH number at W yields the equation

$$d_\pi(W) = \ell(\lambda_i) + \ell(\lambda_j) + \ell(\lambda) - 2 = d_\pi(W) + d_\pi(W) + \ell(\lambda) - 2.$$

  From this it follows that $d_\pi(W) + \ell(\lambda) = 2$, and since $\ell(\lambda), d_\pi(W) \geq 1$, we necessarily have that $d_\pi(W) = 1$ and $\lambda = (1)$.

- Suppose that both $\lambda_i = \lambda_j = (2, 1^{d_\pi(W)-2})$. The vanishing of the RH number at W yields the equation

$$d_\pi(W) = \ell(\lambda_i) + \ell(\lambda_j) + \ell(\lambda) - 2 = 2(d_\pi(W) - 1) + \ell(\lambda) - 2.$$

  This implies that $d_\pi(W) + \ell(\lambda) = 4$. Observe that $d_\pi(W) \geq 2$ due to λ_i and λ_j, and $\ell(\lambda) \geq 1$. Therefore, either $d_\pi(W) = 3$ and $\lambda = (3)$ or $d_\pi(W) = 2$ and $\lambda = (1,1)$. We claim that the former is impossible. Indeed, in this case every other vertex of $\pi^{-1}(V)$ would necessarily have local degree 1. This would imply that in the matrix $\left(\eta_{\mathrm{ft}_{J^c}, \mathrm{src}(\pi)} \circ F_\pi\right)$ the column corresponding to the edge e would be trivial, but this is not the case as this matrix is invertible. In conclusion, in this case $d_\pi(W) = 2$ and $\lambda = (1,1)$.

- Suppose that $\lambda_i = (2^{d_\pi(W)}, 1^{d_\pi(W)-2})$ and $\lambda_j = (1^{d_\pi(W)})$ (or vice versa, the argument is the same). We argue that this is not a possibility. The vanishing of the RH number at W yields the equation

$$d_\pi(W) = \ell(\lambda_i) + \ell(\lambda_j) + \ell(\lambda) - 2 = (d_\pi(W) - 1) + d_\pi(w) + \ell(\lambda) - 2.$$

  In particular, we obtain that $3 = d_\pi(W) + \ell(\lambda)$. Observe that $d_\pi(W) \geq 2$ due to λ_i. Since λ is a partition, necessarily we have that $\ell(\lambda) \geq 1$. These conditions force that $d_\pi(W) = 2$, $\lambda = (2)$. In this case, there would be another vertex $W' \in \pi^{-1}(V)$ with $d_\pi(W') = 2$ but the partitions above the ℓ_i and ℓ_j legs reversed, and every other vertex of $\pi^{-1}(V)$ would necessarily have local degree 1. This is not possible due to $\left(\eta_{\mathrm{ft}_{J^c},\mathrm{src}(\pi)} \circ F_\pi\right)$ being invertible. Indeed, the column corresponding to the edge e of $\mathrm{trgt}(\pi)$ would be trivial and the matrix could not possibly be invertible.

It readily follows from the above that the only possibilities are $(*)$ and $(**)$. Since the ramification profiles over ℓ_i and ℓ_j are $(2, 1^{d-2})$, we must necessarily run into $(*)$ only once in the fiber $\pi^{-1}(V)$ and at all the other vertices we run into $(**)$. □

We continue with a useful but technical definition. In the context of covers, this is analogous to the dangling edges and vertices of [DV21].

Definition 4.3.11. Let A be an arbitrary finite set, let $I \subset A$, let q be a non-negative integer with $2q + \#(A\backslash I) - 2 \geq 0$ and let G be an object of $\mathbb{G}_{g,A}$. We say that an edge $e \in E(G)$ is *expunged after forgetting the I-marked legs* if the column corresponding to e in the matrix $\eta_{\mathrm{ft}_I,G}$ is trivial. A vertex $V \in V(G)$ is *expunged after forgetting the I-marked legs* if every leg incident to V is I-marked and every edge incident to V is expunged after forgetting the I-marked legs.

The following lemma and its proof consist of a simple adaptation to our context of Remark 51, Lemmas 52 and 53, and their proofs of [DV21]. Our approach is geared toward the bare necessities for our framework, and hence is rather minimalistic and plain. We do not seek to phrase a general combinatorial set-up, but instead push through in our given context.

Lemma 4.3.12. *Following the notation of* (4.24)*, let π be an object of top dimension of $\mathbb{AC}_{d,h}(\vec{\mu})$ with $\left(\eta_{\mathrm{ft}_{J^c},\mathrm{src}(\pi)} \circ F_\pi\right)$ invertible. Any vertex of* $\mathrm{src}(\pi)$ *that is expunged after forgetting the J^c-marked legs has local degree* 1*. Analogously, every edge of* $\mathrm{src}(\pi)$ *that is expunged after forgetting the J^c-marked legs has weight* 1*.*

Proof. Since π is of top dimension, every vertex of $\mathrm{trgt}(\pi)$ is 3-valent. We begin with some observations concerning the structure of vertices of $\mathrm{src}(\pi)$ that are expunged after fogetting the J^c-marked legs. Let $V \in V(\mathrm{src}(\pi))$ be expunged after forgetting the J^c-marked legs, then we have the following:

(•) If $\mathrm{legval}(\pi(V)) = 2$, it necessarily holds $d_\pi(V) = 1$.
Indeed, because of Lemma 4.3.10 we know that $d_\pi(V) = 1$ or $d_\pi(V) = 2$. In fact, every vertex in $\pi^{-1}(\pi(V))$ with local degree 1 is expunged after forgetting the J^c-marked legs. If $d_\pi(V) = 2$, then the column of the matrix $\left(\eta_{\mathrm{ft}_{J^c},\mathrm{src}(\pi)} \circ F_\pi\right)$ correspoding to the unique edge of $\mathrm{trgt}(\pi)$ incident to $\pi(V)$ would be trivial. Since the matrix is invertible, this is impossible and therefore $d_\pi(V) = 1$.

(••) If $\mathrm{legval}(\pi(V)) = 1$ and $e_1, e_2 \in E(\mathrm{trgt}(\pi))$ denote the two edges incident to $\pi(V)$, then every leg of $\mathrm{src}(\pi)$ incident to V has degree 1.
We argue by contradiction and assume that there is a leg of $\mathrm{src}(\pi)$ incident to V with degree 2. Observe that there must be at least one vertex in the fiber $\pi^{-1}(\pi(V))$ that is not expunged after forgetting the J^c-marked legs because the matrix $\left(\eta_{\mathrm{ft}_{J^c},\mathrm{src}(\pi)} \circ F_\pi\right)$ is invertible. If $W \in \pi^{-1}(\pi(V))$ is one of these vertices that are not expunged after forgetting the J^c-marked legs, then every leg of $\mathrm{src}(\pi)$ incident to W must have weight 1. Together with the vanishing of the RH number at W, this implies that there are two edges $e_1^W, e_2^W \in E(\mathrm{src}(\pi))$ incident to W with $\pi(e_i^W) = e_i$ and $d_\pi(e_i^W) = d_\pi(W)$ for $i = 1, 2$. Since the vertex W was arbitrary, this shows that the columns of the matrix $\left(\eta_{\mathrm{ft}_{J^c},\mathrm{src}(\pi)} \circ F_\pi\right)$ corresponding to the edges e_1 and e_2 are identical. But this is impossible because the matrix is invertible. Therefore, there is no leg of $\mathrm{src}(\pi)$ incident to V with degree 2.

(•••) If $A, B \in V(\mathrm{src}(\pi))$ are two vertices that are not expunged after forgetting the J^c-marked legs, then there cannot be a path between A and B consisting of edges that are expunged after forgetting the J^c-marked legs.
This follows from the behavior of forgetting the marking morphisms.

After these previous observations, we turn our attention toward edges. We will show that every edge that is expunged after forgetting the J^c-marked legs must have weight 1. We argue by contradiction and suppose that $a_1 \in E(\mathrm{src}(\pi))$ is expunged after forgetting the J^c-marked legs with $d_\pi(e) > 1$.
Let us set $a_1 = e$ and $\partial a_1 = \{W_1, V_1\}$. Because of (•••) there is a vertex of ∂a_1 that is expunged after forgetting the J^c-marked legs, say V_1. Due to (•) and (••) we can assume that every leg (if there are any) of $\mathrm{src}(\pi)$ that is incident to V_1 has weight 1. Observe that the vanishing of the RH number at V_1 implies that there must exist an edge $a_2 \in E(\mathrm{src}(\pi))$ incident to A_1 with $d_\pi(a_2) > 1$. Let V_2 denote the other vertex of $\mathrm{src}(\pi)$ that is incident to a_2. Either V_2 is expunged after forgetting the J^c-marked legs or not. In the former case, clearly $V_2 \neq V_1$ and we repeat the argument to produce a_3 and V_3. Again, either V_3 is expunged after forgetting the J^c-marked leg or not. In the former situation, we additionally have that $V_3 \neq V_2$ and $V_3 \neq V_1$ (because forgetting the marking preserves the genus of the graphs). Since there are only finitely many edges and

vertices, reiterating this process we produce a sequence of edges $a_1,\dots,a_s \in E(\mathrm{src}(\pi))$ and different vertices $V_1, V_2,\dots, V_s, V_{s+1} \in V(\mathrm{src}(\pi))$ such that

- All the edges a_i and vertices V_i are expunged after forgetting the J^c-marked legs and $d_\pi(a_i) > 1$ (for $1 \le i \le s$).
- The vertex V_{s+1} is incident to a_s and is not expunged after forgetting the J^c-marked leg.

Now, (•••) implies that W_1 is expunged after forgetting the J^c-marked legs. Reiterating the process above we obtain a sequence of edges $b_1,\dots,b_t \in E(\mathrm{src}(\pi))$ (with $b_1 = a_1 = e$) and different vertices $W_1,\dots,W_t, W_{t+1} \in V(\mathrm{src}(\pi))$ such that

- All the edges b_i and vertices W_i are expunged after forgetting the J^c-marked legs and $d_\pi(b_i) > 1$ (for $1 \le i \le t$).
- The vertex W_{s+1} is incident to b_t and is not expunged after forgetting the J^c-marked leg.

If $\{W_1,\dots,W_t\} \cap \{V_1,\dots,V_s\} = \varnothing$, we arrive at a contradiction because of (•••). Otherwise, we would also arrive at a contradiction. Because if $\{W_1,\dots,W_t\} \cap \{V_1,\dots,V_s\} \neq \varnothing$, then a cycle of $\mathrm{src}(\pi)$ is expunged after forgetting the J^c-marked legs, but this is not possible. In conclusion, if $e \in E(\mathrm{src}(\pi))$ is expunged after forgetting the J^c-marked legs, then $d_\pi(e) = 1$.
To finish the proof of the lemma we must show that a vertex $V \in V(\mathrm{src}(\pi))$ that is expunged after forgetting the J^c-marked legs has local degree 1. If $\mathrm{legval}(\pi(V)) = 2$, then this readily follows from (•). If $\mathrm{legval}(\pi(V)) < 2$, then we have shown that every edge (and leg) of $\mathrm{src}(\pi)$ incident to V has weight 1. In this case, the vanishing of the RH number at V forces that $d_\pi(V) = 1$. □

Before finally proving Lemma 4.3.2, we introduce, in the spirit of clarity and conciseness, one last definition.

Definition 4.3.13. Following the notation of (4.24), let π denote an object of $\mathbb{AC}_{d,h}(\vec{\mu})$ and let $h \in E(\mathrm{ft}_{J^c}(\mathrm{src}(\pi))$. We say that an edge $e \in E(\mathrm{src}(\pi))$ *adds* to the edge e if the (h,e)-entry of the matrix $\eta_{\mathrm{ft}_{J^c},\mathrm{src}(\pi)}$ is non-zero.

For the convenience of the reader, we restate Lemma 4.3.2. This lemma and its proof are a minimalistic adaptation to our context of Lemmas 60 and 79 (as well as their proofs) of [DV21]. As in the previous lemma, we aim towards straightforwardness and the bare necessities for our tools. The price to pay is a lengthy, somewhat terse, proof.

Lemma 4.3.2. *Following the notation of* (4.24)*, let* π *be an object of* $\mathbb{AC}_{d,h}(\vec{\mu})$ *of top dimension and such that the composition* $\left(\eta_{\mathrm{ft}_{J^c},\mathrm{src}(\pi)} \circ F_\pi\right)$ *is invertible, where* F_π *is as in* (1.13)*. If* $W \in V(\mathrm{src}(\pi))$ *is such that*

- W *is also a vertex of* $\mathrm{ft}_J(\mathrm{src}(\pi))$*,*
- $\mathrm{val}_{\mathrm{ft}_{J^c}(\mathrm{src}(\pi))}(W) = 3$*,*
- W *is incident to a loop and an edge in* $\mathrm{ft}_{J^c}(\mathrm{src}(\pi))$*,*

then $\pi(W)$ *is incident to a leaf of* $\mathrm{trgt}(\pi)$ *and* $\mathrm{legval}(\pi(W)) = 1$*. Furthermore, if* e_1 *denotes the leaf of* $\mathrm{trgt}(\pi)$ *that* $\pi(W)$ *is incident to,* V *denotes the other vertex of* $\mathrm{trgt}(\pi)$ *incident to* e_1*, and* e_2 *denotes the edge of* $\mathrm{trgt}(\pi)$ *that is not incident to* $\pi(W)$*, then we have the following:*

- *The ramification profile above* e_1 *is* (1^d)*.*
- *The ramification profile above* V *is* $(2, 1^{d-2})$*.*
- *The ramification profile above* e_2 *is* $(2, 1^{d-2})$*.*

Proof. As we have mentioned before, every vertex of $\mathrm{trgt}(\pi)$ is 3-valent because π is of top dimension. We examine all the possibilities for $\mathrm{legval}(\pi(W))$. We first show that $\mathrm{legval}(\pi(W)) = 2$ and $\mathrm{legval}(\pi(W)) = 0$ are impossible. Then we show that the only possibility is $\mathrm{legval}(\pi(W)) = 1$ together with the ensuing statements from the lemma.

(○) We observe that if $\mathrm{legval}(\pi(W)) = 2$, then $d_\pi(W) \geq 3$. This is impossible because of Lemma 4.3.2.

(○○) Suppose that $\mathrm{legval}(\pi(W)) = 0$, and let us argue by contradiction. Let $a, b, c \in E(\mathrm{trgt}(\pi))$ denote the three edges that are incident to $\pi(W)$. Let S denote the set of edges of $\mathrm{src}(\pi)$ that are incident to W. Recall that Lemma 4.3.12 shows that every edge of $\mathrm{src}(\pi)$ that is expunged after forgetting the J^c-marked legs must have weight 1.

- If every edge of $\pi^{-1}(a) \cap S$ and every edge of $\pi^{-1}(b) \cap S$ is expunged after forgetting the J^c-marked legs, then $d_\pi(W) \geq 3$ and $\pi^{-1}(c) \cap S \geq 3$. This is impossible due to the vanishing of the RH number at W.
- If every edge of $\pi^{-1}(a) \cap S$ is expunged after forgetting the J^c-marked legs, then $\left(\pi^{-1}(b) \cup \pi^{-1}(c)\right) \cap S \geq 3$. Again, this is impossible due to the vanishing of the RH number at W.

This shows that there are edges $e_a \in \pi^{-1}(a)$, $e_b \in \pi^{-1}(b)$, and $e_c \in \pi^{-1}(c)$ that are not expunged after forgetting the J^c-markings and such that $e_a, e_b, e_c \in S$. In fact,

e_a (resp. e_b and e_c) is the unique edge of $\pi^{-1}(a) \cap S$ (resp. $\pi^{-1}(b) \cap S$ and $\pi^{-1}(c) \cap S$) that is not expunged after forgetting the J^c-marked legs. Two of these, say e_a and e_b must add to the loop of $\mathrm{ft}_{J^c}(\mathrm{src}(\pi))$ incident to W, and the other, in this case e_c, adds to the other edge. Most importantly, since e_a and e_b add to the loop, each of these edges traces a path in $\mathrm{src}(\pi)$ that eventually leads to different edges h_a and h_b (respectively) of $\mathrm{src}(\pi)$. These edges h_a and h_b add to the loop after forgetting the J^c-marked legs, and $\pi(h_a)$ and $\pi(h_b)$ are different leaves of $\mathrm{trgt}(\pi)$. In this case, the columns of the matrix $\left(\eta_{\mathrm{ft}_{J^c},\mathrm{src}(\pi)} \circ F_\pi\right)$ corresponding to the leaves $\pi(h_a)$ and $\pi(h_b)$ would be scalar multiples of each other, and the matrix would not be invertible. This is a contradiction, and therefore $\mathrm{legval}(W) = 0$ is impossible.

From the above, it follows that $\mathrm{legval}(W) = 1$ is the only possibility.
For the remaining statements from the lemma, let e_1 and e_2 denote the edges of $\mathrm{trgt}(\pi)$ that are incident to $\pi(W)$. Let $S \subset E(\mathrm{src}(\pi))$ denote the set of edges that are incident to W. Since the vertex W is 3-valent in $\mathrm{ft}_{J^c}(\mathrm{src}(\pi))$, it follows that there are exactly three edges in $\left(\pi^{-1}(e_1) \cup \pi^{-1}(e_2)\right) \cap S$ that are not expunged after forgetting the J^c-marked legs. Let us denote these edges by a, b and c. Furthermore, the vanishing of the RH number at W implies that $\{a, b, c\} \not\subset \pi^{-1}(e_i)$ for $i = 1, 2$. In addition, two of these edges, say a and b, add to the loop of $\mathrm{ft}_{J^c}(\mathrm{src}(\pi))$ incident to W, and the other edge, in this case c, adds to the other edge. The same argument at the end of $(\circ\circ)$ shows that a and b have to lie in the same fiber. Furthermore, since these edges add to the loop, each of these edges traces a path in $\mathrm{src}(\pi)$ that eventually leads to edges h_a and h_b (respectively) of $\mathrm{src}(\pi)$ such that: these edges add to the loop after forgetting the J^c-marked legs, and $\pi(h_a)$ and $\pi(h_b)$ are leaves of $\mathrm{trgt}(\pi)$. Once again, the same argument at the end of $(\circ\circ)$ shows that it is impossible to have $\pi(h_a) \neq \pi(h_b)$. Therefore, it necessarily holds that $\pi(h_a) = \pi(h_b)$.
We will show that $h_a = a$ and $h_b = b$. Let $p = \pi(h_a) = \pi(h_b)$ and $\partial p = \{P_1, P_2\}$ with $\mathrm{legval}(P_2) = 2$. In addition, we have that $\mathrm{legval}(P_2) < 2$ (there is at least an additional edge since $\mathrm{val}_{\mathrm{ft}_{J^c}}(\mathrm{src}(\pi)) = 3$). It follows from Lemma 4.3.10 that the ramification profile of p is (1^d), and that there is a unique vertex $\hat{P}_1 \in \pi^{-1}(P_1)$ with $d_\pi(\hat{P}_1) = 2$. Furthermore, $\hat{P}_1$ is incident to both h_a and h_b, and we also have that $d_\pi(h_a) = d_\pi(h_b) = 1$. Let $P_{2,a}$ denote the other vertex of h_a, and $P_{2,b}$ denote the other vertex of h_b. It will be first shown that $\mathrm{legval}(P_2) \neq 0$.
For the sake of contradiction, assume that $\mathrm{legval}(P_2) = 0$, and let q and z denote the other edges of $\mathrm{trgt}(\pi)$ that are incident to P_2. Since $\mathrm{trgt}(\pi)$ is a tree and the edges h_a and h_b add to a loop of $\mathrm{ft}_{J^c}(\mathrm{src}(\pi))$, only one of the following is possible:

- Every edge of $\pi^{-1}(q)$ incident to $P_{2,a}$ is expunged after forgetting the J^c-marked legs, and the same happens for every edge of this fiber incident to $P_{2,b}$.
- Every edge of $\pi^{-1}(z)$ incident to $P_{2,a}$ is expunged after forgetting the J^c-marked

legs, and the same happens for every edge of this fiber incident to $P_{2,b}$.

Without loss of generality, we assume the latter and carry on with the argument. In this case there is a unique edge $q_a \in \pi^{-1}(q)$ that is incident to $P_{2,a}$ and a unique edge $q_b \in \pi^{-1}(q)$ that is incident to $P_{2,b}$. Both q_a and q_b must have weight 1. We also observe that the ramification profile of p implies that every vertex $X \in \pi^{-1}(P_2)$ different from $P_{2,a}$ and $P_{2,b}$ has a unique incident edge $q_X \in \pi^{-1}(q)$ and a unique incident edge $z_X \in \pi^{-1}(z)$. In addition, the vanishing of the RH number at X implies that $d_\pi(q_X) = d_\pi(z_X) = d_\pi(X)$. At this point we have arrived at a contradiction, since the previous conclusions give rise to a non-trivial linear relation among the columns of the matrix $\left(\eta_{\mathrm{ft}_{J^c},\mathrm{src}(\pi)} \circ F_\pi\right)$. To be much more precise, the column corresponding to the edge q of $\mathrm{trgt}(\pi)$ is the sum of the columns corresponding to the edges z and p.

The previous work implies that $\mathrm{legval}(P_2) = 1$. Let $q \in E(\mathrm{trgt}(\pi))$ denote the additional edge incident to P_2. We observe that every vertex of $\pi^{-1}(P_2)$ different from $P_{2,a}$ and $P_{2,b}$ does not give rise to a vertex of $\mathrm{ft}_{J^c}(\mathrm{src}(\pi))$. Indeed, such a vertex is either expunged after forgetting the J^c-marked legs or incident to a J-marked leg (in which case it gets "absorbed" by the leg). If $P_{2,a} \neq P_{2,b}$, then the columns of the matrix $\left(\eta_{\mathrm{ft}_{J^c},\mathrm{src}(\pi)} \circ F_\pi\right)$ corresponding to the edges p and q are identical. This is not possible since the matrix is invertible. Therefore, we have the equality $P_{2,a} = P_{2,b}$, which means that h_a and h_b define a cycle, and hence they must equal a and b respectively. Furthermore, we have that $W = P_{2,a} = P_{2,b}$, $e_2 = q$, and e_1 is the leaf p. The vanishing of the RH numbers at all the other vertices of $\pi^{-1}(P_2) = \pi^{-1}(\pi(W))$ implies that $d_\pi(c) = 2$, and the ramification profile above $q = e_2$ is $(2, 1^{d-2})$. □

4.4 Dictionary

In this section we establish a dictionary between our work and that of [DV21] and [VDL22]. In the spirit of concreteness, we avoid excessive length and complication, and simply refer to the specific results and terminology, while kindly pointing the curious reader towards the precise reference in the corresponding manuscript. In contrast to our discrete admissible covers, in [DV21] the authors consider *DT-morphisms* (Definition 10 in loc. cit.) between a loopless unmarked graph and an unmarked tree, and these are allowed to have vertices of arbitrary positive valency. In our terms, a DT-morphism is simply a harmonic morphism with non-negative RH numbers. It turns out that any DT-morphism can be obtained by forgetting the marked legs of a discrete admissible cover. We motivate the comparison by illustrating in Figure 4.13 an example of [DV21] (Case 7a of Section 4 in loc. cit.) and in Figure 4.14 the same example but within our context (namely, with markings).

In loc. cit. the authors restrict their attention to *full-dimensional combinatorial types of DT-morphisms* (Definitions 21 and 34 of loc. cit.), and to approach this class of

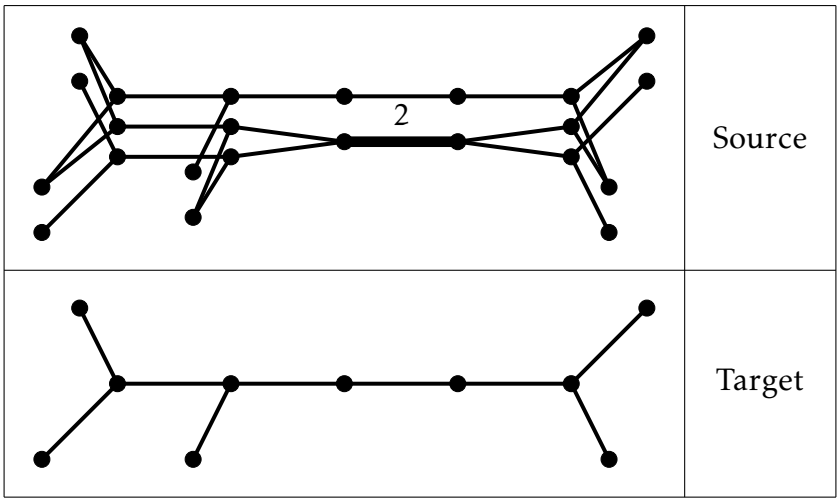


Figure 4.13: An example from [DV21]

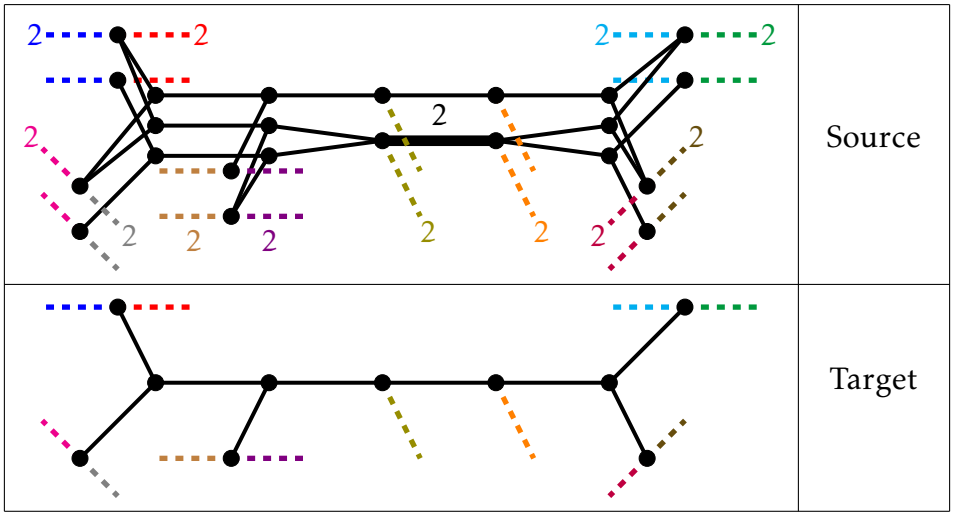


Figure 4.14: Same example as Figure 4.13, but with our markings

morphisms they are compelled to introduce (Section 3.3 of loc. cit.) an assortment of combinatorial properties (which are of course attested by every morphism of this class). Among these properties lies *change-minimality* (Definitions 47 and 48 of loc. cit.), which gives rise to their dimension formula (Proposition 29 of loc. cit.) for a DT-morphism $\phi\colon G \to T$:

$$\#E(T) = 2g(G) + 2\deg\phi - 5.$$

In addition, this condition also forces the target tree T to be at most 3-valent, and furthermore:

(A) If $V \in V(T)$ is 1-valent, then $\sum_{W\in\phi^{-1}(V)} r_\phi(W) = 2$.

(B) If $V \in V(T)$ is 2-valent, then $\sum_{W\in\phi^{-1}(V)} r_\phi(W) = 1$. In particular, there is a unique $W \in \phi^{-1}(V)$ with $r_\phi(W) = 1$, and all other have vanishing RH number.

(C) If $V \in V(T)$ is 3-valent, then $\sum_{W \in \phi^{-1}(V)} r_\phi(W) = 0$. In particular, the RH number of every $W \in \phi^{-1}(W)$ vanishes.

Proposition 4.4.1. *Suppose T is a tree, whose vertices have valency at most 3. Any change-minimal degree-d DT-morphism $\phi : G \to T$ can be obtained by forgetting the marked legs from an admissible cover (G', T', π_ϕ) in $\mathbb{AC}_{d,0}(\vec{\mu})$ where*

- $m := 3 + \#E(T)$.
- $\vec{\mu} := ((2, 1^{d-2})^m)$.

Remark. As hinted by the statement of the proposition, in the proof we describe a process of obtaining a discrete admissible cover from a DT-morphism. In Figure 4.13 we have depicted a DT-morphism from [DV21], and in Figure 4.14 we have depicted the discrete admissible cover produced out of this DT-morphism through the process we describe in the proof.

Proof. We first describe how to obtain T' out of T, then we explain how to obtain G' out of G, and finally we construct π_ϕ. For the first part, we let T' denote a tree obtained by:

- Grafting two legs at every 1-valent vertex of T.
- Grafting one leg at every 2-valent vertex of T.

Of course T' has genus 0, and since T was at most 3-valent, it follows that every vertex of T' is 3-valent. To show that T' is actually an object of $\mathbb{G}_{0,m}$, we first observe that

$$\#V(T) = \#\operatorname{val}_1(T) + \#\operatorname{val}_2(T) + \#\operatorname{val}_3(T).$$

The fact that T is a tree implies that

$$\#E(T) + 1 = \#\operatorname{val}_1(T) + \#\operatorname{val}_2(T) + \#\operatorname{val}_3(T). \tag{4.31}$$

Additionally, we remark that

$$2 \cdot \#E(T) = \sum_{v \in V(T)} \operatorname{val}(v) = \#\operatorname{val}_1(T) + 2 \cdot \#\operatorname{val}_2(T) + 3 \cdot \#\operatorname{val}_3(T), \tag{4.32}$$

and then multiplying (4.31) by 3 and substracting (4.32) from the result we obtain that

$$\#E(T) + 3 = 2 \cdot \#\operatorname{val}_1(T) + \#\operatorname{val}_2(T). \tag{4.33}$$

We observe that the right-hand side of (4.33) coincides, by construction, with $\#L(T')$. As $m := \#E(T) + 3$, we have thus shown that T' is actually an object of $\mathbb{G}_{0,m}$.

We now describe how to produce G' out of G. The idea here is to graft the legs lying above the newly grafted legs of T'. Before we describe the process, it is worth remarking that by definition of the objects of $\mathbb{AC}_{d,0}(\vec{\mu})$, the legs lying above $\ell_i(T')$ (for $1 \leq i \leq m$) must be $\ell_{1+(i-1)\cdot(d-1)}, \ell_{1+(i-1)\cdot(d-1)+1} \ldots, \ell_{1+(i-1)\cdot(d-1)-1}$. Additionally, the leg $\ell_{1+(i-1)\cdot(d-1)}$ must have weight 2 and all the others must have weight 1. We proceed as follows:

- Let $V \in V(T')$ be such that legval$(V) = 1$, and let $\ell_i(T')$ denote the leg incident to V. From (B) it follows that there is a unique $W \in \phi^{-1}(V)$ with $r_\phi(W) = 1$. At every vertex $W' \in \phi^{-1}(V)$ with $W' \neq W$ we graft $d_\phi(W')$ many legs to W' from the set

$$\{\ell_{1+(i-1)\cdot(d-1)+1} \ldots, \ell_{1+i\cdot(d-1)-1}\}. \tag{4.34}$$

  This forces the vanishing of RH number (now with the legs) at these vertices. At the vertex W we graft the leg $\ell_{1+i\cdot(d-1)}$, and graft the remaining $d_\phi(W) - 2$ legs from (4.34). As before, this also forces the vanishing of the RH number (now with the legs) at these vertices.

- Let $V \in V(T')$ be such that legval$(V) = 2$, and let $\ell_i(T')$ and $\ell_j(T')$ denote the legs incident to V. Because of (A) there are only two possiblities:

  - There is a unique vertex $W \in \phi^{-1}(V)$ with $r_\phi(W) = 2$, and the RH number vanishes at all the other vertices in this fiber.
    In this case, we graft the two weight 2 legs $\ell_{1+(i-1)\cdot(d-1)}$ and $\ell_{1+(j-1)\cdot(d-1)}$ to this vertex. If necessary, we graft weight 1 edges until the weights in both directions sum to $d_\phi(W)$. This forces the vanishing of the RH number at the vertex W. At all the other vertices $W' \in \phi^{-1}(V)$ with $W' \neq W$, we just graft $d_\phi(W')$ many weight 1 edges from each of the sets

$$\{\ell_{1+(i-1)\cdot(d-1)+1} \ldots, \ell_{1+i\cdot(d-1)-1}\}, \text{ and } \{\ell_{1+(j-1)\cdot(d-1)+1} \ldots, \ell_{1+j\cdot(d-1)-1}\}.$$

    A routine check shows that this causes the vanishing of the RH number at all these vertices.

  - There are two vertices $W_1, W_2 \in \phi^{-1}(V)$ with $r_\phi(W_1) = r_\phi(W_2) = 1$, and the RH number vanishes at all the other vertices in this fiber. The vertices W_1 and W_2 are necessarily 1-valent, and therefore $d_\phi(W_1) = d_\phi(W_2) = 2$. We graft to W_1 the weight 2 leg $\ell_{1+(i-1)\cdot(d-1)}$ and two weight 1 legs from

$$\{\ell_{1+(j-1)\cdot(d-1)+1} \ldots, \ell_{1+j\cdot(d-1)-1}\}.$$

    This implies the vanishing of the RH number at W_1. Similarly, we graft to

W_2 the weight 2 leg $\ell_{1+(j-1)\cdot(d-1)}$ and two weight 1 legs from

$$\{\ell_{1+(i-1)\cdot(d-1)+1}\ldots,\ell_{1+i\cdot(d-1)-1}\}.$$

As before, this implies the vanishing of the RH number at W_2. We observe that every vertex $W \in \phi^{-1}(V)$ with $W \neq W_1$ and $W \neq W_2$ must satisfy $d_\phi(W) = 1$. Therefore, we graft to every one of these vertices one weight 1 edge (of the remaining) from each of the sets

$$\{\ell_{1+(i-1)\cdot(d-1)+1}\ldots,\ell_{1+i\cdot(d-1)-1}\}, \text{ and } \{\ell_{1+(j-1)\cdot(d-1)+1}\ldots,\ell_{1+j\cdot(d-1)-1}\}.$$

This forces the vanishing of the RH number at the vertices $W \in \phi^{-1}(V)$ with $W \neq W_1$ and $W \neq W_2$.

The previous concoction gives rise to a discrete graph G' which is forcibly $m\cdot(d-1)$-marked. Moreover, we obtain a harmonic morphism $\pi_\phi\colon G' \to T'$ by just using ϕ and specifying for each $0 \leq i \leq m-1$

$$\pi_\phi\left(\left\{\ell_{1+i\cdot(d-1)},\ldots,\ell_{1+i\cdot(d-1)+(d-2)}\right\}\right) := \ell_{i+1}(T').$$

To finalize, we claim that this is, in fact, an admissible cover. We have already remarked the vanishing at the RH numbers at vertices of G' that lie over vertices of T' arising from 1- or 2-valent vertices of T. Since we are not modifying the behavior of ϕ at vertices of G' lying over vertices of T' that arise from 3-valent vertices of T, the vanishing of the RH numbers readily follows from (C). □

Remark. The previous argument can be carefully extended to arbitrary DT-morphisms. The change-minimality condition only simplified the execution of the argument, but was by no means fundamental to the whole procedure. The underlying idea is clear from an algebrogeometric perspective: A positive RH number represents the number of marked legs with simple ramification (weight 2) that have to be grafted. In addition, we observe that grafting legs to vertices of the target and modifying the corresponding vertices of the source by grafting weight 1 legs according to the local degrees preserves the harmonicity of the morphism as well as the RH numbers.

Remark. Following Theorem 4.27, we can recover Theorem 2 of [DV21] through Proposition 4.4.1 by means of the following instances:

- Let g be even, $r = 0$, $d = \frac{g}{2} + 1$, and $J = \varnothing$.
- Let g be odd, $r = 1$, $d = \frac{g+1}{2}$, and $J = \{1\}$.

In both cases, Theorem 4.27 shows that any genus-g tropical curve has a tropical modification that appears as the source of a degree-d DT-morphism (in the second case

we rely on the surjective of the forgetting the marking morphism between the realizations of the corresponding moduli spaces). Following Corollary 32 of [DV21], this is sufficient for Theorem 2 of loc. cit..

Proposition 4.4.2. *Let $\phi\colon G \to T$ be a change-minimal full-rank DT-morphism and let (G', T', π_ϕ) be an object of $\mathbb{AC}_{d,h}(\vec{\mu})$ that gives ϕ after forgetting the marked legs as in Proposition 4.4.1. For any object (T_s, T_t, π_ϕ) of $\mathbb{EAC}^{\varnothing}_{d,h}(\vec{\mu})$, we have that*

$$\operatorname{mult}(T_s, T_t, \pi_\phi) = \mathrm{m}(\phi), \tag{4.35}$$

where $\mathrm{m}(\phi)$ is the Vargas-Draisma multiplicity of the DT-morphism ϕ (Definition 13.18 of [VDL22]). In particular, for a generic genus-g tropical curve Γ, there are $C_{\frac{g}{2}}$ many degree-d $\varnothing$-marked discrete admissible covers (counted with multiplicity (4.35)) that have a tropical modification of Γ as a source.

Remark. The second statement of this proposition is the same enumerative result as that of Theorem 13.20 of [VDL22].

Proof. The second statement of the proposition is a consequence of Theorem 4.3.9 and the first statement. Hence, we focus on the first statement of the proposition. It is shown in Lemma 13.23 of [VDL22] that $\mathrm{m}(\phi) = D_\phi \left|\det(\hat{A}_\phi)\right|$, where $\hat{A}_\phi$ is the reduced edge-length matrix and the D_ϕ is the product of all the smallest positive integers d_h that make the h-row of $\hat{A}_\phi$ integral for $h \in E(\mathrm{ft}_{J^c}(G_{\pi_\phi})$. The reduced edge-length matrix is just the edge-length matrix A_ϕ (Section 2.4 of [DV21]) with the columns corresponding to leaves of T' divided by 2. In other words, $\left|\det(\hat{A}_\phi)\right| = \frac{1}{2^N}\left|\det(A_\phi)\right|$, where N is the number of vertices of T' with leg valency 2. Observe that the product of $\left|\det\left(\eta_{\mathrm{ft}_{J^c},\mathrm{src}(\pi)} \circ F_\pi\right)\right|$ with the denominator of $\omega^J_{d,0,\vec{\mu}}(\pi_\phi)$ from (4.1), give precisely $\left|\det\left(A_\phi\right)\right|$. More interestingly, $\mathrm{HS}(\pi_\phi) = 2^N$, so that if G_{π_ϕ} denotes $\mathrm{src}\left(\pi_\phi\right)$, then we just have to show that

$$\frac{\left(\prod_{e\in E(G_{\pi_\phi})} d_\pi(e)\right)\cdot\left(\prod_{V\in V(G_{\pi_\phi})} H(V)\cdot \mathrm{CF}(V)\right)}{\mathrm{VS}(\pi_\phi)} = D_\phi. \tag{4.36}$$

We first seek to understand the local Hurwitz numbers, we follow an exhaustive classification of all the local possibilities arising at the vertices of G_{π_ϕ} following the classification preceding Proposition 4.4.1 of the vertices of ϕ. Consider a vertex $V \in V\left(G_{\pi_\phi}\right)$ with $D = d_{\pi_\phi}(V)$. The coloring of the following pictures describes the edges that lie in the same fiber.

(I) If $\pi_\phi(V)$ is incident to three edges and no legs, then we get the unique possibility for G_{π_ϕ} around V depicted in Figure 4.15.

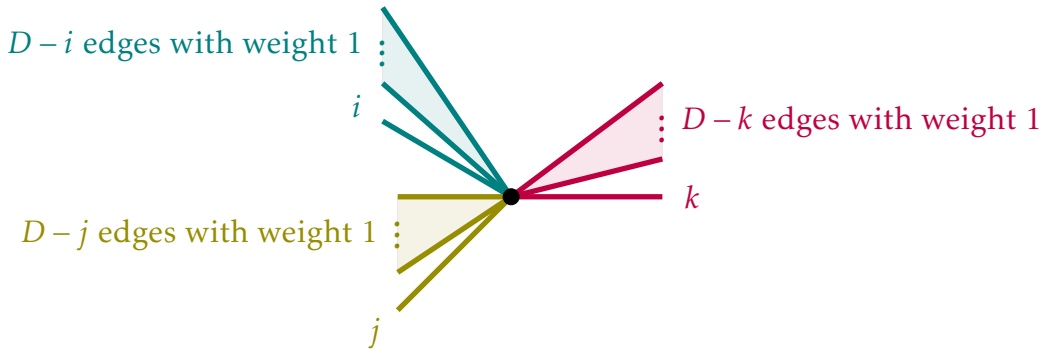


Figure 4.15: Unique possiblity of G_{π_ϕ} around V in case (I)

Since the RH number at V is 0, it follows from Figure 4.15 that $i+j+k = 2D+1$. If one of i, j, or k is 1, then the other two must necessarily be equal to D and therefore $H(V) = \frac{1}{D}$. In this special situation, the combinatorial factor is killed by $\mathrm{VS}(\pi_\phi)$. We now assume that $\min\{i,j,k\} > 1$ and observe that the $H(V)$ is the number of permutations σ_i, σ_j, σ_k of S_D with cycle types $(i,1^{D-i})$, $(j,1^{D-j})$, and $(k,1^{D-k})$ respectively, that multiply to the identity and generate a transitive group, divided by $D!$. In this case, we observe that since $\langle \sigma_i, \sigma_j \rangle$ must generate a transitive group, then the i-cycle of σ_i and the j-cycle of σ_j must intersect in $(i+j-D)$ numbers. Separately, the permutation σ_k (and hence $\sigma_i\sigma_j = \sigma_k^{-1}$) has to fix $(D-k)$ numbers. Observe that a number $q \in \{1,\dots,D\}$ is fixed by $\sigma_i\sigma_j$ if and only if the permutations intersect in both q and $\sigma_j(q)$, and contain these in opposite order. Since $D-k = i+j-D-1$, it follows that the cycles of σ_i and σ_j must consist of $(i+j-D)$ common numbers in opposite order, followed by $(D-j)$ and $(D-i)$ other numbers respectively. There are $\binom{D}{i+j-D}\cdot(i+j-D)!$ different possibilities for these $(i+j-D)$ common numbers because the order matters. Similarly, there are $\binom{D-(i+j-D)}{(D-j)}(D-j)!$ different possibilities arising from the ensuing $(D-j)$ numbers, and there are $\binom{D-(i+j-D)-(D-j)}{(D-i)}(D-i)! = (D-i)!$ different possibilities arising from the remaining $(D-i)$ numbers. In conclusion, we obtain that

$$\begin{aligned} H(V) &= \frac{1}{D!}\cdot\left(\binom{D}{i+j-D}\cdot(i+j-D)!\cdot\binom{D-(i+j-D)}{(D-j)}(D-j)!\cdot(D-i)!\right) \\ &= \frac{1}{D!}\cdot\left(\frac{D!}{(D-(i+j-D))!}\cdot\frac{(D-(i+j-D))!}{(D-i)!}\cdot(D-i)!\right) \\ &= 1. \end{aligned}$$

(II) If $\pi_\phi(V)$ is incident to only one leg of $T_t = \mathrm{trgt}\left(\pi_\phi\right)$, then we get the two possibilities depicted in Figures 4.16 and 4.17. In the second possibility, it is well known that $H(V) = \frac{1}{D}$. Hence, we focus on computing the Hurwitz numbers appearing in the first possibility. Here, we get the additional condition that $D = i+j$. To

compute $H(V)$ we compute the number of transpositions τ and permutations σ of cycle type $(i)(j)$ of S_D, such that $\sigma\tau$ is a D-cycle (the transitivity of $\langle\sigma,\tau\rangle$ is therefore immediate). Assume without loss of generality that $D-i \geq j$. If $i \neq j$, then the number of different possibilities for σ is simply $\frac{D!}{i\cdot(D-i)!}\cdot\frac{(D-i)!}{j\cdot(D-i-j)!} = \frac{D!}{i\cdot j}$. If $i = j$, then we have to divide the previous computation by 2. Since $\sigma\tau$ has to be a transposition, after fixing σ there are only $i\cdot j$ different possibilities for τ satisfying this condition. In conclusion:

(i) If $i \neq j$, then $H(V) = 1$.

(ii) If $i = j$, then $H(V) = \frac{1}{2}$.

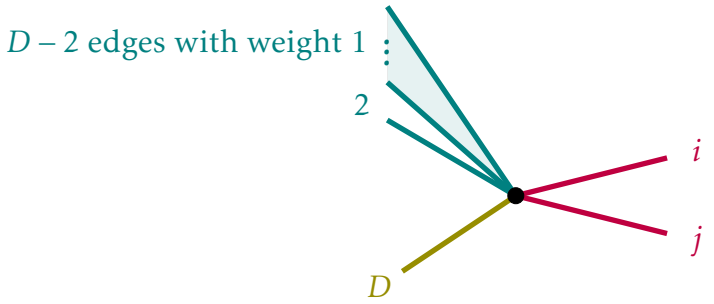


Figure 4.16: First possiblity of G_{π_ϕ} around V in case (II)

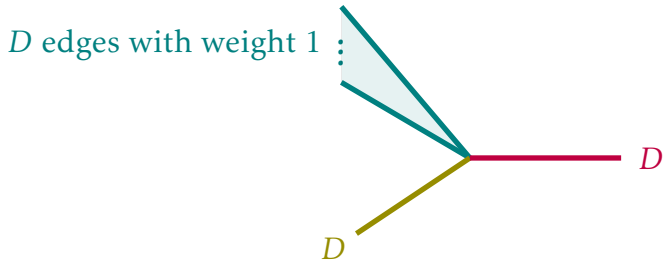


Figure 4.17: Second possibility of G_{π_ϕ} around V in case (II)

(III) If $\pi_\phi(V)$ is a node, then either $\mathrm{val}(V) = 3$ or $\mathrm{val}(V) = 4$. This means that we get the two possibilities depicted in Figures 4.18 and 4.19. In the former we get $H(V) = 1$, and in the latter we get $H(V) = \frac{1}{2}$.

We use this same classification and Lemma 13.24 of [VDL22] to continue with the proof. We observe that if $e \in E(\mathrm{ft}_{J^c} G_{\pi_\phi})$, then the edges of G_{π_ϕ} that add (Definition 4.3.13) to e form a path of G_{π_ϕ}. In this sense, edges of $\mathrm{ft}_{J^c} G_{\pi_\phi}$ are given by paths of G_{π_ϕ}, and this lemma classifies all the possibilities and explains the contribution of such an edge to D_ϕ. It consists of 3 cases, which we now explicit, and, additionally, we also explain the behavior of the contributions of the vertices in the interior of these paths:

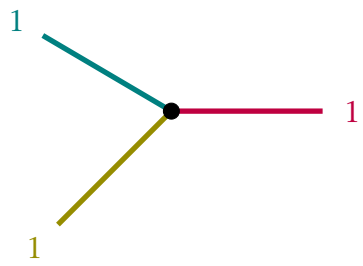


Figure 4.18: Possibility of G_{π_ϕ} around V with $\mathrm{val}(V) = 3$

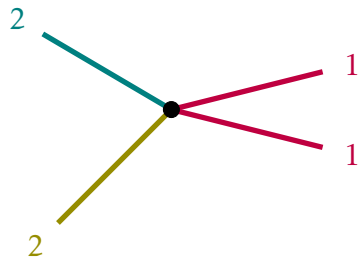


Figure 4.19: Possibility of G_{π_ϕ} around V with $\mathrm{val}(V) = 4$

($\triangle$) If the edge comes from a path passing through through a leaf, then it contributes a factor of 1. In this case the only non-trivial contribution that we see from a vertex in the path is that of the Hurwitz number of the vertex above a node, which gets killed by the corresponding CF($\bullet$). In this case, the VS(π_ϕ) does nothing, and the contributions from our multiplicity coincide.

($\triangle\triangle$) If the edge comes from a path that passes through vertices of G_{π_ϕ} whose number r_ϕ vanishes, then all edges of G_{π_ϕ} in this path have the same weight D and the edge given by the path contributes a factor of D. It is perhaps a good moment to recall that r_ϕ referes to the RH numbers of ϕ, whereas r_{π_ϕ} refers to the RH numbers of π_ϕ. Since π_ϕ is a discrete admissible covers, the latter always vanish. In our terms, the vanishing of the r_ϕ corresponds to the following two possibilities for interior vertices of the path (i. e. vertices not in the boundary of the path):

- we are placed in case (I) from our previous classification, where (without loss of generality) $i = 1$ and every teal edge is expunged after forgetting the marked legs,
- we are placed in the second possibility of case (II) from our previous classification, where every teal edge actually is a marked leg of G_{π_ϕ}, and hence is expunged after forgetting the marked legs.

In these cases, the factors CF($\bullet$) (which are all just D!) coming from the edges (or legs) are killed by the VS(π_ϕ). All the edges of G_{π_ϕ} in this path contribute

a factor of D, and all but one are killed by the local Hurwitz numbers. So, the contributions coming from our multiplicity coincide.

($\triangle\triangle\triangle$) Otherwise the edge comes from two paths of G_{π_ϕ} joined at a vertex with a local picture as in the first possibility of our case (II) from the previous classification, and each of these paths passes through vertices whose images are vertices with trivial leg valency. Moreover, for each of these paths, the edges have all the same weight, and the difference between the weights from one path and the other is just 1. In this case, the edges of each path have the same weight, the contribution of this edge to the D_ϕ is the product of both weights. In our terms, this means that the vertex where these paths are joined has a local picture as in the first possibility of our case (II) with $i = 1$, $j = D - 1$, and all the edges are actually marked legs of G_{π_ϕ}. Here, the edge with weight i is erased after forgetting the marking, and the same analysis as in the previous case yields that we get the same contribution from each path, and we just have the explain the contribution from the vertex where these paths join. The factor CF(•) coming from the possible repetitions in the marked legs is killed by the VS(π_ϕ). The local Hurwitz number is either 1 and there are no accompanying factors CF(•), or $\frac{1}{2}$ (in case $D = 2$) and the factor CF(•) kills this contribution. So, in other words, the local contributions coming from our multiplicity coincide.

The previous reasoning shows that the products of the weights actually give rise to the D_ϕ, taking into account the local Hurwitz numbers.
To finish the proof it is necessary to show that the contribution of the extremal vertices of the paths given by the cases ($\triangle$) to ($\triangle\triangle\triangle$) is trivial or cancels with the what remains of VS(π_ϕ). A change-minimal full-rank DT-morphism satisfies the dangling-no-glue condition (Definition 48 and Lemma 53 of loc. cit.), which implies that these extremal vertices are only of the following form:

- The local picture is the unique possibility of case (I), where every edge except the three edges with weights i, j, and k is expunged after forgetting the marked legs. If $\min\{i,j,k\} > 1$, then the local Hurwitz number is 1 and the factors CF(•) are cancelled by the remaining factors of VS(π). If (without loss of generality) $i = 1$, then the local Hurwitz number is $\frac{1}{D}$ and the combinatorial factors give $D!$, but in this case VS(π_ϕ) is precisely $(D-1)!$ coming from the permutations of the teal edges that are erased after forgetting the marking. So, the contribution of the extremal vertices having this form is always 1.

- The local picture is the first possibility of case (II), where the teal edges are actually marked legs, and none of the purple edges are erased after forgetting the marking. If $i \neq j$, then the local Hurwitz number is 1, and the only factor CF(•)

arises from the teal marked legs which actually gets killed by what remains of $\mathrm{VS}(\pi_\phi)$. If $i = j$, then the local Hurwitz number is $\frac{1}{2}$, which gets killed by the factor $\mathrm{CF}(\bullet)$ coming from the purple edges, and the factor $\mathrm{CF}(\bullet)$ coming from the teal marked legs is, once again, killed by the remainings of $\mathrm{VS}(\pi_\phi)$.

After exhausting all the possibilities, our comprehensive analysis shows that the extremal vertices contribute trivially, and hence the theorem follows. □

Chapter 5

Otherworldly echoes

As hinted by the title, this chapter connects our work with that of other places and outlines a natural construction tracing future developments. More precisely, tropical intersection theory has been a subject area developed throughout this century and as such has had multiple re-interpretations and extensions (ours being one of them). The chapter begins with an explanation of the relationship between our methods and those of [AR10] and [Gro18]. We then relate our work to that of [CGM22] with their description of tropical cycles for the moduli space of Mumford curves, and close with the construction of clutching morphisms for the poic-fibrations of spanning trees.

5.1 Relation to classical tropical intersection theory

We explain how the classical tropical intersection theory of polyhedral complexes in a tropical vector space (as in [AR10]) can be obtained in terms of our framework. Perhaps it is worth noting that the main difference (or diffculty) lies in the description of general polyhedral cycles. To be more precise, the usual description of tropical cycles in a tropical vector space permits the description of general polyhedral cycles, and not only fan cycles (these are, of course, closer to our endeavors). However, by adding an extra dimension and subdividing enough, we circumvent this difficulty by a usual trick concerning recession fans. For this we begin with a construction. Let N be a free abelian group of finite rank and consider N as a full rank lattice of the vector space $N_{\mathbb{R}}$. The pair $(N_{\mathbb{R}}, N)$ is then a tropical vector space.

Construction 5.1.1. Suppose $\mathcal{X}$ is a rational polyhedral complex of this tropical vector space, such that the collection of recession cones of $\mathcal{X}$ is a rational fan in $N_{\mathbb{R}}$. Let $N_X = N \oplus \mathbb{Z}$ and $(N_X)_{\mathbb{R}} = N_{\mathbb{R}} \oplus \mathbb{R}$, with last coordinate denoted by z. For a cell $\xi \in \mathcal{X}$, we consider the partially open integral cone σ_ξ of $(N_X)_{\mathbb{R}}$ given by adjoining the origin to the intersection of: the cone generated by $\xi \times 1 \subset M_{\mathbb{R}}$ and $\mathrm{rc}(\xi) \times \{0\} \subset M_{\mathbb{R}}$, and the half space H^0_z. It is clear that the collection $\Phi_{\mathcal{X}} = \{\sigma_\xi : \xi \in \mathcal{X}\}$ is a partially open rational fan

in $(N_X)_{\mathbb{R}}$, and we regard $\Phi_{\mathcal{X}}$ as a linear poic-complex in the standard way.

In the following example we use some imagery to convey the previous construction.

Example 5.1.2. Consider the 2-dimensional polyhedral complex $\mathcal{X}$ of $\mathbb{R}^2$ depicted in Figure 5.1. Observe that its recession cones produce a fan $\mathrm{rf}(\mathcal{X})$ which we depict in Figure 5.2. We illustrate then the previous construction in Figure 5.3 by producing a partially open fan in $\mathbb{R}^3$, such that the intersection with the affine subspace given by $z = 1$ gives rise to $\mathcal{X}$. To avoid overburdening the illustration, we only depict the cones of dimension ≤ 2 of this construction, and the red plane is the affine hyperplane given by $z = 1$. We highlight the intersections of this hyperplane with the cones by coloring them red.

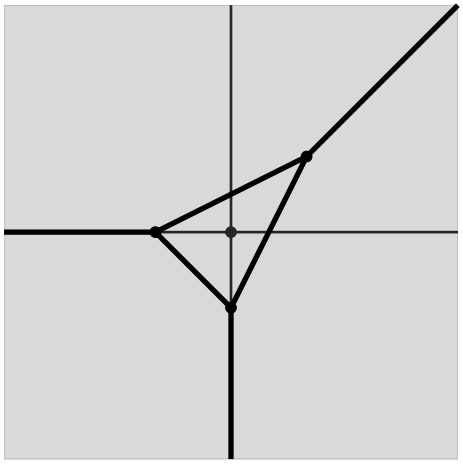


Figure 5.1: The polyhedral complex $\mathcal{X}$ of $\mathbb{R}^2$

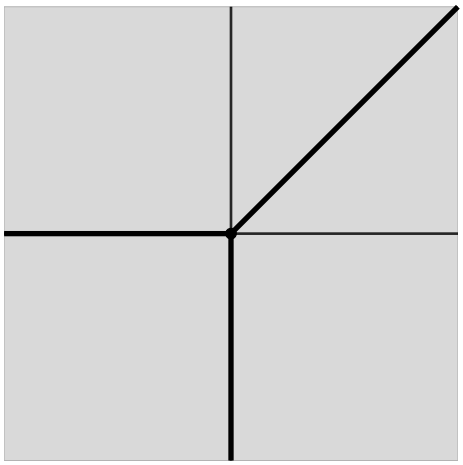


Figure 5.2: The recession fan $\mathrm{rf}(\mathcal{X})$ of $\mathcal{X}$

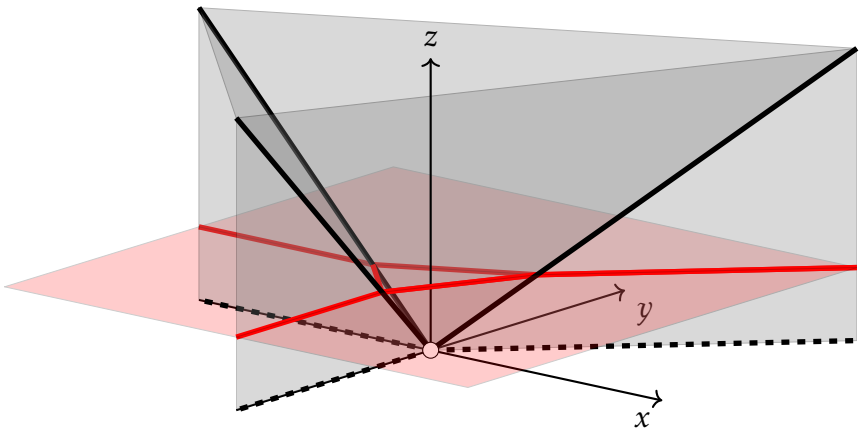


Figure 5.3: Depiction of (the 2-dimensional cones of) Construction 5.1.1

With the previous construction at hand, we have the following lemma. For the moment, we content ourselves with just the statement and briefly postpone its proof, as this makes use of technicalities and constructions better suited in the whole atmosphere of our last chapter. Hence we postpone it to the end of Section 6.2.

Lemma 5.1.3. *The tropical cycles of the linear poic-complex $\Phi_{\mathcal{X}}$ correspond with the classical tropical cycles of $\mathcal{X}$.*

It is always possible to subdivide a polyhedral complex into one whose recession cones form a fan (see [BGS11]), and classical tropical cycles are stable under subdivisions. Therefore, following our construction above, we can compute the classical tropical cycles of any polyhedral complex $\mathcal{X}$ in our framework. We now turn our heads towards the tropical intersection theory for weakly embedded cone complexes described in [Gro18]. The rehashing in this case is much more direct, and we begin by recalling some notions from them.

Following [Gro18] and [ACP15], a *cone complex* is a pair (σ, M) where σ is just a topological space and M is a finitely generated free abelian group of continuous real-valued functions on σ, such that the natural evaluation map $\sigma \to \mathrm{Hom}(M, \mathbb{R})$ is a homeomorphism onto a (closed) rational cone of $(M^\vee)_{\mathbb{R}}$. A face of an integral cone (σ, M) is an integral cone (τ, M) where $\tau \subset \sigma$ is given by the vanishing of finitely many functions on M that are non-negative on σ. A subcone of (σ, M) is a subset $\tau \subset \sigma$ such that the restriction of functions makes (τ, M) into an integral cone. A morphism of integral cones $(\sigma_1, M_1) \to (\sigma_2, M_2)$ is a continuous function $\sigma_1 \to \sigma_2$ that pulls back functions of M_2 to functions on M_1. Now, a cone complex Σ consists of a pair (slightly abusing notation) $(\Sigma, |\Sigma|)$ where:

- $|\Sigma|$ is a topological space,
- Σ is a finite collection of integral cones which are closed subspaces of $|\Sigma|$, such that:

- their union is all of $|\Sigma|$,
- every face of a cone of Σ is also a cone of Σ,
- the intersection of two cones in Σ is a union of common faces.

A *morphism of cone complexes* $f\colon \Sigma_1 \to \Sigma_2$ is simply a continuous map $|\Sigma_1| \to |\Sigma_2|$ such that for every integral cone $(\sigma_1, M_1) \in \Sigma_1$ there is a cone $(\sigma_2, M_2) \in \Sigma_2$ with

- $f(\sigma_1) \subset \sigma_2$,
- the restriction of f to σ_1 is a morphism of integral cones $(\sigma_1, M_1) \to (\sigma_2, M_2)$.

A *proper subdivision* (Section 2.1 of [Gro18]) of a cone complex Σ is a cone complex Σ' such that $|\Sigma'| = |\Sigma|$ and every integral cone of Σ' is a subcone of an integral cone of Σ. A weakly embedded cone complex consists of a cone complex Σ with a continuous map $\phi^\Sigma\colon |\Sigma| \to N^\Sigma_{\mathbb{R}}$, where N^Σ is a free abelian group of finite rank, that is integral linear on every integral cone of Σ.

The reinterpretation in terms of our framework is most direct. More precisely, out of an integral cone (σ, M) (abusing notation slightly) we produce a poic σ, where N^σ is simply the intersection $\mathrm{Lin}(\sigma) \cap (M^\vee)$. It is clear that a morphism of integral cones induces directly a poic-morphism (the condition on pullbacks immediately implies that the function is integral linear with respect to the lattices). From a cone complex Σ we produce a poic-complex by taking Σ as a poset (with respect to inclusions), and taking the tautological association $(\sigma, M) \to \sigma$. We emphasize the difference between the two by denoting this poic-complex $\mathrm{p}(\Sigma)$. It is clear that the realization of $\mathrm{p}(\Sigma)$ comes with a map to $|\Sigma|$ which is, in fact, a homeomorphism (by construction it gives a bijection that is also a local homeomorphism). Under this guise, if Σ is furthermore a weakly embedded cone complex, then the map ϕ^Σ defines naturally a morphism of poic-complexes $\mathrm{p}(\Sigma) \to \underline{\left(N^\Sigma\right)_{\mathbb{R}}}$, and in particular $\mathrm{p}(\Sigma)$ is a linear poic-complex. It is immediately seen that the Minkowski weights (see section 3.1 of [Gro18]) of a weakly embedded cone complex Σ correspond to our Minkowski weights on $\mathrm{p}(\Sigma)$. Such comparison actually extends to the notion of tropical cycles, which are just the corresponding limit with respect to proper subdivisions. We record this as a lemma and, just as before, postpone its proof to section 6.2, due to the involvement of future constructions.

Lemma 5.1.4. *The tropical cycles of a weakly embedded cone complex Σ correspond with the tropical cycles of the linear poic-complex $\mathrm{p}(\Sigma)$.*

5.2 Mumford curves, cycle rigidification, and beyond

In this section we relate our description of tropical cycles in the moduli spaces of tropical curves to that of [CGM22]. It is worth mentioning that in loc. cit., the authors

consider discrete graphs endowed with a genus function on the vertices, in contrast to our situation where these functions are presumed to be trivial. Separately, the authors define the notion of a tropical space and the category of tropical spaces, which they then further endow with the structure of a site. In this guise, they make sense of the notion of a *family of n-marked genus-g stable tropical curves* over a fixed tropical space, and use this to introduce the category $\overline{\mathscr{M}}_{g,n}$ fibered in groupoids over the category of tropical spaces given by

$$\overline{\mathscr{M}}_{g,n} = \{ \text{ families of } n\text{-marked genus-}g \text{ stable tropical curves over } B\},$$

where B is a tropical space. This fibered category is actually a stack with respect to the site structure on the category of tropical spaces, but unfortunately this stack is not geometric and it is not possible to provide it with an atlas. The issue in this context is rather topological (see Examples 3.13, 3.19, 4.29, and Remark 6.12 of [CGM22]), as there cannot exist a local isomorphism onto $\overline{\mathscr{M}}_{g,n}$: the n-marked graph with a single vertex of genus g determines a point that has trivial isotropy group but any combinatorial type has a curve close to it. In addition, using Example 3.13 they show in Example 4.29, why looking for different nice classes of morphisms of tropical spaces (instead of local isomorphisms) is a "fruitless endeavor". The latter implies that the intersection theory they later define in Section 6 for tropical spaces cannot be directly applied to $\overline{\mathscr{M}}_{g,n}$ (there is no atlas after all!), but they are able to introduce divisors and line bundles on this stack. By these means, they settle by considering tropical cycles on tropical spaces T that are equipped with a morphism $T \to \overline{\mathscr{M}}_{g,n}$.

These issues vanish when we consider Mumford curves, as these provide an open substack $\mathscr{M}_{g,n}^{\mathrm{Mf}}$ which does admit an atlas. A *Mumford curve* in [CGM22] is a tropical curve, whose genus function on the underlying combinatorial type (the discrete graph) is identically zero (this is what we call a tropical curve): in terms of spaces our $\mathcal{M}_{g,n}^{\mathrm{trop}}$ corresponds identically to their $\mathscr{M}_{g,n}^{\mathrm{Mf}}$. They construct a geometric atlas for $\mathscr{M}_{g,n}^{\mathrm{Mf}}$ by means of cycle rigidifications. A cycle rigidified graph is simply a graph together with an oriented cycle basis consisting of primitive cycles (those that are not subsets of another cycle), and by means of these they define a space $\mathscr{V}_{g,n}^{\mathrm{Mf}}$ whose points parametrize cycle rigidified tropical curves. The space $\mathscr{V}_{g,n}^{\mathrm{Mf}}$ of cycle rigidified Mumford curves is interpreted as follows in terms of our framework:

Construction 5.2.1. Suppose $g, n \geq 0$ are such that $n + 2g - 2 > 0$. Let $\mathbb{V}_{g,n}$ denote the category specified by:

- The objects of $\mathbb{V}_{g,n}$ consist of tuples $(G, c_1^G, \ldots, c_g^G)$, where G is an object of $\mathbb{G}_{g,n}$ and $(c_i^G)_{1 \leq i \leq g}$ is an oriented cycle basis of $H_1(G, \mathbb{Z})$ consisting of primitive cycles.
- For two objects (G, c_i^G) and (H, c_i^H), the set of morphisms $\mathrm{Hom}_{\mathbb{V}_{g,n}}((G, c_i^G), (H, c_i^H))$

consists of the contractions $f\colon G \to H$ such that $f_* c_i^G = c_i^H$ (here f_* denotes the induced morphism between homology groups).

- Composition of morphisms is just composition of maps.

It is readily observed that $\mathbb{V}_{g,n}$ is essentially finite, but it is not thin (for instance the theta graph and its contractions). On the other hand, there are no non-trivial automorphisms at objects of $\mathbb{V}_{g,n}$ (see Lemma 4.2 of [CGM22]). We can make a poic-space out of this category by means of the association (here cr stands for cycle rigidification)

$$\mathrm{cr}_{g,n}\colon \mathbb{V}_{g,n} \to \mathbb{G}_{g,n}, (G, (c_i^G)) \mapsto G,$$

which is actually a functor. We can then set $\mathrm{V}_{g,n} := \mathrm{M}_{g,n}^{\mathrm{trop}} \circ \mathrm{cr}_{g,n}$ and obtain a poic-space $\mathrm{V}_{g,n}$ that immediately comes with a morphism of poic-spaces $\mathrm{cr}\colon \mathrm{V}_{g,n} \to \mathrm{M}_{g,n}^{\mathrm{trop}}$. As we mentioned before, the category is not thin, so $\mathrm{V}_{g,n}$ is just a poic-space. From this construction it is clear that the space $\mathscr{V}_{g,n}^{\mathrm{Mf}}$ is the realization of this poic-space.

Just as we did with the categories of discrete admissible covers, we can obtain a linear poic-fibration over $\mathrm{V}_{g,n}$ by pulling back $\mathrm{st}_{g,n}$ through the natural morphism of poic-spaces $\mathrm{V}_{g,n} \to \mathrm{M}_{g,n}^{\mathrm{trop}}$. This is the matter of the ensuing construction, but first some notation.

Notation 5.2.2. Let $\mathbb{R}_{\neq 0}$ denote the linear poic-complex given as $\mathbb{R}_{>0} \sqcup \mathbb{R}_{<0}$, where we regard these poics as subcones of $\underline{\mathbb{R}}$. The underlying category of this poic-complex consists of two objects, which for the sake of simplicity we will just denote by ± 1.

Construction 5.2.3. Consider the linear poic-complex $\mathrm{M}_{0,n+2g}^{\mathrm{trop}} \times (\mathbb{R}_{\neq 0})^g$, so that an object of the underlying category is a tuple $(T, (s_i)_{1 \leq i \leq g})$ where $s_i = \pm 1$. To such an object we associate the cycle rigidified graph $(\mathrm{st}_{g,n}(T), (c^{s_i})_{1 \leq i \leq g})$, where c^{s_i} is the oriented primitive cycle given by the primitive cycle generated by the added edge $e_i \in E(\mathrm{st}_{g,n}(T))$ oriented as follows:

- if $s_i = 1$, then it is oriented from the vertex given by $\partial \ell_{n+i}(T)$ towards the vertex incident to $\ell_{n+g+i}(T)$,
- if $s_i = -1$, then it has the reverse orientation.

The morphisms are just assigned as with $\mathrm{st}_{g,n}$, and this association can be seen to give rise to a well-defined functor. Further, this functor can be improved to a morphism of poic-spaces

$$\mathrm{rst}_{g,n}\colon \mathrm{M}_{0,n+2g}^{\mathrm{trop}} \times (\mathbb{R}_{\neq 0})^g \to \mathrm{V}_{g,n}, \tag{5.1}$$

by letting the natural transformation be just like in (3.10) (here rst stands for rigidification of spanning trees).

Proposition 5.2.4. *The morphism of poic-spaces* (5.1) *is a poic-fibration.*

Proof. The proof is analogous to that of Proposition 3.5.8, so we prefer to omit it. □

To trace back to tropical spaces, we remark that the realization of $\mathrm{M}^{\mathrm{trop}}_{0,n+2g} \times \mathbb{R}^g_{\neq 0}$ is a partially open fan embedded in a vector space, and as such has a natural tropical space structure. If we now take the realization of (5.1), then we obtain a continuous map

$$|\mathrm{rst}_{g,n}| \colon \mathcal{M}^{\mathrm{trop}}_{0,n+2g} \times (\mathbb{R}_{\neq 0})^g \to \mathscr{V}^{\mathrm{Mf}}_{g,n}, \tag{5.2}$$

which is a morphism of tropical spaces. Under this morphism, we can identify tropical cycles of the tropical space $\mathscr{V}^{\mathrm{Mf}}_{g,n}$ as tropical cycles of the tropical space $\mathcal{M}^{\mathrm{trop}}_{0,n+2g} \times \mathbb{R}^g_{\neq 0}$ that are invariant with respect to the map $|\mathrm{rst}_{g,n}|$. This coincides epistemologically with our whole enterprise, so to understand how we can compute the tropical cycles on the tropical space $\mathscr{V}^{\mathrm{Mf}}_{g,n}$ in terms of our framework, it suffices to explain how to compute the tropical cycles on the tropical space $\mathcal{M}^{\mathrm{trop}}_{0,n+2g} \times \mathbb{R}^g_{\neq 0}$ in our terms. But this tropical space $\mathcal{M}^{\mathrm{trop}}_{0,n+2g} \times \mathbb{R}^g_{\neq 0}$ is embedded in a real vector space, and the tropical cycles of this tropical space coincide with the classical tropical cycles (see Example 6.2 of loc. cit.). Therefore, following the same exact procedure as in Construction 5.1.1, we can modify $\mathcal{M}^{\mathrm{trop}}_{0,n+2g} \times \mathbb{R}^g_{\neq 0}$ by adding an extra dimension, and produce naturally a linear poic-complex whose cycles are the same as the tropical cycles of this tropical space.

Remark. The realization (5.1) corresponds to the fibered product $(\mathcal{M}^{\mathrm{trop}}_{0,n+2g} \times \mathbb{R}^g_{>0}) \times_{\mathscr{M}^{\mathrm{Mf}}_{g,n}} \mathscr{V}^{\mathrm{Mf}}_{g,n}$. In particular, in guise of Theorem 4.27 of [CGM22] the map obtained from this space to $\mathcal{M}^{\mathrm{trop}}_{0,2g+n} \times \mathbb{R}^g_{>0}$ is a local isomorphism of tropical spaces and a 2^g-sheeted topological covering map.

We close this section with a few words on an additional construction and some speculation. Following the construction of the poic-fibration $\mathrm{rst}_{g,n}$, one might be tempted to follow a similar pattern for $\mathrm{M}^{\mathrm{trop}}_{0,n+2g} \times \underline{\mathbb{R}}^g$ and $\mathscr{M}_{g,n}$[1]. As anticipated, this would naturally not solve the previously mentioned issue of providing the stack $\mathscr{M}_{g,n}$ with an atlas and the underlying map is by no means a local isomorphism. However, we can understand this construction in a similar combinatorial vein as what we have done all throughout with $\mathcal{M}^{\mathrm{trop}}_{g,n}$. To be more precise, instead of $\mathbb{G}_{g,n}$ we can consider the category $\mathbb{SG}_{g,n}$ of genus-g n-marked stable graphs. The objects of this category consist of pairs (G, g) where:

- G is a discrete graph of genus $\leq g$.
- $b \colon V(G) \to \mathbb{Z}_{\geq 0}$ is a function such that $g = g(G) + \sum_{V \in V(G)} b(V)$.
- Every vertex of G satisfies the stability condition on valency. Namely, for $V \in V(G)$ if $b(V) = 0$ then $\mathrm{val}(V) \geq 3$, if $b(V) = 1$ then $\mathrm{val}(V) \geq 1$.

[1] perhaps it is best to say $\mathscr{M}_{g,n}(\mathrm{pt})$.

If (G,b) is a genus-g n-marked stable graph and $f\colon G \to H$ is a contraction, then we set $f_*b\colon V(H) \to V(G)$ as

$$f_*b(W) := g(f^{-1}(W)) + \sum_{V \in V(f^{-1}(W))} b(V),$$

where $W \in V(H)$ and $f^{-1}(W)$ denotes the subgraph of G defined by $f^{-1}(W)$ (this subgraph is necessarily connected). Now, for an object (G,b), we let $\sigma_{(G,b)}$ denote the poic given by $\mathbb{R}_{\geq 0}^{E(G)}$, and this produces a functor

$$\overline{\mathrm{M}}_{g,n}^{\mathrm{trop}}\colon \mathbb{SG}_{g,n}^{\mathrm{op}} \to \mathrm{POIC},$$

which is in fact a poic-space. If $g = 0$ we just obtain $\mathrm{M}_{0,n}^{\mathrm{trop}}$, but if $g > 0$ we obtain a different poic-space. We could adapt a similar construction to that of $\mathrm{st}_{g,n}$ to produce a poic-fibration

$$\overline{\mathrm{st}}_{g,n}\colon \mathrm{M}_{0,n+2g}^{\mathrm{trop}} \times \underline{\mathbb{R}}^g \to \overline{\mathrm{M}}_{g,n}^{\mathrm{trop}}.$$

This would allow us to describe $\overline{\mathrm{st}}_{g,n}$-equivariant tropical cycles, and one can only help but wonder how to adapt our previous constructions with admissible covers to the more general case of stable graphs. This is, at the present time, unbeknown to us. However, J. Li's framework (as it might be apparent from our description of it) also allows for the introduction of the non-trivial genus functions, and this hints that one can follow one's nose through our core argument for the balancing of the standard weight (once one already has made sense of the analogous constructions for discrete admissible covers), and hence produce a tropical cycle of the corresponding poic-fibration.

5.3 Clutching morphisms

In this section we explain a natural construction involving the spanning tree poic-fibrations. Namely, we want to join two discrete graphs at a point. For this we use the follwing standing notation through the section

Notation 5.3.1. Let $g,h \geq 0$ be integers, and let A and B be finite sets with $2g+\#A-2 > 0$ and $2h+\#B-2 > 0$. Additionally, we assume that $A \cap B = \{c\}$, so that $A\Delta B = (A \cup B)\backslash\{c\}$.

We construct a proper morphism of linear poic-fibrations $\mathrm{st}_{g,A} \times \mathrm{st}_{h,B} \to \mathrm{st}_{g+h,A\Delta B}$, given by joining two graphs at the vertices incident to their c-marked legs.

Construction 5.3.2. Suppose G is an object of $\mathbb{G}_{g,A}$ and H is an object of $\mathbb{G}_{h,B}$. We will describe an object $\kappa(G,H)$ of $\mathbb{G}_{g+h,A\Delta B}$ motivated by our description. Let $V_c \in V(G)$ denote the vertex incident to $\ell_c(G)$, and let $F(\kappa(G,H))$ denote the set obtained from

$F(G) \sqcup F(H)$ by identifying the sets $\ell_c(G) \cup \partial\ell_c(G) \subset F(G)$ and $\ell_c(H) \cup \partial\ell_c(H) \subset F(H)$ into the single element V_c, that is:

$$F(\kappa(G,H)) = (F(G) \backslash \ell_c(G)) \sqcup (F(H) \backslash (\ell_c(H) \cup \partial\ell_c(H))).$$

Observe that the following maps are well-defined

$$r_{\kappa(G,H)} \colon F(\kappa(G,H)) \to F(\kappa(G,H)), \qquad X \mapsto \begin{cases} r_G(X), & \text{if } X \in F(G), \\ r_H(X), & \text{if } X \in F(H) \backslash r_H^{-1}(\partial\ell_c(H)), \\ V_c, & \text{if } X \in r_H^{-1}(\partial\ell_c(H)), \end{cases}$$

$$\iota_{\kappa(G,H)} \colon F(\kappa(G,H)) \to F(\kappa(G,H)), \qquad X \mapsto \begin{cases} \iota_G(X), & \text{if } X \in F(G), \\ \iota_H(X), & \text{if } X \in F(H), \end{cases}$$

and, furthermore, satisfy $\iota_{\kappa(G,H)} \circ r_{\kappa(G,H)} = r_{\kappa(G,H)}$. This means then that the triple $(F(\kappa(G,H)), r_{\kappa(G,H)}, \iota_{\kappa(G,H)})$ actually defines a graph $\kappa(G,H)$. By construction, we have that the edges of this graph are just the union of those of G and H (that is $E(\kappa(G,H)) = E(G) \cup E(H)$). Furthermore, if we identify $\mathbb{R}^{E(\kappa(G,H))}$ with the product $\mathbb{R}^{E(G)} \times \mathbb{R}^{E(H)}$, then the cone of metrics $\sigma_{\kappa(G,H)}$ naturally coincides with the product $\sigma_G \times \sigma_H$, so we let $\eta_{\kappa(G,H)}$ denote the identity map $\sigma_G \times \sigma_H \to \sigma_{\kappa(G,H)}$. In addition, we remark that the markings of G and H clearly give rise to a marking

$$\ell \colon A \Delta B \to L(\kappa(G,H)).$$

If we now turn our attention towards morphisms, let $f_1 \colon G \to G'$ and $f_2 \colon H \to H'$ be morphisms of their respective categories. Then the map

$$F(\kappa(G,H)) \to F(\kappa(G',H')), X \mapsto \begin{cases} f_1(X), & \text{if } X \in F(G), \\ f_2(X), & \text{if } X \in F(H), \end{cases}$$

is well-defined and gives rise to a morphism $\kappa(f_1, f_2) \colon \kappa(G,H) \to \kappa(G',H')$.

Definition 5.3.3. Following the notation of the above construction, we call the graph $\kappa(G,H)$ *the clutching of G and H at c*.

Lemma 5.3.4. *The associations of the above construction (and following the notation thereof) give rise to a functor*

$$\kappa \colon \mathbb{G}^{\mathrm{op}}_{g,A} \times \mathbb{G}^{\mathrm{op}}_{h,B} \to \mathbb{G}^{\mathrm{op}}_{g+h,A\Delta B}, (G,H) \mapsto \kappa(G,H).$$

In addition, the maps $\eta_{\kappa,(\bullet,\bullet)}$ define a natural transformation

$$\eta_\kappa \colon \mathrm{M}^{\mathrm{trop}}_{g,A} \times \mathrm{M}^{\mathrm{trop}}_{h,B} \implies \sigma_{g+h,A\Delta B} \circ \kappa.$$

Proof. Both statements of the lemma are routine checks, that we omit for the sake of brevity. □

Definition 5.3.5. Following our standing notation 5.3.1, the previous lemma shows that we obtain a morphism of poic-spaces

$$\kappa \colon \mathrm{M}^{\mathrm{trop}}_{g,A} \times \mathrm{M}^{\mathrm{trop}}_{h,B} \to \mathrm{M}^{\mathrm{trop}}_{g+h,A\Delta B}. \tag{5.3}$$

We call this morphism of poic-spaces the *clutching morphism*.

Following our motivation, we must now describe the behaviour of the morphism at the linear poic-complexes involved in the poic-fibrations. First, we explain the linear maps between vector spaces.

Construction 5.3.6. We briefly separate ourselves from our notation to avoid confusion. Let I and J denote distinct finite sets both of size ≥ 3. Consider the following integral linear map

$$K_{I,J} \colon \mathbb{R}^{\binom{I}{2}} \oplus \mathbb{R}^{\binom{J}{2}} \to \mathbb{R}^{\binom{I\Delta J}{2}}, (\vec{v}, \vec{w}) \mapsto K_{I,J}(\vec{v}, \vec{w}),$$

where $K_{I,J}(\vec{v}, \vec{w})$ is the vector of $\mathbb{R}^{\binom{I\Delta J}{2}}$ with $\{a,b\}$-coordinate given by

$$K_{I,J}(\vec{v}, \vec{w})_{\{a,b\}} := \begin{cases} v_{\{a,b\}}, & \{a,b\} \subset I, \\ w_{\{a,b\}}, & \{a,b\} \subset J, \\ v_{\{a,c\}} + w_{\{c,b\}}, & a \in I, b \in J. \end{cases}$$

This map naturally descends to an integral linear map $K_{I,J} \colon Q_I \oplus Q_J \to Q_{I\Delta J}$, and in particular to a morphism of linear poic-complexes $K_{I,J} \colon \underline{\left(N_{\mathrm{dist}_I} \oplus N_{\mathrm{dist}_J}\right)_{\mathbb{R}}} \to \underline{\left(N_{\mathrm{dist}_{I\Delta J}}\right)_{\mathbb{R}}}$. Furthermore, $K_{I,J} \circ (\mathrm{dist}_I \times \mathrm{dist}_J) = \mathrm{dist}_{I\Delta J} \circ \kappa$, so that letting $\kappa_{\mathrm{int}} := K_{I,J}$, we obtain a morphism of linear poic-complexes

$$\kappa \colon \mathrm{M}^{\mathrm{trop}}_{0,I} \times \mathrm{M}^{\mathrm{trop}}_{0,J} \to \mathrm{M}^{\mathrm{trop}}_{0,I\Delta J}. \tag{5.4}$$

Notation 5.3.7. Let $j = g + h$ and $C = A\Delta B$, we set

$$\mathbb{g} = \{1, \dots, g, 1^*, \dots, g^*\}, \qquad \mathbb{h} = \{1, \dots, h, 1^*, \dots, h^*\}, \qquad \mathbb{j} = \{1, \dots, j, 1^*, \dots, j^*\},$$

and regard $\mathbb{g}$ and $\mathbb{h}$ as disjoint. We identify $\mathbb{j}$ as the disjoint union $\mathbb{g} \sqcup \mathbb{h}$, by viewing

$\mathbb{g} \subset \mathbb{j}$ and identifying $\mathbb{h}$ with the subset (by $n \mapsto g+n$ and $n^* \mapsto (g+n)^*$, for $1 \leq n \leq h$)

$$\{g+1, \dots, g+h, (g+1)^*, \dots, (g+h)^*\} \subset \mathbb{j}.$$

With the above, we regard the morphism (5.4) as

$$\kappa \colon \mathrm{M}^{\mathrm{trop}}_{0,A\sqcup \mathbb{g}} \times \mathrm{M}^{\mathrm{trop}}_{0,B\sqcup \mathbb{h}} \to \mathrm{M}^{\mathrm{trop}}_{0,C\sqcup \mathbb{j}}.$$

Additionally, we denote the natural map $\mathbb{R}^g \times \mathbb{R}^h \to \mathbb{R}^{g+h}$ by $\mathrm{Id}_g \oplus \mathrm{Id}_h$.

Lemma 5.3.8. *Let $\Bbbk^{\mathrm{c}}$ denote the morphism of linear poic-complexes $\kappa \times (\mathrm{Id}_g \oplus \mathrm{Id}_h) \colon \mathrm{ST}_{g,A} \times \mathrm{ST}_{h,B} \to \mathrm{ST}_{j,C}$, and let $\Bbbk^{\mathrm{s}}$ denote the morphisms of poic-spaces $\kappa \colon \mathrm{M}^{\mathrm{trop}}_{g,A} \times \mathrm{M}^{\mathrm{trop}}_{h,B} \to \mathrm{M}^{\mathrm{trop}}_{j,C}$. These define a proper morphism of poic-fibrations*

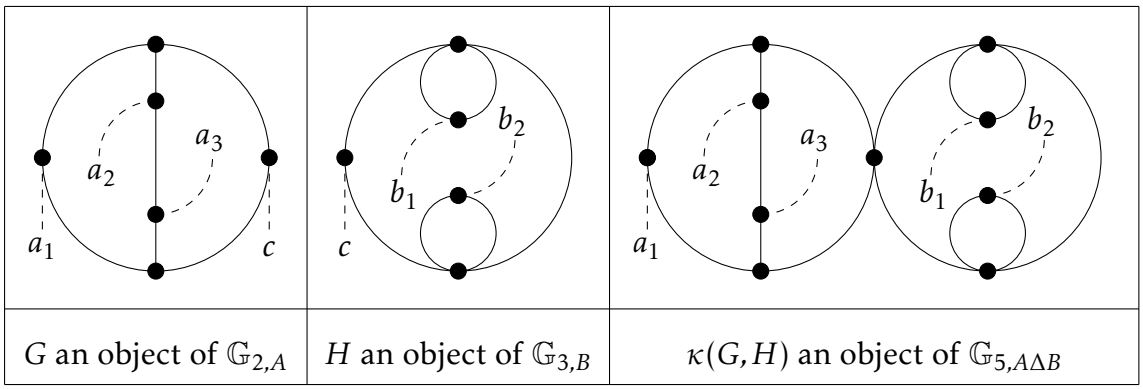
$$\Bbbk \colon \mathrm{st}_{g,A} \times \mathrm{st}_{h,B} \to \mathrm{st}_{j,C}. \tag{5.5}$$

Proof. The properness of $\Bbbk^{\mathrm{c}}$ just depends on the properness of $\kappa \colon \mathrm{M}^{\mathrm{trop}}_{0,A\sqcup \mathbb{g}} \times \mathrm{M}^{\mathrm{trop}}_{0,B\sqcup \mathbb{h}} \to \mathrm{M}^{\mathrm{trop}}_{0,C\sqcup \mathbb{j}}$. But this is clear, since the $\eta_{\Bbbk^{\mathrm{c}},\bullet}$ are isomorphisms of poics. The lifting property of $\Bbbk^{\mathrm{s}}$ on isomorphisms follows by keeping track of the images of the subgraphs of $\kappa(G,H)$ generated by G and H (and reconstructing the respective isomorphisms in this way). To close, we observe that the relation $\mathrm{st}_{j,C} \circ \Bbbk^{\mathrm{c}} = \Bbbk^{\mathrm{s}} \circ \left(\mathrm{st}_{g,A} \times \mathrm{st}_{h,B}\right)$ is readily checked. □

Definition 5.3.9. We also call the proper morphism of linear poic-fibrations (5.5), *the clutching morphism*.

We close this section with a pictorial example of the above constructions.

Example 5.3.10. An example of the clutching morphism is explicitly depicted in Figure 5.4, where we let $A = \{a_1, a_2, a_3, c\}$ and $B = \{b_1, b_2, c\}$. In this case, we have that $A \Delta B = \{a_1, a_2, a_3, b_1, b_2\}$, and from objects G of $\mathbb{G}_{2,A}$ and H of $\mathbb{G}_{3,B}$, we obtain an object $\kappa(G,H)$ of $\mathbb{G}_{5,A\Delta B}$. For size considerations we omit the labelling of vertices and edges in these depictions.



G an object of $\mathbb{G}_{2,A}$	H an object of $\mathbb{G}_{3,B}$	$\kappa(G,H)$ an object of $\mathbb{G}_{5,A\Delta B}$

Figure 5.4: The morphism κ in action

Chapter 6

Remaining Technicalities

This closing chapter ties the loose ends. We conclude with the proofs of the lemmas we have postponed until now (of course, we recall these in the due moment), and build toward them by several constructions and technicalities. It is perhaps best to just plunge into these.

6.1 Equivalent poic-complexes and subdivisions

We have extensively discussed poic-complexes and constructions stemming out of them, but it has seldom been the case that we have examined isomorphisms or equivalences of them. As in the case of categories, the notion of isomorphism of poic-complexes is quite restrictive, and it is often the case that we can change the underlying category by an equivalent one. For our purposes, the motivation for this would be in order to simplify some computations (or constructions!). Let $\Phi\colon C_\Phi \to \mathrm{POIC}$ denote a poic-complex. We first begin with a lemma exemplifying the spirit of equivalences of poic-complexes.

Lemma 6.1.1. *If D is a category with a functor $F : D \to C_\Phi$ that is an equivalence, then $\Phi \circ F$ is also a poic-complex. Moreover, the functor F induces a morphism of poic-complexes $\Phi \circ F \to \Phi$ whose realization is a homeomorphism, which is piecewise linear at the cones.*

Proof. An equivalence is given by an essentially surjective fully faithful functor. The functor being fully faithful implies that D is thin, and, together with essential surjectivity, it also implies that D is essentially finite. Hence the tuple $\Phi \circ F$ is a poic-complex, and the identity natural transformation $\mathrm{Id} : (\Phi \circ F) \implies \Phi \circ F$ gives rise to a morphism of poic-complexes $\Phi \circ F \to \Phi$ where at each cone of $\Phi \circ F$ it is an (integral linear) isomorphism. Since F is essentially surjective and fully faithful, it follows from the previous fact that the induced map between the realizations is a homemorphism with the prescribed properties. □

We now define the notion of equivalence of poic-complexes. Subsequently, we show that equivalences induce isomorphisms between Minkowski weights and state how equivalences behave with respect to subdivisions.

Definition 6.1.2. We say that a morphism of poic-complexes $E : \Sigma \to \Phi$ is an *equivalence* if the underlying functor E is an equivalence, and η_E consists of isomorphisms. In this case, the morphism E induces an isomorphism between the corresponding realizations.

Lemma 6.1.3. *If $E\colon \Sigma \to \Phi$ is an equivalence and Φ_X is a linear poic-complex, then E is a proper morphism and for every integer $k \geq 0$ the linear map*

$$E_*\colon M_k(\Sigma_{X\circ E}) \to M_k(\Phi_X)$$

is an isomorphism. Moreover, if $S\colon \Sigma' \to \Sigma$ is a subdivision, then $S \circ E : \Sigma' \to \Phi$ is also a subdivision.

Proof. An equivalence $E\colon \Sigma \to \Phi$ establishes a bijection between isomorphism classes, and therefore the corresponding weight groups are isomorphic. Since $\eta_E\colon \Sigma \implies \Phi\circ F_E$ consists of isomorphisms, it follows that the balancing conditions are the same, hence the Minkowski weights are also isomorphic. The statement on subdivisions follows from the observation that an equivalence $E\colon \Sigma \to \Phi$ is in particular a subdivision, and the fact that composition of subdivisions is a subdivision. □

The following lemma is important and we will use it several times for several simplifications of future statements.

Lemma 6.1.4. *Any poic-complex Φ is equivalent to a poic-complex whose underlying category is a finite poset.*

Proof. Let Φ be an arbitrary poic-complex. By definition, the isomorphism classes $[\Phi]$ form a finite set. For each class $x \in [\Phi]$, let

$$\mathrm{S} = \{s_x : s_x \text{ is an object of } C_\Phi \text{ and } [s_x] = x\}$$

be a full set of representatives[1] of $[\Phi]$. Let $C_{\mathrm{S}(\Phi)}$ denote the full subcategory of C_Φ given by S. By construction, this category is actually a poset and the inclusion $I : C_{\mathrm{S}(\Phi)} \to C_\Phi$ is a fully faithful essentially surjective functor. Therefore, $\Phi \circ I$ is a poic-complex, and the inclusion morphism $\Phi \circ I \to \Phi$ is an equivalence . □

[1] This is a strong instance of the axiom of choice that can be justified in a usual way (see for instance [Ham]). If the reader prefers, we can also assume that every essentially finite category (that we consider) comes with a specified choice of representatives. This is something that can be assumed in our cases of interest.

Finally, we close this section by showing that equivalences between poic-complexes induce categorical equivalences between the corresponding subdivision categories. We use this to compare tropical cycles, as colimits between categorical equivalences are naturally isomorphic.

Lemma 6.1.5. *Suppose Σ and Φ are poic-complexes, where the underlying category of Σ is a given by a (finite) poset. A morphism $E\colon \Sigma \to \Phi$ that is an equivalence induces an equivalence of categories* $\mathrm{Subd}(\Sigma) \to \mathrm{Subd}(\Phi)$. *In addition, if Φ_X is a linear poic-complex, then the morphism induces an isomorphism $E_*\colon Z_k(\Sigma_{X\circ E}) \to Z_k(\Phi_X)$ for any integer $k \geq 0$.*

Proof. Observe that the following association is a well defined functor:

$$E_*\colon \mathrm{Subd}(\Sigma) \to \mathrm{Subd}(\Phi), S \mapsto S \circ E.$$

We want to show that E_* is an essentially surjective fully faithful functor. The fully faithfulness follows from the fact that E is an equivalence, so we focus on the essential surjectivity. For this, assume a subdivision $S'\colon \Phi' \to \Phi$ is given. We will define a subdivision $E^{-1}S'\colon E^{-1}\Phi' \to \Sigma$ such that $E^*\left(E^{-1}S'\right) \cong \Phi'$ in $\mathrm{Subd}(\Phi)$. Since E is an equivalence and C_Σ is a poset, the functor E establishes a bijection $b_E : C_\Sigma \to [\Phi]$. We let $b_G : [\Phi] \to C_\Sigma$ denote its inverse. Observe that for a cone s of Φ', there is a unique object $p \in C_\Sigma$ with $E(p) \cong S(s)$, namely $p = b_G([S(s)])$. In fact, this defines a functor

$$G : C_{\Phi'} \to C_\Sigma, s \mapsto b_G([S(s)]). \tag{6.1}$$

To construct the poic-complex $E^{-1}\Phi'$, we set $C_{E^{-1}\Phi'} = C_{\Phi'}$, and for an object $s \in C_{E^{-1}\Phi'}$, we define the poic $E^{-1}\Phi'(s)$ as the following fibre square

$$\begin{array}{ccc} & & \Phi'(s) \\ & & \downarrow{\scriptstyle \eta_{S,s}} \\ \Sigma(G(s)) & \xrightarrow[\eta_{E,G(s)}]{} & \Phi(S(s)). \end{array}$$

In other words, the poic $E^{-1}\Phi'(s)$ is the subcone of $E(G(s))$ given by the image of $\Phi'(s)$ under $\eta_{S,s}$. This definition is naturally functorial, so that $E^{-1}\Phi'$ is a poic-complex. The functor $G : C_{E^{-1}\Phi'} \to C_\Sigma$ and the natural inclusions define a subdivision $S^{-1}E : E^{-1}\Phi' \to \Sigma$. To finalize, we remark that the identity functor $C_{\Phi'} \to C_{\Phi'} = C_{E^{-1}\Phi'}$ and the natural transformation η_S (and its inverses defined on the corresponding image cones) define an isomorphism of subdivisions $E_*\left(E^{-1}\Phi'\right) \cong \Phi'$. The isomorphism between the cycle groups follows from Lemma 6.1.3 and this equivalence. □

6.2 Poic-complexes coming from partially open fans

In this section we seek to tie the loose ends of comparing the classical tropical cycles with our framework. Suppose N is a finite rank lattice and let V denote the real vector space $V := N \otimes_{\mathbb{Z}} \mathbb{R}$. A *partially open fan* Φ *in* V is a partially open polyhedral complex Φ in V (see [GO17] and [GMO17]) whose polyhedra consist of partially open integral cones of V. Naturally, any partially open fan in V gives rise to a poic-complex (with the tautological association), and, furthermore, to a linear poic-complex through the embedding map to V. We call this linear poic-complex structure on the partially open fan Φ in V the *natural linear poic-complex structure in* Φ. The support of a partially open fan Φ in V, is the subset $|\Phi| \subset V$ given by

$$|\Phi| := \bigcup_{\sigma \in \Phi} \sigma \subset V.$$

Definition 6.2.1. Suppose Φ is a partially open fan in V. A partially open fan Ψ in V is called an *honest subdivision of* Φ if any cone of Φ is a union of cones of Ψ and $|\Psi| = |\Phi|$. If Ψ is an honest subdivision of Φ, then mapping a cone of Ψ to the minimal cone of Φ where it lies determines a subdivision of the corresponding poic-complexes. We let HnstSubd(Φ) denote the category of honest subdivisions of Φ, where the maps are commuting subdivision maps.

Lemma 6.2.2. *Suppose Φ is a partially open fan in V. Any subdivision $S : \Phi' \to \Phi$ (where Φ' is an arbitrary poic-complex) is equivalent to an honest subdivision $\underline{\Phi'}$ of Φ. Furthermore, this association gives rise to an equivalence of categories*

$$\underline{\bullet} : \mathrm{Subd}(\Phi) \to \mathrm{HnstSubd}(\Phi).$$

Proof. A cone s of the poic-complex Φ' determines a subcone of $S(s) \in \Phi$, namely $\eta_{S,s}(\Phi'(s))$. We remark that if $s \cong s'$, then the corresponding cones are the same, and the collection

$$\{\eta_{S,s}(\Phi'(s)) : s \text{ is a cone of } \Phi'\}$$

is a finite set of partially open integral cones of V. The conditions on subdivisions of poic-complexes imply that this collection is a partially open fan $\underline{\Phi'}$ in V that is also an honest subdivision of Φ. Let $I : \underline{\Phi'} \to \Phi$ denote the subdivision map induced by the honest subdivision. The association

$$E_{\Phi'} : C_{\Phi'} \to \underline{\Phi'}, s \mapsto \eta_{S,s}(\Phi'(s)),$$

where the morphisms of Φ' are mapped to the corresponding face relations, is a func-

tor, which must be an equivalence of categories. It can readily be observed that $I \circ E_{\Phi'} = S$, and, in addition, the identity maps give rise to a natural transformation $\eta_{E_{\Phi'}} : \Phi' \Longrightarrow \underline{\Phi'} \circ E_{\Phi'}$, which actually consists of isomorphisms. Hence, the tuple $E_{\Phi'} \colon \Phi' \to \underline{\Phi'}$ is the sought-after equivalence. Now, by construction, this association gives actually a functor

$$\underline{\bullet} \colon \mathrm{Subd}(\Phi) \to \mathrm{HnstSubd}(\Phi),$$

that is full and faithful (hence an equivalence of categories). In addition, if $S : \Phi' \to \Phi$ was an honest subdivision, then $\underline{\Phi'}$ is this same honest subdivision. Thus it is also essentially surjective, and therefore an equivalence. □

It is clear that if Φ is a partially open fan, then HnstSubd(Φ) is a small category (its objects consist of different finite sets of subsets of a vector space). We close by finally proving Lemma 5.1.3. The proof of Lemma 5.1.4 is postponed a bit further.

Lemma 5.1.3. *The tropical cycles of the linear poic-complex $\Phi_{\mathcal{X}}$ correspond with the classical tropical cycles of $\mathcal{X}$.*

Proof. It is sufficient to consider honest subdivisions of $\Phi_{\mathcal{X}}$ to compute its tropical cycles. An arbitrary honest subdivision of $\Phi_{\mathcal{X}}$ gives rise to a subdivision of $\mathcal{X}$ by taking its intersection with the affine hyperplane $z = 1$. Honest subdivisions of $\Phi_{\mathcal{X}}$ are given by partially open fans in the underlying vector space, therefore intersecting and keeping the corresponding weights produces a Minkowski weight of one dimension less. For the other direction, we recall the well known fact that the recession cones of an arbitrary polyhedral complex do not necessarily form a fan. Nevertheless, it is always possible to produce a honest subdivision (see [BGS11]) of it, whose recession cones actually form a fan. Taking such an appropriate subdivision and extending the weights correspondingly (namely, following the notation of the construction we can set $\omega(\sigma_\xi) := \omega(\xi)$), we produce in this manner a Minkowski weight of $\Phi_{\mathcal{X}}$ of one dimension higher. These previous correspondences are inverse to each other. □

6.3 The ord(Φ) construction

For the sake of simplicity, we assume momentarily that any two zero-dimensional cones of a poic-complex are isomorphic. This is not really a restrictive condition, it really simplifies our work in this subsection, and later at the end we explain what happens when this condition is dropped. We will explain how to construct, for an arbitrary poic-complex whose underlying category is a poset, a subdivision that is a partially open simplicial fan in a specific vector space. This subdivision has the property that at each cone of the poic-complex we would, basically, end up with the barycentric

subdivision. We first deal with the case where Φ consists of closed convex cones, and then explain how to do the general case.

Notation 6.3.1. For a closed simplicial cone[2] σ, we let n_σ denote the vector lying in the interior of the underlying polyhedral cone given by the sum of the integral generators of the rays of σ.

Construction 6.3.2. Suppose Φ is a poic-complex of closed convex polyhedral cones, and whose underlying category is a poset (in this case the underlying set is the set of isomorphism classes). We let $\inf\Phi$ denote the unique zero dimensional cone of Φ. Let $C_{\mathrm{ord}(\Phi)}$ denote the poset of totally ordered chains of C_Φ of non-trivial cones, and let $N_{\mathrm{ord}(\Phi)}$ denote the free abelian group on Φ, that is $N_{\mathrm{ord}(\Phi)} := \mathbb{Z}^\Phi$. We set $V_{\mathrm{ord}(\Phi)} := N_{\mathrm{ord}(\Phi)} \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^\Phi$ and construct a simplicial fan in this vector space. Consider the following closed integral polyhedral cones of $V_{\mathrm{ord}(\Phi)}$ defined by the chains in $C_{\mathrm{ord}(\Phi)}$:

- At chains $\{s\}$ of length 1, we define σ_s as the one dimensional closed integral polyhedral cone of $V_{\mathrm{ord}(\Phi)}$ generated by the basis vector corresponding to s.
- At chains $c = \{s_1 < s_2 < \cdots < s_n\}$, we define $\sigma_{s_1<\cdots<s_n}$ as the closed integral polyhedral cone of $V_{\mathrm{ord}(\Phi)}$ generated by the rays $\sigma_{s_1}, \ldots, \sigma_{s_n}$.

Containment relations of chains in $C_{\mathrm{ord}(\Phi)}$ become face relations of the corresponding cones. Let $\underline{o}$ denote the closed integral polyhedral cone of $V_{\mathrm{ord}(\Phi)}$ given by the origin. By construction, the collection

$$\mathrm{ord}(\Phi) := \{\sigma_c : c \in C_{\mathrm{ord}(\Phi)}\} \cup \{\underline{o}\}$$

is a fan in $V_{\mathrm{ord}(\Phi)}$. Notice that for positive k, the k-dimensional cones of $\mathrm{ord}(\Phi)$ correspond to the k-length chains of Φ. We remark that the function

$$\max : \mathrm{ord}\,\Phi \to \Phi, \begin{cases} \sigma_c \mapsto \max c, & c \in C_{\mathrm{ord}(\Phi)}, \\ \underline{o} \mapsto \inf\Phi, \end{cases}$$

is actually a functor. We upgrade this functor to a morphism of poic-complexes by defining the natural transformation as follows:

- At the trivial cone $\underline{o}$, it is defined as the only possible morphism between zero dimensional poics.
- At the one-dimensional cones σ_s with $s \in \Phi$, it is defined as the linear map $\eta_{\max,\sigma_s} : \sigma_s \to \Phi(s)$ given by mapping the generator of this ray to the vector $n_{\Phi(s)}$.

[2] We refrain from using our term poic, as to not convey the partially openness.

- At chains $c = \{s_1 < \cdots < s_n\}$ with $n > 1$, it is defined as the linear map

$$\eta_{\max,\sigma_c} : \sigma_c \to \Phi(\max c),$$

defined by the compositions $\sigma_{s_i} \xrightarrow{\eta_{\max,s_i}} \Phi(s_i) \xrightarrow{\Phi(s_i \leq \max c)} \Phi(\max c)$.

With the above, we obtain a morphism of poic-complexes $\max : \mathrm{ord}(\Phi) \to \Phi$.

Lemma 6.3.3. *If Φ is a poic-complex of closed convex polyhedral cones whose underlying category is a poset, then the morphism $\max : \mathrm{ord}(\Phi) \to \Phi$ is a subdivision.*

Proof. The above construction provides the barycentric subdivison at each cone of Φ. □

We use the above construction to obtain a similar result for poic-complexes whose underlying categories are posets. The idea is to first construct a poic-complex of closed convex polyhedral cones whose underlying category is also a poset that additionally contains our starting poic-complex as a full subcategory (and as subcones), then perform the previous construction in the new one, and finally just remove faces accordingly.

Construction 6.3.4. Suppose Φ is a poic-complex whose underlying category is a poset. For each cone $s \in \Phi$, let $\overline{\Phi(s)}$ denote the closed poic defined by $\Phi(s)$. For a cone s of Φ, we define $\mathrm{Add}(s)$ as the set of faces $\tau \leq \overline{\Phi(s)}$ ($\mathrm{Add}(s)$ stands for additional faces of s) subject to the condition: there does not exist $t \lneq s$ such that τ is a face of the image of $\overline{\Phi(t)}$ under $\Phi(t \lneq s)$. These come with a natural relation given by inclusion of faces.
We now define the poset $C_{\overline{\Phi}}$ inductively as follows. Let $n = \max\{\dim s : s \in C_\Phi\}$, set $C_0 = C_\Phi$ and

$$C_{i+1} := C_i \cup \bigcup_{s \in C_\Phi, d(s)=i+1} \mathrm{Add}(s),$$

so that $C_{\overline{\Phi}} := C_n$, and its order relation is defined by extension of the following:

- the order relation of C_Φ.
- the order relation (face relation) of each $\mathrm{Add}(s)$, for each cone s of Φ .
- setting that $\tau \leq s$, for every $\tau \in \mathrm{Add}(s)$ and $s \in C_\Phi$.

We now set

$$\overline{\Phi} : C_{\overline{\Phi}} \to \mathrm{POIC}, p \mapsto \begin{cases} \overline{\Phi(p)}, & p \in C_\Phi, \\ p, & p \in \mathrm{Add}(s), \text{ for some } s \in C_\Phi, \end{cases}$$

this preserves the face relations, and hence makes the tuple $\overline{\Phi} := (C_{\overline{\Phi}}, \overline{\Phi})$ into a poic-complex. Furthermore, this is a poic-complex of closed polyhedral cones whose underlying category is a poset. To finalize, we consider $\mathrm{ord}(\overline{\Phi})$, which is a fan that subdivides $\overline{\Phi}$, and take the corresponding partially open integral cones to obtain a partially open fan that subdivides Φ.

In general, when Φ is a poic-complex whose underlying category is a poset, we will refer to this construction as the $\mathrm{ord}(\Phi)$ *construction*. For arbitrary poic-complexes, namely if we drop the assumption on zero dimensional cones, we obtain an identical result by looking at the maximal poic-subcomplexes that satisfy this property. There are finitely many, and for each of these we would obtain such a subdivision.

6.4 Proofs of Lemmas 2.4.6, 5.1.4, 2.5.9, and 3.3.4

Lemma 2.4.6. *For any poic-complex Φ there exists a subdivision $S\colon \Phi' \to \Phi$ such that the category* $\mathrm{Subd}(\Phi')$ *is essentially small.*

Proof. From Lemmas 6.1.4 and 6.1.5, we can assume without loss of generality that Φ is a poic-complex whose underlying category is a poset. To finalize, we apply the $\mathrm{ord}(\Phi)$ construction and Lemma 6.2.2 (we must look at the maximal poic-subcomplexes that satisfy the assumption on zero dimensional cones, but there can only be finitely many) and obtain that $\mathrm{Subd}(\mathrm{ord}(\Phi))$ is essentially small. □

Lemma 5.1.4. *The tropical cycles of a weakly embedded cone complex Σ correspond with the tropical cycles of the linear poic-complex* $\mathrm{p}(\Sigma)$.

Proof. We observe that $\mathrm{ord}(\mathrm{p}(\Sigma))$ is a closed fan, so in particular it is a cone complex. The same morphism ϕ^{Σ} makes it also into a weakly embedded cone complex. We observe that $\mathrm{ord}(\mathrm{p}(\Sigma))$ is isomorphic (as weakly embedded cone complexes) to the barycentric subdivision of Σ, and hence have isomorphic groups of tropical cycles of weakly embedded cone complexes. Now, proper subdivisions of this weakly embedded cone complex $\mathrm{ord}(\mathrm{p}(\Sigma))$ are the same as honest subdivisions of the underlying fan. Therefore, it follows from Lemma 6.2.2 that the group of tropical cycles of this weakly embedded cone complex is isomorphic to the group of tropical cycles of this fan. □

Lemma 2.5.9. *If $P\colon \Phi \to \Psi$ is a morphism, then there exists a P-fine subdivision $Q\colon \Psi' \to \Psi$.*

Proof. For this we first apply Lemma 6.1.4 to Ψ, so that we can assume without loss of generality that the underlying category of Ψ is a poset. In this case we can proceed exactly as in Construction 2.24 of [GKM09], where we kindly point the reader towards. □

Lemma 3.3.4. *Suppose $\pi\colon \Phi \to \mathcal{X}$ is a poic-fibration. If $S : \Phi' \to \Phi$ is a subdivision, then there exists a subdivision $S' : \Phi'' \to \Phi'$ such that $S \circ S' : \Phi'' \to \Phi$ is π-compatible. In other words, π-compatible subdivison are final as subdivisions of Φ.*

Proof. Observe first that if $E : \Sigma \to \Phi$ is an equivalence, then:

- $\pi \circ E\colon \Sigma \to \mathcal{X}$ is a poic-fibration.
- Any $(\pi \circ E)$-compatible subdivision is, after pre-composition with E (see Lemma 6.1.3) a π-compatible subdivision.

From Lemma 6.1.4, it follows that there is an equivalence $E\colon \Sigma \to \Phi$ where the underlying category of Σ is a poset. We can then apply the functor E^* from the proof of Lemma 6.1.5 to obtain a subdivision $E^*S\colon E^*\Phi' \to \Sigma$ such that $(E^*S) \circ E \cong S$, and apply again Lemma 6.1.4 to obtain an equivalence $E'\colon \Sigma' \to E^*\Phi'$. Notice that the composition $E^*S \circ E'$ is then a subdivision $S_E : \Sigma' \to \Sigma$ where the underlying categories of both poic-complexes are finite posets. Since any subdivision $S'_E : \Sigma'' \to \Sigma'$ gives rise to a subdivision $E' \circ S'_E\colon \Sigma'' \to E^*\Phi'$ and $S'_E \circ E \cong S$, it follows that it suffices to show the lemma when the underlying categories of both Φ' and Φ are finite posets.
We proceed as follows in this case. Let $n = \max_{s\in\Phi} d(s)$, and observe that for $s \in \Phi$, the subdivision $S^{-1}(s)$ can be regarded as a full subcategory of Φ'. We use these identifications indiscriminately throughout the proof, and is best to keep them in mind. Since the underlying category of Φ is a poset, the set of isomorphism classes $[\Phi]$ is simply Φ. The poic-fibration $\pi\colon \Phi \to \mathcal{X}$ induces a partition of Φ indexed by $[\mathcal{X}]$:

$$\Phi = \bigsqcup_{[x]\in\mathcal{X}} \Phi_{[x]},$$

where for $[x] \in \mathcal{X}$, the set $\Phi_{[x]}$ consists of the $s \in \Phi$ with $[\pi(s)] = [x]$. Now, we make the following choices:

- For each $[x] \in [\mathcal{X}]$ a $s_{[x]} \in \Phi_{[x]}$.
- For every $s \in \Phi_{[x]}$ a fixed isomorphism $f_s\colon \pi(s_{[x]}) \to \pi(s)$ of $\mathcal{X}$.

We will construct subdivisions $S_i : \Phi'_i \to \Phi(\leq i)$ (for $2 \leq i \leq \max_{s\in\Phi} d(s)$) inductively by dimension, with the goal that Φ'_i is stable under isomorphisms between cones of $\mathcal{X}$ of dimension $\leq i$.

- For $i = 2$, we consider $[x] \in [\mathcal{X}](2)$. The automorphisms of $\pi(s_{[x]})$ act on $\Phi(s_x)^0$, and the subdivision $S^{-1}(s_x)$ can then be pushed through any of these automorphisms of $\pi(s_{[x]})$. This gives multiple new subdivisions of $\Phi(s_{[x]})^0$, and taking the intersection of all of these produces a subdivision (of $\Phi(s_{[x]})^0$) that will be stable under the automorphisms of $\pi(s_{[x]})$. We do this for every $[x] \in [\mathcal{X}](2)$, and

then transport these subdivisions to all the $s \in \Phi$ with $d(s) = 2$ by means of the maps induced by the isomorphisms $f_s : \pi(s) \to \pi(s_{[\pi(s)]})$. Keeping track of the corresponding face relations gives rise to a poset $C_{\Phi_2'}$, and by taking the corresponding subcones we obtain from this poset a poic-complex Φ_2' with a subdivision $S_2 \colon \Phi_2' \to \Phi'(\leq 2)$. This subdivision is already stable with respect to isomorphisms between 2-dimensional cones of $\mathcal{X}$ by construction (any isomorphism will factor as a composition of the f_s with some automorphism of the $\pi(s_x)$). The reason for dimension 2 is just that this is the smallest where the previous construction is non-trivial.

- Given $S_i \colon \Phi_i' \to \Phi'(\leq i)$, we construct Φ_{i+1}'. First, we extend S_i to a subdivision of $\Phi'(\leq i+1)$ by making a stellar subdivision[3] (along arbitrary rays) on every $(i+1)$-dimensional cone of Φ' and linearly extending from the faces inwards. This produces a subdivision $\hat{S}_i \colon \hat{\Phi}_i' \to \Phi'(\leq i+1)$. Since stellar subdivisions do not change the boundary of the cones, this $\hat{S}_i$ is still stable under isomorphisms between objects of $\mathcal{X}$ of dimension $\leq i$. Consider now $[x] \in [\mathcal{X}](i+1)$ and let us mimic our previous procedure. The automorphisms of $\pi(s_{[x]})$ act on $\Phi(s_{[x]})^0$, and we push through the subdivision $(S \circ \hat{S}_i)^{-1}(s_{[x]})$. This gives multiple new subdivisions of $\Phi(s)^0$, which we intersect and obtain a subdivision of $\Phi(s_{[x]})^0$ stable under the automorphisms of $\pi(s_{[x]})$. We repeat this for every $[x] \in [\mathcal{X}](i+1)$, and then transport these subdivisions to all the $s \in \Phi$ with $d(s) = i+1$ by means of the maps induced by the isomorphisms $f_s : \pi(s) \to \pi(s_{[\pi(s)]})$. Since $\hat{\Phi}_i'$ is stable under isomorphisms between objects of $\mathcal{X}$ of dimension $\leq i$, it follows that these subdivisions can be extended to the whole $\Phi(s)$. By keeping track of the corresponding face relations we obtain a finite poset $C_{\Phi_{i+1}'}$, and taking corresponding subcones gives rise to a poic-complex Φ_{i+1}' with a subdivision $S_{i+1} \colon \Phi_{i+1}' \to \Phi'(\leq i+1)$. By construction, this subdivision is necessarily stable under isomorphisms between objects of $\mathcal{X}$ of dimension $\leq i+1$.

□

[3] Also called star subdivisions in the literature.

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